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Higher cycles on the moduli space of stable curves

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Abstract. We construct a number of analytic cycles on the moduli space of stable curves by using three moduli spaces: the moduli space of tori with one marked point, that of spheres with four marked points, and that of tori with two marked points. We then prove the linear independence of the cycles in the rational homology groups in order to improve Wolpert's estimates for even degree Betti numbers of the moduli space of stable curves.

§1. Introduction and results.

We shall denote by $\bar{\mathcal{M}}_g$ the moduli space of stable curves of genus g and assume $g \geq 3$. It is known that $\bar{\mathcal{M}}_g$ is a compactification of the classical moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g and that $\bar{\mathcal{M}}_g$ is a complex V-manifold of dimension $3g-3$ [**D-M**]. It is also known that the compactification locus $\mathcal{D} = \bar{\mathcal{M}}_g - \mathcal{M}_g$, which is the set of points in $\bar{\mathcal{M}}_g$ represented by stable curves with nodes, is a divisor on $\bar{\mathcal{M}}_g$ stratified by the number of nodes which representing stable curves have, and that the closure of a component of the stratum of $\mathcal{D}$ with k nodes is a subvariety of $\bar{\mathcal{M}}_g$ of complex dimension $3g-3-k$.

Wolpert [**W**] constructed $2+[g/2]$ analytic 2-cycles on $\bar{\mathcal{M}}_g$ and showed by using a result of Harer [**H**] that they span $H_2(\bar{\mathcal{M}}_g; \boldsymbol{Q})$. He further constructed analytic $2k$-cycles on $\bar{\mathcal{M}}_g$ for $k < g$ and verified the non-degeneracy of the intersection pairing of the $2k$-cycles and certain components of the strata of $\mathcal{D}$ with k nodes so as to prove the linear independence of the $2k$-cycles in $H_{2k}(\bar{\mathcal{M}}_g; \boldsymbol{Q})$, thereby obtaining the following estimate for the Betti number $b_{2k}(\bar{\mathcal{M}}_g)$ of $\bar{\mathcal{M}}_g$:

$$b_{2k}(\bar{\mathcal{M}}_g) = b_{6g-6-2k}(\bar{\mathcal{M}}_g) \geq \frac{1}{2}\binom{g-1}{k}. \tag{1.1}$$

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His estimate (1.1) is available only for $k < g$, and his method to give (1.1) unfortunately does not apply to $b_{2k}(\bar{\mathcal{M}}_g)$, $2g-2 < 2k < 4g-4$.

The purpose of this paper is to give a better estimate for each Betti number of $\bar{\mathcal{M}}_g$ of even degree so as to improve Wolpert's estimate (1.1).

We fix a Riemann surface S of genus g and a set of $3g-3$ cutting curves

$$d_1, c_1, d_2, d_2', c_2, \dots, d_{g-1}, d_{g-1}', c_{g-1}, d_g$$

on S for a certain pants decomposition of S (§2, Figure 2). For $k \le 2g-2$, we define a *k-selection* to be a selection σ of k from the $3g-3$ cutting curves above satisfying the following two conditions: (1) $d_i' \in \sigma$ if and only if $d_i \in \sigma$; (2) if $d_i, d_i' \in \sigma$, then c_{i-1}, $c_i \notin \sigma$. We call two k-selections σ, τ *conjugate* if there exists an orientation-preserving self-homeomorphism of S which permutes the $3g-3$ cutting curves on S and transforms σ to τ. We shall show that two k-selections σ, τ are conjugate if and only if τ is equal either to σ itself or to the k-selection $\bar{\sigma}$ such that $d_i \in \bar{\sigma}$ if and only if $d_{g-i+1} \in \sigma$ and that $c_j \in \bar{\sigma}$ if and only if $c_{g-j} \in \sigma$. We denote by $\alpha_{g,k}$ the number of conjugacy classes of k-selections.

Our main result is

THEOREM A. *When* $k \ge 2$,

$$b_{2k}(\bar{\mathcal{M}}_g) = b_{6g-6-2k}(\bar{\mathcal{M}}_g) \ge \max(\alpha_{g,k}, \alpha_{g,3g-3-k}). \tag{1.2}$$

One of our additional results to supplement Theorem A is

$$\alpha_{g,k} > \frac{1}{2}\binom{g-1}{k} + \frac{1}{2}\sum_{k',l}\binom{k'-1}{l}\cdot\binom{k-2k'+1}{l+1}\cdot\binom{g-l-2}{k-k'}. \tag{1.3}$$

This inequality (1.3) indicates that, for most of the pairs (g,k), our estimate (1.2) is more than the square of Wolpert's one (1.1).

We have to remark here that for $g=3$ or 4 Faber [**F**] constructed many more linearly independent analytic cycles on $\bar{\mathcal{M}}_g$ and that for $g=3$ he completely computed the Chow ring of $\bar{\mathcal{M}}_3$.

Our proof of (1.2) is outlined as follows.

For each k-selection σ, we construct an analytic $2k$-cycle $[\mathcal{A}_\sigma]$ on $\bar{\mathcal{M}}_g$ as follows. We collapse to nodes the cutting curves on S adjacent to selected curves in σ to obtain a stable curve S_σ of genus g. It turns out that each component of $S_\sigma - \{\text{nodes}\}$ containing selected curves is a torus with one puncture, a sphere with four punctures, or a torus with two punctures. There exist analytic fiber spaces $\mathcal{Q}_\ell$, $\mathcal{U}_2$, and $\tilde{\mathcal{Q}}_\ell$ which contain as 'fibers minus sections' all stable curves of genus 1, 0, and 1 with one, four, and two punctures respectively. We deform S_σ to obtain another stable curve of genus g by replacing each component of $S_\sigma - \{\text{nodes}\}$ containing selected curves with an arbitrary 'fiber minus sections' of $\mathscr{Q}_\ell$, $\mathscr{U}_2$, or $\tilde{\mathscr{Q}}_\ell$ in accordance with the number of punctures on the component. Let $\mathscr{A}_\sigma$ be the family of such deformed stable curves. $\mathscr{A}_\sigma$ turns out to be an analytic fiber space whose fibers are stable curves of genus g and whose base space is a compact complex manifold of dimension k. $\mathscr{A}_\sigma$ hence determines a classifying map from its base space to $\bar{\mathscr{M}}_g$, which is an analytic $2k$-cycle $[\mathscr{A}_\sigma]$ on $\bar{\mathscr{M}}_g$.

For each k-selection σ, we further construct an analytic $(6g-6-2k)$-cycle $[\mathscr{V}_\sigma]$ on $\bar{\mathscr{M}}_g$ as follows. We collapse to nodes the selected cutting curves themselves in σ to obtain a stable curve S'_σ of genus g with k nodes. The set of points in $\bar{\mathscr{M}}_g$ represented by stable curves homeomorphic to S'_σ forms a component of the stratum of $\mathscr{D}$ with k nodes. Let $\mathscr{V}_\sigma$ be the closure of the component. $\mathscr{V}_\sigma$ hence is a subvariety of $\bar{\mathscr{M}}_g$ of complex dimension $3g-3-k$, which itself is an analytic $(6g-6-2k)$-cycle $[\mathscr{V}_\sigma]$ on $\bar{\mathscr{M}}_g$.

We then see that if σ and τ are conjugate, then $[\mathscr{A}_\tau]$ is homologous to $[\mathscr{A}_\sigma]$ and $[\mathscr{V}_\tau]$ is equal to $[\mathscr{V}_\sigma]$. We thus obtain, for each $k \le 2g-2$, $\alpha_{g,k}$ $2k$-cycles $[\mathscr{A}_\sigma]$'s and as many $(6g-6-2k)$-cycles $[\mathscr{V}_\sigma]$'s on $\bar{\mathscr{M}}_g$.

In conclusion, we verify by calculation that the intersection pairing of $[\mathscr{A}_\sigma]$'s and $[\mathscr{V}_\sigma]$'s is non-degenerate, hence (1.2).

This paper is organized as follows. In §2 we construct the cycles $[\mathscr{A}_\sigma]$'s and $[\mathscr{V}_\sigma]$'s on $\bar{\mathscr{M}}_g$ assuming the existence of the three analytic fiber spaces $\mathscr{Q}_\ell$, $\mathscr{U}_2$, and $\tilde{\mathscr{Q}}_\ell$. In §3 we construct $\mathscr{Q}_\ell$, $\mathscr{U}_2$, and $\tilde{\mathscr{Q}}_\ell$ to prove their existence. In §4 we show that the cycles $[\mathscr{A}_\sigma]$'s and $[\mathscr{V}_\sigma]$'s represent linearly independent classes in $H_*(\bar{\mathscr{M}}_g; \boldsymbol{Q})$ by verifying that the intersection pairing of $[\mathscr{A}_\sigma]$'s and $[\mathscr{V}_\sigma]$'s is non-degenerate, thereby proving Theorem A. In §5 we give algorithm to compute $\alpha_{g,k}$ and in particular prove (1.3).

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§2. Construction of cycles.

Let $\bar{\mathscr{M}}_g$ denote the moduli space of stable curves of genus g and assume $g \ge 3$. $\bar{\mathscr{M}}_g$ is a compactification of the classical moduli space $\mathscr{M}_g$ of Riemann surfaces of genus g, and is a complex V-manifold of dimension $3g-3$. The compactification locus $\mathscr{D} = \bar{\mathscr{M}}_g - \mathscr{M}_g$, which is the set of points in $\bar{\mathscr{M}}_g$ represented by stable curves with nodes, is a divisor on $\bar{\mathscr{M}}_g$, the sum of $1+[g/2]$ irreducible ones $\mathscr{D}_0, \dots, \mathscr{D}_{[g/2]}$. $\mathscr{D}$ is stratified by the number of nodes which representing stable curves have: the k-stratum of $\mathscr{D}$ is the set of points in $\bar{\mathscr{M}}_g$ represented by stable curves with k nodes. The closure of a component of the k-stratum of $\mathscr{D}$ is a subvariety of $\bar{\mathscr{M}}_g$ of dimension $3g-3-k$, which itself is a cycle on $\bar{\mathscr{M}}_g$ of degree $6g-6-2k$. For example, the 1-stratum of $\mathscr{D}$ has exactly $1+[g/2]$ components, which correspond to the irreducible divisors $\mathscr{D}_0, \dots, \mathscr{D}_{[g/2]}$, and determines as many cycles on $\bar{\mathscr{M}}_g$ of degree $6g-8$.

Wolpert [**W**] showed that the $1+[g/2]$ classes represented by the $(6g-8)$-cycles above and the Poincaré dual $[\omega]$ of the Weil-Petersson Kähler form ω on $\bar{\mathscr{M}}_g$ span $H_{6g-8}(\bar{\mathscr{M}}_g; \boldsymbol{Q})$. For $k<g$, he further showed that many components of the k-stratum of $\mathscr{D}$ become linearly independent classes in $H_{6g-6-2k}(\bar{\mathscr{M}}_g; \boldsymbol{Q})$ and that as many analytic $2k$-cycles on $\bar{\mathscr{M}}_g$ which he constructed represent linearly independent classes in $H_{2k}(\bar{\mathscr{M}}_g; \boldsymbol{Q})$. He at the same time proved the two facts above by actually showing that the intersection pairing of those components of the k-stratum of $\mathscr{D}$ and those analytic $2k$-cycles on $\bar{\mathscr{M}}_g$ is non-degenerate. The number of those analytic $2k$-cycles on $\bar{\mathscr{M}}_g$ which he constructed was equal to the number of choices, counted up to symmetry, of k from such a specific set of $g-1$ curves $c_1, \dots, c_{g-1}$ on a Riemann surface of genus g as illustrated in Figure 1.

In this section, we construct many more analytic cycles on $\bar{\mathscr{M}}_g$ than Wolpert did, and moreover, we work for all even degrees while he did only for even degrees $\leq 2g-2$ or $\geq 4g-4$.

We begin by taking a larger set of curves on a Riemann surface. We fix a Riemann surface S of genus g and such a set of $3g-3$ cutting curves

$$d_1, c_1, d_2, d_2', c_2, \dots, d_{g-1}, d_{g-1}', c_{g-1}, d_g$$

on S for a pants decomposition of S as in Figure 2 (cf. Figure 1).

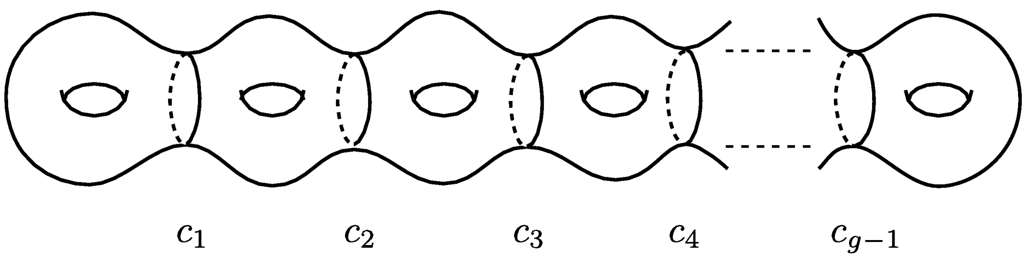


Figure 1. The $g-1$ cutting curves used by Wolpert

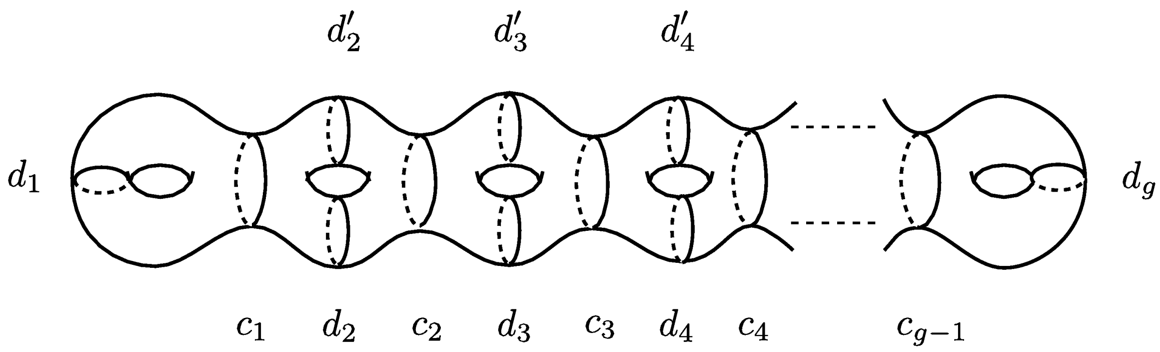


Figure 2. The $3g-3$ cutting curves on S for k-selections

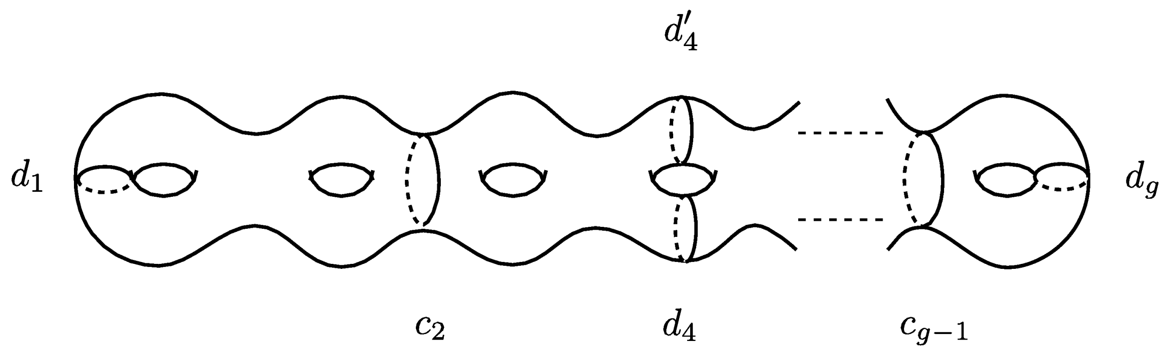


Figure 3. A 6-selection $\sigma = \{d_1, c_2, d_4, d_4', c_{g-1}, d_g\}$

Definition 2.1. A *k-selection* is a selection σ of k from the $3g-3$ cutting curves

$$d_1, c_1, d_2, d_2', c_2, \dots, d_{g-1}, d_{g-1}', c_{g-1}, d_g$$

above on S satisfying the following two conditions.

