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Solutions to the discrete Boltzmann equation with general boundary conditions

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Abstract. We study the initial-boundary value problems and the corresponding stationary problems of the one-dimensional discrete Boltzmann equation in a bounded region. The boundary conditions considered are of mixed type and involve both the reflection and diffusion parts. It is shown that a unique solution to the initial-boundary value problem exists globally in time under the general situation that the reflection parts of both the boundary conditions do not increase the number of gas particles. Furthermore, it is proved that stationary solutions exist under the restriction that the reflection part of the boundary condition on one side really decreases the number of gas particles. This restriction plays an essential role in proving the existence result.

1. Introduction.

This paper is concerned with the study of the global existence and uniqueness of solutions to the initial-boundary value problems for the one-dimensional discrete Boltzmann equation in a bounded region $0 < x < 1$, and also of the existence of solutions to the corresponding stationary problems.

We consider the following initial-boundary value problem for the one-dimensional discrete Boltzmann equation in the region $0 < x < 1$:

$$(1.1)\quad \begin{cases} \dfrac{\partial F}{\partial t} + V\dfrac{\partial F}{\partial x} = A(F), \quad 0 < x < 1, \quad t > 0, & (1.1)_1 \\ F(x,0) = F_0(x), \quad 0 < x < 1, & (1.1)_2 \\ F^+(0,t) = B^+F^-(0,t) + b^+(t), \quad t > 0, & (1.1)_3 \\ F^-(1,t) = B^-F^+(1,t) + b^-(t), \quad t > 0, & (1.1)_4 \end{cases}$$

where

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_m \end{pmatrix}, \quad A(F) = \begin{pmatrix} A_1(F) \\ \vdots \\ A_m(F) \end{pmatrix}, \quad F_0 = \begin{pmatrix} F_{10} \\ \vdots \\ F_{m0} \end{pmatrix}.$$

Here each $F_i = F_i(x,t)$ denotes the mass density of gas particles with the (constant) i-th velocity at time t and position x, v_i is the x-component of the i-th velocity, and $A_i(F)$ is a collision term given explicitly as

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$$A_i(F) = \frac{1}{c_i}\sum_{jkl}(A_{kl}^{ij}F_kF_l - A_{ij}^{kl}F_iF_j), \tag{1.2}$$

where the summation is taken over all $j,k,l = 1,\dots,m$; c_i is a positive constant and A_{kl}^{ij} are nonnegative constants. Let $\Lambda = \{1,\dots,m\}$ and put

$$\Lambda_+ = \{i \in \Lambda; v_i > 0\}, \quad \Lambda_- = \{i \in \Lambda; v_i < 0\}, \quad \Lambda_0 = \{i \in \Lambda; v_i = 0\}. \tag{1.3}$$

Then $F^{\pm} = (F_i)_{i\in\Lambda_\pm}$, $B^{\pm} = (B_{ij})_{i\in\Lambda_\pm, j\in\Lambda_\mp}$ with B_{ij} being nonnegative constants, and $b^{\pm} = (b_i)_{i\in\Lambda_\pm}$ with $b_i = b_i(t)$ being nonnegative functions. In what follows, without loss of generality, we may assume that

$$\Lambda_+ = \{1,\dots,m_+\}, \quad \Lambda_- = \{m_+ + 1,\dots,m_+ + m_-\}, \quad \Lambda_0 = \{m_+ + m_- + 1,\dots,m\}$$

and put $m' = m_+ + m_-$ and $m = m_+ + m_- + m_0$. Also, we put $\Lambda' = \Lambda_+ \cup \Lambda_-$ and

$$B = \begin{pmatrix} O & B^+ \\ B^- & O \end{pmatrix}, \quad b = \begin{pmatrix} b^+ \\ b^- \end{pmatrix}, \quad F' = \begin{pmatrix} F^+ \\ F^- \end{pmatrix}. \tag{1.4}$$

Throughout the paper we assume the following conditions (A) and (B) for the problem (1.1):

(A) The A_{kl}^{ij} are nonnegative constants satisfying

(A1) $A_{kl}^{ji} = A_{kl}^{ij} = A_{lk}^{ij}$ for any $i,j,k,l \in \Lambda$,

(A2) $A_{kl}^{ij} = A_{ij}^{kl}$ for any $i,j,k,l \in \Lambda$,

(A3) $v_i + v_j - v_k - v_l = 0$ for any $i,j,k,l \in \Lambda$ such that $A_{kl}^{ij} \neq 0$.

(B) The B_{ij} are nonnegative constants such that

$$(c_1v_1,\dots,c_{m'}v_{m'})\begin{pmatrix} B^+ \\ I \end{pmatrix} \le O, \quad (c_1v_1,\dots,c_{m'}v_{m'})\begin{pmatrix} I \\ B^- \end{pmatrix} \ge O.$$

Here, for a $k \times n$ matrix A, $A \ge O$ (resp. $A > O$) means that every element of A is nonnegative (resp. strictly positive). $A \ge B$ and $A > B$ are defined similarly. We note that $(A1)$ represents the indistinguishability of gas particles, $(A2)$ the reversibility of collisions, and $(A3)$ the conservation of momentum (in the x-direction) in the collision process (or, in other words, $v = (v_i)_{i\in\Lambda}$ is a collision invariant). The inequalities in (B) are rewritten as

$$c_jv_j + \sum_{i\in\Lambda_+} c_iv_iB_{ij} \le 0 \quad \text{for any } j \in \Lambda_-, \tag*{$(1.5)_1$}$$

$$c_jv_j + \sum_{i\in\Lambda_-} c_iv_iB_{ij} \ge 0 \quad \text{for any } j \in \Lambda_+. \tag*{$(1.5)_2$}$$

These inequalities imply that the reflection parts of the boundary conditions $(1.1)_3$ and $(1.1)_4$ do not increase the number of gas particles.

The stationary problem corresponding to the above problem (1.1) is

$$(1.6)\qquad \begin{cases} V\dfrac{dF}{dx} = A(F), \quad 0 < x < 1, & (1.6)_1 \\ F^+(0) = B^+F^-(0) + b^+, & (1.6)_2 \\ F^-(1) = B^-F^+(1) + b^-. & (1.6)_3 \end{cases}$$

For this stationary problem we assume, in addition to (A), the following condition that is stronger than (B).

$$(\mathrm{B}')\qquad (c_1v_1, \ldots, c_{m'}v_{m'})\begin{pmatrix} B^+ \\ I \end{pmatrix} < O, \quad (c_1v_1, \ldots, c_{m'}v_{m'})\begin{pmatrix} I \\ B^- \end{pmatrix} \geq O.$$

The case where

$$(c_1v_1, \ldots, c_{m'}v_{m'})\begin{pmatrix} B^+ \\ I \end{pmatrix} \leq O, \quad (c_1v_1, \ldots, c_{m'}v_{m'})\begin{pmatrix} I \\ B^- \end{pmatrix} > O$$

hold can be treated similarly. We note that the first inequality in (B$'$) implies that the reflection part of $(1.6)_2$ really decreases the number of gas particles. We also impose the following technical condition that is introduced in [**10**].

(C) There exists a mapping π, which is a C^∞-mapping from $(0,\infty)^{m'}$ to $(0,\infty)^{m_0}$ and is a C^0-mapping from $[0,\infty)^{m'}$ to $[0,\infty)^{m_0}$, such that $F^0 = \pi(F')$ satisfies $A_i(F) = 0$ for any $i \in \Lambda_0$.

Here $F^0 = (F_i)_{i\in\Lambda_0}$. This condition ensures the solvability of $A_i(F) = 0, i \in \Lambda_0$, w.r.t. F^0.

The initial-boundary value problems for the one-dimensional discrete Boltzmann equation in a bounded region were discussed in [**8**], [**9**] when the boundary condition is given by

$$F^+(0,t) = b^+(>0), \quad F^-(1,t) = b^-(>0)$$

or

$$F^+(0,t) = b^+(>0), \quad F^-(1,t) = B^-F^+(1,t)$$

or

$$F^+(0,t) = B^+F^-(0,t), \quad F^-(1,t) = B^-F^+(1,t).$$

It is proved that for any case of these boundary conditions the corresponding problem admits a unique global solution ([**8**], [**9**]). Similar global existence results are known also for the half-space problems ([**9**]). The stationary problems are studied by Cercignani, Illner, Shinbrot [**3**]. They proved the existence of stationary solutions in the region $0 < x < 1$ when the boundary condition is given by

$$F^+(0) = b^+(>0), \quad F^-(1) = b^-(>0)$$

or

$$F^+(0) = b^+(>0), \quad F^-(1) = B^-F^+(1).$$

They proved the result under the assumption that Λ_0 is an empty set but this technical assumption was removed by Kawashima [**10**]. In this paper we study the case where the boundary condition is in a general form. Our boundary conditions are of mixed type and involve both the reflection and diffusion parts. We prove the global existence and uniqueness of solutions to the initial-boundary value problem (1.1) in the general situation that the reflection parts of both the boundary conditions do not increase the number of gas particles (B). This is an improvement on the results in [**8**], [**9**] and the precise statement is given in Theorem 2.1. On the other hand, for the stationary problem (1.6), we prove the existence (without uniqueness) of solutions under the restriction that the reflection part of the boundary condition on one side really decreases the number of gas particles (B$'$). This is a generalization of the results in [**3**], [**10**] and is stated in Theorem 2.2.

