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Dirichlet finite harmonic measures on topological balls

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Abstract. Based upon an intuition from electrostatics one might suspect that there is no topological ball in Euclidean space of dimension $d \geq 2$ which carries a nonconstant Dirichlet finite harmonic measure. This guess is certainly true for $d = 2$. However, contrary to the above intuition, it is shown in this paper that there does exist a topological ball in Euclidean space of every dimension $d \geq 3$ on which there exists a nonconstant Dirichlet finite harmonic measure.

The purpose of this paper is to show rather unexpectedly the existence of a topological ball in Euclidean space of every dimension greater than or equal to three on which there exists a nonconstant Dirichlet finite harmonic measure. In order to clarify the significance of our result we begin with explaining definitions of harmonic measures, topological balls, and related notion.

Consider a bounded domain Ω in the Euclidean space $\boldsymbol{R}^d$ of dimension $d \geq 2$ and an arbitrary subset E of the boundary $\partial\Omega$ of Ω. We denote by 1_E the characteristic function of the set E. The upper class $\mathscr{U}_E(\Omega)$ of E on Ω is the upper class $\mathscr{U}_{1_E}(\Omega)$, which consists of all positive superharmonic functions s on Ω such that $\liminf_{x \to y} s(x) \geq 1_E(y)$ for all $y \in \partial\Omega$. Then the harmonic function $x \mapsto \omega(x; E, \Omega)$ on Ω given by

$$(1) \qquad \omega(x; E, \Omega) := \overline{H}{}^{\Omega}_{1_E}(x) := \inf_{s \in \mathscr{U}_E(\Omega)} s(x) \quad (x \in \Omega)$$

is referred to as the *harmonic measure* of E with respect to Ω (cf. e.g. [**1**]). It is known that one of the following three exclusive cases occurs: $\omega(\cdot\,; E, \Omega) \equiv 0$ on Ω; $\omega(\cdot\,; E, \Omega) \equiv 1$ on Ω; $\omega(\cdot\,; E, \Omega)$ is not constant and $0 < \omega(\cdot\,; E, \Omega) < 1$ on Ω. We say that $\omega = \omega(\cdot\,; E, \Omega)$ is *Dirichlet finite* (*infinite*, resp.) on Ω if its Dirichlet integral $\int_\Omega |\nabla\omega(x)|^2\,dx$ is finite (infinite, resp.). We denote by

$$O_{HmD}$$

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the class of all bounded Euclidean domains Ω such that every harmonic measure $\omega(\cdot\,; E, \Omega)$ is Dirichlet infinite on Ω for every $E \subset \partial\Omega$ unless $\omega(\cdot\,; E, \Omega)$ is constant on Ω. Electrostatically speaking, especially in the case $d = 3$, $\Omega \notin O_{HmD}$ means that there is a subdivision of $\partial\Omega$ into two parts E and $\partial\Omega\backslash E$ such that if the electrode $\partial\Omega\backslash E$ is grounded and the other electrode E is positively charged suitably with a finite energy, then there produces a unit potential difference between these two electrodes so that the configuration $(\Omega; E, \partial\Omega\backslash E)$ functions as an electric condenser. Thus $\Omega \in O_{HmD}$ means that, no matter how we decompose $\partial\Omega$ into two parts E and $\partial\Omega\backslash E$, the configuration $(\Omega; E, \partial\Omega\backslash E)$ does not function as an electric condenser.

We say that a bounded domain M in $\boldsymbol{R}^d$ $(d \geq 2)$ is a *topological ball* if there is a homeomorphism h of $\overline{M} = M \cup \partial M$ onto the unit closed ball $\bar{B}^d = B^d \cup S^{d-1}$ such that $h(M) = B^d$ and $h(\partial M) = S^{d-1}$, where B^d is the unit ball $\{x \in \boldsymbol{R}^d : |x| < 1\}$ in $\boldsymbol{R}^d$ and $S^{d-1} = \partial B^d$ is the unit sphere $\{x \in \boldsymbol{R}^d : |x| = 1\}$ in $\boldsymbol{R}^d$. Our study has been motivated by the feeling that topological balls must belong to O_{HmD}. This feeling comes from the following electrostatical guess in the case $d = 3$. Consider the decomposition of the boundary ∂M of a topological ball M in $\boldsymbol{R}^3$ into two electrodes E and $\partial M\backslash E$ and let $\partial M\backslash E$ be grounded. Since E and $\partial M\backslash E$ are put together very tightly no matter how we choose $E \subset \partial M$, all charges put on the electrode E must instantly go to the earth through the electrode $\partial M\backslash E$ so that any configuration $(M; E, \partial M\backslash E)$ cannot function as an electric condenser. The first evidence backing up the above feeling is the following result obtained in [**6**], [**7**], and Herron-Koskela [**3**]:

Theorem A. *If the topological ball $M = B^d$ $(d \geq 2)$, then M belongs to the class O_{HmD}.*

We soon realized that what is important in the proof of the above result is, in addition to that B^d is a topological ball, the smoothness of $\partial B^d = S^{d-1}$. We then obtained the following result in [**11**]:

Theorem B. *If a topological ball M in $\boldsymbol{R}^d$ $(d \geq 2)$ has a C^2 boundary ∂M, then M belongs to the class O_{HmD}.*

As a response to the criticism that the C^2 assumption in the above result is too strong, we succeeded in weakening it to the C^1 condition or rather the Lipschitz condition. Actually these are special cases of the following more general result. We say that a boundary point $y \in \partial\Omega$ of a bounded domain Ω in $\boldsymbol{R}^d$ $(d \geq 2)$ is *graphic* if one of the following two conditions is satisfied: there are a neighborhood U of y, a Cartesian coordinate $x = (x^1, \dots, x^{d-1}, x^d) = (x', x^d)$, and a continuous function $\varphi(x')$ of x' such that $(\partial\Omega) \cap U$ is represented as the graph of $x^d = \varphi(x')$ and $\Omega \cap U$ is situated on only one side of the graph; there are a neighborhood U of y, a polar coordinate (r, ξ) $(r \geq 0,$ $\xi \in S^{d-1})$, and a continuous function $\varphi(\xi) \geq 0$ of ξ such that $(\partial\Omega) \cap U$ is represented as the graph of $r = \varphi(\xi)$ and $\Omega \cap U$ is situated on only one side of the graph. A bounded domain $\Omega \subset \boldsymbol{R}^d$ is referred to as a *continuous domain* if every boundary point of Ω is graphic. Clearly C^1-domains or more generally Lipschitz domains are special continuous domains. Then we have the following result (cf. [**9**], [**10**]):