(1) d_i is in σ if and only if d_i' is in σ.

(2) If d_i and d_i' are in σ, then neither c_{i-1} nor c_i is in σ (for $2 \le i \le g-1$).

The selection $\sigma = \{d_1, c_2, d_4, d_4', c_{g-1}, d_g\}$ in Figure 3 $(g \ge 6)$ is an example of a 6-selection. (In Figures 4, 8, and 9, we take the same selection σ as in Figure 3.)

For each k-selection σ, we construct an analytic fiber space $\mathscr{A}_\sigma$ whose fibers are stable curves of genus g and whose base space is a compact complex manifold of dimension k. Since such an analytic fiber space $\mathscr{A}_\sigma$ determines a classifying map from its base space to $\overline{\mathscr{M}}_g$, we shall obtain an analytic $2k$-cycle $[\mathscr{A}_\sigma]$ on $\overline{\mathscr{M}}_g$.

We collapse to nodes the cutting curves on S adjacent to selected curves in σ to obtain a stable curve S_σ of genus g. It is easy to see that each component of $S_\sigma - \{\text{nodes}\}$ containing selected curves is a torus with one puncture, a sphere with four punctures, or a torus with two punctures, while a component of $S_\sigma - \{\text{nodes}\}$ not containing selected curves is not necessarily one of the three above (Figure 4).

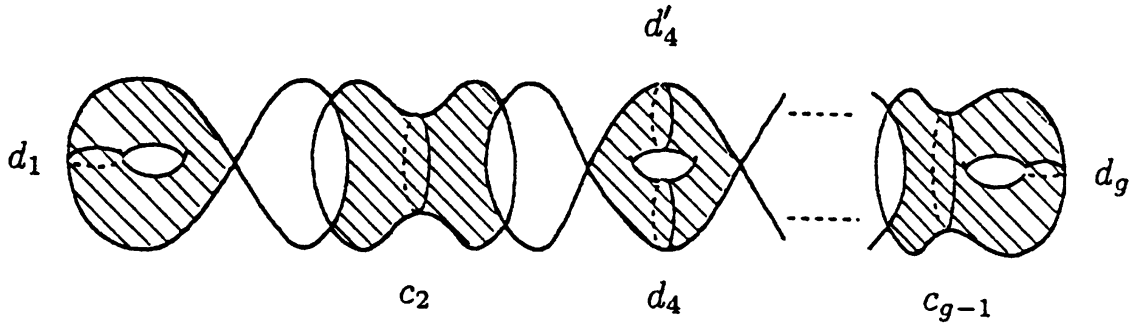


Figure 4. S_σ for $\sigma = \{d_1, c_2, d_4, d_4', c_{g-1}, d_g\}$

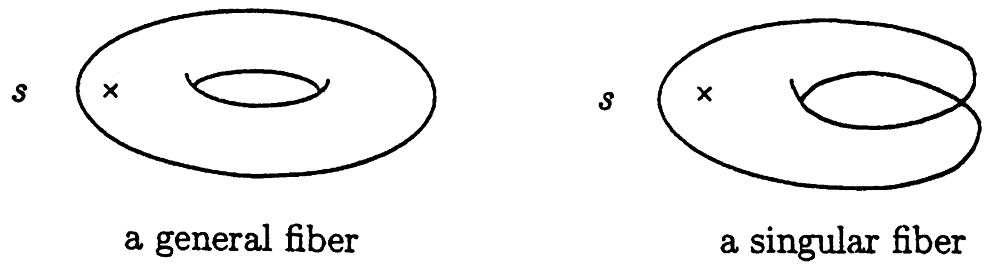


Figure 5. Fibers of $\mathscr{Q}_\ell$

We wish to make the conformal structures of the components of $S_\sigma - \{\text{nodes}\}$ containing selected curves vary over all the possible ones while keeping those of the other components fixed. The analytic fiber space we are constructing shall be the family of all the stable curves made from S_σ in this way.

We therefore prepare three analytic fiber spaces

$$\pi_\ell' : \mathscr{Q}_\ell \to \widehat{\boldsymbol{H}/\Gamma_\ell}, \quad \pi_2 : \mathscr{U}_2 \to \widehat{\boldsymbol{H}/\Gamma_2}, \quad \tilde{\pi}_\ell : \tilde{\mathscr{Q}}_\ell \to \mathscr{U}_\ell$$

which contain as 'fibers minus sections' all stable curves of genus 1, 0, and 1 with one, four, two punctures respectively. Indeed such analytic fiber spaces exist, but we only make a sketch of these fiber spaces in this section, assuming their existence; we shall construct these fiber spaces in the next section, proving their existence.

The base space $\widehat{\boldsymbol{H}/\Gamma_\ell}$ of $\mathscr{Q}_\ell$ is a closed Riemann surface. A general fiber of $\mathscr{Q}_\ell$ is an elliptic curve. $\mathscr{Q}_\ell$ has finitely many singular fibers each of which is a projective line which intersects itself at a double point. $\mathscr{Q}_\ell$ has one analytic section s which does not attain as a value the node on any singular fiber (Figure 5). The classifying map from $\widehat{\boldsymbol{H}/\Gamma_\ell}$ to the moduli space $\bar{\mathscr{M}}_{1,1}$ of stable curves of genus 1 with 1 marked point is surjective.

The base space $\widehat{\boldsymbol{H}/\Gamma_2}$ of $\mathscr{U}_2$ is a projective line. A general fiber of $\mathscr{U}_2$ is also a projective line. $\mathscr{U}_2$ has three singular fibers each of which is a union of two projective lines which intersect at a double point. $\mathscr{U}_2$ has four disjoint analytic sections s_1, s_2, s_3, s_4

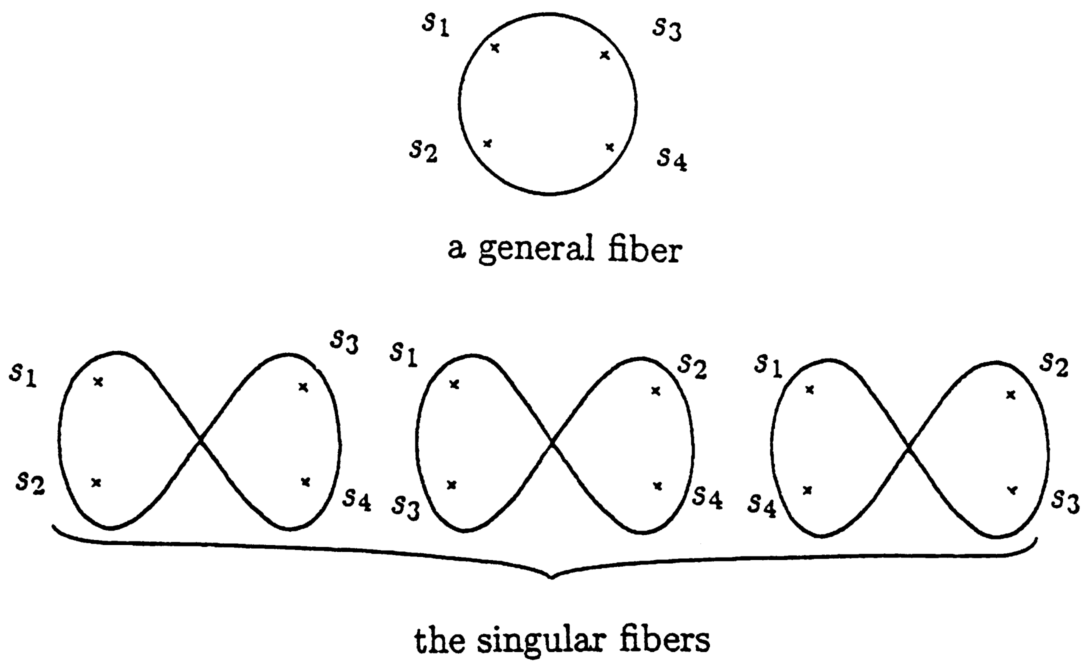


Figure 6. Fibers of $\mathscr{U}_2$

each of which does not attain as a value the node on any singular fiber (Figure 6). The classifying map from $\widehat{\boldsymbol{H}/\Gamma_2}$ to the moduli space $\overline{\mathscr{M}}_{0,4}$ of stable curves of genus 0 with 4 marked points is surjective.

The base space $\mathscr{U}_\ell$ of $\tilde{\mathscr{Q}}_\ell$ is a compact elliptic surface with several singular fibers with only nodes as singularities, which has some disjoint analytic sections each of which does not attain as a value any node on any singular fiber. A general fiber of $\tilde{\mathscr{Q}}_\ell$ is an elliptic curve. Let $\mathscr{S}$, $\mathscr{N}$, and Σ denote the union of singular fibers of $\mathscr{U}_\ell$, the set of nodes of singular fibers of $\mathscr{U}_\ell$, and the union of sections of $\mathscr{U}_\ell$ respectively. The singular fibers of $\tilde{\mathscr{Q}}_\ell$ then lie over $\mathscr{S} \cup \Sigma$, and are classified into the following four types (Figure 7):

(1) each singular fiber of $\tilde{\mathscr{Q}}_\ell$ over $\mathscr{S} - \mathscr{N} - \Sigma$ is a projective line which intersects itself at a double point;

(2) each singular fiber of $\tilde{\mathscr{Q}}_\ell$ over $\Sigma - \mathscr{S}$ is a union of a projective line and an elliptic curve which intersect at a double point;

(3) each singular fiber of $\tilde{\mathscr{Q}}_\ell$ over $\mathscr{S} \cap \Sigma$ is a union of a projective line and a singular fiber of type (1) which intersect at a double point;

(4) each singular fiber of $\tilde{\mathscr{Q}}_\ell$ over $\mathscr{N}$ is a union of two projective lines which intersect at two double points.

$\tilde{\mathscr{Q}}_\ell$ has two disjoint analytic sections $\tilde{s}_1$, $\tilde{s}_2$ either of which does not attain as a value any node on any singular fiber (Figure 7). The classifying map from $\mathscr{U}_\ell$ to the moduli space $\overline{\mathscr{M}}_{1,2}$ of stable curves of genus 1 with 2 marked points is surjective.

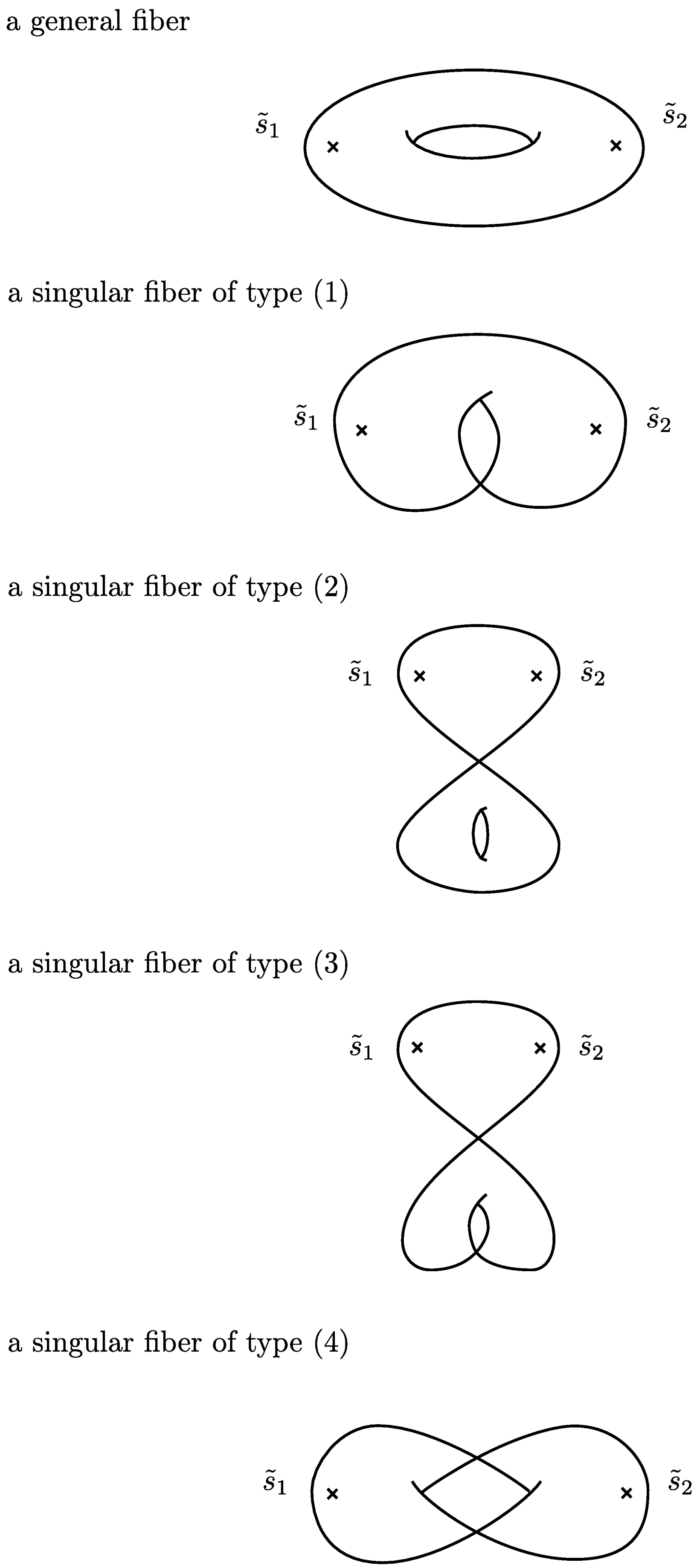


Figure 7. Fibers of $\tilde{\mathscr{Q}}_\ell$

We now replace each of the components of S_σ containing selected curves in the k-selection σ with a fiber of $\mathscr{Q}_\ell$, $\mathscr{U}_2$, or $\tilde{\mathscr{Q}}_\ell$ as follows.

Case 1. If $d_j, d_j' \in \sigma$, then the component of S_σ containing d_j, d_j' is a torus with two marked points, which is homeomorphic to a general fiber of $\tilde{\mathscr{Q}}_\ell$, an elliptic curve with two sections; replace the component with an arbitrary fiber of $\tilde{\mathscr{Q}}_\ell$ identifying the nodes to which c_{j-1}, c_j collapse with the sections $\tilde{s}_1, \tilde{s}_2$ respectively.

Case 2. If $c_j \in \sigma$ $(2 \le j \le g-2)$, then the component of S_σ containing c_j is a sphere with four marked points, which is homeomorphic to a general fiber of $\mathscr{U}_2$, a projective line with four sections; replace the component with an arbitrary fiber of $\mathscr{U}_2$ identifying the nodes to which $d_j, d_j', d_{j+1}, d_{j+1}'$ collapse with the sections s_1, s_2, s_3, s_4 respectively.

Case 3. If $d_1, c_1 \in \sigma$, then the component of S_σ containing d_1, c_1 is a torus with two marked points, which is homeomorphic to a general fiber of $\tilde{\mathscr{Q}}_\ell$, an elliptic curve with two sections; replace the component with an arbitrary fiber of $\tilde{\mathscr{Q}}_\ell$ identifying the nodes to which d_2, d_2' collapse with the sections $\tilde{s}_1, \tilde{s}_2$ respectively.