The contents of this paper are as follows. In section 2, we state our main theorems. In section 3, we prove the global existence and uniqueness of solutions to the problem (1.1) by a combination of the local existence result and the a priori estimates. Our a priori estimates are derived essentially in the same way as in [**8**], [**9**] and are based on the conservation equations and the Boltzmann H-theorem. In section 4, we show the existence of solutions to the stationary problem (1.6) by applying the fixed point theorem of the Leray-Schauder type. The proof is similar to the one in [**10**] but a new technical consideration is needed in solving the linearized stationary problem.

Notations. For a nonnegative integer k and a region Ω, we denote by $C^k(\Omega)$ the space of k-times continuously differentiable functions on Ω. We denote by $\mathbf{1}_{k,n}$ the $k \times n$ matrix whose entries are all equal to one, i.e., $\mathbf{1}_{k,n} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$. We use the abbreviation $\mathbf{1}_k = \mathbf{1}_{k,1}$. Let $f = (f_i)$ be an n - vector and let $A = (a_{ij})$ be a $k \times n$ matrix. We put

$$|f| = \max_i |f_i|, \quad |A| = \max_{f \neq 0} \frac{|Af|}{|f|}.$$

Note that when $A \geq O$, we have $|A| = |A\mathbf{1}_n| = \max_i \sum_j a_{ij}$.

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2. Main results.

First we formulate the compatibility conditions up to order one for the initial-boundary value problem (1.1). The said conditions are

$$F_0^+(0) = B^+F_0^-(0) + b^+(0), \quad F_0^-(1) = B^-F_0^+(1) + b^-(0), \tag{2.1$_1$}$$

$$F_1^+(0) = B^+F_1^-(0) + \frac{db^+}{dt}(0), \quad F_1^-(1) = B^-F_1^+(1) + \frac{db^-}{dt}(0), \tag{2.1$_2$}$$

where

$$F_1^{\pm} = -V^{\pm}\frac{dF_0^{\pm}}{dx} + A^{\pm}(F_0),$$

where $V^{\pm} = \mathrm{diag}(v_i)_{i\in\Lambda_{\pm}}$, and $A^{\pm}(F) = (A_i(F))_{i\in\Lambda_{\pm}}$. The initial-boundary value problem (1.1) is then solved globally in time as follows.

Theorem 2.1. *We assume* (A) *and* (B). *Suppose that* $F_0 \in C^1[0,1]$ *and* $F_0(x) > O$ *for* $x \in [0,1]$ *and that* $b \in C^1[0,\infty)$ *and* $b(t) \geq O$,

$$B\mathbf{1}_{m'} + b(t) > O \tag{2.2}$$

for $t \in [0,\infty)$, *where* B *and* $b(t)$ *are given in* (1.4). *Moreover, we assume the compatibility conditions* (2.1). *Then the problem* (1.1) *has a unique global solution* $F = (F_i)_{i\in\Lambda}$ *satisfying* $F \in C^1([0,1]\times[0,\infty))$ *and* $F(x,t) > O$ *for* $(x,t) \in [0,1]\times[0,\infty)$.

Next, we state an existence theorem for the stationary problem (1.6), which is as follows.

Theorem 2.2. *We assume* (A), (B') *and* (C). *Suppose that* $b \geq O$ *and*

$$(I-B)^{-1}b > O. \tag{2.3}$$

Then the stationary problem (1.6) *has a solution* $F = (F_i)_{i\in\Lambda}$ *satisfying* $F \in C^{\infty}[0,1]$ *and* $F(x) > O$ *for* $x \in [0,1]$.

Remark 2.1. That the matrix $I - B$ is invertible follows from (B'). In fact, when (B') is assumed, the absolute value of any eigenvalue of B is strictly less than 1, which will be proved in Lemma 4.2.

Remark 2.2. Let $b \geq O$. Then the condition (2.3) implies $B\mathbf{1}_{m'} + b > O$ so that (2.3) is stronger than (2.2).

In the proof of Theorems 2.1 and 2.2 we make use of several basis properties in discrete kinetic theory. Here we review these properties (see [**2**], [**5**], [**7**]). Let $A_i(F,G)$ be the bilinear form associated with the collision term $A_i(F)$ in (1.2), that is,

$$A_i(F,G) = \frac{1}{2c_i}\sum_{jkl}\{A_{kl}^{ij}(F_kG_l + F_lG_k) - A_{ij}^{kl}(F_iG_j + F_jG_i)\}, \quad i \in \Lambda.$$

Put $A(F,G) = (A_i(F,G))_{i\in\Lambda}$. Then $A(F,F) = A(F)$. We denote by $\mathscr{M}_0$ the space of collision invariants; $\mathscr{M}_0$ consists of vectors $\phi = (\phi_i)_{i\in\Lambda} \in \boldsymbol{R}^m$ satisfying

$$\phi_i + \phi_j - \phi_k - \phi_l = 0 \quad \text{for any } i,j,k,l \in \Lambda \quad \text{such that } A_{kl}^{ij} \neq 0.$$

Note that $\mathbf{1}_m$ and $v \equiv (v_i)_{i\in\Lambda}$ belong to $\mathscr{M}_0$, which is a consequence of (A3).

The following lemma is due to [**5**], [**7**].

Lemma 2.3. *We assume* (A). *Then the following three conditions are equivalent to each other.*

(i) $\phi = (\phi_i)_{i\in\Lambda} \in \mathscr{M}_0$.

(ii) $\langle \phi, I_c A(F,G) \rangle = 0$ *for all* $F, G \in \boldsymbol{R}^m$.

(iii) $\langle \phi, I_c A(F) \rangle = 0$ *for all* $F \in \boldsymbol{R}^m$.

Here $I_c = diag(c_i)_{i \in \Lambda}$ *and* $\langle \, , \, \rangle$ *denotes the standard inner product of* $\boldsymbol{R}^m$.

3. Global solutions.

In this section, we prove Theorem 2.1 by showing a local existence theorem (Proposition 3.8) and the a priori L^∞-estimates (Proposition 3.7) to the problem (1.1).

3.1. Entropy estimates and L^1-estimates on characteristics.

We first note that, since $\mathbf{1}_m$ and $v = (v_i)_{i \in \Lambda}$ are collision invariants, by virtue of Lemma 2.3, we have the conservation equations:

$$\frac{\partial}{\partial t}\left(\sum_i c_i F_i\right) + \frac{\partial}{\partial x}\left(\sum_i c_i v_i F_i\right) = 0, \tag{3.1}$$

$$\frac{\partial}{\partial t}\left(\sum_i c_i v_i F_i\right) + \frac{\partial}{\partial x}\left(\sum_i c_i {v_i}^2 F_i\right) = 0. \tag{3.2}$$

Making use of these conservation equations, we have the following L^1-estimates.

LEMMA 3.1. *We assume* (*A*) *and* (*B*). *Let* $T > 0$ *and let* $F = (F_i)_{i \in \Lambda}$ *be a solution to the problem* (1.1) *such that* $F \in C^1([0,1] \times [0,T])$ *and* $F(x,t) > O$ *for* $(x,t) \in [0,1] \times [0,T]$. *Then we have*

$$\int_0^1 |F(x,t)|\,dx + \int_0^t |F'(0,\tau)|\,d\tau + \int_0^t |F'(1,\tau)|\,d\tau$$
$$\leq C_1(1+T)\left(\int_0^t |b(\tau)|\,d\tau + \int_0^1 |F_0(x)|\,dx\right), \quad t \in [0,T], \tag{3.3}$$

where C_1 *is a positive constant independent of* T.