Theorem C. *If a topological ball M in $\boldsymbol{R}^d$ $(d \geq 2)$ is a continuous domain, then M belongs to the class O_{HmD}.*

In view of these results we are tempted to suspect that every topological ball belongs to the class O_{HmD}. Actually this is true for all topological balls in the two dimensional Euclidean space $\boldsymbol{R}^2$. In fact, by the Riemann mapping theorem there is a conformal homeomorphism h of any topological ball (i.e. Jordan domain) $M \subset \boldsymbol{R}^2$ onto the unit disc B^2 and this mapping h can be extended to a homeomorphism of $\overline{M} = M \cup \partial M$ onto the closed unit disc $\bar{B}^2 = B^2 \cup S^1$ by the Carathéodory theorem. This with the conformal invariance of the harmonicity and that of the Dirichlet finiteness and Theorem A instantaneously implies the following result (cf. e.g. [**6**]).

Theorem D. *Any topological ball in $\boldsymbol{R}^2$ belongs to the class O_{HmD}.*

By virtue of this result, hereafter in this paper, we may and will assume that the dimension d of the base Euclidean space $\boldsymbol{R}^d$ is at least three: $d \geq 3$. To continue the study in the direction of Theorems A, B, and C, it is therefore of compelling importance to determine whether or not there is a topological ball M in $\boldsymbol{R}^d$ $(d \geq 3)$ that does not belong to O_{HmD}. Contrary to our intuition mentioned thus far it turned out that the following rather surprising result holds, to prove which is the chief object of this paper.

2. Main Theorem. *For every dimension $d \geq 3$ there exists a topological ball M in $\boldsymbol{R}^d$ that does not belong to the class O_{HmD}.*

A harmonic function w is said to be a *harmonic measure* on M in the sense of Heins [**2**] if the greatest harmonic minorant of w and $1 - w$ is the constant function zero. It is easy to see that a harmonic measure $\omega(\cdot\,; E, M)$ of any boundary set $E \subset \partial M$ with respect to M is a harmonic measure on M in the sense of Heins. It is known (cf. e.g. [**6**]) that the *Royden harmonic boundary* $\Delta(M)$ of M is connected if and only if there are no nonconstant Dirichlet finite harmonic measures on M in the sense of Heins. Thus the main theorem 2 above implies the following: *for every dimension $d \geq 3$ there exists* *a topological ball M in $\boldsymbol{R}^d$ whose Royden harmonic boundary $\Delta(M)$ is disconnected.* Actually it is known (cf. [**6**], [**7**], [**9**], [**10**], [**11**]) that topological balls M in Theorem A, B, C, and D all have connected Royden harmonic boundaries $\Delta(M)$. To prove the above main theorem 2 we only have to exhibit an example of an $M \notin O_{HmD}$. We will construct an example of M with a bit more properties than really required, which is inspired by the so called Keldysh ball obtained in the celebrated paper [**4**] to show a phenomenon related to the stability of the Dirichlet problem.

3. Example. *For each dimension $d \geq 3$ there exist a topological ball M in $\boldsymbol{R}^d$ and a compact subset E of the boundary ∂M of M with the following properties*:

(a) *every point of the boundary ∂M of M is regular with respect to the harmonic Dirichlet problem on M*;

(b) *the surface area $|\partial M|$ of ∂M is finite*;

(c) *the surface areas $|E|$ of E and $|\partial M \backslash E|$ of $\partial M \backslash E$ are both strictly positive*;

(d) *the harmonic measure $\omega(\cdot\,; E, M)$ of E relative to M is Dirichlet finite and is not constant, i.e.*

$$0 < \int_M |\nabla\omega(x; E, M)|^2\, dx < \infty. \tag{4}$$

To construct an M and an E in the above example we need two simple lemmas concerning harmonic and superharmonic functions. We fix an $\boldsymbol{R}^d$ $(d \geq 3)$ and identify the hyperplane $\boldsymbol{R}^{d-1} \times \{0\}$ in $\boldsymbol{R}^d$ with $\boldsymbol{R}^{d-1}$. Let $a = (a^1, \dots, a^{d-1})$ be a point in $\boldsymbol{R}^{d-1}$ $(= \boldsymbol{R}^{d-1} \times \{0\})$ and r a positive number. We call the open set

$$Q(a,r) = \{x = (x^1, \dots, x^{d-1}) \in \boldsymbol{R}^{d-1} : |x^i - a^i| < r \ (1 \leq i \leq d-1)\}$$

in $\boldsymbol{R}^{d-1}$ a flat cube in $\boldsymbol{R}^d$ or simply a *cube* in $\boldsymbol{R}^{d-1}$ and a its *center* and r its interior radius or simply *radius*. The number $(d-1)^{1/2}r$ may be called the exterior radius of $Q(a,r)$. We denote by $B(a,r) = B^d(a,r)$ the open ball in $\boldsymbol{R}^d$ with radius r centered at a. Then

$$B^{d-1}(a,r) \subset Q(a,r) \subset B^{d-1}(a,(d-1)^{1/2}r).$$

We denote by $\bar{Q}(a,r)$ the closure of $Q(a,r)$. We single out the particular boundary point $b = (a^1 + r, a^2, \dots, a^{d-1})$ of $Q(a,r)$ in $\boldsymbol{R}^{d-1}$, which will be referred to as the *distinguished boundary point* of $Q(a,r)$.

Let G be an arbitrary domain in $\boldsymbol{R}^d$ containing a $\bar{Q}(a,r)$. We will seriously use the following fact: every point of $\bar{Q}(a,r)$ is a regular boundary point of the domain $G \backslash \bar{Q}(a,r)$ with respect to the Dirichlet problem on $G \backslash \bar{Q}(a,r)$. This is assured by the following criterion of regularity (cf. e.g. Kuran [**5**] and also [**8**; Appendix]): a boundary point y of a domain Ω in $\boldsymbol{R}^d$ is regular for the Dirichlet problem if there is a truncated flat cone (i.e. $(d-1)$-dimensional cone) with vertex y contained in the complement $\boldsymbol{R}^d \backslash \Omega$ of Ω.

