J. Math. Soc. Japan
Vol. 52, No. 3, 2000

Holomorphic vertical line bundle of the twistor space over a quaternionic manifold

By Toshimasa Kobayashi

(Received Jun. 18, 1998)
(Revised Nov. 11, 1998)

Abstract. The vertical bundle of the twistor fibration over a 4-dimensional self-dual manifold is a holomorphic line bundle and plays an important role in a study of the twistor space. On the other hand, the vertical bundle of the twistor space over a quaternionic manifold is not a holomorphic line bundle, in general. We shall give the condition for a vertical bundle to be a holomorphic line bundle.

1. Introduction.

We are concerned with holomorphic structures on the vertical bundle of the twistor fibration over a quaternionic manifold.

For an oriented m-dimensional conformal manifold M, we may consider a Weyl structure D on M, which is a symmetric linear connection preserving the conformal structure of M. Over M, there is a line bundle L associated to the $CO(m)$-principal bundle of M and the representation $A \mapsto |\det A|^{1/m}$ of the linear group. Thus a Weyl structure D on M induces a linear connection D^L on L. In the case of $m = 4$, if the curvature of D^L is a self-dual 2-form, then D is called a self-dual Weyl structure. While it is known that if M is a 4-dimensional self-dual manifold, then there is a complex 3-manifold Z fibered over M by a family of projective lines. Z is called the twistor space of M. The vertical bundle Θ of Z is considered as a complex line bundle over Z and has a natural Hermitian metric. We choose a Weyl structure D on M, then a linear connection ∇ on Θ is induced by D. If the curvature of ∇ is of type $(1,1)$ relative to the complex structure on Z, then we call ∇ a *Chern connection*. A Chern connection on Θ induces a holomorphic structure that renders Θ a holomorphic line bundle over Z. In particular, if D is the Levi-Civita connection of a self-dual metric on M, then the induced connection ∇ on Θ is a Chern connection, and $\otimes^2 \Theta$ is isomorphic to the dual bundle of the canonical bundle of Z as a holomorphic bundle.

2000 *Mathematics Subject Classification*. Primary 53C07; Secondary 53C10, 53C15.
Key Words and Phrases. Quaternionic manifold, twistor space, Obata connection.

Gauduchon showed that for a 4-dimensional self-dual manifold, a linear connection ∇ on Θ is a Chern connection if and only if a Weyl structure D that induces ∇ is self-dual. Furthermore, if M is compact, he classified the types of the conformal structures admitting holomorphic sections on $\otimes^p \Theta$. Using these results and a vanishing theorem, he proved that if the conformal class of M contains a metric with negative scalar curvature then the twistor space of M does not contain any nontrivial divisor.

A $4n$-dimensional manifold ($n \geq 2$) is called *quaternionic* if it has a $GL(n,\boldsymbol{H})Sp(1)$-structure preserved by a torsion-free connection. We note that if $n = 1$ then $GL(1,\boldsymbol{H})Sp(1) = CO(4)$. Salamon showed that there is a twistor space Z over a quaternionic manifold M. The fiber Z_x over each point $x \in M$ is a real 2-sphere, which parametrizes almost complex structures on T_xM, and the total space of Z admits a complex structure. Therefore, we regard the notion of quaternionic manifold as a generalization of that of self-dual manifold and examine quaternionic manifolds and their twistor spaces.

In the next section, we recall the twistor space of a quaternionic manifold. We express a twistor space and its vertical bundle as associated bundles with the $GL(n,\boldsymbol{H})Sp(1)$-principal bundle and representations of $GL(n,\boldsymbol{H})Sp(1)$. Thus we see that a connection D on a quaternionic manifold induces a connection ∇ on a vertical bundle. Further, we may describe the curvature R^{∇} of ∇ explicitly, and see the relation between the curvatures R^{∇} and R^{D}. In Section 3, we recall representations of the structure group $GL(n,\boldsymbol{H})Sp(1)$ and the first prolongation of its Lie algebra. Combining the Clebsch-Gordan formula and the formulas of irreducible decompositions of $GL(n,\boldsymbol{H})$-modules, we describe the first prolongation as a $GL(n,\boldsymbol{H})Sp(1)$-module. In Section 4, we shall study a curvature of a quaternionic manifold by means of representation theory. We consider R^{D} as a 2-form with values in the Lie algebra $\mathfrak{gl}(n,\boldsymbol{H}) \oplus \mathfrak{sp}(1)$ of $GL(n,\boldsymbol{H})Sp(1)$. From the first Bianchi identity, we see that R^{D} determines an element of a Spencer cohomology. By using some irreducible decompositions of $GL(n,\boldsymbol{H})Sp(1)$-modules, we have the irreducible decomposition of a curvature of a quaternionic manifold. In Section 5, we have the main theorem. From the results in Sections 3 and 4, we may describe a curvature of a quaternionic manifold explicitly. We shall obtain the condition for the vertical bundle of the twistor space of a quaternionic manifold to have a Chern connection. We also find that this condition corresponds to the condition for a Weyl structure to be self-dual in the case of a 4-dimensional self-dual manifold. In Section 6, we deal with hypercomplex manifolds. A $4n$-dimensional manifold that has a $GL(n,\boldsymbol{H})$-structure with a torsion-free connection is called a *hypercomplex manifold.* We note that the class of hypercomplex manifolds is included in that of quaternionic manifolds. It is known that a hypercomplex manifold has a unique torsion-free connection. It is called the *Obata connection*. Applying the theorem in Section 5 to the case of a hypercomplex manifold, we see that an Obata connection induces a Chern connection on a vertical line bundle.

The author would like to express his gratitude to Professors Takashi Nitta and Shin Nayatani for their valuable advice and kind encouragement.