Case 4. If $c_{g-1}, d_g \in \sigma$, then as in Case 3 replace the component of S_σ containing c_{g-1}, d_g with an arbitrary fiber of $\tilde{\mathscr{Q}}_\ell$ identifying the nodes to which d_{g-1}, d_{g-1}' collapse with the sections $\tilde{s}_1, \tilde{s}_2$ respectively.

Case 5. If $c_1 \in \sigma$ but $d_1 \notin \sigma$, then the component of $S_\sigma - \{\text{nodes}\}$ containing c_1 is a sphere with four punctures, which is homeomorphic to a general fiber of $\mathscr{U}_2 - \{\text{sections}\}$, a projective line with four sections deleted; replace the component of S_σ containing c_1 with an arbitrary fiber of $\mathscr{U}_2$ identifying the nodes to which d_2, d_2' collapse with the sections s_3, s_4 respectively and identifying s_1 with s_2 to form a double point.

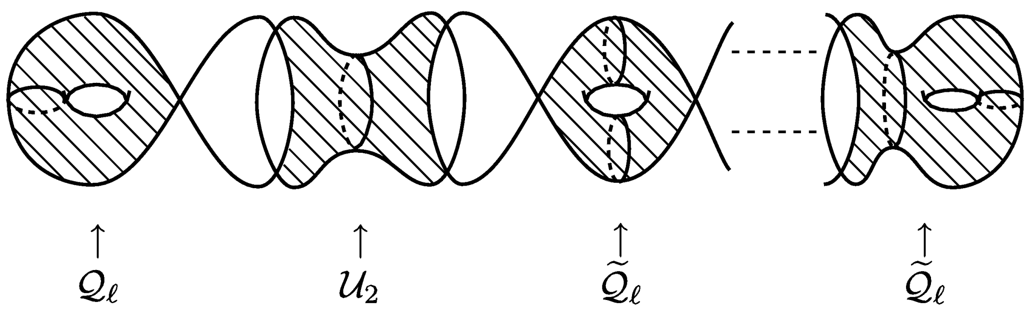
Case 6. If $c_{g-1} \in \sigma$ but $d_g \notin \sigma$, then as in Case 5 replace the component of S_σ containing c_{g-1} with an arbitrary fiber of $\mathscr{U}_2$ identifying the nodes to which d_{g-1}, d_{g-1}' collapse with the sections s_1, s_2 respectively and identifying s_3 with s_4 to form a double point.

Case 7. If $d_1 \in \sigma$ but $c_1 \notin \sigma$, then the component of S_σ containing d_1 is a torus with one marked point, which is homeomorphic to a general fiber of $\mathscr{Q}_\ell$, an elliptic curve with one section; replace the component with an arbitrary fiber of $\mathscr{Q}_\ell$ identifying the node to which c_1 collapses with the section s.

Case 8. If $d_g \in \sigma$ but $c_{g-1} \notin \sigma$, then as in Case 7 replace the component of S_σ containing d_g with an arbitrary fiber of $\mathscr{Q}_\ell$ identifying the node to which c_{g-1} collapses with the section s. (Figure 8).

The curve constructed from S_σ in this way is again a stable curve of genus g.

Let $\mathscr{A}_\sigma$ be the family of such stable curves. $\mathscr{A}_\sigma$ is an analytic fiber space which



$$\downarrow$$

$$(\mathcal{U}_\ell)^2 \times (\widehat{\mathbf{H}/\Gamma_\ell}) \times (\widehat{\mathbf{H}/\Gamma_2})$$

Figure 8. $\mathscr{A}_\sigma$ for $\sigma = \{d_1, c_2, d_4, d_4', c_{g-1}, d_g\}$

fibers over the Cartesian product $(\widehat{\boldsymbol{H}/\Gamma_\ell})^a \times (\widehat{\boldsymbol{H}/\Gamma_2})^b \times (\mathscr{U}_\ell)^c$ for some non-negative integers a, b, c. Note that the complex dimension $a + b + 2c$ of the base space of $\mathscr{A}_\sigma$ is equal to k if σ is a k-selection. $\mathscr{A}_\sigma$ is also regarded as the collection of conformal structures of S_σ which are the same as those of S_σ on the components not containing selected curves in the k-selection σ. In this sense, $\mathscr{A}_\sigma$ contains all the possible conformal structures of the components of S_σ containing selected curves in σ, since the classifying maps

$$\widehat{\boldsymbol{H}/\Gamma_\ell} \to \bar{\mathscr{M}}_{1,1}, \quad \widehat{\boldsymbol{H}/\Gamma_2} \to \bar{\mathscr{M}}_{0,4}, \quad \mathscr{U}_\ell \to \bar{\mathscr{M}}_{1,2}$$

are surjective.

Since each fiber of $\mathscr{A}_\sigma$ is a stable curve of genus g, the analytic fiber space $\mathscr{A}_\sigma$ determines a classifying map from $(\widehat{\boldsymbol{H}/\Gamma_\ell})^a \times (\widehat{\boldsymbol{H}/\Gamma_2})^b \times (\mathscr{U}_\ell)^c$ to $\bar{\mathscr{M}}_g$. Let $[\mathscr{A}_\sigma]$ be the $2k$-cycle of $\bar{\mathscr{M}}_g$ represented by this classifying map. In this way, we have constructed an analytic $2k$-cycle $[\mathscr{A}_\sigma]$ on $\bar{\mathscr{M}}_g$ for each k-selection σ (Figure 8).

For each k-selection σ, we further construct a subvariety $\mathscr{V}_\sigma$ of $\bar{\mathscr{M}}_g$ of complex dimension $3g - 3 - k$, which itself is an analytic $(6g - 6 - 2k)$-cycle $[\mathscr{V}_\sigma]$ on $\bar{\mathscr{M}}_g$.

We collapse to nodes the selected cutting curves themselves in the k-selection σ to obtain a stable curve S_σ' of genus g with k nodes (Figure 9). The set of points in $\bar{\mathscr{M}}_g$ represented by stable curves homeomorphic to S_σ' forms a component of the k-stratum of the compactification locus $\mathscr{D}$ of $\bar{\mathscr{M}}_g$. Let $\mathscr{V}_\sigma$ be the closure in $\bar{\mathscr{M}}_g$ of the component. $\mathscr{V}_\sigma$ hence is a subvariety of $\bar{\mathscr{M}}_g$ of complex dimension $3g - 3 - k$. $\mathscr{V}_\sigma$ itself is an

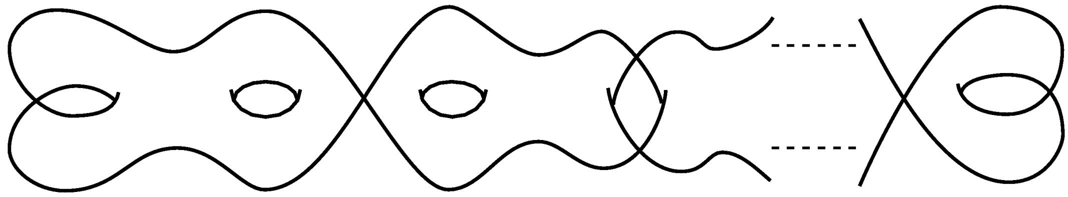


Figure 9. S'_σ for $\sigma = \{d_1, c_2, d_4, d'_4, c_{g-1}, d_g\}$

analytic $(6g-6-2k)$-cycle $[\mathscr{V}_\sigma]$ on $\bar{\mathscr{M}}_g$. In this way, we have constructed an analytic $(6g-6-2k)$-cycle $[\mathscr{V}_\sigma]$ on $\bar{\mathscr{M}}_g$ for each k-selection σ.

To conclude this section, we consider when two of the analytic cycles on $\bar{\mathscr{M}}_g$ we have constructed represent the same homology class of $\bar{\mathscr{M}}_g$. We remark that there exist orientation-preserving self-homeomorphisms of S which permute the $3g-3$ cutting curves

$$d_1, c_1, d_2, d'_2, c_2, \ldots, d_{g-1}, d'_{g-1}, c_{g-1}, d_g$$

on S. Let G be the group of such self-homeomorphisms of S. It is easy to see that G acts on the set of k-selections, and that if a k-selection τ is in the G-orbit of a k-selection σ, then the cycle $[\mathscr{A}_\tau]$ is homologous to the cycle $[\mathscr{A}_\sigma]$ and the subvariety $\mathscr{V}_\tau$ is the same as the subvariety $\mathscr{V}_\sigma$. We thus call two k-selections σ, τ *conjugate* if a self-homeomorphism of S in G transforms σ to τ. We further observe that G acts on the set of c_j's since the cutting curves c_j's separate the surface S while the other cutting curves do not, and that, similarly, G acts on the set of pairs $\{d_i, d'_i\}$'s: more precisely, an orientation-preserving self-homeomorphism of S in G transforms $\{c_j, d_i, d'_i\}$ either to $\{c_j, d_i, d'_i\}$ or to $\{c_{g-j}, d_{g-i+1}, d'_{g-i+1}\}$.

That is why we give

DEFINITION 2.2. (1) For a k-selection σ, let $\bar{\sigma}$ denote the k-selection which contains d_i and c_j if and only if σ contains d_{g-i+1} and c_{g-j} respectively.

(2) A k-selection σ is called *symmetric* if and only if $\bar{\sigma} = \sigma$.

Note that two k-selections σ, τ are conjugate if and only if τ is equal either to σ itself or to $\bar{\sigma}$, and that the conjugacy class of a k-selection is composed of one k-selection or two according as the k-selection is symmetric or not.

NOTATION 2.3. Let $\alpha_{g,k}$ denote the number of conjugacy classes of k-selections.

Note that $\alpha_{g,k}$ is the number of distinct k-selections modulo the action of G.

We have thus constructed $\alpha_{g,k}$ $2k$-homology classes of $\bar{\mathscr{M}}_g$ represented by $[\mathscr{A}_\sigma]$'s and $\alpha_{g,k}$ $(6g-6-2k)$-homology classes of $\bar{\mathscr{M}}_g$ represented by $[\mathscr{V}_\sigma]$'s. In §4 we shall show their linear independence in $H_*(\bar{\mathscr{M}}_g; \boldsymbol{Q})$.

§3. Construction of fiber spaces of stable curves.

In the previous section, we introduced, and made a sketch of, three fiber spaces $\mathcal{Q}_\ell$, $\mathcal{U}_2$, and $\tilde{\mathcal{Q}}_\ell$, assuming their existence, to construct an analytic $2k$-cycle $[\mathcal{A}_\sigma]$ on $\bar{\mathcal{M}}_g$ for each k-selection σ. In this section, we actually construct $\mathcal{Q}_\ell$, $\mathcal{U}_2$, and $\tilde{\mathcal{Q}}_\ell$, demonstrating their existence.

We begin by reviewing the construction of $\mathcal{Q}_\ell$ and $\mathcal{U}_2$ by Wolpert [**W**].

The construction of $\mathcal{Q}_\ell$ is divided into three steps as follows.

The first step to construct $\mathcal{Q}_\ell$ is to construct a fiber bundle of elliptic curves. Assume $\ell \geq 3$. Let $\boldsymbol{H}$ be the upper half plane, and Γ_ℓ the principal congruence subgroup of $\mathrm{SL}(2;\boldsymbol{Z})$ of level ℓ:

$$\Gamma_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\boldsymbol{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \ell \right\}.$$

Γ_ℓ freely acts on $\boldsymbol{H}$ as linear fractional transformations:

$$z \mapsto g(z) = \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\ell, \quad z \in \boldsymbol{H}.$$

Γ_ℓ also acts on $\boldsymbol{Z}^2$ as right linear transformations:

$$(m,n) \mapsto (m,n)g = (am+cn, bm+dn) \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\ell, \quad (m,n) \in \boldsymbol{Z}^2.$$

Let $\Gamma_\ell L$ be the semiproduct $\Gamma_\ell \ltimes \boldsymbol{Z}^2$ determined by the action above. $\Gamma_\ell L$ freely and discontinuously acts on the product $\boldsymbol{H} \times \boldsymbol{C}$ as follows:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} (m,n) \right] (z,\xi) = \left(\frac{az+b}{cz+d}, \frac{\xi + mz + n}{cz+d} \right) \quad \text{for } (z,\xi) \in \boldsymbol{H} \times \boldsymbol{C}.$$

Since the projection $\boldsymbol{H} \times \boldsymbol{C} \to \boldsymbol{H}$ is equivariant with respect to the two actions above, of $\Gamma_\ell L$ on $\boldsymbol{H} \times \boldsymbol{C}$ and of Γ_ℓ on $\boldsymbol{H}$, the quotient

$$\boldsymbol{H} \times \boldsymbol{C}/\Gamma_\ell L \to \boldsymbol{H}/\Gamma_\ell$$

$$[(z,\xi)] \mapsto [z]$$

is a well-defined analytic fiber bundle of elliptic curves, called the universal elliptic curve with level ℓ structure. The base space $\boldsymbol{H}/\Gamma_\ell$ is a Riemann surface with i/ℓ punctures, where i is a multiple of ℓ equal to the index $[\mathrm{PSL}(2;\boldsymbol{Z}) : \Gamma_\ell]$.

The second step to construct $\mathcal{Q}_\ell$ is to compactify the fiber bundle above. In order to compactify the base space $\boldsymbol{H}/\Gamma_\ell$, it is natural to fill in each of the i/ℓ punctures in $\boldsymbol{H}/\Gamma_\ell$ with a point. Let $\widehat{\boldsymbol{H}/\Gamma_\ell}$ be the compact Riemann surface thus compactified. Each filled-in puncture in $\widehat{\boldsymbol{H}/\Gamma_\ell}$ shall be called a *cusp*. The analytic fiber bundle $\boldsymbol{H} \times \boldsymbol{C}/\Gamma_\ell L \to \boldsymbol{H}/\Gamma_\ell$ of elliptic curves then is extended to an analytic fiber space

$$\pi_\ell : \mathscr{U}_\ell \to \widehat{\boldsymbol{H}/\Gamma_\ell}.$$

The total space $\mathscr{U}_\ell$ is a complex surface, called an elliptic surface: a general fiber of $\mathscr{U}_\ell$ is an elliptic curve; $\mathscr{U}_\ell$ has i/ℓ singular fibers each of which is an ℓ-gon of projective lines over a cusp in $\widehat{\boldsymbol{H}/\Gamma_\ell}$. $\mathscr{U}_\ell$ has ℓ^2 disjoint natural analytic sections $s_1, \dots, s_{\ell^2}$ each of which does not attain as a value any node on any singular fiber, described as

$$[z] \mapsto \left[\left(z, \frac{mz+n}{\ell}\right)\right] \quad \text{where } (m,n) \in \boldsymbol{Z}/\ell\boldsymbol{Z} \times \boldsymbol{Z}/\ell\boldsymbol{Z}.$$

These sections form a group identified with $\boldsymbol{Z}/\ell\boldsymbol{Z} \times \boldsymbol{Z}/\ell\boldsymbol{Z}$, which analytically acts on $\mathscr{U}_\ell$ as translation along fiber. This action is the extension of the following action of $\boldsymbol{Z}/\ell\boldsymbol{Z} \times \boldsymbol{Z}/\ell\boldsymbol{Z}$ on $\boldsymbol{H} \times \boldsymbol{C}/\Gamma_\ell L$:

$$[(z,\xi)] \mapsto \left[\left(z, \xi + \frac{mz+n}{\ell}\right)\right] \quad \text{where } (m,n) \in \boldsymbol{Z}/\ell\boldsymbol{Z} \times \boldsymbol{Z}/\ell\boldsymbol{Z}.$$

The last step to construct $\mathscr{Q}_\ell$ is to take the quotient of $\mathscr{U}_\ell$ by the action above of $\boldsymbol{Z}/\ell\boldsymbol{Z} \times \boldsymbol{Z}/\ell\boldsymbol{Z}$ on $\mathscr{U}_\ell$. Let $\mathscr{Q}_\ell$, π'_ℓ be the quotients of $\mathscr{U}_\ell, \pi_\ell$ by the action respectively. The quotient

$$\pi'_\ell : \mathscr{Q}_\ell \to \widehat{\boldsymbol{H}/\Gamma_\ell}$$

is an analytic fiber space over the compact Riemann surface $\widehat{\boldsymbol{H}/\Gamma_\ell}$ again. A general fiber of $\mathscr{Q}_\ell$ is an elliptic curve. $\mathscr{Q}_\ell$ has i/ℓ singular fibers each of which, lying over a cusp in $\widehat{\boldsymbol{H}/\Gamma_\ell}$, is a projective line which intersects itself at a double point; the double point of a singular fiber of $\mathscr{Q}_\ell$ comes from the ℓ double points of a singular fiber of $\mathscr{U}_\ell$, an ℓ-gon of projective lines. $\mathscr{Q}_\ell$ has an analytic section s which comes from the ℓ^2 sections $s_1, \dots, s_{\ell^2}$ of $\mathscr{U}_\ell$. The section s does not attain as a value the double point of any singular fiber of $\mathscr{Q}_\ell$ (Figure 5 in §2). It turns out that the classifying map from the base space $\widehat{\boldsymbol{H}/\Gamma_\ell}$ of $\mathscr{Q}_\ell$ to the moduli space $\overline{\mathscr{M}}_{1,1}$ of stable curves of genus 1 with 1 marked point is surjective.