PROOF. Put

$$I_1(t) = \int_0^t |b(\tau)|\,d\tau + \int_0^1 |F_0(x)|\,dx. \tag{3.4}$$

We integrate (3.1) over $[0,1] \times [0,t]$ and use the boundary conditions $(1.1)_3$ and $(1.1)_4$, obtaining

$$\int_0^1 \sum_i c_i F_i(x,t)\,dx + \int_0^t \sum_{j \in \Lambda_-}\left(c_j|v_j| - \sum_{i \in \Lambda_+} c_i v_i B_{ij}\right) F_j(0,\tau)\,d\tau$$
$$+ \int_0^t \sum_{j \in \Lambda_+}(c_j v_j - \sum_{i \in \Lambda_-} c_i |v_i| B_{ij}) F_j(1,\tau)\,d\tau$$
$$= \int_0^t \sum_{i \in \Lambda'} c_i |v_i| b_i(\tau)\,d\tau + \int_0^1 \sum_i c_i F_{i0}(x)\,dx.$$

Since the second and third terms on the left hand side are nonnegative by (1.5), we have

$$\int_0^1 |F(x,t)|\,dx \le CI_1(t), \tag{3.5}$$

where and what follows C denotes a positive constant. Next, we choose a function $\chi(x) \in C_0^\infty(R)$ such that

$$\chi(x) = \begin{cases} 1 & \left(|x| \le \dfrac{1}{3}\right) \\ 0 & (|x| \ge 1), \end{cases}$$

and multiply (3.2) by $\chi(x)$. This yields

$$\frac{\partial}{\partial t}\left(\chi \sum_i c_i v_i F_i\right) + \frac{\partial}{\partial x}\left(\chi \sum_i c_i v_i^2 F_i\right) = \frac{\partial \chi}{\partial x} \sum_i c_i v_i^2 F_i. \tag{3.6}$$

Integrating (3.6) over $[0,1] \times [0,t]$ gives

$$\begin{aligned} \int_0^t |F'(0,\tau)|\,d\tau &\le C\left\{\int_0^1 |F_0(x)|\,dx + \int_0^1 |F(x,t)|\,dx + \int_0^t \int_0^1 |F(x,\tau)|\,dx d\tau\right\} \\ &\le C(1+T)I_1(t), \end{aligned} \tag{3.7}$$

where we used (3.5). Furthermore, integrating (3.2) over $[0,1] \times [0,t]$ and using (3.5) and (3.7), we obtain

$$\begin{aligned} \int_0^t |F'(1,\tau)|\,d\tau &\le C\left\{\int_0^1 |F_0(x)|\,dx + \int_0^1 |F(x,t)|\,dx + \int_0^t |F'(0,\tau)|\,d\tau\right\} \\ &\le C(1+T)I_1(t). \end{aligned} \tag{3.8}$$

Thus we have shown the desired estimate (3.3). This completes the proof. □

Secondly, we derive the entropy estimates associated with the Boltzmann H-function. We multiply the i-th equation of $(1.1)_1$ by $c_i(1+\log F_i)$ and take the sum over $i \in \Lambda$. Then, using (A1) and (A2), we obtain the equation for the Boltzmann H-function:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\sum_i c_i F_i \log F_i\right) + \frac{\partial}{\partial x}\left(\sum_i c_i v_i F_i \log F_i\right) \\ &\qquad = -\frac{1}{4}\sum_{i,j,k,l} A_{kl}^{ij}(F_iF_j - F_kF_l)\left(\log \frac{F_iF_j}{F_kF_l}\right) \le 0. \end{aligned} \tag{3.9}$$

Making use of this inequality, we have the following entropy estimates.

Lemma 3.2. *Under the same condition of Lemma* 3.1 *we have*

$$\begin{aligned}\int_0^1 \max_{i\in\Lambda} F_i(x,t)|\log F_i(x,t)|\,dx \\ &\le C_2(1+T)^2\biggl(\int_0^t |b(\tau)|\,|\log b(\tau)|\,d\tau \\ &\quad + \int_0^1 |F_0(x)|\,|\log F_0(x)|\,dx + 1\biggr), \quad t\in[0,T], \end{aligned}\tag{3.10}$$

where C_2 is a positive constant independent of T.

Proof. Put

$$I_2(t) = \int_0^t |b(\tau)|\,|\log b(\tau)|\,d\tau + \int_0^1 |F_0(x)|\,|\log F_0(x)|\,dx + 1.\tag{3.11}$$

Because of the inequality $f \le f|\log f| + 1$ for $f > 0$ we see that $I_1(t) \le (1+T)I_2(t)$, where $I_1(t)$ is given by (3.4). We now integrate (3.9) over $[0,1]\times[0,t]$, obtaining

$$\begin{aligned}\int_0^1 \sum_i c_i g(F_i(x,t))\,dx + \int_0^t \sum_i c_i v_i g(F_i(1,\tau))\,d\tau - \int_0^t \sum_i c_i v_i g(F_i(0,\tau))\,d\tau \\ \le \int_0^1 \sum_i c_i g(F_{i0}(x))\,dx,\end{aligned}\tag{3.12}$$

where we put $g(f) = f\log f$; $g(f)$ is a strictly convex function of $f > 0$. We estimate the second and third terms on the left hand side of (3.12). To this end, we put $\alpha_i = \sum_{j\in\Lambda_-} B_{ij} + 1$ for $i \in \Lambda_+$. This implies that

$$\frac{1}{\alpha_i}\sum_{j\in\Lambda_-} B_{ij} + \frac{1}{\alpha_i} = 1, \quad i\in\Lambda_+.$$

Now, applying the relation $g(\alpha f) = \alpha g(f) + g(\alpha) f$ and making use of the convexity of $g(f)$, we see that

$$\begin{aligned}\sum_{i\in\Lambda_+} c_i v_i g(F_i(0,t)) &= \sum_{i\in\Lambda_+} c_i v_i \left\{\alpha_i g\left(\frac{1}{\alpha_i}F_i(0,t)\right) + g(\alpha_i)\cdot\frac{1}{\alpha_i}F_i(0,t)\right\} \\ &= \sum_{i\in\Lambda_+} c_i v_i \left\{\alpha_i g\left(\frac{1}{\alpha_i}\sum_{j\in\Lambda_-} B_{ij}F_j(0,t) + \frac{1}{\alpha_i}b_i(t)\right) + g(\alpha_i)\cdot\frac{1}{\alpha_i}F_i(0,t)\right\} \\ &\le \sum_{i\in\Lambda_+} c_i v_i \left\{\sum_{j\in\Lambda_-} B_{ij}\cdot g(F_j(0,t)) + g(b_i(t)) + |\log\alpha_i| F_i(0,t)\right\},\end{aligned}$$

where we used $(1.1)_3$. By virtue of this inequality, we find that

$$
\begin{aligned}
\sum_i c_i v_i g(F_i(0,t)) &\leq \sum_{j\in\Lambda_-}\left(c_j v_j + \sum_{i\in\Lambda_+} c_i v_i B_{ij}\right) g(F_j(0,t)) \\
&\quad + \sum_{i\in\Lambda_+} c_i v_i g(b_i(t)) + \sum_{i\in\Lambda_+} c_i v_i |\log\alpha_i| F_i(0,t) \\
&\leq C(1 + |b^+(t)|\,|\log b^+(t)| + |F^+(0,t)|),
\end{aligned}
\tag{3.13}
$$

where we used $(1.5)_1$ and $g(f) \geq -1/e$. Similarly,

$$
-\sum_i c_i v_i g(F_i(1,t)) \leq C(1 + |b^-(t)|\,|\log b^-(t)| + |F^-(1,t)|). \tag{3.14}
$$

Substituting (3.13) and (3.14) into (3.12) gives

$$
\int_0^1 \sum_i c_i F_i(x,t) \log F_i(x,t)\,dx \leq C(1+T)(I_1(t) + I_2(t)), \tag{3.15}
$$

where we used (3.7) and (3.8). Since $f|\log f| \leq g(f) + 2/e$, we have from (3.15) that

$$
\int_0^1 \max_{i\in\Lambda} F_i(x,t)|\log F_i(x,t)|\,dx \leq C(1+T)(I_1(t) + I_2(t)),
$$

which together with $I_1(t) \leq (1+T)I_2(t)$ gives the desired estimate (3.10). This completes the proof. □

Thirdly, we make a preparation for showing the L^1-estimates on the characteristics. Let us fix constants V_0 and h_0 such that $\max_i |v_i| < V_0$ and $2V_0 h_0 = 1$. Letting $t_0 \geq 0$ and $0 < h \leq h_0$, we consider the rectangle

$$
\Omega = [0,1] \times [t_0, t_0 + h]
$$

contained in $[0,1] \times [0,T]$. Let (x_1, t_1) be an arbitrary point in Ω, and consider the straight lines $l_\pm$ with the slope $\pm V_0$, respectively, which pass through the point (x_1, t_1), that is, $l_\pm : x = \pm V_0(t - t_1) + x_1$. Let $(0, t_*)$ be the intersection of l_+ and $x = 0$, and let $(1, t_{**})$ be the intersection of l_- and $x = 1$. Then we have the following L^1-estimates on $l_\pm$.