2. Twistor spaces.

Let M be a quaternionic manifold, which is a real $4n$-dimensional manifold, $n \geq 2$, with a $GL(n,\boldsymbol{H})Sp(1)$-structure admitting a torsion-free connection. We choose a connection D out of such connections. We denote by E, H the standard complex representations of $GL(n,\boldsymbol{H})$, $Sp(1)$ on $\boldsymbol{C}^{2n}$, $\boldsymbol{C}^2$ respectively. The complex vector spaces E and H possess antilinear structure maps $v \mapsto \tilde{v}$ commuting with the action of the respective groups and satisfying $\tilde{\tilde{v}} = -v$. Such representations are called quaternionic. Then the complexified cotangent bundle of M has the form

$$(2.1) \qquad (T^*M)^{\boldsymbol{C}} \cong \mathbf{E} \otimes_{\boldsymbol{C}} \mathbf{H},$$

where $\mathbf{E}$, $\mathbf{H}$ are vector bundles associated to representations E, H respectively. The symmetric powers S^kH $(k \geq 0)$ are the irreducible complex representations of $Sp(1)$. If k is even, then S^kH has a real structure induced from the structure map of H, so we regard it as a real vector space. In particular, S^2H is the adjoint representation of $Sp(1)$. There is an $Sp(1)$-invariant skew form $\omega_H \in \Lambda^2H^*$ which induces an isomorphism $H \cong H^*$. Using the inclusion $S^2H \hookrightarrow H \otimes H \cong_{\omega_H} H \otimes H^* = \operatorname{End} H$, we may identify $\mathfrak{sp}(1)$ with S^2H. Let $\langle\,,\,\rangle$ be the inner product on $S^2H \subset H \otimes H$ induced by ω_H. If $J, K \in S^2H$, then as endomorphisms of TM,

$$(2.2) \qquad J \circ K + K \circ J = -\langle J, K\rangle 1.$$

We consider the bundle

$$Z = \{J \in S^2\mathbf{H} \mid \langle J, J\rangle^{1/2} = \sqrt{2}\}$$

whose fiber Z_x over a point $x \in M$ is a real 2-sphere. From (2.2), an element $J \in Z_x$ defines an almost complex structure on T_xM. The bundle Z is called the twistor space of M. Let π be the natural projection from Z to M and Θ the vertical tangent bundle on Z. For any point $J \in Z_x$, we have a natural identification

$$\Theta_J = \{A \in S^2\mathbf{H} \mid J \circ A = -A \circ J\},$$

where $\Theta_J = T_J Z_x$ is the fiber of Θ at J. The bundle Θ admits a complex structure determined by

$$\mathscr{J}A = J \circ A, \qquad A \in \Theta_J.$$

An inner product $\langle\ ,\ \rangle$ on Θ_J is induced by the embedding of Θ_J in $S^2\mathbf{H}$. $\mathscr{J}$ is compatible with $\langle\ ,\ \rangle$, so Θ has a canonical Hermitian structure. We denote by $\Omega^{(x)}$ the Kähler form on Θ_J $(J \in Z_x)$ induced by $\langle\ ,\ \rangle$. Let v^D denote the vertical projection from TZ to Θ with respect to D. Any vector U on Z, at a point J, is represented by

$$U = (v^D(U), X),$$

where $X = \pi_*(U)$ is the projection of U in T_xM. Thus we obtain an almost complex structure $\mathscr{J}$ on Z defined by

$$\mathscr{J}U = (J \circ v^D(U), JX).$$

Salamon showed that $\mathscr{J}$ is integrable when M is a quaternionic manifold. We define Π the orthogonal projection of $\pi^* S^2\mathbf{H}$ onto Θ such that for any point J of Z_x,

$$\Pi^J(A) = A - \frac{1}{2}\langle A, J\rangle J, \qquad A \in S^2\mathbf{H}.$$

A connection D on M induces a connection D^{Ad} on $S^2\mathbf{H}$ via the adjoint representation of $Sp(1)$. We denote by $\pi^* D^{\mathrm{Ad}}$ the pull back connection on $\pi^* S^2\mathbf{H}$. We may define a Hermitian connection ∇ on Θ as follows:

$$\nabla = \Pi \circ \pi^* D^{\mathrm{Ad}},$$

more explicitly,

$$\nabla_U \tilde{A} = \widetilde{D_X^{\mathrm{Ad}} A} - \frac{1}{2}\langle A, J\rangle v^D(U), \qquad U \in T_J Z,$$

where $\tilde{A}$ is a vertical vector field on Z defined by

$$\tilde{A}(J) = \Pi^J(A), \qquad A \in S^2\mathbf{H}, \quad J \in Z_x.$$

We may compute the curvature of ∇ as follows.

Lemma 2.1 ([**3**]). *Let R^∇ denote the curvature of the Hermitian connection ∇ on Θ induced by a connection D of M. Then we have*

(1) $$R^\nabla_{B,C} A = \frac{1}{2}\Omega^{(x)}(C, B)\mathscr{J}A,$$

(2) $$R^\nabla_{B,\tilde{X}} A = 0,$$

(3) $$R^\nabla_{\tilde{X},\tilde{Y}} A = \Pi^J[R^D(X, Y), A],$$

where $A, B, C \in \Theta_J$, $X, Y \in T_xM$, $\tilde{X}$, $\tilde{Y}$ *is the horizontal lift of* X, Y *respectively, and* R^D *is the curvature of* D.