The construction of $\mathscr{U}_2$ is almost parallel to that of $\mathscr{U}_\ell$ as follows.

Consider the case when $\ell = 2$ in the construction of $\mathscr{U}_\ell$, $\ell \geq 3$, i.e. the first and second steps of the construction of $\mathscr{Q}_\ell$. In a way similar to the first step to construct $\mathscr{Q}_\ell$, an analytic fiber bundle $\boldsymbol{H} \times \boldsymbol{C}/\Gamma_2 L \to \boldsymbol{H}/\Gamma_2$ is constructed. The fibers of $\boldsymbol{H} \times$

$\boldsymbol{C}/\Gamma_2 L$, however, are not elliptic curves but projective lines, since the principal congruence subgroup Γ_2 of $\mathrm{SL}(2;\boldsymbol{Z})$ of level 2 contains the torsion element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. (The action of Γ_2 on $\boldsymbol{H}$ is not free.) In a way similar to the second step to construct $\mathscr{Q}_\ell$, the fiber bundle $\boldsymbol{H} \times \boldsymbol{C}/\Gamma_2 L \to \boldsymbol{H}/\Gamma_2$ extends to an analytic fiber space $\pi_2 : \mathscr{U}_2 \to \widehat{\boldsymbol{H}/\Gamma_2}$. The base space $\widehat{\boldsymbol{H}/\Gamma_2}$ of $\mathscr{U}_2$ is a projective line with three cusps 0, 1, and ∞. A general fiber of $\mathscr{U}_2$ is a projective line. $\mathscr{U}_2$ has three singular fibers each of which, lying over a cusp in $\widehat{\boldsymbol{H}/\Gamma_2}$, is a union of two projective lines which intersect at a double point. $\mathscr{U}_2$ has four disjoint analytic sections s_1, s_2, s_3, s_4, called the Weierstrass points, each of which does not attain as a value the node of any singular fiber (Figure 6 in §2). It turns out that the classifying map from the base space $\widehat{\boldsymbol{H}/\Gamma_2}$ of $\mathscr{U}_2$ to the moduli space $\overline{\mathscr{M}}_{0,4}$ of stable curves of genus 0 with 4 marked points is surjective.

REMARK 3.1. Wolpert [**W**] further showed the following. Let $\mathscr{S}$ be either $\mathscr{Q}_\ell$ or $\mathscr{U}_2$. Let T' be the line bundle over $\mathscr{S} - \{\text{double points}\}$ which consists of all the vectors tangent to fibers of $\mathscr{S}$. Then $c_1(s^*T') = -i/12$ for $\mathscr{S} = \mathscr{Q}_\ell$ ($i = [\mathrm{PSL}(2;\boldsymbol{Z}) : \Gamma_\ell]$); $c_1(s_\nu^* T') = -1$ for all $\nu = 1,2,3,4$ for $\mathscr{S} = \mathscr{U}_2$. We shall need these facts in the next section.

As for the details of $\mathscr{U}_\ell$ ($\ell \geq 2$) and $\mathscr{Q}_\ell$ ($\ell \geq 3$), refer to [**W**].

Using the fiber spaces $\mathscr{U}_\ell$ and $\mathscr{Q}_\ell$ above, we construct the last fiber space

$$\tilde{\pi}_\ell : \tilde{\mathscr{Q}}_\ell \to \mathscr{U}_\ell.$$

We first take the pull-back of $\pi'_\ell : \mathscr{Q}_\ell \to \widehat{\boldsymbol{H}/\Gamma_\ell}$ by $\pi_\ell : \mathscr{U}_\ell \to \widehat{\boldsymbol{H}/\Gamma_\ell}$, which we denote by $\hat{\pi}_\ell : \hat{\mathscr{Q}}_\ell \to \mathscr{U}_\ell : \hat{\mathscr{Q}}_\ell = \{(x,\xi) \in \mathscr{U}_\ell \times \mathscr{Q}_\ell \mid \pi_\ell(x) = \pi'_\ell(\xi)\}$, $\hat{\pi}_\ell(x,\xi) = x$, and

$$\begin{array}{ccc} \hat{\mathscr{Q}}_\ell & \longrightarrow & \mathscr{Q}_\ell \\ {\scriptstyle \hat{\pi}_\ell}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi'_\ell} \\ \mathscr{U}_\ell & \underset{\pi_\ell}{\longrightarrow} & \widehat{\boldsymbol{H}/\Gamma_\ell}. \end{array}$$

Note that the space $\hat{\mathscr{Q}}_\ell$ inherits the following from $\mathscr{Q}_\ell$: $\hat{\mathscr{Q}}_\ell$ is an analytic fiber space; the base space $\mathscr{U}_\ell$ of $\hat{\mathscr{Q}}_\ell$ is an elliptic surface, explained in the second step of the construction of $\mathscr{Q}_\ell$; a general fiber of $\hat{\mathscr{Q}}_\ell$ is an elliptic curve; the singular fibers of $\hat{\mathscr{Q}}_\ell$ lie over the singular fibers of $\mathscr{U}_\ell$. Moreover, $\hat{\mathscr{Q}}_\ell$ has two analytic sections

$$\hat{s}_1 : x \mapsto (x, s(\pi_\ell(x))),$$

$$\hat{s}_2 : x \mapsto (x, p(x)),$$

where p is a natural projection from $\mathscr{U}_\ell$ to $\mathscr{Q}_\ell = \mathscr{U}_\ell / \boldsymbol{Z}_\ell \times \boldsymbol{Z}_\ell$. Unfortunately, at any point in the subset $\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})$ of $\mathscr{U}_\ell$ the sections $\hat{s}_1$ and $\hat{s}_2$ attain the same value, and at any node of any singular fiber of $\mathscr{U}_\ell$ $\hat{s}_2$ attains as a value the node of a singular fiber of $\hat{\mathscr{Q}}_\ell$: accordingly, we need to modify the fiber space $\hat{\mathscr{Q}}_\ell$ in order to have two disjoint analytic sections each of which does not attain as a value the node of any singular fiber of $\hat{\mathscr{Q}}_\ell$.

We then blow up the fiber space $\hat{\mathscr{Q}}_\ell$ along the divisor $\hat{s}_1(\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell}))$ and at each of the points $\hat{s}_2$(nodes), and denote by $\tilde{\mathscr{Q}}_\ell$ (resp. $\tilde{\pi}_\ell$) the resulting fiber space (resp. projection map). It is easy to blow up $\hat{\mathscr{Q}}_\ell$ along the divisor $\hat{s}_1(\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell}))$, since $\hat{s}_1(\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell}))$ itself is smooth and does not intersect any singlarity of $\hat{\mathscr{Q}}_\ell$. It is not so easy to blow up $\hat{\mathscr{Q}}_\ell$ at each of the points $\hat{s}_2$(nodes), since $\hat{s}_2$(nodes) are singular. In fact, a neighborhood of a point $\hat{s}_2$(a node) in $\hat{\mathscr{Q}}_\ell$ is identified with the analytic space

$$U' = \{(u, v, u', v') \in \boldsymbol{C}^4; |u| < 1, |v| < 1, |u'| < 1, |v'| < 1, u^\ell v^\ell = u'v'\},$$

where the point $\hat{s}_2$(a node) is identified with $(0, 0, 0, 0)$. In this neighborhood U', $\hat{\pi}_\ell$ and $\hat{s}_2$ are described as

$$\hat{\pi}_\ell(u, v, u', v') = (u, v),$$

$$\hat{s}_2(u, v) = (u, v, u^\ell, v^\ell),$$

where (u, v) is a point in the neighborhood $U = \hat{\pi}_\ell(U') \subset \mathscr{U}_\ell$ of the node on a singular fiber of $\mathscr{U}_\ell$. We define an analytic map f from $U' - \{(0, 0, 0, 0)\}$ to $\boldsymbol{P}^1$ as

$$f : U' - \{(0, 0, 0, 0)\} \to \boldsymbol{P}^1,$$

$$(u, v, u', v') \mapsto \begin{cases} (v^\ell : v') & \text{if } v \neq 0 \quad \text{or} \quad v' \neq 0, \\ (u' : u^\ell) & \text{if } u \neq 0 \quad \text{or} \quad u' \neq 0. \end{cases}$$

The well-definedness of this map is easy to see. We denote by $\tilde{U}'$ the closure of the graph of f in $U' \times \boldsymbol{P}^1$. That is to say, $\tilde{U}'$ is an analytic space obtained by blowing up U' at the point $(0, 0, 0, 0)$. In this way, we blow up the fiber space $\hat{\mathscr{Q}}_\ell$, so as to obtain $\tilde{\pi}_\ell : \tilde{\mathscr{Q}}_\ell \to \mathscr{U}_\ell$.

Note that the space $\tilde{\mathscr{Q}}_\ell$ inherits the following from $\hat{\mathscr{Q}}_\ell : \tilde{\mathscr{Q}}_\ell$ is an analytic fiber space; the base space $\mathscr{U}_\ell$ of $\tilde{\mathscr{Q}}_\ell$ is the same elliptic surface as that of $\hat{\mathscr{Q}}_\ell$; a general fiber of $\tilde{\mathscr{Q}}_\ell$ is an elliptic curve. It is easy to check that the singular fibers of $\tilde{\mathscr{Q}}_\ell$ lie over the union of the singular fibers of $\mathscr{U}_\ell$ and the sections $s_1, \dots, s_{\ell^2}$ of $\mathscr{U}_\ell$, and are classified into the four types mentioned in §2 (Figure 7 in §2). As a result of blowing up, $\tilde{\mathscr{Q}}_\ell$ has two disjoint analytic sections $\tilde{s}_1$ and $\tilde{s}_2$ coming from $\hat{s}_1$ and $\hat{s}_2$, either of which does not attain as a value any node of any singular fiber of $\tilde{\mathcal{Q}}_\ell$. It turns out that the classifying map from the base space $\mathcal{U}_\ell$ of $\tilde{\mathcal{Q}}_\ell$ to the moduli space $\overline{\mathcal{M}}_{1,2}$ of stable curves of genus 1 with 2 marked points is surjective.

§4. Proof of linear independence of cycles.

In this section we prove that the cycles $[\mathcal{A}_\sigma]$'s, $[\mathcal{V}_\sigma]$'s we construct in §2 represent linearly independent classes in $H_*(\overline{\mathcal{M}}_g; \boldsymbol{Q})$ by showing that the intersection pairing of $[\mathcal{A}_\sigma]$'s and $[\mathcal{V}_\sigma]$'s is non-degenerate.

In order to calculate the intersection number of $[\mathcal{A}_\sigma]$ and $[\mathcal{V}_\tau]$ for k-selections σ and τ, we slightly push $[\mathcal{A}_\sigma]$ off the compactification locus $\mathcal{D}$ of $\overline{\mathcal{M}}_g$, which contains $[\mathcal{V}_\tau]$, by opening up those nodes of the fibers of $\mathcal{A}_\sigma$ which correspond to the collapsed curves on S_σ, as illustrated in Figure 10.

Recall that in the construction (§2) of $\mathcal{A}_\sigma$ the collapsed curves on S_σ were identified with the sections

$$s : \widehat{\boldsymbol{H}/\Gamma_\ell} \to \mathcal{Q}_\ell,$$

$$s_j : \widehat{\boldsymbol{H}/\Gamma_2} \to \mathcal{U}_2,$$

$$\tilde{s}_1, \tilde{s}_2 : \mathcal{U}_\ell \to \tilde{\mathcal{Q}}_\ell.$$

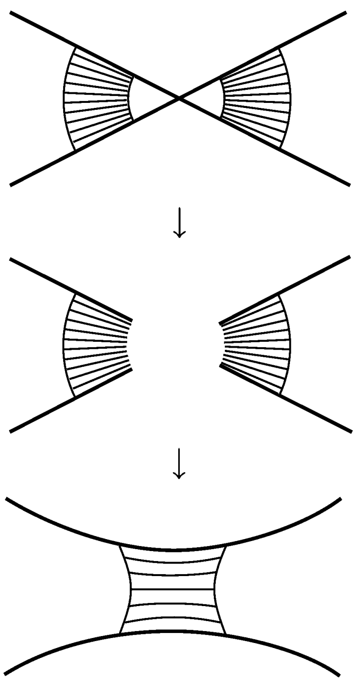


Figure 10. Opening up a node of a fiber of $\mathcal{A}_\sigma$

To open up the nodes of the fibers of $\mathscr{A}_\sigma$ corresponding to collapsed curves on S_σ, we smoothly and slightly perturb the sections above.

We construct perturbations of s, s_j as in [**W**]. Since $c_1(s^*T') = -i/12$, we choose a smooth perturbation s' of s such that at only one point $p \in \mathscr{Q}_\ell$ s' intersects s as

$$\begin{cases} s(z) = (z, 0), \\ s'(z) = (z, \bar{z}^{i/12}), \end{cases}$$

where z is a coordinate around $\pi'_\ell(p)$ in $\widehat{\boldsymbol{H}/\Gamma_\ell}$ and (z, ξ) are coordinates around p in $\mathscr{Q}_\ell$ such that $\pi'_\ell(z, \xi) = z$. Since $c_1(s_j^*T') = -1$, we choose a smooth perturbation s'_j of s_j such that at only one point $q_j \in \mathscr{U}_2$ s'_j intersects s_j as

$$\begin{cases} s_j(z) = (z, 0), \\ s'_j(z) = (z, \bar{z}), \end{cases}$$

where z is a coordinate around $\pi_2(q_j)$ in $\widehat{\boldsymbol{H}/\Gamma_2}$ and (z, ξ) are coordinates around q_j in $\mathscr{U}_2$ such that $\pi_2(z, \xi) = z$. The points $q_1, \dots, q_4$ are on mutually distinct general fibers. (See Remark 3.1 in §3.)