Lemma 3.3. *Under the same condition of Lemma* 3.1 *we have*:
(1) *In case when* $t_* > t_0$,

$$
\begin{aligned}
&\int_{t_*}^{t_1} |F(V_0(t-t_1) + x_1, t)|\,dt \\
&\qquad + \int_{t_0}^{t_1} |F(-V_0(t-t_1) + x_1, t)|\,dt + \int_{t_0}^{t_*} |F'(0,t)|\,dt \\
&\qquad \leq C\left(\int_{t_0}^{t_*} |b^+(t)|\,dt + \int_0^{-V_0(t_0-t_1)+x_1} |F(x,t_0)|\,dx\right).
\end{aligned}
\tag{3.16$_1$}
$$

(2) *In case when* $t_{**} > t_0$,

$$\int_{t_0}^{t_1} |F(V_0(t-t_1)+x_1,t)|\,dt + \int_{t_{**}}^{t_1} |F(-V_0(t-t_1)+x_1,t)|\,dt + \int_{t_0}^{t_{**}} |F'(1,t)|\,dt \le C\left(\int_{t_0}^{t_{**}} |b^-(t)|\,dt + \int_{V_0(t_0-t_1)+x_1}^{1} |F(x,t_0)|\,dx\right). \tag{3.16$_2$}$$

(3) *In case when* $t_* \le t_0$ *and* $t_{**} \le t_0$,

$$\int_{t_0}^{t_1} |F(V_0(t-t_1)+x_1,t)|\,dt + \int_{t_0}^{t_1} |F(-V_0(t-t_1)+x_1,t)|\,dt \le C\left(\int_{V_0(t_0-t_1)+x_1}^{-V_0(t_0-t_1)+x_1} |F(x,t_0)|\,dx\right). \tag{3.16$_3$}$$

Proof. We prove the estimates only in the case (1) because the other cases can be treated similarly. First we apply the Green formula to (3.1) over the trapezium bounded by l_+, l_-, $x=0$ and $t=t_0$, and then use the boundary condition $(1.1)_3$, obtaining

$$\begin{aligned}
&\int_{t_*}^{t_1} \sum_i c_i(V_0-v_i)F_i(V_0(t-t_1)+x_1,t)\,dt \\
&\quad + \int_{t_0}^{t_1} \sum_i c_i(V_0+v_i)F_i(-V_0(t-t_1)+x_1,t)\,dt \\
&\quad + \int_{t_0}^{t_*} \sum_{j\in\Lambda_-}\left(c_j|v_j| - \sum_{i\in\Lambda_+} c_iv_iB_{ij}\right)F_j(0,t)\,dt - \int_{t_0}^{t_*} \sum_{i\in\Lambda_+} c_iv_ib_i(t)\,dt \\
&= \int_0^{-V_0(t_0-t_1)+x_1} \sum_i c_iF_i(x,t_0)\,dx.
\end{aligned}$$

Since the third term on the left hand side is nonnegative by $(1.5)_1$, we have

$$\int_{t_*}^{t_1} |F(V_0(t-t_1)+x_1,t)|\,dt + \int_{t_0}^{t_1} |F(-V_0(t-t_1)+x_1,t)|\,dt \le C\left(\int_{t_0}^{t_*} |b^+(t)|\,dt + \int_0^{-V_0(t_0-t_1)+x_1} |F(x,t_0)|\,dx\right). \tag{3.17}$$

Next, we apply the Green formula to (3.2) over the same trapezium used above. This yields

$$\int_{t_*}^{t_1} \sum_i c_i(V_0 - v_i)v_iF_i(V_0(t-t_1)+x_1,t)\,dt$$
$$+\int_{t_0}^{t_1} \sum_i c_i(V_0 + v_i)v_iF_i(-V_0(t-t_1)+x_1,t)\,dt$$
$$=\int_{t_0}^{t_*} \sum_i c_iv_i^2F_i(0,t)\,dt + \int_0^{-V_0(t_0-t_1)+x_1} \sum_i c_iv_iF_i(x,t_0)\,dx.$$

This equality together with (3.17) shows that

$$\int_{t_0}^{t_*} |F'(0,t)|\,dt \le C\left(\int_{t_0}^{t_*} |b^+(t)|\,dt + \int_0^{-V_0(t_0-t_1)+x_1} |F(x,t_0)|\,dx\right). \tag{3.18}$$

Therefore the proof of Lemma 3.3 is complete. □

Finally in this subsection we derive the L^1-estimates on the characteristics. Let $\alpha \in \Lambda$ be arbitrary. We consider the v_α-characteristics l_α which pases through the point (x_1, t_1), that is, $l_\alpha : x = v_\alpha(t-t_1)+x_1$. When $v_\alpha \neq 0$, we denote by $(0, t_{\alpha*})$ the intersection of l_α and $x=0$, and by $(1, t_{\alpha**})$ the intersection of l_α and $x=1$. We define τ_α by

$$\tau_\alpha = \begin{cases} \max(t_{\alpha*}, t_0) & (v_\alpha > 0) \\ \max(t_{\alpha**}, t_0) & (v_\alpha < 0) \\ t_0 & (v_\alpha = 0). \end{cases}$$

We use the following equation

$$\frac{\partial}{\partial t}\left(\sum_i c_i(v_\alpha - v_i)F_i\right) + \frac{\partial}{\partial x}\left(\sum_i c_i(v_\alpha - v_i)v_iF_i\right) = 0, \tag{3.19}$$

that is obtained by subtracting (3.2) from v_α times (3.1).

Lemma 3.4. *Assume the same condition of Lemma* 3.1. *For any* $\alpha \in \Lambda$, *we put* $\Lambda_\alpha = \{i \in \Lambda; v_i = v_\alpha\}$. *Then we have*:

(1) *In case when* $t_* > t_0$,

$$\int_{\tau_\alpha}^{t_1} \sum_{i \notin \Lambda_\alpha} F_i(v_\alpha(t-t_1)+x_1,t)\,dt$$
$$\le C\left(\int_{t_0}^{t_*} |b^+(t)|\,dt + \int_0^{-V_0(t_0-t_1)+x_1} |F(x,t_0)|\,dx\right). \tag{3.20$_1$}$$

(2) *In case when* $t_{**} > t_0$,

$$\int_{\tau_\alpha}^{t_1} \sum_{i \notin \Lambda_\alpha} F_i(v_\alpha(t-t_1)+x_1,t)\,dt$$
$$\le C\left(\int_{t_0}^{t_{**}} |b^-(t)|\,dt + \int_{V_0(t_0-t_1)+x_1}^1 |F(x,t_0)|\,dx\right). \tag{3.20$_2$}$$

(3) *In case when* $t_* \le t_0$ and $t_{**} \le t_0$,

$$\int_{t_0}^{t_1} \sum_{i \notin \Lambda_\alpha} F_i(v_\alpha(t - t_1) + x_1, t)\, dt$$

$$\le C\left(\int_{V_0(t_0-t_1)+x_1}^{-V_0(t_0-t_1)+x_1} |F(x, t_0)|\, dx\right). \tag{3.20$_3$}$$

PROOF. We show the estimates only in the case (1). First we consider the case where $v_\alpha(t_0 - t_1) + x_1 < 0$. Notice that $v_\alpha > 0$ and $t_* > \tau_\alpha = t_{\alpha*} > t_0$ in this case. We apply the Green formula to (3.19) over the trapezium defined by l_α, l_-, $x = 0$ and $t = t_0$, obtaining

$$\int_{t_{\alpha*}}^{t_1} \sum_i c_i(v_\alpha - v_i)^2 F_i(v_\alpha(t - t_1) + x_1, t)\, dt$$
$$+ \int_{t_0}^{t_1} \sum_i c_i(v_\alpha - v_i)(V_0 + v_i) F_i(-V_0(t - t_1) + x_1, t)\, dt$$
$$= \int_{t_0}^{t_{\alpha*}} \sum_i c_i(v_\alpha - v_i) F_i(0, t)\, dt$$
$$+ \int_0^{-V_0(t_0-t_1)+x_1} \sum_i c_i(v_\alpha - v_i) F_i(x, t_0)\, dx.$$

This together with $(3.16)_1$ gives the desired estimates $(3.20)_1$. Next, we consider the case where $v_\alpha(t_0 - t_1) + x_1 > 0$. Note that $\tau_\alpha = t_0 < t_*$ in this case. We again apply the Green formula to (3.19) over the triangle defined by l_α, l_- and $t = t_0$. Then we have

$$\int_{t_0}^{t_1} \sum_i c_i(v_\alpha - v_i)^2 F_i(v_\alpha(t - t_1) + x_1, t)\, dt$$
$$+ \int_{t_0}^{t_1} \sum_i c_i(v_\alpha - v_i)(V_0 + v_i) F_i(-V_0(t - t_1) + x_1, t)\, dt$$
$$= \int_{v_\alpha(t_0-t_1)+x_1}^{-V_0(t_0-t_1)+x_1} \sum_i c_i(v_\alpha - v_i) F_i(x, t_0)\, dx,$$

which together with $(3.16)_1$ gives $(3.20)_1$. Therefore the proof of Lemma 3.4 is complete. □

3.2. L^∞-estimates.

We want to show the a priori L^∞-estimates. We define

$$E(t) = \sup_{0 \le x \le 1,\, 0 \le \tau \le t} |F(x, \tau)|. \tag{3.21}$$

For the data $F_0(x)$ and $b(t)$, we put

$$D(t) = E_0 + B_0(t), \tag{3.22}$$

$$E_0 = \sup_{0 \le x \le 1} |F_0(x)|, \quad B_0(t) = \sup_{0 \le \tau \le t} |b(\tau)|. \tag{3.23}$$

First, we drive a difference inequality for $E(t)$.