Proof. (1) We note that $[B, C] = \Omega^{(x)}(B, C)J$ and (2.2), we have

$$\begin{aligned}
R^{\nabla}_{B,C}A &= \nabla_B\nabla_{\tilde{C}}\tilde{A} - \nabla_C\nabla_{\tilde{B}}\tilde{A} - \nabla_{[\tilde{B},\tilde{C}]}\tilde{A} \\
&= \nabla_B\left(-\frac{1}{2}\langle A, J\rangle\tilde{C}\right) - \nabla_C\left(-\frac{1}{2}\langle A, J\rangle\tilde{B}\right) \\
&= -\frac{1}{2}\{(\langle A, \nabla_B J\rangle)C - (\langle A, \nabla_C J\rangle)B\} \\
&= \frac{1}{2}(\langle A, C\rangle B - \langle A, B\rangle C) \\
&= \frac{1}{2}\Omega^{(x)}(C, B)\mathscr{J}A.
\end{aligned}$$

(2) We note that $[\tilde{X}, \tilde{B}]$ is vertical, we have

$$\begin{aligned}
R^{\nabla}_{B,\tilde{X}}A &= \nabla_B\nabla_{\tilde{X}}\tilde{A} - \nabla_{\tilde{X}}\nabla_{\tilde{B}}\tilde{A} - \nabla_{[\tilde{B},\tilde{X}]}\tilde{A} \\
&= \nabla_B(\widetilde{D^{\mathrm{Ad}}_X A}) - \nabla_{\tilde{X}}\left(-\frac{1}{2}\langle A, J\rangle\tilde{B}\right) \\
&= -\frac{1}{2}\langle D^{\mathrm{Ad}}_X A, J\rangle B + \frac{1}{2}D^{\mathrm{Ad}}_X\widetilde{(\langle A, J\rangle B)} \\
&= -\frac{1}{2}\langle D^{\mathrm{Ad}}_X A, J\rangle B + \frac{1}{2}(\langle D^{\mathrm{Ad}}_X A, J\rangle + \langle A, D^{\mathrm{Ad}}_X J\rangle)B \\
&= 0.
\end{aligned}$$

(3) We note that $R^{D^{\mathrm{Ad}}} = d(Ad)(R^D) = ad(R^D)$, we have

$$\begin{aligned}
R^{\nabla}_{\tilde{X},\tilde{Y}}A &= \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{A} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{A} - \nabla_{[\tilde{X},\tilde{Y}]}\tilde{A} \\
&= \nabla_{\tilde{X}}(\widetilde{D^{\mathrm{Ad}}_Y A}) - \nabla_{\tilde{Y}}(\widetilde{D^{\mathrm{Ad}}_X A}) - \widetilde{D^{\mathrm{Ad}}_{[X,Y]}}A \\
&= D^{\mathrm{Ad}}_X\widetilde{D^{\mathrm{Ad}}_Y}A - D^{\mathrm{Ad}}_Y\widetilde{D^{\mathrm{Ad}}_X}A - \widetilde{D^{\mathrm{Ad}}_{[X,Y]}}A \\
&= \Pi^J(R^{D^{\mathrm{Ad}}}(X, Y)A) \\
&= \Pi^J[R^D(X, Y), A]. \qquad \square
\end{aligned}$$

From this lemma, we see that R^{∇} is $\mathscr{J}$-invariant in cases of (1) and (2). In (3), [,] is the bracket of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \boldsymbol{H}) \oplus \mathfrak{sp}(1)$ of the structure group $GL(n, \boldsymbol{H})Sp(1)$. R^D is a 2-form with values in $\mathfrak{g}$ and A is in $\Theta_J \subset S^2H \cong \mathfrak{sp}(1)$, so we take notice of the component on $\mathfrak{sp}(1)$ of R^D in Section 5. By virtue of representation theory, we examine the curvature of a connection on a quaternionic manifold.

3. Representations of $GL(n, \boldsymbol{H})Sp(1)$.

We denote by G the structure group $GL(n, \boldsymbol{H})Sp(1)$ of M. Let $\mathfrak{g}^{(1)}$ be the first prolongation of the Lie algebra $\mathfrak{g}$ of G and T the representation of G corresponding to the tangent bundle. We have

$$\mathfrak{g} \subset \operatorname{End} T = T \otimes T^*,$$

then $\mathfrak{g}^{(1)}$ is defined to be the kernel of the skewing mapping

$$\partial : \mathfrak{g} \otimes T^* \to T \otimes \Lambda^2 T^*.$$

We shall determine the above homomorphism for $\mathfrak{g} = \mathfrak{gl}(n, \boldsymbol{H}) \oplus \mathfrak{sp}(1) \cong E^*E \oplus S^2H$. Tensor products are indicated either in the usual way or simply by juxtaposition. From (2.1), we have

$$\mathfrak{g} \otimes T^* \cong (E^*E \oplus S^2H) \otimes EH,$$

and

$$\begin{aligned} T \otimes \Lambda^2 T^* &\cong E^*H \otimes \Lambda^2(EH) \\ &\cong E^*H \otimes (S^2E \oplus \Lambda^2 E S^2 H). \end{aligned}$$

There is a contraction $\varphi : E^* \otimes S^2E \to E$, so by Schur's lemma, E appears in $E^* \otimes S^2E$, and we have

$$E^* \otimes S^2E \cong E \oplus C, \tag{3.1}$$

where $C = \ker \varphi$. In a similar fashion, we see

$$E^* \otimes \Lambda^2 E \cong E \oplus D. \tag{3.2}$$

C and D are both irreducible. Combining the above isomorphisms and the Clebsch-Gordan formula

$$S^jH \otimes S^kH \cong \bigoplus_{r=0}^{\min(j,k)} S^{j+k-2r}H, \tag{3.3}$$

we have

Lemma 3.1 ([8]).

$$\mathfrak{g} \otimes T^* \cong 3EH \oplus CH \oplus DH \oplus ES^3H,$$

$$T \otimes \Lambda^2 T^* \cong 2EH \oplus CH \oplus DH \oplus ES^3H \oplus DS^3H,$$

where nEH denotes an isotypic component isomorphic to the direct sum of n copies of EH.