The hyperplanes $\{x \in \boldsymbol{R}^{d-1} : x^i = a^i\}$ $(1 \le i \le d-1)$ divide $Q(a,r)$ into 2^{d-1} congruent small cubes $Q(a_k, r/2)$ $(1 \le k \le 2^{d-1})$. The points a_k $(1 \le k \le 2^{d-1})$ are referred to as *subcenters* of $Q(a,r)$. Let $\{a_k : 1 \le k \le 2^{d-1}\}$ be subcenters of a cube $Q = Q(a,r)$ and let $Q_k = Q(a_k, (r/2)\lambda)$ $(1 \le k \le 2^{d-1})$, where $0 < \lambda < 1$. Then the family $\{Q_1, \dots, Q_{2^{d-1}}\}$ is said to be *regularly distributed* in Q with *index* $\lambda \in (0,1)$.

We fix a cube $Q_0 := Q(0,1)$ in $\boldsymbol{R}^{d-1} (= \boldsymbol{R}^{d-1} \times \{0\})$ and a ball $B_0 := B^d(0, 3(d-1)^{1/2})$ in $\boldsymbol{R}^d$, which contains $\overline{Q_0}$. The first auxiliary result is the following:

5. Lemma. *For any number $\varepsilon > 0$ there exists a $\lambda_\varepsilon \in (0,1)$ with the following property: for any cube $Q \subset Q_0$ and for any family $\{Q_i : 1 \le i \le 2^{d-1}\}$ of congruent cubes Q_i regularly distributed in Q with any index $\lambda \in [\lambda_\varepsilon, 1)$, any continuous positive superharmonic function s on B_0 such that $s \ge 1$ on $\bigcup_{1 \le i \le 2^{d-1}} \overline{Q_i}$ satisfies $s \ge 1 - \varepsilon$ on $\bar{Q}$.*

Proof. Take a ball $B_1 = B^d(0, 2(d-1)^{1/2})$. Let $w \in C(\overline{B_1}) \cap H(B_1 \backslash \overline{Q_0})$ such that $w|\partial B_1 = 0$ and $w|\overline{Q_0} = 1$ as a result of every point in $\overline{Q_0}$ being regular. Here $H(G)$ denotes the class of all harmonic functions on an open set G in $\boldsymbol{R}^d$. We set $Q(\lambda) = Q(0, \lambda^{-1})$ for $\lambda \in (1/2, 1)$ so that $\overline{Q_0} \subset Q(\lambda) \subset \overline{Q(\lambda)} \subset B_1$. Since $w|\overline{Q_0} = 1$, there is a $\lambda_\varepsilon \in (2/3, 1)$ such that $w|\overline{Q(\lambda_\varepsilon)} \ge 1 - \varepsilon$. Fix an arbitrary $\lambda \in [\lambda_\varepsilon, 1)$ so that $Q(\lambda_\varepsilon) \supset Q(\lambda)$ and $w|\overline{Q(\lambda)} \ge 1 - \varepsilon$. Let $r \in (0,1]$ be the radius of Q and a_k be the center of Q_k and set $Q_k' := Q(a_k, r/2)$. We consider a function w_k on $(r/2)\lambda\overline{B_1} + a_k \subset B_0$ given by

$$w_k(x) = w((r/2)^{-1}\lambda^{-1}(x - a_k)).$$

Observe that $(r/2)\lambda Q(\lambda) + a_k = Q_k'$ and $(r/2)\lambda Q_0 + a_k = Q_k$. Therefore $w_k \in C(\overline{B'}) \cap H(B' \backslash \overline{Q_k})$ such that $w_k|\partial B' = 0$, $w_k|\overline{Q_k} = 1$, and $w_k|\overline{Q_k'} \ge 1 - \varepsilon$, where $B' = (r/2)\lambda B_1 + a_k$. Since $s \ge w_k$ on the boundary of $B' \backslash \overline{Q_k}$, the minimum (comparison) principle assures that $s \ge w_k$ on $B' \backslash \overline{Q_k}$. Thus we can conclude that $s|\overline{Q_k'} \ge 1 - \varepsilon$ for every k. However $\bigcup_{1 \le k \le 2^{d-1}} \overline{Q_k'} = \bar{Q}$ and therefore we deduce $s|\bar{Q} \ge 1 - \varepsilon$ as desired. □

With every $\varepsilon > 0$ we associate a number $\lambda(\varepsilon)$ which is the infimum of the set of λ_ε appeared in the above lemma.

Let G_0 be a bounded *regular domain* in the sense that every boundary point of G_0 is regular for the Dirichlet problem on G_0. Take a compact subset K of G_0 such that $G := G_0 \backslash K$ is again a regular domain. Suppose there is a union L of a finite number of polygonal line segments contained in G except possibly for their end points such that

$\mathrm{dis}(L, K) > 0$. Let $(T_i)_{i\geq 1}$ be a sequence of closed sets T_i which is the closure of an open set $T_i^\circ \subset G$ with piecewise smooth boundary ∂T_i° such that $T_i \supset T_{i+1} \supset L$ $(1 \leq i < \infty)$, $\mathrm{dis}(T_1, K) > 0$, and $\bigcap_{i\geq 1} T_i = L$. The second auxiliary result we need is the following:

6. Lemma. *Let $u \in C(\overline{G_0}) \cap H(G)$ with $u|\partial G_0 = 0$ and $u|K = 1$ and let $u_i \in C(\overline{G_0}) \cap H(G \backslash T_i)$ with $u_i|T_i \cup \partial G_0 = 0$ and $u_i|K = 1$ $(1 \leq i < \infty)$. Then $(u_i)_{i\geq 1}$ converges to u on G in the Dirichlet integral $D_G(f) := \int_G |\nabla f(x)|^2\, dx$:*

$$\lim_{i\to\infty} D_G(u_i - u) = 0. \tag{7}$$

Proof. We denote by $W^{1,2}(G)$ the Sobolev space on G with exponent 2, i.e. $W^{1,2}(G)$ is the space of functions $f \in L^2(G)$ having distributional gradients ∇f with $|\nabla f| \in L^2(G)$ so that the Dirichlet integral $D_G(f) = \int_G |\nabla f(x)|^2\, dx = \| \, |\nabla f|; L^2(G)\|^2$ of f can be defined. The space $W^{1,2}(G)$ is a Banach space with the norm $\|f; W^{1,2}(G)\| := (\|f; L^2(G)\|^2 + D_G(f))^{1/2}$. We denote by $W_0^{1,2}(G)$ the Sobolev null space on G, i.e. the closure of $C_0^\infty(G)$ in the Banach space $W^{1,2}(G)$. For convenience we also use the mutual Dirichlet integral $D_G(f, g) := \int_G \nabla f(x) \cdot \nabla g(x)\, dx$ for two functions f and g in $W^{1,2}(G)$. It is easy to see that $u_i \in W^{1,2}(G)$ for every $i \geq 1$. Let $i < j$ and observe that $u_i - u_j = 0$ on $\partial(G \backslash T_j)$. Hence we easily see that $u_i - u_j \in W_0^{1,2}(G \backslash T_j)$. Since $u_j \in W^{1,2}(G \backslash T_j) \cap H(G \backslash T_j)$, u_j is a weak solution of $\Delta u_j = 0$ on $G \backslash T_j$ and a fortiori

$$\int_{G\backslash T_j} \nabla u_j(x) \cdot \nabla(u_i - u_j)(x)\, dx = 0$$

or $D_{G\backslash T_j}(u_j, u_i - u_j) = 0$. Clearly $D_G(u_j, u_i - u_j) = D_{G\backslash T_j}(u_j, u_i - u_j)$ since $u_i = u_j = 0$ on T_j and ∂T_j is piecewise smooth. Hence we have $D_G(u_j, u_i) = D_G(u_j)$. Observe that

$$D_G(u_i - u_j) = D_G(u_i) - 2D_G(u_i, u_j) + D_G(u_j) = D_G(u_i) - D_G(u_j).$$

This shows that $(D_G(u_i))_{i\geq 1}$ is a decreasing convergent sequence and so is $(D_G(u_i - u_j))_{j\geq i}$ and

$$\lim_{i\to\infty}\left(\lim_{j\to\infty} D_G(u_i - u_j)\right) = \lim_{i\to\infty}\left(\lim_{j\to\infty}(D_G(u_i) - D_G(u_j))\right) = 0. \tag{8}$$

By the minimum principle we see that $(u_i)_{i\geq 1}$ is an increasing sequence dominated by u on G_0. Hence $u_\infty := \lim_{i\to\infty} u_i \in H(G \backslash L)$ and $0 \leq u_i \leq u_\infty \leq u$ on $G_0 \backslash L$, which shows that $u_\infty \in C(\overline{G_0} \backslash L)$, $u_\infty|\partial G_0 = 0$, and $u_\infty|K = 1$. Since the Newtonian capacity of L is zero because of $d \geq 3$, there is a continuous map V of the one point compactification $\overline{\boldsymbol{R}^d}$ of $\boldsymbol{R}^d$ to the extended half interval $[0, \infty]$ such that $V \in H(\boldsymbol{R}^d \backslash L)$ with $V|L = \infty$ and

$V(\infty) = \lim_{x\to\infty} V(x) = 0$. The maximum principle yields $u \le u_\infty + \varepsilon V$ on $G\backslash L$ for any $\varepsilon > 0$. Hence $u_\infty = u$ on $G\backslash L$ so that $\lim_{j\to\infty} u_j = u$ locally uniformly on $G\backslash L$. This shows that $(|\nabla u_j - \nabla u_i|)_{j\ge i}$ converges to $|\nabla u - \nabla u_i|$ a.e. on G. Hence by the Fatou lemma

$$D_G(u_i - u) = \int_G \left(\liminf_{j\to\infty} |\nabla u_i(x) - \nabla u_j(x)|^2 \right) dx$$

$$\le \liminf_{j\to\infty} \int_G |\nabla u_i(x) - \nabla u_j(x)|^2 \, dx = \lim_{j\to\infty} D_G(u_i - u_j).$$

This with (8) implies (7), which is to be shown. □

We turn now to the construction of M and $E \subset \partial M$ in Example 3. Reversing the process we first construct E before determining M. These sets E and M will be subsets of $B_0 := B^d(0, 3(d-1)^{1/2})$. We will construct cubes $Q_{i_1\cdots i_n}$ in $B_0 \cap (\boldsymbol{R}^{d-1} \times \{0\})$ for each $n \ge 1$, where $i_1 = 1$ and $1 \le i_k \le 2^{d-1}$ $(2 \le k \le n)$. The construction is by induction. First let $Q_{i_1} = Q(0,1)$, where i_1 runs over $\{1\}$ so that $i_1 = 1$. Consider a sequence $(\lambda_n)_{n\ge 2}$ given by

$$\lambda_n = \max\{\lambda(2^{-n}), 1 - 2^{-n}\} \quad (n \ge 2).$$

Here recall that the number $\lambda(\varepsilon)$ is introduced right after the proof of Lemma 5. Next take the family $\{Q_{i_1 i_2} : 1 \le i_2 \le 2^{d-1}\}$ of congruent cubes $Q_{i_1 i_2}$ regularly distributed in Q_{i_1} with index λ_2. Suppose congruent cubes $Q_{i_1\cdots i_k}$ $(i_1 = 1, 1 \le i_j \le 2^{d-1}$ $(2 \le j \le k))$ have been constructed for each $1 \le k \le n-1$. Then let $\{Q_{i_1\cdots i_{n-1} i_n} : 1 \le i_n \le 2^{d-1}\}$ be regularly distributed in $Q_{i_1\cdots i_{n-1}}$ with index λ_n for each $i_1 \cdots i_{n-1}$. Now we define the set E by

$$E = \bigcap_{1\le n<\infty} \left(\bigcup_{i_1\cdots i_n} \bar{Q}_{i_1\cdots i_n} \right),$$