Lemma 3.5. *We assume the same condition of Lemma* 3.1. *Then, for any* $0 < h \le h_0$ *and* $t \ge 0$ *with* $t + h \le T$, *we have*

$$E(t+h) \le C_3[E(t) + B_0(t+h) + E(t+h)\{M(t, 2V_0h) + hB_0(t+h)\}], \tag{3.24}$$

where $C_3 > 1$ *is a constant independent of* T *and* h. *Here we put*

$$M(t, r) = \sup_{|J| \le r} \int_J |F(x,t)|\,dx, \tag{3.25}$$

where the supremum is taken over all the interval $J \subset [0,1]$ *with the length* $|J| \le r$.

Proof. We multiply the i-th equation of $(1.1)_1$ by c_i and add the resulting equation over $i \in \Lambda_\alpha$. This yields

$$\frac{\partial \tilde{F}_\alpha}{\partial t} + v_\alpha \frac{\partial \tilde{F}_\alpha}{\partial x} = \tilde{A}_\alpha(F), \tag{3.26}$$

where

$$\tilde{F}_\alpha(x,t) = \sum_{i \in \Lambda_\alpha} c_i F_i(x,t), \quad \tilde{A}_\alpha(F) = \sum_{i \in \Lambda_\alpha} \sum_{k \notin \Lambda_\alpha} \sum_{j,l} A_{kl}^{ij}(F_k F_l - F_i F_j).$$

Note that $\tilde{A}_\alpha(F)$ does not contain the summation over $k \in \Lambda_\alpha$. We consider the v_α-characteristics $l_\alpha : x = x_\alpha(t) \equiv v_\alpha(t - t_1) + x_1$. Here we only discuss the case where $v_\alpha > 0$ and $t_0 < t_{\alpha*} < t_1$. We integrate (3.26) along the characteristics l_α to obtain

$$\tilde{F}_\alpha(x_1, t_1) = \tilde{F}_\alpha(0, t_{\alpha*}) + \int_{t_{\alpha*}}^{t_1} \tilde{A}_\alpha(F)(x_\alpha(\tau), \tau)\,d\tau. \tag{3.27}$$

From the boundary condition $(1.1)_3$ we have

$$\tilde{F}_\alpha(0, t_{\alpha*}) = \sum_{i \in \Lambda_\alpha} c_i \left(\sum_{j \in \Lambda_-} B_{ij} F_j(0, t_{\alpha*}) + b_i(t_{\alpha*}) \right). \tag{3.28}$$

Let $v_\beta < 0$ and $l_{\alpha\beta}^*$ be the v_β-characteristics passing through the point $(0, t_{\alpha*})$, that is, $l_{\alpha\beta}^* : x = x_{\alpha\beta}(t) \equiv v_\beta(t - t_{\alpha*})$. Noting that $0 < x_{\alpha\beta}(t_0) < 1$, we integrate (3.26) with α replaced by β along $l_{\alpha\beta}^*$ to obtain

$$\tilde{F}_\beta(0, t_{\alpha*}) = \tilde{F}_\beta(x_{\alpha\beta}(t_0), t_0) + \int_{t_0}^{t_{\alpha*}} \tilde{A}_\beta(F)(x_{\alpha\beta}(\tau), \tau)\,d\tau. \tag{3.29}$$

Now, estimating each term in (3.27), (3.28) and (3.29), we see that

$$\begin{aligned}
&\tilde{F}_\alpha(x_1, t_1) \\
&\quad \le C(E(t_0) + B_0(t_0 + h)) + CE(t_0 + h) \\
&\quad \times \left\{ \int_{t_{\alpha *}}^{t_1} \sum_{k \notin \Lambda_\alpha} F_k(x_\alpha(\tau), \tau)\, d\tau + \sum_{\beta \in \Lambda_-} \int_{t_0}^{t_{\alpha *}} \sum_{k \notin \Lambda_\beta} F_k(x_{\alpha\beta}(\tau), \tau)\, d\tau \right\}. \qquad (3.30)
\end{aligned}$$

On the other hand, it follows from Lemma 3.4 that both the integrals in (3.30) are majorized by

$$C\left(\int_{t_0}^{t_*} |b^+(\tau)|\, d\tau + \int_0^{-V_0(t_0 - t_1) + x_1} |F(x, t_0)|\, dx \right) \le C\{M(t_0, 2V_0 h) + hB_0(t_0 + h)\}.$$

Substituting this estimate into (3.30), we arrive at the desired difference inequality (3.24). Thus we have proved Lemma 3.5. □

The following lemma concerning the estimate for $M(t, r)$ in (3.25) is well known so that the proof is omitted (see [**1**], [**12**], for example).

LEMMA 3.6. *Under the same condition of Lemma* 3.1 *there exists a continuous function* $\delta(r) > 0$ *of* $r \in (0, 1]$ *with the property that* $\delta(r) \to 0$ *as* $r \to 0$, *such that*

$$M(t, r) \le \delta(r) \cdot M(t), \quad t \in [0, T], \qquad (3.31)$$

where

$$M(t) = \int_0^1 \max_{i \in \Lambda} F_i(x, t) |\log F_i(x, t)|\, dx + 1. \qquad (3.32)$$

The estimate (3.31) is crucial in solving the difference inequality (3.24). In fact, making use of (3.31) together with the entropy estimate (3.10), we can solve (3.24), obtaining the following a priori L^∞-estimate.

PROPOSITION 3.7. *Assume the same condition of Lemma* 3.1. *Then there exists a constant* $K(T)$ *depending only on* T, E_0 *and* $B_0(T)$ *such that*

$$E(T) \le K(T). \qquad (3.33)$$

PROOF. We have from (3.10) that

$$M(t) \le C_2(1 + T)^2 I_2(T) + 1 \le C(1 + T)^3 (D(T) |\log D(T)| + 1),$$

where $D(T)$ is given by (3.22). Substituting (3.31) together with the above estimate into (3.24), we obtain

$$E(t + h) \le C_3(E(t) + B_0(T)) + E(t + h) K_0(T)(\delta(2V_0 h) + h),$$

where $K_0(T) = C(1 + T)^3 (D(T) |\log D(T)| + 1)$ with a suitable constant C independent of T. Now we take $h = h(T)$ with $0 < h \le h_0$, such that $K_0(T)(\delta(2V_0 h) + h) \le 1/2$. For this choice of h, we have

$$E(t + h) \le 2C_3(E(t) + B_0(T)).$$

Solving this simplified difference inequality, we obtain

$$E(t) \leq (2C_3)^{1+T/h}(E_0 + 2B_0(T)), \quad t \in [0, T].$$

This completes the proof of Proposition 3.7. $\square$

3.3. Proof of Theorem 2.1.

We need to show a suitable local existence result, which is as follows.

PROPOSITION 3.8. *We assume the same condition of Theorem* 2.1. *Let h_0 be a positive constant determined in subsection* 3.1. *Then there exist positive constants $T_0 \leq h_0$ and K_0, which depend only on E_0 and $B_0(h_0)$, such that the problem* (1.1) *admits a unique solution $F = (F_i)_{i \in \Lambda}$ satisfying $F \in C^1([0,1] \times [0, T_0])$, $F(x,t) > O$ for $(x,t) \in [0,1] \times [0, T_0]$, and*

$$E(T_0) \leq K_0. \tag{3.34}$$

PROOF. We give an outline of the proof. For a positive number ν, we introduce a new unknown function $f = (f_i)_{i \in \Lambda}$ by $f_i = F_i \exp(\nu t)$ and transform the problem (1.1) into

$$(3.35) \qquad \begin{cases} \dfrac{\partial f}{\partial t} + V \dfrac{\partial f}{\partial x} = \nu f + e^{-\nu t} A(f), \quad 0 < x < 1, \quad t > 0, & (3.35)_1 \\ f(x,0) = F_0(x), \quad 0 < x < 1, & (3.35)_2 \\ f^+(0,t) = B^+ f^-(0,t) + e^{\nu t} b^+(t), \quad t > 0, & (3.35)_3 \\ f^-(1,t) = B^- f^+(1,t) + e^{\nu t} b^-(t), \quad t > 0. & (3.35)_4 \end{cases}$$