From this lemma, we obtain

Proposition 3.1 ([8]).

$$\mathfrak{g}^{(1)} = \ker \partial \cong EH.$$

We represent the isomorphism in Proposition 3.1 more precisely. There is one copy of EH in each of the three terms on the right-hand side of

$$\mathfrak{g} \otimes T^* \cong (\boldsymbol{C} \oplus \mathfrak{sl}(n, \boldsymbol{H}) \oplus \mathfrak{sp}(1)) \otimes EH.$$

We take a basis $\{e_i\}_{i=1}^{2n}$ of E, such that $\tilde{e}_j = e_{j+n}, \widetilde{e_{j+n}} = -e_j$ $(j = 1, \dots, n)$, and an $SU(2)$-basis $\{h, \tilde{h}\}$ of H $(\omega_H(h, \tilde{h}) = 1)$, where $v \mapsto \tilde{v}$ are antilinear structure maps commuting with the action of $GL(n, \boldsymbol{H})$ or $Sp(1)$ and satisfying $\tilde{\tilde{v}} = -v$. Let $\{e^i\}_{i=1}^{2n}$ denote the dual basis of E^*, then

$$\alpha_1 = \sum_{i=1}^{2n} (e^i h e_i \tilde{h} - e^i \tilde{h} e_i h) e_1 h \in \boldsymbol{C} \otimes EH,$$

$$\alpha_2 = \sum_{i=1}^{2n} (e^i h e_1 \tilde{h} - e^i \tilde{h} e_1 h) e_i h - \frac{1}{2n} \alpha_1 \in \mathfrak{sl}(n, \boldsymbol{H}) \otimes EH,$$

$$\alpha_3 = \sum_{i=1}^{2n} \{2 e^i h e_i h e_1 \tilde{h} - (e^i \tilde{h} e_i h + e^i h e_i \tilde{h}) e_1 h\} \in \mathfrak{sp}(1) \otimes EH,$$

are representatives of the element $e_1 h$ in each of the three copies of EH, and $\ker \partial$ is spanned by the element

$$\alpha = \frac{n+1}{n} \alpha_1 + 2\alpha_2 + \alpha_3 \tag{3.4}$$

$$= \sum_{i=1}^{2n} \{(e^i h e_i \tilde{h} - e^i \tilde{h} e_i h) e_1 h + 2(e^i h e_1 \tilde{h} - e^i \tilde{h} e_1 h) e_i h$$

$$+ 2 e^i h e_i h e_1 \tilde{h} - (e^i \tilde{h} e_i h + e^i h e_i \tilde{h}) e_1 h\}.$$

By using (3.4), in Section 5, we may describe a curvature of a quaternionic manifold concretely.

4. Curvature of a quaternionic manifold.

We consider the Spencer complex

$$\cdots \to \mathfrak{g}^{(r)} \otimes \Lambda^{s-1} T^* \to \mathfrak{g}^{(r-1)} \otimes \Lambda^s T^* \to \mathfrak{g}^{(r-2)} \otimes \Lambda^{s+1} T^* \to \cdots,$$

where $\mathfrak{g}^{(r)}$ denotes the r-th prolongation of $\mathfrak{g}$, where $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = T$. The cohomology at the point $\mathfrak{g}^{(r-1)} \otimes \Lambda^s T^*$ is denoted by $H^{r,s}(\mathfrak{g})$.

For a quaternionic manifold M with a torsion-free connection D, the curvature R^D of D lies in $\mathfrak{g} \otimes \Lambda^2 T^*$. The first Bianchi identity implies that $\partial R = 0$, and hence R^D represents the cohomology class in $H^{1,2}(\mathfrak{g})$ of the sequence

$$\mathfrak{g}^{(1)} \otimes T^* \to \mathfrak{g} \otimes \Lambda^2 T^* \to T \otimes \Lambda^3 T^*.$$

In order to decompose these spaces, we introduce some irreducible decompositions of $GL(n, \boldsymbol{H})$-modules. First,

$$\begin{cases} E \otimes S^2 E \cong S^3 E \oplus F, \\ E \otimes \Lambda^2 E \cong \Lambda^3 E \oplus F', \end{cases} \tag{4.1}$$

where modules F and F' are irreducible, and $F \cong F'$ via Schur's lemma. Secondly,

$$\begin{cases} E^* \otimes S^3 E \cong S^2 E \oplus U, \\ E^* \otimes \Lambda^3 E \cong \Lambda^2 E \oplus V, \end{cases} \tag{4.2}$$

with U, V irreducible, and from (4.1) and (4.2),

$$\begin{cases} E^* \otimes E \otimes S^2 E \cong S^2 E \oplus U \oplus E^* F, \\ E^* \otimes E \otimes \Lambda^2 E \cong \Lambda^2 E \oplus V \oplus E^* F. \end{cases} \tag{4.3}$$

We see that both left-hand members in (4.3) contain $E \otimes E$ from (3.1) and (3.2), thus we have that

$$E^* F \cong S^2 E \oplus \Lambda^2 E \oplus W,$$

for some irreducible module W. Thirdly,

$$\Lambda^3(EH) \cong \Lambda^3 E S^3 H \oplus FH.$$

Combining the above decompositions and the Clebsch-Gordan formula (3.3), we have

LEMMA 4.1 ([8]).

$$\mathfrak{g}\otimes\Lambda^2T^*\cong 2S^2E\oplus 2\Lambda^2E\oplus U\oplus W\oplus(2S^2E\oplus 3\Lambda^2E\oplus V\oplus W)S^2H\oplus\Lambda^2ES^4H,$$

$$T\otimes\Lambda^3T^*\cong S^2E\oplus\Lambda^2E\oplus W\oplus(S^2E\oplus 2\Lambda^2E\oplus V\oplus W)S^2H\oplus(\Lambda^2E\oplus V)S^4H.$$

On the other hand, from (2.1) and Proposition 3.1, we have

$$\mathfrak{g}^{(1)}\otimes T^*\cong EH\otimes EH\cong S^2E\oplus\Lambda^2E\oplus(S^2E\oplus\Lambda^2E)S^2H. \tag{4.4}$$

Thus we see that the components of $\mathfrak{g}\otimes\Lambda^2T^*$ minus those of $\partial(\mathfrak{g}^{(1)}\otimes T^*)$ all occur in $T\otimes\Lambda^3T^*$ with the exception of U. Using Schur's lemma, we may check that $\partial:\mathfrak{g}\otimes\Lambda^2T^*\to T\otimes\Lambda^3T^*$ has full rank. Hence we obtain

PROPOSITION 4.1 ([8]).

$$H^{1,2}(\mathfrak{g})\cong U.$$

Therefore, the curvature R^D has the form

$$R^D=\partial\left(\sum_i v_i\otimes t^i\right)+R_U, \tag{4.5}$$

where $v_i\in\mathfrak{g}^{(1)}, t^i\in T^*$, and $R_U\in U$, i.e., R^D decomposes into irreducible $GL(n,\boldsymbol{H})Sp(1)$-components in $S^2E,\Lambda^2E,S^2ES^2H,\Lambda^2ES^2H$, and U.