where the union is taken over $i_1 = 1$ and $1 \le i_k \le 2^{d-1}$ $(2 \le k \le n)$. The set E is a *compact*, *totally disconnected* and *perfect* subset of $B_0 \cap (\boldsymbol{R}^{d-1} \times \{0\})$. We compute the area (i.e. the $(d-1)$-dimensional Hausdorff measure in essence) $|E|$ of E and show that

$$0 < |E| < \infty. \tag{9}$$

To see this let r be the radius of $Q_{i_1\cdots i_{n-1}}$. Then the radius of $Q_{i_1\cdots i_{n-1} i_n}$ is $(r/2)\lambda_n$ and thus $|Q_{i_1\cdots i_{n-1}}| = (2r)^{d-1}$ and $|Q_{i_1\cdots i_{n-1} i_n}| = (r\lambda_n)^{d-1}$. Hence

$$\left|\bigcup_{1\le i_n\le 2^{d-1}} Q_{i_1\cdots i_{n-1}i_n}\right| = \lambda_n^{d-1}|Q_{i_1\cdots i_{n-1}}|.$$

Since $|Q_{i_1}| = 2^{d-1}$, we conclude that

$$\left|\bigcup_{i_1\cdots i_n} Q_{i_1\cdots i_n}\right| = 2^{d-1}\left(\prod_{2\le j\le n}\lambda_j\right)^{d-1}$$

and a fortiori we deduce

$$|E| = \left|\bigcap_{1\le n<\infty}\left(\bigcup_{i_1\cdots i_n} Q_{i_1\cdots i_n}\right)\right| = 2^{d-1}\left(\prod_{2\le j<\infty}\lambda_j\right)^{d-1}.$$

By the choice of λ_n we have $1 \ge \prod_{2\le j<\infty}\lambda_j \ge \prod_{2\le j<\infty}(1-2^{-j}) > 0$, which assures the validity of (9).

For each $n \ge 1$ let u_n be such that $u_n \in C(\overline{B_0}) \cap H\big(B_0 \backslash \bigcup_{i_1\cdots i_n}\bar{Q}_{i_1\cdots i_n}\big)$ and $u_n|\partial B_0 = 0$ and $u_n|\bigcup_{i_1\cdots i_n}\bar{Q}_{i_1\cdots i_n} = 1$. Observe that u_n is positive and superharmonic on B_0. Since $\lambda_n \in [\lambda(2^{-n}), 1)$, Lemma 5 assures that $u_n \ge 1 - 2^{-n}$ on every $\bar{Q}_{i_1\cdots i_{n-1}}$ on which $u_{n-1} = 1$. By the maximum principle we conclude that $(u_n)_{n\ge 1}$ is decreasing on B_0 and

$$|u_n(x) - u_{n-1}(x)| \le 2^{-n} \quad (x \in \overline{B_0}, n \ge 2).$$

Hence $(u_n)_{n\ge 1}$ is uniformly convergent on $\overline{B_0}$ and we denote by u the limit function of $(u_n)_{n\ge 1}$ on $\overline{B_0}$. Then $u \in C(\overline{B_0}) \cap H(B_0\backslash E)$, $u|\partial B_0 = 0$, and $u|E = 1$. The function $1-u$ plays the role of barrier (in the wider sense) at each point of E for the region $B_0\backslash E$ with respect to the harmonic Dirichlet problem on $B_0\backslash E$.

We next define a system of polygonal line segments $l_{i_1\cdots i_n}$ as follows: l_{i_1} is the straight line segment joining the point $b_0 = (3(d-1)^{1/2}, 0, 0, \dots, 0)$ of the boundary of B_0 with the distinguished boundary point b_{i_1} of Q_{i_1}, where $i_1 = 1$; $l_{i_1 i_2}$ is a simple polygonal line segment joining the point b_{i_1} with the distinguished boundary point $b_{i_1 i_2}$ of $Q_{i_1 i_2}$. The arcs $l_{i_1 i_2}$ $(1 \le i_2 \le 2^{d-1})$ lie on $Q_{i_1}\backslash\bigcup_{i_2}\bar{Q}_{i_1 i_2}$ except for their end points; the arcs $l_{i_1 i_2}$ do not intersect one another anywhere except at b_{i_1}. The simple polygonal line segment $l_{i_1\cdots i_{n-1}i_n}$ connect the distinguished boundary point $b_{i_1\cdots i_{n-1}}$ of $Q_{i_1\cdots i_{n-1}}$ with the distinguished boundary point $b_{i_1\cdots i_{n-1}i_n}$ of $Q_{i_1\cdots i_{n-1}i_n}$. Here the arcs $l_{i_1\cdots i_{n-1}i_n}$ $(1 \le i_n \le 2^{d-1})$ remain within the domain $Q_{i_1\cdots i_{n-1}}\backslash\bigcup_{i_n}\bar{Q}_{i_1\cdots i_{n-1}i_n}$ except for their end points and they do not have points of intersection apart from $b_{i_1\cdots i_{n-1}}$. Moreover we assume that $\{l_{i_1\cdots i_{n-1}i_n} : 1 \le i_n \le 2^{d-1}\}$ is congruent with $\{l_{j_1\cdots j_{n-1}j_n} : 1 \le j_n \le 2^{d-1}\}$ for every pair of $i_1\cdots i_{n-1}$ and $j_1\cdots j_{n-1}$.

