Groupoids and the integration of Lie algebroids

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Abstract. We show that a Lie algebroid on a stratified manifold is integrable if, and only if, its restriction to each strata is integrable. These results allow us to construct a large class of algebras of pseudodifferential operators. They are also relevant for the definition of the graph of certain singular foliations of manifolds with corners and the construction of natural algebras of pseudodifferential operators on a given complex algebraic variety.

Introduction.

Differentiable groupoids appear in geometry in various instances, for example in the theory of connections and parallel transport on fiber bundles, or in the theory of pseudogroups of transformations. The concept of groupoid is a generalization of the concept of group, the main difference being that not any two elements of a groupoid are composable. Intuitively, it is convenient to think of a groupoid as a set of arrows between various points, called units, two arrows being composable if, and only if, their ends match. (See Section 1 for precise definitions.)

One of the main features of differential groupoids is that they are geometric objects that interpolate between differentiable manifolds and Lie groups: differentiable manifolds have many units and few arrows; whereas Lie groups have many arrows and few units— actually only one. This "interpolation" property is valid also at the level of algebras: to a compact smooth manifold M one associates the commutative algebra $\mathscr{C}^\infty(M)$ of its differentiable functions; whereas to a Lie group G one associates the convolution algebra $\mathscr{C}_c^\infty(G)$ of compactly supported smooth functions on the group, which is usually highly non-commutative. In this way, differentiable groupoids provide a link between geometry and harmonic analysis.

The algebras $\mathscr{C}^\infty(M)$ and $\mathscr{C}_c^\infty(G)$ are particular cases of the convolution algebra of a differential groupoid, and this feature makes groupoids a favorite toy model in non-commutative geometry. From this point of view, the results of this paper are a first step towards a generalization of the results of [2], from étale groupoids to general differential groupoids.

Recall that a *Lie algebroid* is a vector bundle $A \to M$ on a differentiable manifold M together with a Lie algebra structure on the space $\Gamma(A)$ of its smooth sections and a Lie algebra morphism

$$\Gamma(A) \to \Gamma(TM),$$

defined by a structural vector bundle morphism $q : A \to TM$, called "the anchor map." The Lie algebroid A is called *regular* [resp. *transitive*] if, and only if, the anchor map $q : A \to TM$ has locally constant rank [resp. it is onto]. From a classical, differential geometric point of view, to a differential groupoid there is associated a "Lie algebroid," which is some sort of "infinitesimal form" of the groupoid, generalizing both the Lie algebra of a group and the tangent space to a differentiable manifold.

In [5] it was proved that any Lie algebroid such that $q = 0$ is the Lie algebroid of a differentiable groupoid, in other words, it is *integrable*. This result can be regarded as a generalization of "Lie's third theorem," which states that every finite dimensional Lie algebra is the Lie algebra of a Lie group. This is relevant because, unlike (finite dimensional) Lie algebras, Lie algebroids do not always correspond to differential groupoids, that is, they are not always integrable. Actually, for a transitive groupoid, one can define an obstruction to integrability, see [8]. Nevertheless, it is still interesting to construct differentiable groupoids that integrate specific Lie algebroids. Some examples can be found in [14]. In that paper, to a differentiable groupoid $\mathscr{G}$ there was associated an algebra of pseudodifferential operators $\Psi^*(\mathscr{G})$, such that, for $\mathscr{G}^{(0)}$ compact, all first order differential operators in $\Psi^*(\mathscr{G})$ are linear combinations of sections of $A(\mathscr{G})$ and operators of multiplication by functions. The integration of Lie algebroids is thus a first step toward constructing algebras of pseudodifferential operators, and this explains why we are interested in the problem of integrating Lie algebroids. See [3], [4], [11], [13], [14], [21] for more on the question of constructing pseudodifferential operators.

In this paper, we approach the problem of integrating Lie algebroids from an abstract point of view, looking for some general methods to integrate Lie algebroids. Since arbitrary Lie algebroids $\pi : A \to M$ behave rather wildly, we make two assumptions. First, we assume that the manifold M has a stratification

$$M = \bigcup S$$

into disjoint strata, each of which is invariant with respect to the diffeomorphisms generated by the sections of A and, second, we assume that the restrictions

$$A_S = A|_S,$$

which are Lie algebroids precisely because the strata are invariant, are regular. (If M satisfies the first assumption, we say that it has an "A-invariant regular stratification.")

For the class of groupoids with A-invariant regular stratifications, one can approach the "integration problem" in two steps. First, a necessary condition for the integrability of A is the integrability of each of the restrictions A_S, and hence the first step will be to integrate each of these restrictions (see also below). Assume then that we can find a differentiable groupoid $\mathscr{G}_S$ that integrates A_S, for each S. The second step is to "glue" the resulting groupoids $\mathscr{G}_S$. Surprisingly enough, this naïve approach actually works in most cases. It works for example if we choose the integrating groupoids $\mathscr{G}_S$ to be maximal in a suitable sense (d-simply connected), and this is our main general result on the integration of differential groupoids. As in the paper of Douady and Lazard [5], we obtain in general non-Hausdorff groupoids.