REMARK. In the case of a 4-dimensional conformal manifold, we see that $\mathfrak{g}^{(1)}\otimes T^*\cong S^2E\oplus\boldsymbol{C}\oplus S^2ES^2H\oplus S^2H$ and $H^{1,2}(\mathfrak{g})\cong U\oplus S^4H$. Thus a curvature has its components in S^2E, $\boldsymbol{C}$, S^2ES^2H, S^2H, U and S^4H. If M is self-dual, then the S^4H-component vanishes. The components lying in $\boldsymbol{C}$, S^2ES^2H, and U correspond to the parts of the scalar curvature, the traceless Ricci curvature, and the self-dual Weyl tensor, respectively. And the S^2E-component and the S^2H-component correspond to the self-dual part and the anti-self-dual part of the curvature of D^L respectively.

5. Chern connections.

Let X be a complex manifold and $\mathscr{L}$ a Hermitian line bundle over X. A Hermitian connection on $\mathscr{L}$ is called a *Chern connection*, if its curvature is of type $(1,1)$ with respect to the complex structure on X. It is well-known that for any fixed Hermitian structure on $\mathscr{L}$, there is a natural bijection between Chern connections and holomorphic structures on $\mathscr{L}$, obtained by identifying a Chern connection with its $(0,1)$-part. In Section 2, we have seen that the twistor space of a quaternionic manifold is a complex manifold and its vertical bundle is a Hermitian line bundle. In this section, we shall obtain the condition for a Hermitian connection on the vertical bundle to be a Chern connection.

We extend the curvature R of a torsion-free connection on a quaternionic manifold to a complex bilinear form, also denote it by R, on $TM^{\boldsymbol{C}}$. We see that the U-component R_U of R is $\mathfrak{gl}(n,\boldsymbol{H})$-valued. So from (4.5), we also see that the $\mathfrak{sp}(1)$-component of R is constructed by the vectors $e_p h e_q h$, $e_p h e_q \tilde{h}$, $e_p \tilde{h} e_q h$, and $e_p \tilde{h} e_q \tilde{h}$ in $\mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH$. We denote the coefficients of these vectors by α_{pq}, $\alpha_{p\tilde{q}}$, $\alpha_{\tilde{p}q}$, and $\alpha_{\tilde{p}\tilde{q}}$ respectively. On the other hand, from (3.4), we may express the component on $\mathfrak{sp}(1)$ of R as follows:

$$R(e^p h, e^q h)_{S^2H} = a_{pq} h \cdot h + b_{pq} \tilde{h} \cdot h,$$

$$R(e^p h, e^q \tilde{h})_{S^2H} = a_{p\tilde{q}} h \cdot h + b_{p\tilde{q}} \tilde{h} \cdot h + c_{p\tilde{q}} \tilde{h} \cdot \tilde{h},$$

$$R(e^p \tilde{h}, e^q h)_{S^2H} = a_{\tilde{p}q} h \cdot h + b_{\tilde{p}q} \tilde{h} \cdot h + c_{\tilde{p}q} \tilde{h} \cdot \tilde{h},$$

$$R(e^p \tilde{h}, e^q \tilde{h})_{S^2H} = b_{\tilde{p}\tilde{q}} \tilde{h} \cdot h + c_{\tilde{p}\tilde{q}} \tilde{h} \cdot \tilde{h},$$

where $a \cdot b$ means the symmetric product of a and b. We note that coefficients a_{pq}, $a_{p\tilde{q}}$, $a_{\tilde{p}q}$, b_{pq}, $b_{p\tilde{q}}$, $b_{\tilde{p}q}$, $b_{\tilde{p}\tilde{q}}$, $c_{p\tilde{q}}$, $c_{\tilde{p}q}$, $c_{\tilde{p}\tilde{q}}$ and α_{pq}, $\alpha_{\tilde{p}q}$, $\alpha_{p\tilde{q}}$, $\alpha_{\tilde{p}\tilde{q}}$ satisfy the following relations:

$$(5.1)\qquad \begin{cases} a_{pq} = \alpha_{p\tilde{q}} - \alpha_{q\tilde{p}},\ a_{p\tilde{q}} = -\alpha_{pq},\ a_{\tilde{p}q} = \alpha_{qp}, \\ b_{pq} = \alpha_{\tilde{p}\tilde{q}} - \alpha_{\tilde{q}\tilde{p}},\ b_{p\tilde{q}} = -\alpha_{\tilde{p}q} - \alpha_{q\tilde{p}},\ b_{\tilde{p}q} = \alpha_{p\tilde{q}} + \alpha_{\tilde{q}p},\ b_{\tilde{p}\tilde{q}} = \alpha_{qp} - \alpha_{pq}, \\ c_{p\tilde{q}} = -\alpha_{\tilde{q}\tilde{p}},\ c_{\tilde{p}q} = \alpha_{\tilde{p}\tilde{q}},\ c_{\tilde{p}\tilde{q}} = \alpha_{\tilde{q}p} - \alpha_{\tilde{p}q} \qquad (p,q = 1,\dots,2n). \end{cases}$$