Let Δ_0 be the set of points not further away from l_{i_1} than a number $\delta_1 \in (0,1)$ and belonging to $\overline{B_0} \backslash Q_{i_1} \times (-\infty, \infty)$; let $\Delta_{i_1 \cdots i_{n-1}}$ $(n \geq 2)$ be the set of points not further away from $\bigcup_{i_n} l_{i_1 \cdots i_{n-1} i_n}$ than a number $\delta_n \in (0,1)$ and belonging to $\overline{B_0} \cap ((\overline{Q}_{i_1 \cdots i_{n-1}} \backslash \bigcup_{i_n} Q_{i_1 \cdots i_{n-1} i_n}) \times (-\infty, \infty))$. By choosing $\delta_n > 0$ further small we may suppose that the following conditions are satisfied: $\Delta_{i_1 \cdots i_{n-1}}$ is contained in $B_0 \cap ((Q_{i_1 \cdots i_{n-1}} \backslash \bigcup_{i_n} \overline{Q}_{i_1 \cdots i_{n-1} i_n}) \times (-\infty, \infty))$ except for the points of $\Delta_{i_1 \cdots i_{n-1}}$ lying on the hyperplanes $x^1 = (b_{i_1 \cdots i_{n-1}})^1$ and $x^1 = (b_{i_1 \cdots i_{n-1} i_n})^1$ $(1 \leq i_n \leq 2^{d-1})$; the set $\Delta_{i_1 \cdots i_{n-1}}$ is the closure of a topological ball; the surface area $|\partial \Delta_{i_1 \cdots i_{n-1}}|$ of $\partial \Delta_{i_1 \cdots i_{n-1}}$ is not greater than $(2^{-d+1})^{n-2} 2^{-n}$ $(n \geq 2)$ and also $|\partial \Delta_0| \leq 2^{-1}$. Finally we set $F_0 := \Delta_0$, $F_1 := F_0 \cup \Delta_{i_1}$, and $F_n := F_{n-1} \cup (\bigcup_{i_1 \cdots i_n} \Delta_{i_1 \cdots i_n})$ $(n \geq 2)$. We also set $F_\infty := \bigcup_{0 \leq n < \infty} F_n$. Clearly $E = \overline{F_\infty} \backslash F_\infty$.

We choose $(\delta_n)_{n \geq 1}$ further small so as to have the following situation. Set $v := u$, the function defined above. Recall that $v \in C(\overline{B_0}) \cap H(B_0 \backslash E)$ with $v | \partial B_0 = 0$ and $v | E = 1$. Clearly $0 < \delta < \infty$ for $\delta = D_{B_0}(v)$. Let $v_n \in C(\overline{B_0}) \cap H(B_0 \backslash E \cup F_n)$ with $v_n | F_n \cup \partial B_0 = 0$ and $v_n | E = 1$ $(n \geq 0)$. We maintain that $(\delta_n)_{n \geq 1}$ can be made so small that

$$D_{B_0}(v_n - v_{n-1}) \leq \delta / 4^{n+2} \quad (n \geq 0), \tag{10}$$

where we understand that $v_{-1} = v$. We first use Lemma 6 for $G = B_0$, $L = l_{i_1}$, and $T_i = \Delta_0$ with $\delta_1 < 1/i$ to conclude that by choosing $\delta_1 > 0$ small enough we can deduce that $D_{B_0}(v_0 - v) \leq \delta / 4^2$. Again we use Lemma 6 for $G = B_0 \backslash \Delta_0$, $L = \bigcup_{i_1 i_2} l_{i_1 i_2}$, and $T_i = \Delta_{i_1}$ with $\delta_2 < 1/i$ to conclude that by choosing $\delta_2 > 0$ sufficiently small we have $D_{B_0}(v_1 - v_0) = D_{B_0 \backslash F_0}(v_1 - v_0) \leq \delta / 4^3$. Assume that by repeating the same process we have chosen positive numbers $\delta_1, \ldots, \delta_n$ so small that $D_{B_0}(v_k - v_{k-1}) \leq \delta / 4^{k+2}$ $(0 \leq k \leq n-1)$. Then, by making $\delta_{n+1} > 0$ smaller, using Lemma 6 again for $G = B_0 \backslash F_{n-1}$, $L = \bigcup_{i_1 \cdots i_{n+1}} l_{i_1 \cdots i_{n+1}}$, and $T_i = \Delta_{i_1 \cdots i_n}$ with $\delta_{n+1} < 1/i$ we conclude that $D_{B_0}(v_n - v_{n-1}) \leq \delta / 4^{n+2}$. We have thus completed the induction of choosing $(\delta_n)_{n \geq 1}$ further so small as to make (10) valid. We can of course moreover assume that $(\delta_n)_{n \geq 1}$ is a strictly decreasing zero sequence.

We are now ready to define the required topological ball M in Example 3 as follows:

$$M := B_0 \backslash (E \cup F_\infty). \tag{11}$$

It is not difficult to see that M is in fact a topological ball in $\boldsymbol{R}^d$ only by taking a close look at the construction of M. For the sake of completeness, however, we will ascertain in the sequel that M is certainly a topological ball in $\boldsymbol{R}^d$. For each $\xi \in \partial B_0$ and $r \in (0, 6(d-1)^{1/2})$, the set $\{x \in \partial B_0 : |x - \xi| < r\}$ is referred to as a spherical cap on ∂B_0 of chordal radius, or simply radius, r centered at ξ. In the sequel spherical caps considered are all on ∂B_0. We see that $F_\infty \cap \partial B_0$ is the closure of a spherical cap S_0 centered at $(3(d-1)^{1/2}, 0, \ldots, 0) \in \partial B_0 : F_\infty \cap \partial B_0 = \overline{S_0}$. Observe that

$$\partial M = (\partial B_0 \backslash \overline{S}_0) \cup (\partial F_\infty \backslash S_0).$$