We solve this equivalent problem by the standard iteration method. To this end, we define the successive approximation sequence $\{f^n\}$ as follows: Let $f^0(x,t) = F_0(x)$. For $n \geq 1$, let $f^n(x,t)$ be a unique solution to

$$\begin{cases} \dfrac{\partial f^n}{\partial t} + V \dfrac{\partial f^n}{\partial x} = \nu f^{n-1} + e^{-\nu t} A(f^{n-1}), & 0 < x < 1, \quad t > 0, \\ f^n(x,0) = F_0(x), & 0 < x < 1\ , \\ f^{n+}(0,t) = B^+ f^{n-}(0,t) + e^{\nu t} b^+(t), & t > 0, \\ f^{n-}(1,t) = B^- f^{n+}(1,t) + e^{\nu t} b^-(t), & t > 0, \end{cases}$$

where $f^{n-1}(x,t)$ is assumed to be given. To ensure the positivity of $f^n(x,t)$, we choose $\nu > 0$ such that $\nu = 2M_0 M_1 D(h_0)$, where $D(h_0) = E_0 + B_0(h_0)$, and

$$M_0 = \max\left\{1, \max_{i \in \Lambda_-} \sum_{j \in \Lambda_+} B_{ij}, \max_{i \in \Lambda_+} \sum_{j \in \Lambda_-} B_{ij}\right\}, \quad M_1 = \sum_i \frac{1}{c_i} \sum_{j,k,l} A_{kl}^{ij}.$$

Then, for a suitable positive constant $T_0 \leq h_0$ depending only on E_0 and $B_0(h_0)$, it is proved that $f^n(x,t)$ is strictly positive and uniformly bounded on $[0,1] \times [0, T_0]$, and that $f^n(x,t)$ converges to a strictly positive function $f(x,t)$ in the $C^1([0,1] \times [0, T_0])$-topology. It turns out that the limit $f(x,t)$ is a desired solution to the equivalent problem (3.35) and hence the proof of Proposition 3.8 is complete. $\square$

Proof of Theorem 2.1. We have shown a local existence result (Proposition 3.8) and the a priori L^∞-estimate (Proposition 3.7). Therefore, the standard argument of continuating a local solution is applicable and we obtain a desired global solution to the problem (1.1). We omit the details and refer the reader to [**9**]. The proof of Theorem 2.1 is complete. □

4. Stationary solutions.

In this section, we prove Theorem 2.2 by applying a fixed point theorem of the Leray-Schauder type, which is stated in Theorem 4.5 below.

4.1. Linearized problem.

We want to obtain a solution to the stationary problem (1.6) as fixed point of a suitable mapping depending on a parameter $\lambda \in [0, 1]$. To define a desired mapping, we consider the following linearized problem corresponding to the problem (1.6):

$$(4.1)\quad \begin{cases} V'\dfrac{dF'}{dx} = \lambda(q'(G) - r'(G)F'), \quad 0 < x < 1, & (4.1)_1 \\ F^0 = \pi(F'), \quad 0 < x < 1, & (4.1)_2 \\ F^+(0) = B^+F^-(0) + b^+, & (4.1)_3 \\ F^-(1) = B^-F^+(1) + b^-, & (4.1)_4 \end{cases}$$

where λ is a parameter with $\lambda \in [0, 1]$ and π is the mapping in the condition (C). Here, for $\Lambda' = \Lambda_+ \cup \Lambda_- = \{i \in \Lambda; v_i \neq 0\}$, we put $F' = (F_i)_{i \in \Lambda'}$, $V' = diag(v_i)_{i \in \Lambda'}$, $q'(G) = (q_i(G))_{i \in \Lambda'}$ and $r'(G) = diag(r_i(G))_{i \in \Lambda'}$, with

$$q_i(F) = \frac{1}{c_i}\sum_{j,k,l} A^{ij}_{kl} F_k F_l, \quad r_i(F) = \frac{1}{c_i}\sum_{j,k,l} A^{kl}_{ij} F_j. \tag{4.2}$$

When the problem (4.1) has a solution $F = F^\lambda$ depending on the parameter λ, we can define a mapping Φ^λ by $F^\lambda = \Phi^\lambda[G]$. Since $(4.1)_1$ and $(4.1)_2$ with $G = F$ give

$$V\frac{dF}{dx} = \lambda A(F), \tag{4.3}$$

we see that a fixed point of Φ^1 becomes a solution to the stationary problem (1.6).

We prove the solvability of the problem (4.1). First, we note that the general solution of the ordinary differential equations $(4.1)_1$, is given in the form

$$\begin{aligned} F_i(x) = {} & \beta_i \exp\left(-\frac{\lambda}{|v_i|}\int_0^x r_i(G(\xi))\, d\xi\right) \\ & + \frac{\lambda}{|v_i|}\int_0^x q_i(G(\xi)) \exp\left(-\frac{\lambda}{|v_i|}\int_\xi^x r_i(G(\eta))\, d\eta\right) d\xi, \quad i \in \Lambda_+, \end{aligned} \tag*{$(4.4)_1$}$$

$$\begin{aligned} F_i(x) = {} & \beta_i \exp\left(-\frac{\lambda}{|v_i|}\int_x^1 r_i(G(\xi))\, d\xi\right) \\ & + \frac{\lambda}{|v_i|}\int_x^1 q_i(G(\xi)) \exp\left(-\frac{\lambda}{|v_i|}\int_x^\xi r_i(G(\eta))\, d\eta\right) d\xi, \quad i \in \Lambda_-, \end{aligned} \tag*{$(4.4)_2$}$$

where β_i are arbitrary constants. We put (4.4) into the boundary conditions $(4.1)_3$ and $(4.1)_4$. This yields the following system of linear algebraic equations for $\beta = (\beta_i)_{i \in \Lambda'}$:

$$(I - BS^{\lambda}[G])\beta = BT^{\lambda}[G] + b, \tag{4.5}$$

where B and b are given in (1.4), $S^{\lambda}[G] = diag(S_i^{\lambda}[G])_{i \in \Lambda'}$ and $T^{\lambda}[G] = (T_i^{\lambda}[G])_{i \in \Lambda'}$, with

$$S_i^{\lambda}[G] = \exp\left(-\frac{\lambda}{|v_i|}\int_0^1 r_i(G(\xi))\,d\xi\right), \quad i \in \Lambda', \tag{4.6}$$

$$T_i^{\lambda}[G] = \frac{\lambda}{|v_i|}\int_0^1 q_i(G(\xi))\exp\left(-\frac{\lambda}{|v_i|}\int_{\xi}^1 r_i(G(\eta))\,d\eta\right)d\xi, \quad i \in \Lambda_+, \tag*{$(4.7)_1$}$$

$$T_i^{\lambda}[G] = \frac{\lambda}{|v_i|}\int_0^1 q_i(G(\xi))\exp\left(-\frac{\lambda}{|v_i|}\int_0^{\xi} r_i(G(\eta))\,d\eta\right)d\xi, \quad i \in \Lambda_-. \tag*{$(4.7)_2$}$$

Thus we have shown the following lemma.

Lemma 4.1. *In order that there exists a unique solution $F > O$ to the problem* (4.1), *it is necessary and sufficient that there exists a unique solution $\beta > O$ of the equation* (4.5).

The next lemma plays a crucial role in solving the equation (4.5).

Lemma 4.2. (i) *Under the condition* (*B*), *any eigenvalue μ of the matrix B verifies* $|\mu| \leq 1$.

(ii) *When* (B') *is assumed, we have* $|\mu| \leq (1 - \delta_1)^{1/2}$. *Moreover, we have*

$$|B^n| \leq \gamma m'(1 - \delta_1)^{[n/2]}, \quad n = 1, 2, \ldots, \tag*{$(4.8)_1$}$$

$$|(I - B)^{-1}| \leq \frac{2\gamma m'}{\delta_1}, \tag*{$(4.8)_2$}$$

where $\delta_1 = \dfrac{\delta_0}{\max_{i \in \Lambda_-} c_i|v_i|}$ *and* $\gamma = \dfrac{\max_{i \in \Lambda'} c_i|v_i|}{\min_{i \in \Lambda'} c_i|v_i|}$, *with*

$$\delta_0 = \min_{j \in \Lambda_-}\left(c_j|v_j| - \sum_{i \in \Lambda_+} c_i|v_i|B_{ij}\right) > 0. \tag{4.9}$$

Proof. We only prove (ii). Let $P = diag(c_i|v_i|)_{i \in \Lambda'}$ and put $\hat{B} = PBP^{-1}$. We have

$$\hat{B} = \begin{pmatrix} O & \hat{B}^+ \\ \hat{B}^- & O \end{pmatrix}.$$

It then follows from (B') that

$${}^t\mathbf{1}_{m'}\begin{pmatrix} -\hat{B}^+ \\ I \end{pmatrix} \geq \delta_1 \cdot {}^t\mathbf{1}_{m_-}, \quad {}^t\mathbf{1}_{m'}\begin{pmatrix} I \\ -\hat{B}^- \end{pmatrix} \geq O.$$

This implies that ${}^t\hat{B}^+\mathbf{1}_{m_+} \leq (1 - \delta_1)\mathbf{1}_{m_-}$ and ${}^t\hat{B}^-\mathbf{1}_{m_-} \leq \mathbf{1}_{m_+}$, so that we have $|{}^t\hat{B}^+| \leq$