At first, since a curvature is skew-symmetric, its complex coefficients satisfy

$$(5.2)\qquad \begin{cases} a_{pq} = -a_{qp},\ a_{p\tilde{q}} = -a_{\tilde{q}p}, \\ b_{pq} = -b_{qp},\ b_{p\tilde{q}} = -b_{\tilde{q}p},\ b_{\tilde{p}\tilde{q}} = -b_{\tilde{q}\tilde{p}}, \\ c_{p\tilde{q}} = -c_{\tilde{q}p},\ c_{\tilde{p}\tilde{q}} = -c_{\tilde{q}\tilde{p}}, \\ a_{p\tilde{q}} + a_{\tilde{p}q} = b_{\tilde{p}\tilde{q}},\ c_{p\tilde{q}} + c_{\tilde{p}q} = b_{pq}, \\ a_{pq} - b_{p\tilde{q}} - b_{\tilde{p}q} + c_{\tilde{p}\tilde{q}} = 0 \qquad (p,q = 1,\dots,2n). \end{cases}$$

Next, the curvature R is real, i.e., $\overline{R(X,Y)} = R(\overline{X},\overline{Y})$ for $X, Y \in TM^{\boldsymbol{C}}$, where $\overline{\ast}$ is the operation of complex conjugation, so that its coefficients also satisfy the following conditions (5.3):

$$\overline{a_{jk}} = \begin{cases} c_{\widetilde{j+n}\widetilde{k+n}} & (1 \le j, k \le n) \\ -c_{\widetilde{j+n}\widetilde{k-n}} & (1 \le j \le n, n+1 \le k \le 2n) \\ -c_{\widetilde{j-n}\widetilde{k+n}} & (n+1 \le j \le 2n, 1 \le k \le n) \\ c_{\widetilde{j-n}\widetilde{k-n}} & (n+1 \le j, k \le 2n) \end{cases}$$

$$\overline{a_{j\tilde{k}}} = \begin{cases} -c_{\widetilde{j+n}k+n} & (1 \le j, k \le n) \\ c_{\widetilde{j+n}k-n} & (1 \le j \le n, n+1 \le k \le 2n) \\ c_{\widetilde{j-n}k+n} & (n+1 \le j \le 2n, 1 \le k \le n) \\ -c_{\widetilde{j-n}k-n} & (n+1 \le j, k \le 2n) \end{cases}$$

$$\overline{b_{jk}} = \begin{cases} -b_{\widetilde{j+n}\widetilde{k+n}} & (1 \le j, k \le n) \\ b_{\widetilde{j+n}\widetilde{k-n}} & (1 \le j \le n, n+1 \le k \le 2n) \\ b_{\widetilde{j-n}\widetilde{k+n}} & (n+1 \le j \le 2n, 1 \le k \le n) \\ -b_{\widetilde{j-n}\widetilde{k-n}} & (n+1 \le j, k \le 2n) \end{cases}$$

$$\overline{b_{j\tilde{k}}} = \begin{cases} b_{\widetilde{j+n}k+n} & (1 \le j, k \le n) \\ -b_{\widetilde{j+n}k-n} & (1 \le j \le n, n+1 \le k \le 2n) \\ -b_{\widetilde{j-n}k+n} & (n+1 \le j \le 2n, 1 \le k \le n) \\ b_{\widetilde{j-n}k-n} & (n+1 \le j, k \le 2n). \end{cases}$$

Moreover, we assume that R is of type $(1,1)$. From Lemma 2.1, we see that R^{∇} is of type $(1,1)$ if and only if R satisfies the condition

$$(*) \qquad \Pi^J([R^D(JX, JY) - R^D(X, Y), A]) = 0$$

for each $X, Y \in T_xM$ and $A \in \Theta_J$. We take a real basis

$$(5.4) \qquad \begin{cases} X^j = e^j h + e^{j+n}\tilde{h}, \\ Y^j = \sqrt{-1}(e^j h - e^{j+n}\tilde{h}), \\ Z^j = e^{j+n} h - e^j \tilde{h}, \\ W^j = \sqrt{-1}(e^{j+n} h + e^j \tilde{h}) \qquad (j = 1, \dots, n) \end{cases}$$

on $TM^{\boldsymbol{C}}$, and put $J = ah \cdot h + b\tilde{h} \cdot h + c\tilde{h} \cdot \tilde{h}$. Since J is a real operator, i.e., $\bar{J} = J$, and $\langle J, J \rangle = \sqrt{2}$, we have $c = \bar{a}, \bar{b} = -b$ and $4ac - b^2 = 1$. For each $A \in \Theta_J$, $A = dh \cdot h + e\tilde{h} \cdot h + f\tilde{h} \cdot \tilde{h}$, we also have $f = \bar{d}, \bar{e} = -e, 4df - e^2 = 1$, and $2af - be + 2cd = 0$ (*i.e.*, $\langle J, A \rangle = 0$). We compute the condition $(*)$ for the basis (5.4), we obtain the following conditions for coefficients of R (5.5):

$$
\begin{aligned}
&a_{jk} + b_{\tilde{j}\widetilde{k+n}} + b_{\widetilde{j+n}\tilde{k}} - c_{\tilde{j}\tilde{k}} = 0,\\
&a_{jk} - b_{\tilde{j}\widetilde{k+n}} + b_{\widetilde{j+n}\tilde{k}} - c_{\tilde{j}\tilde{k}} = 0,\\
&a_{jk+n} - b_{\tilde{j}\tilde{k}} + b_{\widetilde{j+n}\widetilde{k+n}} - c_{\tilde{j}\widetilde{k+n}} = 0,\\
&a_{jk+n} + b_{\tilde{j}\tilde{k}} + b_{\widetilde{j+n}\widetilde{k+n}} - c_{\tilde{j}\widetilde{k+n}} = 0,\\
&b_{jk} + b_{\widetilde{j+n}\widetilde{k+n}} = 0,\\
&b_{jk} - b_{\widetilde{j+n}\widetilde{k+n}} = 0,\\
&b_{jk+n} - b_{\widetilde{j+n}\tilde{k}} = 0,\\
&b_{jk+n} + b_{\widetilde{j+n}\tilde{k}} = 0,\\
&b_{j+nk+n} + b_{\tilde{j}\tilde{k}} = 0,\\
&b_{j+nk+n} - b_{\tilde{j}\tilde{k}} = 0 \qquad (j,k = 1,\dots,n).
\end{aligned}
$$