We will show that there is a homeomorphism h of $\partial F_\infty \backslash S_0$ onto $\overline{S}_0$ fixing the boundary of S_0 in ∂B_0 so that, by setting $h|(\partial B_0 \backslash \overline{S}_0) = \text{identity}$, h is a homeomorphism of ∂M onto ∂B_0. By the construction of M, we will then see that h is extended to a homeomorphism of $\overline{M}$ onto $\overline{B}_0$ such that $h(M) = B_0$ so that we can conclude that M is a topological ball. In other words, by specifically deforming $\partial F_\infty \backslash S_0$ topologically to $\overline{S}_0$, $\overline{M}$ (M, resp.) is deformed topologically to $\overline{B}_0$ (B_0, resp.). Now we start the construction of a homeomorphism h of $\partial F_\infty \backslash S_0$ onto $\overline{S}_0$ fixing the boundary of S_0 in ∂B_0. For the purpose we choose spherical caps $S_{i_1 \cdots i_n}$ in S_0 for each $n \geq 1$, where $i_1 = 1$ and $1 \leq i_k \leq 2^{d-1}$ $(2 \leq k \leq n)$. The choice is by induction. First let S_{i_1} be a spherical cap of radius $r_1 > 0$ such that $\overline{S}_{i_1} \subset S_0$ where $i_1 = 1$. Next take a family $\{S_{i_1 i_2} : 1 \leq i_2 \leq 2^{d-1}\}$ of spherical caps $S_{i_1 i_2}$ of the same radii $r_2 > 0$ such that $\overline{S}_{i_1 i_2}$ are mutually disjoint and $\overline{S}_{i_1 i_2} \subset S_{i_1}$. Suppose spherical caps $S_{i_1 \cdots i_k}$ $(i_1 = 1, 1 \leq i_j \leq 2^{d-1}$ $(2 \leq j \leq k))$ have been chosen for each $1 \leq k \leq n-1$. Then let $\{S_{i_1 \cdots i_{n-1} i_n} : 1 \leq i_n \leq 2^{d-1}\}$ be a family of spherical caps $S_{i_1 \cdots i_{n-1} i_n}$ of the same radii $r_n > 0$ such that $\overline{S}_{i_1 \cdots i_{n-1} i_n}$ are mutually disjoint and $\overline{S}_{i_1 \cdots i_{n-1} i_n} \subset S_{i_1 \cdots i_{n-1}}$ for each $i_1 \cdots i_{n-1}$. It automatically follows that $r_n \downarrow 0$. We then set $X_0 := S_0$ and $X_n := \bigcup_{i_1 \cdots i_n} S_{i_1 \cdots i_n}$ for each $n \geq 1$, and finally set $Y := \bigcap_{n \geq 1} X_n$. We decompose

$$\partial F_\infty \backslash S_0 = ((\partial F_\infty \backslash S_0) \cap F_0) \cup \left(\bigcup_{1 \leq i < \infty} ((\partial F_\infty \backslash S_0) \cap (F_i \backslash F_{i-1})) \right) \cup E$$

and similarly

$$\overline{S}_0 = (\overline{X}_0 \backslash X_1) \cup \left(\bigcup_{1 \leq i < \infty} (\overline{X}_i \backslash X_{i+1}) \right) \cup Y.$$

Since $(\partial F_\infty \backslash S_0) \cap F_0$ is homeomorphic to $\overline{X}_0 \backslash X_1$ and these two sets have the boundary of S_0 in ∂B_0 in common, we can construct a homeomorphism h of $(\partial F_\infty \backslash S_0) \cap F_0$ onto $\overline{S}_0 \backslash X_1 = \overline{X}_0 \backslash X_1$ fixing the boundary of S_0 in ∂B_0 such that h induces a natural correspondence $\Delta_0 \to s_0 := S_{i_1}$. Since $(\partial F_\infty \backslash S_0) \cap (F_1 \backslash F_0)$ is homeomorphic to $\overline{X}_1 \backslash X_2$, h can be continued to a homeomorphism of $(\partial F_\infty \backslash S_0) \cap F_1$ onto $\overline{S}_0 \backslash X_2$ such that h induces a natural correspondence $\Delta_{i_1} \to s_{i_1} := \bigcup_{1 \leq j \leq 2^{d-1}} S_{i_1 j}$. Suppose h can be continued to a homeomorphism of $(\partial F_\infty \backslash S_0) \cap F_n$ onto $\overline{S}_0 \backslash X_{n+1}$ such that h induces a natural correspondence $\Delta_{i_1 \cdots i_n} \to s_{i_1 \cdots i_n} := \bigcup_{1 \leq j \leq 2^{d-1}} S_{i_1 \cdots i_n j}$ for every $i_1 \cdots i_n$. Then, since $(\partial F_\infty \backslash S_0)$

$\cap (F_{n+1}\backslash F_n)$ is homeomorphic to $\overline{X}_{n+1}\backslash X_{n+2}$, h can be continued to a homeomorphism of $(\partial F_\infty\backslash S_0)\cap F_{n+1}$ onto $\overline{S}_0\backslash X_{n+2}$ such that h induces a natural correspondence $\varDelta_{i_1\cdots i_{n+1}} \to s_{i_1\cdots i_{n+1}} := \bigcup_{1\leq j\leq 2^{d-1}} S_{i_1\cdots i_{n+1}j}$ for every $i_1\cdots i_{n+1}$. Hence we can extend h to a homeomorphism of $(\partial F_\infty\backslash S_0)\backslash E$ onto $\overline{S}_0\backslash Y$ in a special manner described above. By the construction of E there exists a bijective correspondence between points $x\in E$ and sequences $i_1i_2\cdots i_n\cdots$ $(i_1 = 1, 1\leq i_k\leq 2^{d-1}$ $(k\geq 2))$ such that the sequence of sets $\varDelta_{i_1}$, $\varDelta_{i_1i_2}, \ldots, \varDelta_{i_1i_2\cdots i_n}, \ldots$ converges to x. In this case we write $x = x(i_1i_2\cdots i_n\cdots)$. Similarly by the way Y is constructed there exists a bijective correspondence between points $y\in Y$ and sequences $i_1i_2\cdots i_n\cdots$ as above such that the intersection of $s_{i_1}, s_{i_1i_2}, \ldots, s_{i_1i_2\cdots i_n}, \ldots$ is $\{y\}$. In this case we also write $y = y(i_1i_2\cdots i_n\cdots)$. By the fashion h is determined, h induces the natural correspondence $\varDelta_{i_1i_2\cdots i_n} \to s_{i_1i_2\cdots i_n}$. Hence if we define $h: E\to Y$ by

$$h(x(i_1i_2\cdots i_n\cdots)) = y(i_1i_2\cdots i_n\cdots)$$

for every sequence $i_1i_2\cdots i_n\cdots$ $(i_1 = 1, 1\leq i_k\leq 2^{d-1}$ $(k\geq 2))$, then $h:\partial F_\infty\backslash S_0\to\overline{S}_0$ is seen to be a homeomorphism of $\partial F_\infty\backslash S_0$ onto $\overline{S}_0$ fixing the boundary of S_0 in ∂B_0. By extending h to ∂M on setting h as identity on $\partial B_0\backslash\overline{S}_0$, we have thus constructed a homeomorphism h of ∂M onto ∂B_0. Since we have seen that $\partial F_\infty\backslash S_0$ is topologically deformed to $\overline{S}_0$ fixing the boundary of S_0 in ∂B_0, $\partial F_\infty = (\partial F_\infty\backslash S_0)\cup S_0$ is seen to be homeomorphic to a sphere. Hence $\partial(E\cup F_\infty) = \partial F_\infty$ is homeomorphic to a sphere. By the construction of $E\cup F_\infty$, we see that $E\cup F_\infty$ is the closure of a region homeomorphic to a ball bounded by the topological sphere $\partial(E\cup F_\infty) = \partial F_\infty$. Thus $E\cup F_\infty$ is the closure of a topological ball, and again by the construction of $M = B_0\backslash(E\cup F_\infty)$, we see that M is homeomorphic to a ball bounded by the topological sphere ∂M. Because of this we can extend h to a homeomorphism h of $\overline{M}$ onto $\overline{B}_0$ with $h(M) = B_0$. Hence we have ascertained that M is a topological ball.