$1-\delta_1$ and $|{}^t\hat{B}^-| \le 1$. In paticular, we know that

$$|{}^t\hat{B}| \le 1. \tag{4.10}$$

Also, since

$$({}^t\hat{B})^2 = \begin{pmatrix} {}^t\hat{B}^- \cdot {}^t\hat{B}^+ & O \\ O & {}^t\hat{B}^+ \cdot {}^t\hat{B}^- \end{pmatrix},$$

we see that

$$|({}^t\hat{B})^2| \le |{}^t\hat{B}^+|\,|{}^t\hat{B}^-| \le 1-\delta_1. \tag{4.11}$$

Now, let μ be any eigenvalue of B. Then μ is also an eigenvalue of ${}^t\hat{B}$, with the corresponding eigenvector $f : {}^t\hat{B}f = \mu f$. Then $({}^t\hat{B})^2 f = \mu^2 f$. Therefore,

$$|\mu|^2|f| \le |({}^t\hat{B})^2|\,|f| \le (1-\delta_1)|f|$$

by (4.11). Thus we have proved that $|\mu| \le (1-\delta_1)^{1/2}$. Next we show $(4.8)_1$ and $(4.8)_2$. A simple calculation, using (4.10) and (4.11), gives

$$|B^n| \le m'|({}^tB)^n| = m'|P({}^t\hat{B})^n P^{-1}| \le \gamma m'|({}^t\hat{B})^2|^{[n/2]} \le \gamma m'(1-\delta_1)^{[n/2]},$$

which is $(4.8)_1$, where we used the fact that $|P|\,|P^{-1}| \le \gamma$. By virtue of $(4.8)_1$, we obtain

$$|(I-B)^{-1}| = \left|\sum_{n=0}^{\infty} B^n\right| \le 2\gamma m' \sum_{k=0}^{\infty}(1-\delta_1)^k \le \frac{2\gamma m'}{\delta_1}.$$

Thus the proof of Lemma 4.2 is complete. □

Now, we consider the linear algebraic equation (4.5). Let $\lambda \in [0,1]$ and suppose that

$$O \le G(x) \le R\mathbf{1}_m, \quad x \in [0,1], \tag{4.12}$$

for some $R > 0$. Then

$$S(R) \le S_i^\lambda[G] \le 1, \quad 0 \le T_i^\lambda[G] \le T(R), \quad i \in \Lambda', \tag{4.13}$$

where

$$S(R) = \exp\left[-\max_{i\in\Lambda'} \frac{r_i(R\mathbf{1}_m)}{|v_i|}\right], \quad T(R) = \max_{i\in\Lambda'} \frac{q_i(R\mathbf{1}_m)}{|v_i|}.$$

LEMMA 4.3. *We assume* (B'). *Moreover, we assume* $b \ge O$ *and* (2.3). *Let* $\lambda \in [0,1]$ *and let* $G \in C^0[0,1]$ *satisfy* (4.12) *for some* $R > 0$. *Then there exists a unique solution* $\beta > O$ *of the equation* (4.5). *This solution is given explicitly by*

$$\beta = \beta^\lambda[G] \equiv (I - BS^\lambda[G])^{-1}(BT^\lambda[G] + b) \tag{4.14}$$

and verifies the estimate

$$\delta_0(R) \le \beta_i^\lambda[G] \le K_0(R), \quad i \in \Lambda', \tag{4.15}$$

where $\delta_0(R)$ *and* $K_0(R)$ *are positive constants depending on* R, *which is specified below. In particular, when* $\lambda = 0$, *we have*

$$\beta = \beta^0 \equiv (I - B)^{-1} b. \tag{4.16}$$

Proof. Since $BS^\lambda[G] \le B$ by (4.13), we have $(BS^\lambda[G])^n \le B^n$, so that

$$O \le (I - BS^\lambda[G])^{-1} = \sum_{n=0}^{\infty} (BS^\lambda[G])^n \le \sum_{n=0}^{\infty} B^n = (I - B)^{-1}.$$

Therefore, we have from (4.8) and (4.13) that

$$\begin{aligned} |\beta^\lambda[G]| &= |(I - BS^\lambda[G])^{-1}(BT^\lambda[G] + b)| \\ &\le |(I - B)^{-1}|\,|BT^\lambda[G] + b| \\ &\le \frac{2\gamma m'}{\delta_1}(\gamma m' T(R) + |b|) \equiv K_0(R). \end{aligned} \tag{4.17}$$

This proves the upper estimate in (4.15). On the other hand, it follows from (2.3) that

$$\delta_2 \mathbf{1}_{m'} \le (I - B)^{-1} b = \sum_{n=0}^{\infty} B^n b, \tag{4.18}$$

with a positive constant δ_2. By virtue of $(4.8)_1$, we can choose a large integer N such that

$$\left| \sum_{n=N+1}^{\infty} B^n b \right| = \left| B^{N+1} \sum_{n=0}^{\infty} B^n b \right| = |B^{N+1}(I - B)^{-1} b| \le \gamma m' (1 - \delta_1)^{N/2} \cdot \frac{2\gamma m'}{\delta_1} |b| \le \frac{\delta_2}{2},$$

where we also used $(4.8)_2$. Substituting this in (4.18) gives

$$\sum_{n=0}^{N} B^n b \ge \frac{1}{2} \delta_2 \mathbf{1}_{m'}.$$

Also, we have from (4.13) that

$$(I - BS^\lambda[G])^{-1} \ge \sum_{n=0}^{N} (BS^\lambda[G])^n \ge S(R)^N \sum_{n=0}^{N} B^n.$$

Consequently, we obtain

$$\beta^\lambda[G] \ge (I - BS^\lambda[G])^{-1} b \ge S(R)^N \sum_{n=0}^{N} B^n b \ge \frac{\delta_2}{2} S(R)^N \mathbf{1}_{m'} \equiv \delta_0(R) \mathbf{1}_{m'}, \tag{4.19}$$

which gives the lower estimate in (4.15). This completes the proof of Lemma 4.3. □

Now, we put $\beta = \beta^\lambda[G]$ in (4.4). This gives the formula $F_i(x) = \Phi_i^\lambda[G](x)$, $i \in \Lambda'$, where

$$\Phi_i^\lambda[G](x) = \beta_i^\lambda[G]\exp\left(-\frac{\lambda}{|v_i|}\int_0^x r_i(G(\xi))\,d\xi\right)$$
$$+\frac{\lambda}{|v_i|}\int_0^x q_i(G(\xi))\exp\left(-\frac{\lambda}{|v_i|}\int_\xi^x r_i(G(\eta))\,d\eta\right)d\xi, \quad i \in \Lambda_+, \tag{4.20$_1$}$$

$$\Phi_i^\lambda[G](x) = \beta_i^\lambda[G]\exp\left(-\frac{\lambda}{|v_i|}\int_x^1 r_i(G(\xi))\,d\xi\right)$$
$$+\frac{\lambda}{|v_i|}\int_x^1 q_i(G(\xi))\exp\left(-\frac{\lambda}{|v_i|}\int_x^\xi r_i(G(\eta))\,d\eta\right)d\xi, \quad i \in \Lambda_-. \tag{4.20$_2$}$$

Moreover, we put $F_i(x) = \Phi_i^\lambda[G](x)$, $i \in \Lambda'$, in $(4.1)_2$, obtaining $F_i(x) = \Phi_i^\lambda[G](x)$, $i \in \Lambda_0$, where

$$\Phi_i^\lambda[G](x) = \pi_i((\Phi_j^\lambda[G](x))_{j\in\Lambda'}), \quad i \in \Lambda_0. \tag{4.20$_3$}$$

Thus we arrive at the solution formula $F(x) = \Phi^\lambda[G](x) = (\Phi_i^\lambda[G](x))_{i\in\Lambda}$ for the linearized problem (4.1). This is summarized as follows:

Proposition 4.4. *We assume* (B'), (C), $b \geq O$ *and* (2.3). *Let* $\lambda \in [0,1]$ *and let* $G \in C^0[0,1]$ *satisfy* (4.12) *for some* $R > 0$. *Then the linearized problem* (4.1) *admits a unique solution* $F = (F_i)_{i\in\Lambda} > O$ *satisfying* $F \in C^1[0,1]$. *This solution is given by the formula* $F = \Phi^\lambda[G]$ *and verifies the estimates*

$$\delta(R)\mathbf{1}_m \leq F(x) \leq K(R)\mathbf{1}_m, \quad \left|\frac{dF(x)}{dx}\right| \leq K'(R), \tag{4.21}$$

for $x \in [0,1]$, *where* $\delta(R)$, $K(R)$ *and* $K'(R)$ *are positive constants depending on* R.