Next, we construct perturbations of $\tilde{s}_1$, $\tilde{s}_2$ as follows. Let $\widetilde{T'}$ be a line bundle over $\tilde{\mathscr{Q}}_\ell - \{\text{nodes}\}$ which consists of all the tangent vectors of the fibers. The Poincaré dual of the Euler class of $\tilde{s}_k^*\widetilde{T'}$ then is equal to $-i/12[\text{general fiber}] - \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]$ for $k = 1, 2$. We choose a smooth perturbation $\tilde{s}'_1$ of $\tilde{s}_1$ as follows. Let $\hat{s}'_1$ be the pull back of the section s'_1. $\hat{s}'_1$ is a smooth perturbation of $\hat{s}_1$, and intersects $\hat{s}_1$ over a general fiber of $\mathscr{U}_\ell$ with multiplicity $-i/12$. Since we may assume that $\hat{s}'_1$ does not attain as a value the node of any singular fiber of $\hat{\mathscr{Q}}_\ell$, we choose $\tilde{s}'_1$ such that only over the union of a general fiber and $\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})$ $\tilde{s}'_1$ intersects $\tilde{s}_1$ subject to the class $-i/12[\text{general fiber}] - \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]$. (Recall that $\tilde{\mathscr{Q}}_\ell$ is constructed from $\hat{\mathscr{Q}}_\ell$ by blowing up $\hat{\mathscr{Q}}_\ell$ along $\hat{s}_1(\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell}))$ and at $\hat{s}_2(\text{nodes})$.) We similarly choose $\tilde{s}'_2$ such that only over the union of a general fiber and $\bigcup_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})$ $\tilde{s}'_2$ intersects $\tilde{s}_2$ subject to the class $-i/12[\text{general fiber}] - \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]$.

Using the perturbed sections $s', s'_1, \dots, s'_4, \tilde{s}'_1$ and $\tilde{s}'_2$, we open up the nodes of the fibers of $\mathscr{A}_\sigma$ corresponding to collapsed curves on S_σ, as follows. We shall describe only the case of opening up such nodes using $\tilde{s}'_1$ and $\tilde{s}'_2$. (We similarly open up such nodes using $s', s'_1, \dots, s'_4$.)

Case 1: $d_j, d'_j \in \sigma$ but $d_{j-1}, d'_{j-1} \notin \sigma$. Recall that in the construction of $\mathscr{A}_\sigma$ the collapsed curve c_{j-1} is identified with the section $\tilde{s}_1$. We thus open up the node to which c_{j-1} collapses by using $\tilde{s}'_1$ instead of $\tilde{s}_1$. Fix a metric $\|\cdot\|$ on the line bundle $\tilde{s}_k^*\widetilde{T'}$. Choose a neighborhood $\mathscr{U}$ of the zero section of $\tilde{s}_k^*\widetilde{T'}$ which is injectively mapped into the fiber space $\tilde{\mathscr{Q}}_\ell$. Identifying $\mathscr{U}$ with its image in $\tilde{\mathscr{Q}}_\ell$, we assume that $\mathscr{U}$ intersects each fiber of $\tilde{\mathscr{Q}}_\ell$ in a unit disk centered at the origin, that the section $\tilde{s}_1'$ is contained in $\mathscr{U}$, and that $\|\tilde{s}_1'\| < 1$. Choose a local coordinate chart (z_1, z_2, ζ) of $\mathscr{U}$ in $\tilde{\mathscr{Q}}_\ell$ such that $|\zeta| = \|\zeta\|$. The section $\tilde{s}_1'$ is described with (z_1, z_2, ζ) as $\zeta = \tilde{s}_1'(z_1, z_2)$. Let w be a coordinate of a neighborhood of the node to which c_{j-1} collapses in the fixed component of S_σ containing the node. We assume that w maps this neighborhood to the unit disk in $\boldsymbol{C}$. If $\tilde{s}_1'(z_1, z_2) \neq \tilde{s}_1(z_1, z_2)$ (i.e. if $\tilde{s}_1'(z_1, z_2) \neq 0$), remove the disk $\{|w| \leq \|\tilde{s}_1'(z_1, z_2)\|\}$ from S_σ, remove the disk $\{\|\zeta\| \leq \|\tilde{s}_1'(z_1, z_2)\|\}$ from the fiber $F_{(z_1, z_2)}$ of $\tilde{\mathscr{Q}}_\ell$ over the point (z_1, z_2), and form a connected sum of the resulting surfaces by identifying $\{\|\tilde{s}_1'(z_1, z_2)\| < |w| < 1\}$ with $\{\|\tilde{s}_1'(z_1, z_2)\| < |\zeta| < 1\}$ subject to $w\zeta = \tilde{s}_1'(z_1, z_2)$. If $\tilde{s}_1'(z_1, z_2) = \tilde{s}_1(z_1, z_2)$ (i.e. if $\tilde{s}_1'(z_1, z_2) = 0$), we do not open up the node.

Case 2: $d_j, d_j' \in \sigma$ but $d_{j+1}, d_{j+1}' \notin \sigma$. We similarly open up the node to which c_{j+1} collapses by using $\tilde{s}_2'$ instead of $\tilde{s}_2$.

Case 3: $d_j, d_j' \in \sigma$ and $d_{j+1}, d_{j+1}' \in \sigma$. Choose local coordinates (z_1, z_2, ζ_1) around the section $\tilde{s}_1$ in $\tilde{\mathscr{Q}}_\ell$ with which we replaced the d_{j+1}-component, (x_1, x_2, ζ_2) around the section $\tilde{s}_2$ in $\tilde{\mathscr{Q}}_\ell$ with which we replaced the d_j-component. As in Case 1, if $\|\tilde{s}_1'(z_1, z_2)\| \cdot \|\tilde{s}_2'(x_1, x_2)\|$ is not equal to zero, remove the disk neighborhoods $\{|\zeta_1| \leq \|\tilde{s}_1'(z_1, z_2)\| \cdot \|\tilde{s}_2'(x_1, x_2)\|\}$ and $\{|\zeta_2| \leq \|\tilde{s}_2'(x_1, x_2)\|\,\|\tilde{s}_1'(z_1, z_2)\|\}$ of the sections $\tilde{s}_1$ and $\tilde{s}_2$, and form a connected sum of the resulting surfaces by identifying $\{\|\tilde{s}_1'\|\,\|\tilde{s}_2'\| < |\zeta_1| < 1\}$ with $\{\|\tilde{s}_2'\| \cdot \|\tilde{s}_1'\| < |\zeta_2| < 1\}$ subject to $\zeta_1\zeta_2 = \tilde{s}_1'\tilde{s}_2'$. If $\|\tilde{s}_1'(z_1, z_2)\|\,\|\tilde{s}_2'(x_1, x_2)\|$ is equal to zero, we do not open up the node.

It is easy to check that each of the procedures above of opening up a node does not depend on the choice of coordinates around the node. In this way we open up the nodes of the fibers of $\mathscr{A}_\sigma$ corresponding to collapsed curves on S_σ, and denote by $\mathscr{A}_\sigma^\sharp$ the resulting smooth fiber space of stable curves of genus g. Each fiber of $\mathscr{A}_\sigma^\sharp$ is assumed to have at most k nodes. It is obvious that $\mathscr{A}_\sigma^\sharp$ determines a cycle $[\mathscr{A}_\sigma^\sharp]$ on $\bar{\mathscr{M}}_g$ homotopic to the cycle $[\mathscr{A}_\sigma]$.

We shall consider the intersection pairing of $[\mathscr{A}_\sigma^\sharp]$'s and $[\mathscr{V}_\sigma]$'s, instead of that of $[\mathscr{A}_\sigma]$'s and $[\mathscr{V}_\sigma]$'s.

Before calculating the intersection pairing of $[\mathscr{A}_\sigma^\sharp]$'s and $[\mathscr{V}_\sigma]$'s, we give some remarks according to Wolpert [**W**].

A V-manifold such as $\bar{\mathscr{M}}_g$ is a rational homology manifold. If two cycles on a V-manifold intersect at manifold points, then the intersection number of the two cycles is calculated in the standard intuitive way. However, if two cycles on a V-manifold intersect at some non-manifold points, then it is necessary to perturb them so that they intersect at manifold points, in order to calculate the intersection number of them. To perturb a cycle on a V-manifold around a non-manifold point, it is necessary to consider a local manifold cover of the V-manifold around the non-manifold point.

We describe coordinates for the local manifold cover of $\bar{\mathscr{M}}_g$ around a point in $\mathscr{D} = \bar{\mathscr{M}}_g - \mathscr{M}_g$. Let S be a Riemann surface of genus g with nodes $p_1, \dots, p_m$ such that each component of $S - \{p_1, \dots, p_m\}$ is hyperbolic. Let a_i, b_i be the two punctures of $S - \{p_i\}$ which correspond to the node p_i of S. Choose disjoint neighborhoods D_i^1, D_i^2 $(i = 1, 2, \dots, m)$ of the punctures a_i and b_i respectively, and let $z_i : D_i^1 \to D = \{u \in C; |u| < 1\}$ and $w_i : D_i^2 \to D$ be local coordinates with $z_i(a_i) = w_i(b_i) = 0$. Fixing an suitable open set $\mathscr{O}$ disjoint from D_i^1 and D_i^2, choose Beltrami differentials μ_j with support in $\mathscr{O}$ spanning the Teichmüller space of $S - \{p_1, \dots, p_m\}$ (of dimension $3g - 3 - m$). If $t = (t_1, \dots, t_{3g-3-m}) \in \boldsymbol{C}^{3g-3-m}$ is sufficiently close to the origin, the sum $\mu(t) = \sum_j t_j \mu_j$ satisfies $\|\mu(t)\|_\infty < 1$ and thus a μ-conformal solution $\omega^{\mu(t)}$ of the Beltrami equation exists. The Riemann surface $\omega^{\mu(t)}(S) = S_t$ is a quasiconformal deformation of S. The map $\omega^{\mu(t)}$ is conformal on D_i^1 and D_i^2; therefore z_i and w_i serve as coordinates for $\omega^{\mu(t)}(D_i^1)$ and $\omega^{\mu(t)}(D_i^2) \subset S_t$ respectively. Given $\tau = (\tau_1, \dots, \tau_m) \in D^m$, we construct a surface $S_{\tau,t}$ as follows. Remove the disks $\{z_i; |z_i| \leq |\tau_i|\}$ and $\{w_i; |w_i| \leq |\tau_i|\}$ from S_τ. Attach $\{z_i; |\tau_i| < |z_i| < 1\}$ to $\{w_i; |\tau_i| < |w_i| < 1\}$ by identifying z_i and τ_i/w_i to obtain $S_{\tau,t}$. The couple (τ, t) gives holomorphic coordinates for the local manifold cover of $\bar{\mathscr{M}}_g$ around the point represented by S. The automorphism group $\mathrm{Aut}(S)$ locally acts on these coordinates (see also [**B**]).

To calculate the intersection number of $[\mathscr{A}_\tau^\sharp]$ and $[\mathscr{V}_\sigma]$ for k-selections τ and σ, we begin with giving a sufficient condition for $[\mathscr{A}_\tau^\sharp]$ and $[\mathscr{V}_\sigma]$ to intersect at manifold points of $\bar{\mathscr{M}}_g$.

Lemma 4.1. *Let σ be a k-selection. Assume that if $g \geq 4$, not all of $d_2, d_2', \dots, d_{g-1}, d_{g-1}'$ are selected in σ simultaneously. Then, for any k-selection cycle $[\mathscr{A}_\tau]$, the cycle $[\mathscr{A}_\tau^\sharp]$ can be chosen to intersect the k-selection subvariety cycle $[\mathscr{V}_\sigma]$ at manifold points of $\bar{\mathscr{M}}_g$.*

Proof. Let S_∞ be a fiber of $\mathscr{A}_\tau^\sharp$ over an intersection point of $[\mathscr{A}_\tau^\sharp]$ and $[\mathscr{V}_\sigma]$. We consider the automorphism group $\mathrm{Aut}(S_\infty)$ of S_∞.

Assume that, in the complement of those nodes of S_∞ which correspond to the curves selected in σ, S_∞ has m connected components: $S_1, \dots, S_m$. We divide S_j's into the following three types:

i) those S_j's each of which is either of an elliptic curve with one puncture and a projective line with one puncture which intersects itself at a double point;

ii) those S_j's each of which is one of an elliptic curve with two punctures, a projective line with two punctures which intersects itself at a double point, and a projective line with four punctures;

iii) those S_j's each of which is one of a curve of genus at least 2 with one puncture, a curve of genus at least 2 with two punctures, a curve of genus at least 1 with three punctures, and a curve of genus at least 1 with four punctures.

Since we can arbitrarily vary each component S_j in an open set of its Teichmüller space, we may assume $\mathrm{Aut}(S_j)$ as follows:

I) if S_j is of type i), then $\mathrm{Aut}(S_j)$ is generated by an elliptic involution;

II) if S_j is of type ii), then $\mathrm{Aut}(S_j)$ is not trivial but the group $\mathrm{Aut}_f(S_j)$ of automorphisms of S_j fixing the punctures of S_j is trivial;

III) if S_j is of type iii), then $\mathrm{Aut}(S_j)$ is trivial.

As a result of topological consideration, we see that the only homeomorphism of S_∞ which may permute the components $S_1, \dots, S_m$ is the left-right switch map. Since we may assume that the conformal structures for $S_1, \dots, S_m$ are distinct, we hence observe that the left-right switch map cannot be isotopic to any element in $\mathrm{Aut}(S_\infty)$. We therefore conclude that any element in $\mathrm{Aut}(S_\infty)$ does not permute the components $S_1, \dots, S_m$.

When $g \geq 4$, since at least one of the curves $d_2, d_2', \dots, d_{g-1}, d_{g-1}'$ is not selected in σ, there exists a component S_j of type iii) in S_∞. Thus, any element of $\mathrm{Aut}(S_\infty)$ fixes all the punctures and $\mathrm{Aut}(S_\infty) = \prod_j \mathrm{Aut}_f(S_j)$. Hence, we have only to consider components S_j of type i). There are three possibilities for $\mathrm{Aut}(S_\infty)$:

a) $\mathrm{Aut}(S_\infty)$ is trivial;

b) $\mathrm{Aut}(S_\infty) = \boldsymbol{Z}/2\boldsymbol{Z}$;

c) $\mathrm{Aut}(S_\infty) = (\boldsymbol{Z}/2\boldsymbol{Z})^2$.

These three possibilities correspond to the following cases respectively:

A) none of c_1, c_{g-1} are selected in σ;

B) exactly one of c_1, c_{g-1} is selected in σ;

C) both of c_1, c_{g-1} are selected in σ.

In case A) S_∞ certainly represents a manifold point. In case B) assume that c_1 is selected in σ; since the non-trivial element $k \in \mathrm{Aut}(S_\infty)$ is an elliptic involution, introduce coordinates (τ_1, t) of a local manifold cover of $\overline{\mathscr{M}}_g$ around the intersection point, where τ_1 is, say, for the c_1 node; since the elliptic involution k is assumed to be generic for the elliptic curve S_j corresponding to the component of S_σ' with d_1 so that k acts as $k(\tau_1, t) = (-\tau_1, t)$, (τ_1^2, t) give coordinates of $\overline{\mathscr{M}}_g$ around the intersection point. In case C) introduce coordinates (τ_1, τ_{g-1}, t) of a local manifold cover of $\overline{\mathscr{M}}_g$ around the intersection point, where τ_1 is for the c_1 node and τ_{g-1} is for the c_{g-1} node; since

$\mathrm{Aut}(S_\infty)$ has two generators

$$(\tau_1, \tau_{g-1}, t) \mapsto (-\tau_1, \tau_{g-1}, t),$$

$$(\tau_1, \tau_{g-1}, t) \mapsto (\tau_1, -\tau_{g-1}, t).$$

$(\tau_1^2, \tau_{g-1}^2, t)$ give coordinates of $\bar{\mathcal{M}}_g$ around the intersection point.

When $g = 3$ and d_2, d_2' are not selected in σ, we can define coordinates around the intersection point as above.