Since $\partial M\backslash E$ is piecewise smooth, every point in $\partial M\backslash E$ is regular, which is seen by e.g. the cone condition criterion. As before, $1-u = 1-v$ plays the role of barrier on M for every point of E. Thus M is a regular domain and a fortiori the condition (a) of Example 3 is satisfied. Observe that, in addition to (9),

$$|\partial F_\infty| \leq |\partial\varDelta_0| + \sum_{2\leq n<\infty}\left|\bigcup_{i_1\cdots i_{n-1}}\partial\varDelta_{i_1\cdots i_{n-1}}\right|$$

$$\leq 2^{-1} + \sum_{2\leq n<\infty}(2^{d-1})^{n-2}\cdot(2^{-d+1})^{n-2}2^{-n} = 1.$$

Therefore we see that $|\partial M|\leq|\partial B_0| + |\partial F_\infty| + |E| < \infty$ so that the condition (b) of

Example 3 is fulfilled. It is clear that $|\partial M \backslash E| > |\partial B_0|/2 > 0$. This with (9) assures the validity of (c) in Example 3.

To complete the construction for Example 3 only the proof of (4) (i.e. the condition (d) in Example 3) is left. We first claim that

$$\lim_{n\to\infty} v_n(x) = \omega(x; E, M) \quad (x \in M). \tag{12}$$

Observe that $v \geq v_n \geq v_{n+1} \geq 0$ on M $(n \geq 0)$. Therefore $(v_n)_{n\geq 1}$ converges to a function $v_\infty \in C(\overline{B_0}\backslash E) \cap H(M)$ such that $0 \leq v_\infty \leq 1$ on $\overline{B_0}\backslash E$ and $v_\infty | F_\infty \cup \partial B_0 = 0$. To prove (12) we need to recall the definition (1) of $\omega(\cdot\,; E, M)$. Clearly $v_k | M \in \mathcal{U}_E(M)$ and hence $v_k \geq \omega(\cdot\,; E, M)$ on M. On letting $k \uparrow \infty$ we deduce $v_\infty \geq \omega(\cdot\,; E, M)$ on M. To show the reversed inequality, take an arbitrary $s \in \mathcal{U}_E(M)$ and any number $\lambda \in (0, 1)$. Since $\liminf_{x\to y} s(x) \geq 1$ for each $y \in E$, there is a ball $B(y, r_y) = B^d(y, r_y)$ $(r_y > 0)$ in $\boldsymbol{R}^d$ such that $s > \lambda$ on $B(y, r_y) \cap M$. Then the set $U = \bigcup_{y\in E} B(y, r_y)$ is open, $U \supset E$, and $s > \lambda$ on $U \cap M$. Because $E = \bigcap_{1\leq k<\infty} \overline{(\overline{F_\infty}\backslash F_k)}$ is compact and $E \subset U$, there is a number k_0 such that $\overline{F_\infty}\backslash F_k \subset \overline{\overline{F_\infty}\backslash F_k} \subset U$ for each $k \geq k_0$. Fix an arbitrary $k \geq k_0$. If $y \in (\partial M)\backslash U$, then, since $v_k(y) = 0$, $\liminf_{x\in M, x\to y} \lambda^{-1}s(x) \geq 0 = v_k(y)$. If $y \in (\partial M) \cap U$, then, since $\lambda^{-1}s > 1$ and $v_k \leq 1$ on $M \cap U$, $\liminf_{x\in M, x\to y} \lambda^{-1}s(x) \geq 1 \geq v_k(y)$. Hence

$$\liminf_{x\in M, x\to y} \lambda^{-1}s(x) \geq \limsup_{x\in M, x\to y} v_k(x)$$

for every $y \in \partial M$, which implies, in view of the minimum (comparison) principle, that $\lambda^{-1}s \geq v_k$ on M. On letting $k \uparrow \infty$ and then $\lambda \uparrow 1$, we obtain $s \geq v_\infty$ on M. By the arbitrariness of $s \in \mathcal{U}_E(M)$, we finally conclude that $\omega(\cdot\,; E, M) \geq v_\infty$ on M. The proof of (12) is thus over.

Finally we turn to the proof of (4). Since $D_M(v_n - v_{n-1}) \leq D_{B_0}(v_n - v_{n-1}) \leq \delta/4^{n+2}$ $(n \geq 0)$, we have for every $j > 1$ that

$$D_M(v_j - v)^{1/2} \leq \sum_{0\leq i\leq j} D_M(v_i - v_{i-1})^{1/2} \leq \sum_{0\leq i<\infty} \delta^{1/2}/2^{i+2} = \delta^{1/2}/2.$$

In view of (12) the Fatou lemma yields

$$D_M(\omega(\cdot\,; E, M) - v)^{1/2} \leq \liminf_{j\to\infty} D_M(v_j - v)^{1/2} \leq \delta^{1/2}/2$$

and a fortiori we obtain that

$$|D_M(\omega(\cdot\,; E, M))^{1/2} - D_M(v)^{1/2}| \leq D_M(\omega(\cdot\,; E, M) - v)^{1/2} \leq \delta^{1/2}/2.$$

Since $D_M(v) = \delta > 0$, the above inequality implies that

$$\frac{1}{4}D_M(v) \leq D_M(\omega(\cdot\,; E, M)) \leq \frac{9}{4}D_M(v),$$

which yields (4). The construction of M and $E \subset \partial M$ in Example 3 is completed.

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