Proof. We show the estimates in (4.21). It follows from $(4.20)_1$, $(4.20)_2$, (4.13) and (4.15) that

$$\delta_1(R) \leq \Phi_i^\lambda[G](x) \leq K_1(R), \quad i \in \Lambda', \tag{4.22}$$

for $x \in [0,1]$, where $\delta_1(R) = \delta_0(R)S(R)$ and $K_1(R) = K_0(R) + T(R)$. Also, we see that $\pi([\delta_1(R), K_1(R)]^{m'}) \subset [\delta_2(R), K_2(R)]^{m_0}$ by the condition (C), where $\delta_2(R)$ and $K_2(R)$ are some positive constants depending on R. Therefore, substituting (4.22) into $(4.20)_3$ gives

$$\delta_2(R) \leq \Phi_i^\lambda[G](x) \leq K_2(R), \quad i \in \Lambda_0, \tag{4.23}$$

for $x \in [0,1]$. Thus we have shown the first estimate in (4.21). Next we show the estimate for the derivative. Since $F' = (\Phi_i^\lambda[G])_{i\in\Lambda'}$ saisfies $(4.1)_1$, we have from (4.12) and (4.22) that

$$\left|\frac{d}{dx}\Phi_i^\lambda[G](x)\right| \leq K_1'(R), \quad i \in \Lambda', \tag{4.24}$$

for $x \in [0,1]$, where $K_1'(R)$ is a positive constant depending on R. Also, differentiating $(4.20)_3$ with respect to x gives

$$\frac{d}{dx}\Phi_i^\lambda[G](x) = \sum_{j\in\Lambda'} \frac{\partial\pi_i}{\partial F_j}((\Phi_j^\lambda[G](x))_{j\in\Lambda'})\frac{d}{dx}\Phi_j^\lambda[G](x), \quad i\in\Lambda_0.$$

We substitute (4.22) and (4.24) to this expression and conclude that

$$\left|\frac{d}{dx}\Phi_i^\lambda[G](x)\right| \le K_2'(R), \quad i\in\Lambda_0, \tag{4.25}$$

for $x\in[0,1]$, where $K_2'(R)$ is a positive constant depending on R. Thus we have shown the second estimate in (4.21). This completes the proof of Proposition 4.4. □

4.2. Proof of Theorem 2.2.

In this subsection we prove Theorem 2.2 by applying the following fixed point theorem that is due to *Browder-Potter* (see [**11**]).

Theorem 4.5. *Let S be a closed convex subset of a Banach space X. Let $\Phi^\lambda[F]$ be a continuous mapping of $(F,\lambda)\in S\times[0,1]$ into X such that*
(i) $\bigcup_{\lambda\in[0,1]}\Phi^\lambda[S]$ *is contained in a compact set in X,*
(ii) $\Phi^0[\partial S]\subset S$,
(iii) *for $\lambda\in[0,1]$, $\Phi^\lambda[\cdot]$ has no fixed point on ∂S.*
Then $\Phi^1[\cdot]$ has a fixed point in S.

In the application of the fixed point theorem, the following proposition concerning the a priori estimate plays an important role.

Proposition 4.6. *We assume* (A), (B′), (C), $b\ge O$ *and* (2.3). *Let $\lambda\in[0,1]$ and let $F\in C^0[0,1]$ with $F\ge O$ be a fixed point of the mapping $\Phi^\lambda[\cdot]$ defined by* (4.20). *Then we have the regularity $F\in C^1[0,1]$. Moreover, there are positive constants δ_1 and R_1 such that*

$$\delta_1\mathbf{1}_m \le F(x) \le R_1\mathbf{1}_m, \quad x\in[0,1]. \tag{4.26}$$

Proof. Let $F\in C^0[0,1]$ with $F\ge O$ and let $F=\Phi^\lambda[F]$. Then F belongs to $C^1[0,1]$ and is a solution to the problem (4.1) with $G=F$. Consequently, F satisfies (4.3). The conservation equations for (4.3) are

$$\frac{d}{dx}\sum_i c_iv_iF_i = 0, \quad \frac{d}{dx}\sum_i c_i{v_i}^2F_i = 0, \tag{4.27}$$

which correspond to (3.1) and (3.2), respectively. We integrate the first equation of (4.27) over $[0,1]$ and substitute the boundary conditions $(4.1)_3$ and $(4.1)_4$, obtaining

$$\sum_{j\in\Lambda_-}\left(c_j|v_j| - \sum_{i\in\Lambda_+}c_iv_iB_{ij}\right)F_j(0) + \sum_{j\in\Lambda_+}\left(c_jv_j - \sum_{i\in\Lambda_-}c_i|v_i|B_{ij}\right)F_j(1) = \sum_{i\in\Lambda'}c_i|v_i|b_i.$$

This equality together with (B′) gives $F_i(0)\le C_0$, $i\in\Lambda_-$, with a positive constant C_0. It then follows from $(4.1)_3$ that $F_i(0)\le C_1$, $i\in\Lambda_+$, where C_1 is some positive constant. Next, we integrate the second equation of (4.27) over $[0,x]$. This yields

$$\sum_i c_i{v_i}^2F_i(x) = \sum_i c_i{v_i}^2F_i(0),$$

which together with the estimate for $(F_i(0))_{i\in\Lambda'}$ obtained above gives

$$F_i(x) \le C_2, \quad i \in \Lambda', \tag{4.28}$$

for $x \in [0,1]$, where C_2 is some positive constant. On the other hand, the condition (C) implies that $\pi([0,C_2]^{m'}) \subset [0,C_3]^{m_0}$ for some positive constant C_3. Therefore, substituting $0 \le F_i(x) \le C_2$, $i \in \Lambda'$, into $(4.1)_2$, we conclude that

$$F_i(x) \le C_3, \quad i \in \Lambda_0, \tag{4.29}$$

for $x \in [0,1]$. Thus we have $O \le F(x) \le R_1\mathbf{1}_m$ for $x \in [0,1]$, where $R_1 = \max(C_2, C_3)$. Now, we apply Proposition 4.4 with $G = F$ and obtain the lower bound $\delta_1\mathbf{1}_m \le F(x)$, $x \in [0,1]$, where $\delta_1 = \delta(R_1)$. This completes the proof of Proposition 4.6. □

Proof of Theorem 2.2. Let us consider the mapping $\Phi^\lambda = (\Phi_i^\lambda)_{i\in\Lambda}$ defined by (4.20). Let $X = C^0[0,1]$ and

$$S_R = \{F = (F_i)_{i\in\Lambda} \in X;\ O \le F(x) \le R\mathbf{1}_m,\ x \in [0,1], i \in \Lambda\},$$

with $R > R_1$, where R_1 is the constant in (4.26). Obviously, S_R is a closed convex subset of X, and $\Phi^\lambda[F]$ is a mapping of $(F,\lambda) \in S_R \times [0,1]$ into $C^1[0,1] \subset X$. In order to apply Theorem 4.5, we need to verify the conditions (i), (ii) and (iii) in Theorem 4.5.

(i) Let $(F,\lambda) \in S_R \times [0,1]$. Then we have from Proposition 4.4 that $\Phi^\lambda[F] \in C^1[0,1]$ and

$$\delta(R)\mathbf{1}_m \le \Phi^\lambda[F](x) \le K(R)\mathbf{1}_m, \quad \left|\frac{d}{dx}\Phi^\lambda[F](x)\right| \le K'(R),$$

for $x \in [0,1]$. This combined with the *Ascoli–Arzelà* theorem shows that $\bigcup_{\lambda\in[0,1]}\Phi^\lambda[S_R]$ is contained in a compact set in X. Therefore the condition (i) has been verified.

(ii) Let $F \in S_R$. Then

$$\Phi^0[F](x) = \bar{F} = \begin{pmatrix} \beta^0 \\ \pi(\beta^0) \end{pmatrix},$$

where β^0 is defined in (4.16). Since $\bar{F} > O$, we see that $\Phi^0[S_R] \subset S_R$ if R is chosen such that $R > |\bar{F}|$. Thus we have verified the condition (ii).

(iii) To verify the condition (iii), we assume that there exists a fixed point $F \in S_R$ of $\Phi^\lambda[\cdot]$ for some $\lambda \in [0,1]$. Then we have from Proposition 4.6 that

$$\delta_1\mathbf{1}_m \le F(x) \le R_1\mathbf{1}_m,$$

for $x \in [0,1]$, where δ_1 and R_1 are positive constants in (4.26), which are independent of R. This implies that $F \notin \partial S_R$ if $R > R_1$. Thus the condition (iii) is verified.

Now, Theorem 4.5 is applicable and we conclude the existence of a fixed point F of $\Phi^1[\cdot]$ in S_R. This fixed point F is in $C^1[0,1]$, strictly positive, and is a solution to the problem (1.6). The regularity $F \in C^\infty[0,1]$ follows from the smoothing property of $\Phi^\lambda[\cdot]$, and hence F is a desired solution. The proof of Theorem 2.2 is complete. □

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