When $g = 3$ and d_2, d_2' are selected in σ, each component S_j is either an elliptic curve with two punctures or a projective line with two punctures which intersects itself at a double point; it follows that $\mathrm{Aut}(S_\infty) = \boldsymbol{Z}/2\boldsymbol{Z}$; introduce coordinates (τ_1, τ_2, t) of a local manifold cover of $\bar{\mathcal{M}}_g$ around the intersection point, where τ_1 is for the d_2 node and τ_2 is for the d_2' node; since the non-trivial element $k \in \mathrm{Aut}(S_\infty)$ acts as

$$k(\tau_1, \tau_2, t) = (\tau_2, \tau_1, t),$$

$(\tau_1 + \tau_2, (\tau_1 - \tau_2)^2, t)$ give coordinates of $\bar{\mathcal{M}}_g$ around the intersection point. This completes the proof of Lemma 4.1. □

When $g \geq 4$ and all of $d_2, d_2', \dots, d_{g-1}, d_{g-1}'$ occur in a k-selection σ, the k-selection cycle $[\mathcal{A}_\tau^\sharp]$ cannot be chosen not to intersect the k-selection subvariety cycle $[\mathcal{V}_\sigma]$ at any non-manifold point of $\bar{\mathcal{M}}_g$. We thus need to construct a smooth perturbation $\mathcal{V}_\sigma^\sharp$ of $\mathcal{V}_\sigma$ so that $[\mathcal{A}_\tau^\sharp]$ and $[\mathcal{V}_\sigma^\sharp]$ intersect only at manifold points of $\bar{\mathcal{M}}_g$.

We describe a perturbation $\mathcal{V}_\sigma^\sharp$ of $\mathcal{V}_\sigma$ only in the case when

$$\sigma = \{d_2, d_2', \dots, d_{g-1}, d_{g-1}'\}.$$

(We can similarly perturb $\mathcal{V}_\sigma$ in other cases.) Let S_∞ be a fiber of $\mathcal{A}_\tau^\sharp$ over an intersection point with $[\mathcal{V}_\sigma]$. We may assume $\mathrm{Aut}(S_\infty) = \boldsymbol{Z}/2\boldsymbol{Z}$. We introduce a local manifold cover

$$(V, \psi : V \to \bar{\mathcal{M}}_g, \psi(V))$$

and coordinates

$$(\tau_2, \tau_2', \dots, \tau_{g-1}, \tau_{g-1}', t_1, \dots, t_m)$$

for V around the intersection point in $\bar{\mathcal{M}}_g$, where τ_j is for the d_j node and τ_j' is for the d_j' node. The non-trivial element $k \in \mathrm{Aut}(S_\infty)$ acts as

$$k(\tau_2, \tau_2', \dots, \tau_{g-1}, \tau_{g-1}', t_1, \dots, t_m) = (\tau_2', \tau_2, \dots, \tau_{g-1}', \tau_{g-1}, t_1, \dots, t_m).$$

In this local manifold cover, $\mathscr{V}_\sigma$ is given as the locus

$$\{\tau_2 = \tau_2' = \cdots = \tau_{g-1} = \tau_{g-1}' = 0\},$$

and mapped injectively to $\overline{\mathscr{M}}_g$. We introduce a local neighborhood $V' \subset \mathscr{V}_\sigma$ and coordinates $(x_1, \ldots, x_m)$ for V' around the point in $\mathscr{V}_\sigma$ represented by S_∞. $(x_1, \ldots, x_m)$ is mapped to $(0, \ldots, 0, x_1, \ldots, x_m)$ in the local manifold cover. Let $\varepsilon(x)$ be a function on $\mathscr{V}_\sigma$ such that V' contains the support of $\varepsilon(x)$ and $\varepsilon(x) = \varepsilon$ in a smaller neighborhood of S_∞, where ε is a small constant. We perturb $\mathscr{V}_\sigma$ in $\psi(V)$ to obtain $\mathscr{V}_\sigma^\sharp$ subject to

$$(x_1, \ldots, x_m) \to \psi(\varepsilon(x), 0, \ldots, 0, x_1, \ldots, x_m) \in \psi(V).$$

$\mathscr{V}_\sigma^\sharp$ is homotopic to $\mathscr{V}_\sigma$ and we may assume that $[\mathscr{A}_\tau^\sharp]$ intersects $[\mathscr{V}_\sigma^\sharp]$ only at manifold points.

In order to prove that the intersection pairing is non-degenerate, we introduce the concept of $\widehat{\ }$-*intersection number*.

Before we define $\widehat{\ }$-*intersection numbers*, we remark the following. If a k-selection σ is not symmetric and σ is neither $\{d_1\}$ nor $\{d_g\}$, the intersection points of $[\mathscr{A}_\tau^\sharp]$ and $[\mathscr{V}_\sigma]$ are divided into two types for any k-selection τ. For example, if c_i is in σ but c_{g-i} is not in σ, the intersection points are divided into the following two types:

(1) points corresponding to the fibers of $\mathscr{A}_\tau^\sharp$ where c_i is collapsed to a node and c_{g-i} is not collapsed to a node.

(2) points corresponding to the fibers of $\mathscr{A}_\tau^\sharp$ where c_{g-i} is collapsed to a node and c_i is not collapsed to a node.

We thus define $\widehat{\ }$-*intersection numbers* when σ is neither $\{d_1\}$ nor $\{d_g\}$.

Definition 4.2. (1) When a k-selection σ is not symmetric and σ is neither $\{d_1\}$ nor $\{d_g\}$, we define a $\widehat{\ }$-*intersection number* $\widehat{[\mathscr{A}_\tau] \cdot [\mathscr{V}_\sigma]}$ for any k-selection τ as follows:

(a) If c_i is in σ but c_{g-i} is not in σ, we define $\widehat{[\mathscr{A}_\tau] \cdot [\mathscr{V}_\sigma]}$ as the sum of the intersection numbers only on the intersection points corresponding to the fibers of $\mathscr{A}_\tau^\sharp$ each of which has the c_i node but does not have the c_{g-i} node.

(b) If d_i, d_i' are in σ but d_{g-i+1}, d_{g-i+1}' are not in σ, we define $\widehat{[\mathscr{A}_\tau] \cdot [\mathscr{V}_\sigma]}$ as the sum of the intersection numbers only on the intersection points corresponding to the fibers of $\mathscr{A}_\tau^\sharp$ each of which has both the d_i node and the d_i' node but does not have either the d_{g-i+1} node or the d_{g-i+1}' node.

(2) When a k-selection σ is symmetric, we define a $\widehat{\ }$-*intersection number* $\widehat{[\mathscr{A}_\tau] \cdot [\mathscr{V}_\sigma]}$ for any k-selection τ as the usual intersection number $[\mathscr{A}_\tau] \cdot [\mathscr{V}_\sigma]$ in $\overline{\mathscr{M}}_g$.

The well-definedness of $\widehat{\ }$-intersection numbers follows from the fact that c_j, d_i, d_i' are mapped to c_j, d_i, d_i' or $c_{g-j}, d_{g-i+1}, d_{g-i+1}'$ by a homeomorphism of S.

We cannot naturally define $\widehat{\ }$-*intersection numbers* when σ is either $\{d_1\}$ or $\{d_g\}$, since each of these curves may be mapped to any d-curve by some homeomorphism of S. However, for convenience, we will also define $\widehat{\ }$-*intersection numbers* in this case later (Definition 4.6).

Lemma 4.3. (1) $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]} = \widehat{[\mathscr{A}_{\bar\tau}]\cdot[\mathscr{V}_{\bar\sigma}]}$.

(2) *If σ is not symmetric and σ is neither $\{d_1\}$ nor $\{d_g\}$, then*

$$[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma] = [\mathscr{A}_{\bar\tau}]\cdot[\mathscr{V}_\sigma] = \widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]} + \widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_{\bar\sigma}]}.$$

(3) *If σ is symmetric, then*

$$[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma] = [\mathscr{A}_{\bar\tau}]\cdot[\mathscr{V}_\sigma] = \widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}.$$

Proof. As mentioned in §2, the cycle $[\mathscr{A}_\tau]$ is homologous to $[\mathscr{A}_{\bar\tau}]$; hence (3) is immediate from the definition. If σ is not symmetric and σ is neither $\{d_1\}$ nor $\{d_g\}$, the intersection points of $[\mathscr{A}_\tau^\sharp]$ and $[\mathscr{V}_\sigma]$ are divided into the two types mentioned above just before Definition 4.2: the one associated with $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}$, and the other associated with $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_{\bar\sigma}]}$; hence (2) follows. If σ is symmetric, (1) is obvious; if not, (1) follows from the fact that the intersection points associated with $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}$ correspond to the intersection points associated with $\widehat{[\mathscr{A}_{\bar\tau}]\cdot[\mathscr{V}_{\bar\sigma}]}$. □

Lemma 4.4. *If the $\widehat{\ }$-intersection pairing matrix is non-degenerate, then the intersection pairing matrix is non-degenerate.*

Proof. Let $\sigma_1,\dots,\sigma_m$ be all the symmetric k-selections, and $\tau_1,\dots,\tau_n,\bar\tau_1,\dots,\bar\tau_n$ all the non-symmetric k-selections. By Lemma 4.3 (3), the $\widehat{\ }$-intersection pairing matrix is

$$\left(\begin{array}{c|c|c}
[\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\sigma_j}] & \widehat{[\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\tau_j}]} & \widehat{[\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\bar\tau_j}]} \\ \hline
[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & \widehat{[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\tau_j}]} & \widehat{[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\bar\tau_j}]} \\ \hline
[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & \widehat{[\mathscr{A}_{\bar\tau_i}]\cdot[\mathscr{V}_{\tau_j}]} & \widehat{[\mathscr{A}_{\bar\tau_i}]\cdot[\mathscr{V}_{\bar\tau_j}]}
\end{array}\right).$$

By Lemma 4.3, elementary column operations transform it to

$$\left(\begin{array}{c|c|c}
[\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\tau_j}] & \widehat{[\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\bar\tau_j}]} \\ \hline
[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\tau_j}] & \widehat{[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\bar\tau_j}]} \\ \hline
[\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\tau_j}] & \widehat{[\mathscr{A}_{\bar\tau_i}]\cdot[\mathscr{V}_{\bar\tau_j}]}
\end{array}\right).$$

Moreover, elementary row operations transform it to

$$\left(\begin{array}{c|c|c} [\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\tau_j}] & * \\ \hline [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\tau_j}] & * \\ \hline 0 & 0 & * \end{array}\right).$$

Note that the intersection pairing matrix is

$$\left(\begin{array}{c|c} [\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\sigma_i}]\cdot[\mathscr{V}_{\tau_j}] \\ \hline [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\sigma_j}] & [\mathscr{A}_{\tau_i}]\cdot[\mathscr{V}_{\tau_j}] \end{array}\right).$$

Hence, the assertion follows. □

By Lemma 4.4, it is sufficient to show that the $\frown$-intersection pairing matrix is non-degenerate. We prove this by induction.

We first compute the $\frown$-intersection pairing matrix for the two 2-selections $D = \{d_j, d_j'\}$ and $C = \{c_{j-1}, c_j\}$.

Lemma 4.5.

$$\begin{array}{cc} & \begin{array}{cc} 1/m[\mathscr{V}_D] & 1/n[\mathscr{V}_C] \end{array} \\ \begin{array}{r} 1/i\ell^2[\mathscr{A}_D] \\ [\mathscr{A}_C] \end{array} & \left(\begin{array}{cc} 1 & 1/12 \\ 2 & 1 \end{array}\right), \end{array}$$

where

$$i = [\mathrm{PSL}(2,\boldsymbol{Z}) : \Gamma_\ell],$$

$$m = \begin{cases} 2 & \textit{if } \ g = 3, \\ 1 & \textit{if } \ g \neq 3, \end{cases}$$

$$n = \begin{cases} 4 & \textit{if } \ g = 3, \\ 2 & \textit{if } \ C \textit{ contains exactly one of } \ c_1, c_{g-1}, \\ 1 & \textit{otherwise}. \end{cases}$$

Proof. The $\frown$-intersection points of $[\mathscr{A}_D^\sharp]$ and $[\mathscr{V}_D]$ correspond to the nodes of the singular fibers of $\mathscr{U}_\ell$. Let (z_1, z_2) be coordinates of $\mathscr{U}_\ell$ around such a node, and let $(\tau, \tau', t \in \boldsymbol{C}^{3g-5})$ be local coordinates of a local manifold cover of $\bar{\mathscr{M}}_g$ over the $\frown$-intersection point p which corresponds to the node $(z_1, z_2) = (0, 0)$. Then, $[\mathscr{A}_D^\sharp]$ is locally written as

$$[\mathscr{A}_D^\sharp] : \mathscr{U}_\ell \to \bar{\mathscr{M}}_g$$

$$(z_1, z_2) \mapsto [(\tau = z_1^\ell, \tau' = z_2^\ell, t = f(z_1, z_2))]$$

for some smooth function f. When $g \geq 4$, (τ, τ', t) are coordinates of $\bar{\mathscr{M}}_g$ and $\mathscr{V}_D$ is locally given as the locus $\{\tau = \tau' = 0\}$. The $\widehat{\ }$-intersection number at p is ℓ^2, $\mathscr{U}_\ell$ has i/ℓ singular fibers, and each singular fiber of $\mathscr{U}_\ell$ has ℓ nodes. Hence, the $\widehat{\ }$-intersection number of $[\mathscr{A}_D]$ and $[\mathscr{V}_D]$ is equal to $\ell^2 \times i/\ell \times \ell = i\ell^2$. When $g = 3$, $(\sigma_1, \sigma_2, t) := (\tau + \tau', (\tau - \tau')^2, t)$ give coordinates of $\bar{\mathscr{M}}_g$ around p and the $\widehat{\ }$-intersection number of $[\mathscr{A}_D]$ and $[\mathscr{V}_D]$ is equal to $2i\ell^2$.

We calculate the $\widehat{\ }$-intersection number of $[\mathscr{A}_D]$ and $[\mathscr{V}_C]$ as follows. The Poincaré dual of the Euler class of $\tilde{s}_k^*\widetilde{T'}$ $(k = 1, 2)$ is equal to

$$-\frac{i}{12}[\text{general fiber}] - \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})].$$

We remark here that we later use the following computation in [**W**]:

$$[s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})] \cdot [s_k(\widehat{\boldsymbol{H}/\Gamma_\ell})] = \begin{cases} -\dfrac{i}{12} & (j = k), \\ 0 & (j \neq k). \end{cases}$$

The $\widehat{\ }$-intersection points of $[\mathscr{A}_D^\sharp]$ and $[\mathscr{V}_C]$ correspond to the intersection of the zero points of $\tilde{s}_1'$ and those of $\tilde{s}_2'$ in $\mathscr{U}_\ell$. Let (z_1, z_2) be coordinates of $\mathscr{U}_\ell$ around such an intersection point, and (τ_1, τ_2, t) coordinates of a local manifold cover of $\bar{\mathscr{M}}_g$ over the $\widehat{\ }$-intersection point p which corresponds to the point $(z_1, z_2) = (0, 0)$. Then, $[\mathscr{A}_D^\sharp]$ is locally written as

$$\begin{aligned} &[\mathscr{A}_D^\sharp] : \mathscr{U}_\ell \to \bar{\mathscr{M}}_g \\ &(z_1, z_2) \mapsto [(\tau_1 = \tilde{s}_1'(z_1, z_2), \tau_2 = \tilde{s}_2'(z_1, z_2), t = f(z_1, z_2))] \end{aligned}$$

for some smooth function $f(z_1, z_2)$. When neither of c_1, c_{g-1} is contained in C, the $\widehat{\ }$-intersection number of $[\mathscr{A}_D]$ and $[\mathscr{V}_C]$ is equal to the self-intersection number of $-i/12[\text{general fiber}] - \sum [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]$ in $\mathscr{U}_\ell$, since (τ_1, τ_2, t) give coordinates of $\bar{\mathscr{M}}_g$:

$$\begin{aligned} &\left(-\frac{i}{12}[\text{general fiber}] - \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]\right)^2 \\ &\quad = \frac{i}{6}\sum_j [\text{general fiber}] \cdot [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})] + \sum_j [s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})]^2 \\ &\quad = \frac{i}{6}\ell^2 - \frac{i}{12}\ell^2 = \frac{i}{12}\ell^2. \end{aligned}$$

Similar computations give $\widehat{[\mathscr{A}_D]\cdot[\mathscr{V}_C]} = i\ell^2/6$ when exactly one of c_1, c_{g-1} is contained in C, and $\widehat{[\mathscr{A}_D]\cdot[\mathscr{V}_C]} = i\ell^2/3$ when both of c_1, c_{g-1} are contained in C.