For example, we compute $(*)$ for X^j and X^k, then we have

$$
\begin{aligned}
&\Pi^J([R^D(JX^j, JX^k) - R^D(X^j, Y^k), A])\\
&\quad = \{2e(a_{jk} + b_{\tilde{j}\widetilde{k+n}} + b_{\widetilde{j+n}\tilde{k}} - c_{\tilde{j}\tilde{k}}) - 4d(b_{jk} + b_{\widetilde{j+n}\widetilde{k+n}})\}h\cdot h\\
&\qquad + \{4f(a_{jk} + b_{\tilde{j}\widetilde{k+n}} + b_{\widetilde{j+n}\tilde{k}} - c_{\tilde{j}\tilde{k}}) + 4d(a_{j+nk+n} - b_{jk+n} - b_{j+nk} - c_{\widetilde{j+n}\widetilde{k+n}})\}\tilde{h}\cdot h\\
&\qquad + \{2e(a_{j+nk+n} - b_{jk+n} - b_{j+nk} - c_{\widetilde{j+n}\widetilde{k+n}}) + 4f(b_{jk} + b_{\widetilde{j+n}\widetilde{k+n}})\}\tilde{h}\cdot\tilde{h}\\
&\quad = 0,
\end{aligned}
$$

for each A. So we get some equations in (5.5).

From (5.2), (5.3) and (5.5), we obtain

$$a_{pq} = c_{\tilde{p}\tilde{q}} \quad \text{and} \quad b_{pq} = b_{\tilde{p}\tilde{q}} = 0 \quad (p,q = 1,\dots,2n). \tag{5.6}$$

Using the relation (5.1), we may rewrite the conditions (5.3) and (5.6) as the following (5.7):

$$
\overline{\alpha_{pq}} = \begin{cases}
\alpha_{\widetilde{p+n}\widetilde{q+n}} & (1 \le p,q \le n)\\
-\alpha_{\widetilde{p+n}\widetilde{q-n}} & (1 \le p \le n, n+1 \le q \le 2n)\\
-\alpha_{\widetilde{p-n}\widetilde{q+n}} & (n+1 \le p \le 2n, 1 \le q \le n)\\
\alpha_{\widetilde{p-n}\widetilde{q-n}} & (n+1 \le p,q \le 2n)
\end{cases}
$$

$$\overline{\alpha_{p\tilde{q}}} = \begin{cases} -\alpha_{\widetilde{p+nq+n}} & (1 \le p, q \le n) \\ \alpha_{\widetilde{p+nq-n}} & (1 \le p \le n, n+1 \le q \le 2n) \\ \alpha_{\widetilde{p-nq+n}} & (n+1 \le p \le 2n, 1 \le q \le n) \\ -\alpha_{\widetilde{p-nq-n}} & (n+1 \le p, q \le 2n) \end{cases}$$

$$\alpha_{pq} = \alpha_{qp},\ \alpha_{\tilde{p}\tilde{q}} = \alpha_{\tilde{q}\tilde{p}},$$

$$\alpha_{p\tilde{q}} - \alpha_{q\tilde{p}} = \alpha_{\tilde{q}p} - \alpha_{\tilde{p}q} \qquad (p, q = 1, \dots, 2n).$$

In (5.7), we note that the first 2 conditions correspond to (5.3), and the last 3 conditions correspond to (5.6). From (5.7), we see that the curvatures of type $(1,1)$ are constructed by the vectors

$$A_{jk} = e_j h e_k \tilde{h} - e_j \tilde{h} e_k h + e_{j+n} h e_{k+n} \tilde{h} - e_{j+n} \tilde{h} e_{k+n} h,$$

$$B_{jk} = \sqrt{-1}(e_j h e_k \tilde{h} - e_j \tilde{h} e_k h - e_{j+n} h e_{k+n} \tilde{h} + e_{j+n} \tilde{h} e_{k+n} h),$$

$$C_{jk} = e_j h e_{k+n} \tilde{h} - e_j \tilde{h} e_{k+n} h + e_{j+n} \tilde{h} e_k h - e_{j+n} h e_k \tilde{h},$$

$$D_{jk} = \sqrt{-1}(e_j h e_{k+n} \tilde{h} - e_j \tilde{h} e_{k+n} h - e_{j+n} \tilde{h} e_k h + e_{j+n} h e_k \tilde{h}),$$

$$E_{jk} = e_j h e_k h + e_k h e_j h + e_{j+n} \tilde{h} e_{k+n} \tilde{h} + e_{k+n} \tilde{h} e_{j+n} \tilde{h},$$

$$F_{jk} = \sqrt{-1}(e_j h e_k h + e_k h e_j h - e_{j+n} \tilde{h} e_{k+n} \tilde{h} - e_{k+n} \tilde{h} e_{j+n} \tilde{h}),$$

$$G_{jk} = e_j h e_{k+n} h + e_{k+n} h e_j h - e_{j+n} \tilde{h} e_k \tilde{h} - e_k \tilde{h} e_{j+n} \tilde{h},$$

$$H_{jk} = \sqrt{-1}(e_j h e_{k+n} h + e_{k+n} h e_j h + e_{j+n} \tilde{h} e_k \tilde{h} + e_k \tilde{h} e_{j+n} \tilde{h}),$$

$$I_{jk} = e_{j+n} h e_{k+n} h + e_{k+n} h e_{j+n} h + e_j \tilde{h} e_k \tilde{h} + e_k \tilde{h} e_j \tilde{h},$$