We calculate the $\widehat{\ }$-intersection number of $[\mathscr{A}_C]$ and $[\mathscr{V}_D]$ by using $c_1(s_j^* T') = -1$. The following is easily obtained:

$$\widehat{[\mathscr{A}_C]\cdot[V_D]} = \begin{cases} 2 & (g \geq 4), \\ 4 & (g = 3). \end{cases}$$

Since $[\mathscr{A}_C^\sharp]$ and $[\mathscr{V}_C]$ intersect at one point corresponding to (∞, ∞) in $(\widehat{\boldsymbol{H}/\Gamma_2})^2$, we have

$$\widehat{[\mathscr{A}_C]\cdot[\mathscr{V}_C]} = \begin{cases} 1 & \text{when neither of } c_1, c_{g-1} \text{ is contained in } C, \\ 2 & \text{when exactly one of } c_1, c_{g-1} \text{ is contained in } C, \\ 4 & \text{when both of } c_1, c_{g-1} \text{ are contained in } C. \end{cases}$$

This completes the proof of Lemma 4.5. □

We have so far dealt with $\widehat{\ }$-intersection numbers $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}$ only when σ is neither $\{d_1\}$ nor $\{d_g\}$; we now define, for convenience, $\widehat{\ }$-intersection numbers $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}$ even when σ is either $\{d_1\}$ or $\{d_g\}$.

DEFINITION 4.6. When σ is either $\{d_1\}$ or $\{d_g\}$, we define $\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_\sigma]}$ as

$$\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_{d_1}]} = \begin{cases} [\mathscr{A}_\tau]\cdot[\mathscr{V}_{d_1}] & (\tau \neq d_g), \\ 0 & (\tau = d_g), \end{cases}$$

$$\widehat{[\mathscr{A}_\tau]\cdot[\mathscr{V}_{d_g}]} = \begin{cases} 0 & (\tau \neq d_g), \\ [\mathscr{A}_\tau]\cdot[\mathscr{V}_{d_g}] & (\tau = d_g). \end{cases}$$

Let A_{k,d_j} $(1 \leq j \leq g)$ denote the $\widehat{\ }$-intersection matrix of all k-selections from $d_1, c_1, d_2, d_2', c_2, \ldots, d_j, d_j'$, and A_{k,c_j} $(1 \leq j \leq g-1)$ that of all k-selections from $d_1, c_1, d_2, d_2', \ldots, c_j$. In the following series of Lemmas, we inductively prove that A_{k,d_j} and A_{k,c_j} are all non-degenerate.

LEMMA 4.7. *A_{1,d_j}, A_{1,c_j} are all non-degenerate.*

Lemma 4.7 is immediate from the consequences of §5 in [**W**], from which we can easily calculate the $\widehat{\ }$-intersection number of any 1-selection cycle $[\mathscr{A}_\sigma]$ and any 1-selection variety cycle $[\mathscr{V}_\sigma]$.

LEMMA 4.8. *A_{2,d_j}, A_{2,c_j} are all non-degenerate.*

Proof. The $\widehat{\ }$-intersection number of $[\mathscr{A}_\sigma]$ and $[\mathscr{V}_\sigma]$ for $\sigma = \{d_1, c_1\}$ is equal to

$$2\left[\sum_j s_j(\widehat{\boldsymbol{H}/\Gamma_\ell})\right] \cdot [\ell \text{ singular fibers}] = 2i\ell^2,$$

thus

$$A_{2,c_1} = (2i\ell^2),$$

and hence A_{2,c_1} is non-degenerate.

Computation for $\sigma = \{d_1, c_1\}$ and $\tau = \{d_2, d_2'\}$ gives

$$A_{2,d_2} = \begin{cases} \begin{pmatrix} 2i\ell^2 & i\ell^2/12 \\ 0 & i\ell^2 \end{pmatrix} & (g \geq 4), \\ \begin{pmatrix} 2i\ell^2 & i\ell^2/6 \\ 0 & 2i\ell^2 \end{pmatrix} & (g = 3). \end{cases}$$

Computation for $\eta_1 = \{c_1, c_2\}$ and $\eta_2 = \{d_1, c_2\}$ and Lemma 4.5 give

$$A_{2,c_2} = \left(\begin{array}{c|c} A_{2,d_2} & * \\ \hline * & mA_{1,c_1} \end{array}\right) \quad \text{where } m = \begin{cases} 1 & (g \geq 4), \\ 2 & (g = 3), \end{cases}$$

$$= \begin{cases} \begin{pmatrix} 2i\ell^2 & i\ell^2/12 & 0 & 0 \\ 0 & i\ell^2 & i\ell^2/6 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & -i/6 & i \end{pmatrix} & (g \geq 4), \\ \begin{pmatrix} 2i\ell^2 & i\ell^2/6 & 0 & 0 \\ 0 & 2i\ell^2 & i\ell^2/3 & 0 \\ 0 & 4 & 4 & -4 \\ 0 & 0 & -i/3 & 2i \end{pmatrix} & (g = 3). \end{cases}$$

It is easy to check that A_{2,d_2} and A_{2,c_2} are non-degenerate.

Consider the case when $g \geq 4$ for some time.

Assume that the assertion holds up to A_{2,c_j} $(2 \leq j \leq g-2)$. $A_{2,d_{j+1}}$ is given by

$$A_{2,d_{j+1}} = \begin{pmatrix} A_{2,c_j} & 0 \\ 0 & i\ell^2 \end{pmatrix},$$

which then is also non-degenerate.

Assume that the assertion holds up to A_{2,d_j} $(3 \leq j \leq g-1)$. It follows from

Lemma 4.5 that

$$A_{2,c_j} = \left(\begin{array}{c|c} A_{2,d_j} & * \\ \hline * & A_{1,c_{j-1}} \end{array}\right)$$

$$= \begin{cases} \left(\begin{array}{cc|cc} A_{2,c_{j-1}} & 0 & 0 & 0 \\ 0 & i\ell^2 & i\ell^2/12 & 0 \\ \hline 0 & 2 & 1 & * \\ 0 & 0 & 0 & A_{1,d_{j-1}} \end{array}\right) \begin{array}{l} \\ \ldots\sigma \\ \ldots\tau \\ \\ \end{array} & (j \neq g-1), \\ \left(\begin{array}{cc|cc} A_{2,c_{j-1}} & 0 & 0 & 0 \\ 0 & i\ell^2 & i\ell^2/6 & 0 \\ \hline 0 & 2 & 2 & * \\ 0 & 0 & 0 & 2A_{1,d_{j-1}} \end{array}\right) \begin{array}{l} \\ \ldots\sigma \\ \ldots\tau \\ \\ \end{array} & (j = g-1)\ , \end{cases}$$

where $\sigma = \{d_j, d_j'\}$ and $\tau = \{c_{j-1}, c_j\} : A_{2,c_j}$ then is also non-degenerate.

Let $A_{2,d_g'}$ be the $\frown$-intersection matrix of all 2-selections except $\eta = \{c_{g-1}, d_g\}$. Let $\sim$ denote the equivalent relation by elementary row/column transformations. It follows that

$$A_{2,d_g'} = \left(\begin{array}{c|c} A_{2,c_{g-1}} & * \\ \hline * & iA_{1,d_{g-1}} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} A_{2,d_{g-1}} & \begin{matrix} 0 & 0 \\ i\ell^2/6 & 0 \end{matrix} & * \\ \hline \begin{matrix} 0 & 2 \\ 0 & 0 \end{matrix} & 2A_{1,c_{g-2}} & -2A_{1,c_{g-2}} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & -(i/6)A_{1,c_{g-2}} & iA_{1,c_{g-2}} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc|cc} A_{2,c_{g-2}} & 0 & 0 & 0 & * & * \\ 0 & i\ell^2 & i\ell^2/6 & 0 & -i\ell^2 & * \\ \hline 0 & 2 & \multicolumn{2}{c|}{\multirow{2}{*}{$2A_{1,c_{g-2}}$}} & \multicolumn{2}{c}{\multirow{2}{*}{$-2A_{1,c_{g-2}}$}} \\ 0 & 0 & & & & \\ \hline 0 & 0 & \multicolumn{2}{c|}{-(i/6)A_{1,c_{g-2}}} & \multicolumn{2}{c}{iA_{1,c_{g-2}}} \end{array}\right) \begin{array}{l} \\ \ldots d_{g-1} \\ \ldots c_{g-2}, c_{g-1} \\ \\ \\ \end{array}$$

$$
\sim \left(\begin{array}{cc|cc|cc}
A_{2,c_{g-2}} & 0 & * & * & * & * \\
0 & i\ell^2 & 0 & * & -i\ell^2 & * \\
\hline
0 & 2 & \multicolumn{2}{c|}{(5/3)A_{1,c_{g-2}}} & \multicolumn{2}{c}{-2A_{1,c_{g-2}}} \\
0 & 0 & & & & \\
\hline
0 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{iA_{1,c_{g-2}}}
\end{array}\right)
$$

$$
= \left(\begin{array}{cc|cc|c}
A_{2,c_{g-2}} & 0 & * & * & * \\
0 & i\ell^2 & 0 & * & \\
\hline
0 & 2 & 5/3 & * & -2A_{1,c_{g-2}} \\
0 & 0 & 0 & (5/3)A_{1,d_{g-2}} & \\
\hline
0 & 0 & 0 & 0 & iA_{1,c_{g-2}}
\end{array}\right)
$$

$$
\sim \left(\begin{array}{cc|c|c}
A_{2,c_{g-2}} & * & * & * \\
0 & i\ell^2 & & \\
\hline
\multicolumn{2}{c|}{0} & (5/3)A_{1,c_{g-2}} & * \\
\hline
\multicolumn{2}{c|}{0} & 0 & iA_{1,c_{g-2}}
\end{array}\right),
$$

which is non-degenerate: hence, $A_{2,d_g'}$ is also non-degenerate. A_{2,d_g} is given by

$$
A_{2,d_g} = \begin{pmatrix} A_{2,d_g'} & 0 \\ * & 2i\ell^2 \end{pmatrix},
$$

which is also non-degenerate.

In the case when $g = 3$, the assertion is proved in a similar way. This completes the proof of Lemma 4.8. □

Lemma 4.9. *A_{k,c_j} and A_{k,d_j} are all non-degenerate for all k, j.*

Proof. The assertion holds for $k = 1, 2$ by Lemma 4.7 and Lemma 4.8.

Assume that the assertion holds for $k \leq p-1$ $(p \geq 3)$.

If $\{d_1, c_1, d_2, d_2', \dots, c_{j-1}\}$ does not have any p-selection but $\{d_1, c_1, d_2, d_2', \dots, c_{j-1}, d_j, d_j'\}$ has p-selections, then

$$
A_{p,d_j} = i\ell^2 A_{p-2,d_{j-1}},
$$

and hence A_{p,d_j} is non-degenerate.

If $\{d_1, c_1, d_2, d_2', \dots, d_j, d_j'\}$ does not have any p-selection but $\{d_1, c_1, d_2, d_2', \dots, d_j, d_j', c_j\}$ has p-selections, then

$$A_{p,c_j} = A_{p-1,c_{j-1}},$$

and hence A_{p,c_j} is non-degenerate.

A_{p,d_j} is non-degenerate if $A_{p,c_{j-1}}$ is non-degenerate for $j \le g-1$, since A_{p,d_j} is written as

$$A_{p,d_j} = \begin{pmatrix} A_{p,c_{j-1}} & 0 \\ 0 & i\ell^2 A_{p-2,d_{j-1}} \end{pmatrix}.$$

A_{p,c_j} is non-degenerate if A_{p,d_j} is non-degenerate for $j \le g-1$, as follows. Assume that A_{p,d_j} is non-degenerate for some time. If there exist no p-selections containing both c_{j-1} and c_j, then

$$A_{p,c_j} = \begin{pmatrix} A_{p,d_j} & 0 \\ 0 & A_{p-1,c_j} \end{pmatrix},$$

and hence A_{p,c_j} is non-degenerate. Otherwise,

$$A_{p,c_j} = \begin{pmatrix} A_{p,d_j} & * \\ * & A_{p-1,c_{j-1}} \end{pmatrix},$$

which by Lemma 4.5 is written as

$$\begin{pmatrix} B & 0 & 0 & 0 \\ 0 & i\ell^2 A_{p-2,c_{j-2}} & (i\ell^2/12)A_{p-2,c_{j-2}} & 0 \\ 0 & 2A_{p-2,c_{j-2}} & A_{p-2,c_{j-2}} & C \\ 0 & 0 & D & A_{p-1,d_{j-1}} \end{pmatrix}$$

for some B, C, D. Note that B is non-degenerate, since

$$\begin{pmatrix} B & 0 \\ 0 & i\ell^2 A_{p-2,c_{j-2}} \end{pmatrix} = A_{p,d_j}.$$

(Note that the two partitioning above of A_{p,d_j} are different.) More elementary operations transform A_{p,c_j} to

$$\begin{pmatrix} B & 0 & 0 & 0 \\ 0 & A_{p-2,c_{j-2}} & 0 & 0 \\ 0 & 0 & (5/6)A_{p-2,c_{j-2}} & C \\ 0 & 0 & D & A_{p-1,d_{j-1}} \end{pmatrix}.$$

If there exist no p-selections containing all of c_{j-2}, c_{j-1}, c_j, then $C = D = 0$, and hence

A_{p,c_j} is non-degenerate. Otherwise,

$$\begin{aligned}
&\begin{pmatrix} (5/6)A_{p-2,c_{j-2}} & C \\ D & A_{p-1,d_{j-1}} \end{pmatrix} \\
&= \begin{pmatrix} (5/6)A_{p-2,d_{j-2}} & (5/6)E & 0 & 0 \\ (5/6)F & (5/6)A_{p-3,c_{j-3}} & 2A_{p-3,c_{j-3}} & 0 \\ 0 & (i\ell^2/12)A_{p-3,c_{j-3}} & i\ell^2 A_{p-3,c_{j-3}} & 0 \\ 0 & 0 & 0 & G \end{pmatrix}
\end{aligned}$$

for some E, F, G, which is transformed by elementary operations to

$$\begin{pmatrix} (5/6)A_{p-2,d_{j-2}} & (5/6)E & 0 & 0 \\ (5/6)F & (4/6)A_{p-3,c_{j-3}} & 0 & 0 \\ 0 & 0 & A_{p-3,c_{j-3}} & 0 \\ 0 & 0 & 0 & G \end{pmatrix}.$$

Note that G is non-degenerate, since

$$A_{p-1,d_{j-1}} = \begin{pmatrix} i\ell^2 A_{p-3,c_{j-3}} & 0 \\ 0 & G \end{pmatrix}.$$

If there exist no p-selections containing all of c_{j-3}, c_{j-2}, c_{j-1}, c_j, then $E = F = 0$, and hence A_{p,c_j} is non-degenerate. Otherwise, (with rows/columns reordered)

$$\begin{aligned}
&\begin{pmatrix} A_{p-2,d_{j-2}} & E \\ F & (4/5)A_{p-3,c_{j-3}} \end{pmatrix} \\
&= \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & i\ell^2 A_{p-4,c_{j-4}} & (i\ell^2/12)A_{p-4,c_{j-4}} & 0 \\ 0 & 2A_{p-4,c_{j-4}} & (4/5)A_{p-4,c_{j-4}} & (4/5)I \\ 0 & 0 & (4/5)J & (4/5)A_{p-3,d_{j-3}} \end{pmatrix} \\
&\sim \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & A_{p-4,c_{j-4}} & 0 & 0 \\ 0 & 0 & (19/24)A_{p-4,c_{j-4}} & I \\ 0 & 0 & J & A_{p-3,d_{j-3}} \end{pmatrix}
\end{aligned}$$

for some H, I, J with H non-degenerate. If there exist no p-selections containing all of c_{j-4}, c_{j-3}, c_{j-2}, c_{j-1}, c_j, then $I = J = 0$, and hence A_{p,c_j} is non-degenerate. Otherwise, ...