$$J_{jk} = \sqrt{-1}(e_{j+n} h e_{k+n} h + e_{k+n} h e_{j+n} h - e_j \tilde{h} e_k \tilde{h} - e_k \tilde{h} e_j \tilde{h}),$$

$$K_{jk} = e_k \tilde{h} e_j h + e_j \tilde{h} e_k h - e_{k+n} h e_{j+n} \tilde{h} - e_{j+n} h e_{k+n} \tilde{h},$$

$$L_{jk} = \sqrt{-1}(e_k \tilde{h} e_j h + e_j \tilde{h} e_k h + e_{k+n} h e_{j+n} \tilde{h} + e_{j+n} h e_{k+n} \tilde{h}),$$

$$M_{jk} = e_{k+n} h e_j \tilde{h} + e_j \tilde{h} e_{k+n} h + e_k \tilde{h} e_{j+n} h + e_{j+n} h e_k \tilde{h},$$

$$N_{jk} = \sqrt{-1}(e_{k+n} h e_j \tilde{h} + e_j \tilde{h} e_{k+n} h - e_k \tilde{h} e_{j+n} h - e_{j+n} h e_k \tilde{h}) \qquad (j, k = 1, \dots, n)$$

over $\boldsymbol{R}$. We note that $\mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH \cong S^2E \oplus \Lambda^2E \oplus (S^2E \oplus \Lambda^2E)S^2H$. We may see that the above vectors are all in $S^2E \oplus \Lambda^2E \oplus S^2ES^2H$ and span $S^2E \oplus$

$\Lambda^2 E \oplus S^2 E S^2 H$. More precisely, $S^2 E$, $\Lambda^2 E$, and $S^2 E S^2 H$ are spanned by $\{A_{jk} + A_{kj}, B_{jk} + B_{kj}, C_{jk} - C_{kj}, D_{jk} + D_{kj}\}$, $\{A_{jk} - A_{kj}, B_{jk} - B_{kj}, C_{jk} + C_{kj}, D_{jk} - D_{kj}\}$, and $\{E_{jk},\ F_{jk},\ G_{jk},\ H_{jk},\ I_{jk},\ J_{jk},\ A_{jk} + A_{kj} + 2K_{jk},\ B_{jk} + B_{kj} + 2L_{jk},\ C_{jk} + C_{kj} + M_{jk} + M_{kj},\ D_{jk} - D_{kj} + N_{jk} - N_{kj}\}$, respectively. Hence we obtain the following

Theorem 5.1. *Let M be a quaternionic manifold with a torsion-free connection D, and Θ the vertical bundle of the twistor fibration Z. Then the linear connection ∇ on Θ induced by D is a Chern connection if and only if the curvature R^D of D has no component in $\Lambda^2 E S^2 H$.*

Remark. The condition for the curvature R^D of D to have no component in $\Lambda^2 E S^2 H$ in Theorem 5.1 corresponds to the condition for a Weyl structure to be self-dual in the case of a 4-dimensional self-dual manifold (*cf.* Remark in Section 4).

Example. If (M, g) is a quaternionic Kähler manifold with Levi-Civita connection D, then D induces a Chern connection. Because the components of the curvature R^D lie in $\Lambda^2 E \oplus U$.

6. Hypercomplex manifolds.

A $4n$-dimensional manifold M with a $GL(n, \boldsymbol{H})$-structure admitting a torsion-free connection is a *hypercomplex manifold*. Therefore the family of quaternionic manifolds contains that of hypercomplex manifolds. Applying the results of Sections 3 and 4 with the Lie algebra $\mathfrak{gl}(n, \boldsymbol{H})$, we obtain

Theorem 6.1 ([**8**]).

$$\mathfrak{gl}(n, \boldsymbol{H})^{(1)} = 0,$$

$$H^{1,2}(\mathfrak{gl}(n, \boldsymbol{H})) \cong U \oplus S^2 E.$$

For any two torsion-free G-connections $\nabla^{(1)}$ and $\nabla^{(2)}$, the difference $\nabla^{(1)} - \nabla^{(2)}$ belongs to $\mathfrak{g}^{(1)}$. From the first equation in Theorem 6.1, we see that a torsion-free $GL(n, \boldsymbol{H})$-connection is unique if it exists. We call it the *Obata connection*. And we also see that the curvature of an Obata connection has the components in $U \oplus S^2 E$. Hence an Obata connection induces a Chern connection.

References

[1] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in Four-Dimensional Riemannian Geometry, Proc. R. Soc. Lond. Ser. A, Vol. 362 (1978), 425–461.

[2] A. Besse, Einstein Manifolds, Springer Verlag, 1987, Die Ergebnisse der Mathematik 10.
[3] P. Gauduchon, Structures de Weyl et théorèmes d'annulation sur une variété conforme autoduale, Ann. Scuola Norm. Sup. Pisa, Serie 4, Vol. 18, Fasc., **4** (1991), 563–629.
[4] ———, Weyl Structures on Self-Dual Conformal Manifolds, Proceedings of Symposia in Pure Mathematics, Vol. 54 (1993), Part 2, 259–270.
[5] D. Joyce, Compact Hypercomplex and Quaternionic Manifolds, J. Differential Geometry, **35** (1992) 743–761.
[6] M. Obata, Affine Connections on Manifolds with Almost Complex, Quaternion or Hermitian Structure, Japan J. Math., New Ser., Vol. 26 (1956), 353–362.
[7] S. M. Salamon, Quaternionic Kähler Manifolds, Invent. Math., **67** (1982), 143–171.
[8] ———, Differential Geometry of Quaternionic Manifolds, Ann. Sci. École Norm. Sup. (4), **19** (1986), 31–55.

Toshimasa Kobayashi
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, 560-0043 Osaka, Japan
E-mail: toshi@math.sci.osaka-u.ac.jp