The steps above can be repeated since the sequence $\{a_n\}_{n=1,2,\dots}$ such that

$$a_1 = 1, \quad a_{n+1} = 1 - \frac{1}{6a_n}$$

has no zeros; the steps must be terminated because of the finiteness of the number of rows/columns of A_{p,c_j}: hence A_{p,c_j} is non-degenerate.

To show that A_{p,d_g} is non-degenerate, let $A_{p,d_g'}$ be the $\frown$-intersection matrix of p-selections not containing both c_{g-1} and d_g.

$$A_{p,d_g'} = \begin{pmatrix} A_{p,c_{g-1}} & * \\ * & iA_{p-1,d_{g-1}} \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc|cc|c} A_{p,c_{g-2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\ell^2 A_{p-2,c_{g-3}} & (i\ell^2/6)A_{p-2,c_{g-3}} & 0 & -i\ell^2 A_{p-2,c_{g-3}} & 0 & 0 \\ \hline 0 & 2A_{p-2,c_{g-3}} & \multicolumn{2}{c|}{\multirow{2}{*}{$2A_{p-1,c_{g-2}}$}} & \multicolumn{2}{c|}{\multirow{2}{*}{$-\,2A_{p-1,c_{g-2}}$}} & \multirow{2}{*}{0} \\ 0 & 0 & & & & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{-(i/6)A_{p-1,c_{g-2}}} & \multicolumn{2}{c|}{iA_{p-1,c_{g-2}}} & 0 \\ & & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & i^2\ell^2 A_{p-3,d_{g-2}} \end{array}\right)$$

$$\sim \begin{pmatrix} A_{p,c_{g-2}} & & & & \\ & A_{p-2,c_{g-3}} & & * & \\ & & (5/3)A_{p-1,c_{g-2}} & & \\ & 0 & & iA_{p-1,c_{g-2}} & \\ & & & & i^2\ell^2 A_{p-3,d_{g-2}} \end{pmatrix},$$

and hence $A_{p,d_g'}$ is non-degenerate. A_{p,d_g} is written as

$$A_{p,d_g} = \begin{pmatrix} A_{p,d_g'} & 0 \\ * & 2i\ell^2 A_{p-2,c_{g-2}} \end{pmatrix},$$

and hence A_{p,d_g} is non-degenerate. This completes the proof of Lemma 4.9. □

Lemma 4.9 together with Lemma 4.4 completes the proof of

Theorem A. *When* $k \geq 2$,

$$b_{2k}(\bar{\mathcal{M}}_g) = b_{6g-6-2k}(\bar{\mathcal{M}}_g) \geq \max(\alpha_{g,k}, \alpha_{3g-3-k}).$$

Remark. When $k = 1$, Harer's result shows the following equality.

$$b_2(\bar{\mathcal{M}}_g) = b_{6g-8}(\bar{\mathcal{M}}_g) = 2 + \left[\frac{g}{2}\right] \quad (= \alpha_{g,1} + 1)$$

§5. Computation of the number of cycles.

We constructed in §2 $\alpha_{g,k}$ cycles of degree $2k$ on $\bar{\mathscr{M}}_g$ and showed in §4 that they represent linearly independent homology classes in $H_{2k}(\bar{\mathscr{M}}_g; \boldsymbol{Q})$. In this concluding section, we give algorithm to compute the number $\alpha_{g,k}$ of the $2k$-cycles we constructed.

Recall that a k-selection is a selection σ of k from the set

$$\{d_1, c_1, d_2, d_2', c_2, \dots, c_{g-2}, d_{g-1}, d_{g-1}', c_{g-1}, d_g\}$$

of $3g-3$ elements ($g \geq 3$) satisfying the following two conditions: (1) $d_i' \in \sigma$ if and only if $d_i \in \sigma$; (2) if $d_i, d_i' \in \sigma$, then $c_{i-1}, c_i \notin \sigma$ (Definition 2.1). Note that k of a k-selection can be any non-negative integer $\leq 2g-2$. Recall that two k-selections σ, τ are called conjugate if and only if τ is equal either to σ itself or to the k-selection $\bar{\sigma}$ such that $d_i \in \bar{\sigma}$ if and only if $d_{g-i+1} \in \sigma$ and $c_j \in \bar{\sigma}$ if and only if $c_{g-j} \in \sigma$, and that a k-selection σ is called symmetric if $\bar{\sigma} = \sigma$ (Definition 2.2). Recall that $\alpha_{g,k}$ is the number of conjugacy classes of k-selections (Notation 2.3).

Note that the conjugacy class of a k-selection σ is composed of one k-selection or two according as σ is symmetric or not. Let $\beta_{g,k}$ be the number of k-selections and $\beta_{g,k}^s$ the number of symmetric k-selections. The number $\alpha_{g,k}$ then is equal to half of the sum $\beta_{g,k} + \beta_{g,k}^s$.

We shall give an explicit method to calculate polynomials $\sum_k \beta_{g,k} t^k$, $\sum_k \beta_{g,k}^s t^k$ in $\boldsymbol{Z}[t]$ of degree $2g-2$, and hence $\sum_k \alpha_{g,k} t^k \in \boldsymbol{Z}[t]$, where we naturally assume $\beta_{g,0} = \beta_{g,0}^s = \alpha_{g,0} = 1$.

Let C, D, and Q be the following matrices in $M_2(\boldsymbol{Z}[t])$:

$$C = \begin{pmatrix} 0 & t \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & t^2 \\ 1 & 1 \end{pmatrix}, \quad D' = \begin{pmatrix} 0 & 0 \\ 0 & t^2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & t^4 \\ 1 & 1 \end{pmatrix}.$$

Note that C, D, and Q are in $\mathrm{GL}_2(\boldsymbol{Z}(t))$. D' shall be used in Lemma 5.3.

Let $h_g(t)$, $h_g^s(t)$ be the second component of

$$(1,1) \cdot (C(CD)^{g-2}C^2 + t^2(CD)^{g-3}C^2 + t^2C(CD)^{g-3}C + t^4(CD)^{g-4}C),$$

$$(1,1) \cdot \begin{cases} (D(DQ)^{g/2-1}C + t^4(DQ)^{g/2-2}C) & \text{if } n \text{ is even,} \\ (D(DQ)^{(g-3)/2}D^2 + t^4(DQ)^{(g-5)/2}D^2) & \text{if } n \text{ is odd,} \end{cases}$$

respectively, where, for a matrix $A \in \mathrm{GL}_2(\boldsymbol{Z}(t))$ and a negative integer q, A^q denotes $(A^{-1})^{(-q)}$, A^{-1} being the inverse matrix of A in $\mathrm{GL}_2(\boldsymbol{Z}(t))$. It is easily verified that both $h_g(t)$ and $h_g^s(t)$ are polynomials in $\boldsymbol{Z}[t]$. Let $f_g(t) = 1/2(h_g(t) + h_g^s(t))$.

Our goal then is to prove

Proposition B. *It follows that*

$$h_g(t) = \sum_{k=0}^{2g-2} \beta_{g,k} t^k, \quad h_g^s(t) = \sum_{k=0}^{2g-2} \beta_{g,k}^s t^k, \quad \textit{hence } f_g(t) = \sum_{k=0}^{2g-2} \alpha_{g,k} t^k \in \boldsymbol{Z}[t].$$

We prepare some more denotations for our proof of Proposition B. Let $\gamma_{i,k}$, $\delta_{j,k}$, $\delta_{g,k}$ be the numbers of the k-selections included in

$$\{d_1, c_1, d_2, \dots, c_i\}, \quad \{d_1, c_1, \dots, d_j, d_j'\}, \quad \{d_1, c_1, d_2, \dots, c_{g-1}, d_g\}$$

respectively, and $\gamma_{i,k}^s$, $\delta_{j,k}^s$ the numbers of the symmetric k-selections included in

$$\{d_1, c_1, d_2, \dots, c_i\} \cup \{c_{g-i}, \dots, d_{g-1}', c_{g-1}, d_g\},$$

$$\{d_1, c_1, \dots, d_j, d_j'\} \cup \{d_{g-j+1}, d_{g-j+1}', \dots, c_{g-1}, d_g\}$$

respectively. Let

$$c_i(t) = \sum_k \gamma_{i,k} t^k, \quad d_j(t) = \sum_k \delta_{j,k} t^k,$$

$$c_i^s(t) = \sum_k \gamma_{i,k}^s t^k, \quad d_j^s(t) = \sum_k \delta_{j,k}^s t^k.$$

Note that

$$\sum_k \beta_{g,k} t^k = d_g(t), \quad \sum_k \beta_{g,k}^s t^k = \begin{cases} c_{g/2}^s(t) & \text{if } g \text{ is even,} \\ d_{(g+1)/2}^s(t) & \text{if } g \text{ is odd.} \end{cases}$$

We draw three lemmas to establish recurrence formulae among $c_i(t)$'s, $d_j(t)$'s and among $c_i^s(t)$'s, $d_j^s(t)$'s. We omit all the proofs of the lemmas since they are immediate from the definition of k-selections.

Lemma 5.1.

(1) $$(c_1(t), d_2(t)) = (1,1) \cdot (C^2 D + t^2).$$

(2) $$(c_1^s(t), d_2^s(t)) = (1,1) \cdot (D^2 Q + t^4) \quad \textit{if } \; g > 3.$$

Lemma 5.2.

(1) $$\begin{cases} (c_{i-1}(t), d_i(t))C = (d_i(t), c_i(t)) & \textit{if } \; 2 \le i \le g-1, \\ (d_{j-1}(t), c_{j-1}(t))D = (c_{j-1}(t), d_j(t)) & \textit{if } \; 2 \le j \le g-1. \end{cases}$$

$$(2) \qquad \begin{cases} (c^s_{i-1}(t), d^s_i(t))D = (d^s_i(t), c^s_i(t)) & \text{if } 2 \le i < g/2, \\ (d^s_{j-1}(t), c^s_{j-1}(t))Q = (c^s_{j-1}(t), d^s_j(t)) & \text{if } 2 \le j \le g/2. \end{cases}$$

LEMMA 5.3.

$$(1) \qquad (d_{g-2}(t), c_{g-2}(t)) \cdot (DC^2 + D') = (c_{g-1}(t), d_g(t)).$$

$$(2) \qquad \begin{cases} (c^s_{g/2-1}(t), d^s_{g/2}(t))C = (d^s_{g/2}(t), c^s_{g/2}(t)) & \text{if } g \text{ is even}, \\ (d^s_{(g-1)/2}(t), c^s_{(g-1)/2}(t))D = (c^s_{(g-1)/2}(t), d^s_{(g+1)/2}(t)) & \text{if } g \text{ is odd}. \end{cases}$$

PROOF OF PROPOSITION B. On one hand, it follows from (1) of Lemmas 5.1–5.3 that

$$(1,1) \cdot (C^2D + t^2)(CD)^{g-4}C(DC^2 + D') = (c_{g-1}(t), d_g(t)).$$

The second component of the left side is equal to $h_g(t)$; the second component of the right side is equal to $\sum_k \beta_{g,k} t^k$. Hence, $h_g(t) = \sum_k \beta_{g,k} t^k$. On the other hand, it follows from (2) of Lemmas 5.1–5.3 that

$$\begin{cases} (1,1) \cdot (D^2Q + t^4)(DQ)^{(g/2)-2}C = (d^s_{g/2}(t), c^s_{g/2}(t)) & \text{if } g \text{ is even}, \\ (1,1) \cdot (D^2Q + t^4)(DQ)^{(g-5)/2}D^2 = (c^s_{(g-1)/2}(t), d^s_{(g+1)/2}(t)) & \text{if } g \text{ is odd}. \end{cases}$$

The second component of the left side is equal to $h^s_g(t)$; the second component of the right side is equal to $\sum_k \beta^s_{g,k} t^k$. Hence, $h^s_g(t) = \sum_k \beta^s_{g,k} t^k$. □

Instead of computing the number $\alpha_{g,k}$ accurately, we can roughly estimate $\alpha_{g,k}$ from below as follows. Let $\beta'_{g,k}$ be the number of the k-selections without d_1, d_g. We see in light of Lemma 5.2 (2) that $\sum_k \beta'_{g,k} t^k$ is equal to the second component of $(1,1)(CD)^{g-2}C$. We actually compute $\sum_k \beta'_{g,k} t^k$ as follows by induction:

$$\sum_k \beta'_{g,k} t^k = \sum_k \left\{ \binom{g-1}{k} + \sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'} \right\} t^k.$$

We thus obtain the following estimate for $\alpha_{g,k}$:

$$\alpha_{g,k} > \frac{1}{2}\binom{g-1}{k} + \frac{1}{2}\sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'}.$$

We finally give some of our concrete computations of the polynomial $f_g(t)$ and the resulting estimates for the Betti numbers of $\bar{\mathcal{M}}_g$ for genus $g \le 6$.

$(g = 3)$ $\qquad f_3(t) = 1 + 2t + 5t^2 + 3t^3 + 2t^4$:

$$b_2 = b_{10} = 3,$$

$$b_4 = b_8 \ge 5,$$

$$b_6 \ge 3.$$

$(g = 4)$
$$f_4(t) = 1 + 3t + 7t^2 + 9t^3 + 7t^4 + 3t^5 + t^6 :$$
$$b_2 = b_{16} = 4,$$
$$b_4 = b_{14} \geq 7,$$
$$b_6 = b_{12} \geq 9,$$
$$b_8 = b_{10} \geq 7.$$

$(g = 5)$
$$f_5(t) = 1 + 3t + 11t^2 + 16t^3 + 21t^4 + 13t^5 + 8t^6 + 2t^7 + t^8 :$$
$$b_2 = b_{22} = 4,$$
$$b_4 = b_{20} \geq 11,$$
$$b_6 = b_{18} \geq 16,$$
$$b_8 = b_{16} \geq 21,$$
$$b_{10} = b_{14} \geq 13,$$
$$b_{12} \geq 8.$$

$(g = 6)$
$$f_6(t) = 1 + 4t + 14t^2 + 29t^3 + 43t^4 + 43t^5 + 31t^6 + 16t^7 + 7t^8 + 2t^9 + t^{10} :$$
$$b_2 = b_{28} = 5,$$
$$b_4 = b_{26} \geq 14,$$
$$b_6 = b_{24} \geq 29,$$
$$b_8 = b_{22} \geq 43,$$
$$b_{10} = b_{20} \geq 43,$$
$$b_{12} = b_{18} \geq 31,$$
$$b_{14} = b_{16} \geq 16.$$

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