Since transitive Lie algebroids are a particular case of regular Lie algebroids, the first part of the problem—that is, integrating regular Lie algebroids—is similar to the problem of integrating transitive Lie algebroids, and probably can be handled similarly. In particular, it is clear that not all regular Lie algebroids are integrable. It is not our purpose in this paper to study the integration of general regular algebroids, but we do show how to integrate particular classes of regular algebroids. For example, we show that a regular algebroid A is integrable if $\ker(q)$, the kernel of q, consists of semisimple Lie algebras, or if A is a semi-direct product.

As for the second part of the problem, it turns out that there exists at most one way to glue the groupoids $\mathscr{G}_S$ that integrate $A_S = A|_S$, assuming that they exist. The problem is that the resulting glued space (a groupoid) is not always a smooth manifold, so this procedure does not lead directly to a differentiable groupoid. However, we show that this procedure does lead to a differentiable groupoid that integrates A, provided that all groupoid $\mathscr{G}_S$ are d-simply connected. We do not assume that the strata are regular here. We thus obtain the following result.

THEOREM 1. *A Lie algebroid $\pi : A \to M$ on a manifold M with an A-invariant stratification is integrable if, and only if, it is integrable along each stratum.*

These results provide us with an explicit way of integrating many Lie algebroids. As an application of the theorem, we prove the integrability of certain Lie algebroids on foliated manifolds with corners. A *foliated manifold with corners* is a manifold with corners, each of whose open faces is a foliated manifold, the foliations being required to satisfy certain compatibility relations. This result generalizes a construction due to Winkelnkemper [22]. Previously, Melrose [10] and Mazzeo–Melrose [9] have shown how to integrate certain particular algebroids. However, their framework was different from ours.

The results of this paper, together with the results of [14], can be used to construct a natural algebra of pseudodifferential operators on a complex algebraic variety endowed with a "$\mathscr{C}^\infty$-resolution of singularities," thus making a substantial step towards a solution of the problem stated in [11]. Then the methods of [12] can presumably be applied to study the resulting algebras of operators. Algebras of pseudodifferential operators on groupoids are also a natural framework to study adiabatic limits [14], [23]. The problem of associating an algebra of pseudodifferential operators to a groupoid was first formally stated in a paper by Weinstein, [21]. However, before that, in [3], Alain Connes has constructed algebras of pseudodifferential operators on foliations, which in our setting corresponds to the case of a regular groupoid with discrete holonomy (see also [4]). His methods have played a role in inspiring the constructions of [14].

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1. Basic concepts.

We begin this section by fixing notation and recalling some of the basic concepts used in this paper.

In the following, we shall use the framework of [**14**]. In particular, a *manifold* is a smooth manifold, possibly *with corners*. By definition, every point m in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0, 1)^k \times \boldsymbol{R}^{n-k}$, such that the transition functions are smooth (including on the boundary). Let $k(m)$ be the least k such that m is in a set diffeomorphic to $[0, 1)^k \times \boldsymbol{R}^{n-k}$, and let $\partial_k(M)$ be the set of points m for which $k(m) = k$. A component of $\partial_k(M)$ is called an open face of codimension k. A face of M is the closure of an open face (of M). A *hyperface* of M is a face H of codimension one.

It is customary to assume that any hyperface H of a manifold with corners M is of the form $H = \{x_H = 0\}$, where x_H is a smooth positive function on M such that $dx_H \neq 0$ on H. If this is the case, x_H is called a *defining function* of H. We shall also assume in this paper that each hyperface of M has a defining functions, although, most of our results are true even without this assumption. An *interior point* of M is a point of M that belongs to no hyperface.

A *submersion* $f : M \to N$ between two manifolds with corners is a differentiable map with surjective differential at each point, such that a non-zero tangent vector to M points inward if, and only if, its image in TN is non-zero and points inward. By definition, if $f : M \to N$ is a submersion, then the fibers $f^{-1}(y)$ are smooth manifolds *without* corners. A *submanifold with corners* $N \subset M$ is a closed submanifold N of M such that each face of N is locally the transverse intersection of N with a face of M.

We now define groupoids. Recall that a small category is a category whose class of morphisms is a set. By definition, a *groupoid* is a small category $\mathscr{G}$ in which every morphism is invertible. General results on groupoids can be found in [**17**].

We now fix some notation and make the definition of a groupoid more explicit. The set of objects (or *units*) of $\mathscr{G}$ is denoted by $\mathscr{G}^{(0)}$. The set of morphisms (or *arrows*) of $\mathscr{G}$ is denoted by $\mathscr{G}^{(1)} = \mathrm{Mor}(\mathscr{G})$. We shall sometimes write $\mathscr{G}$ instead of $\mathscr{G}^{(1)}$, by abuse of notation. For example, when we consider a space of functions on $\mathscr{G}$, we actually mean a space of functions on $\mathscr{G}^{(1)}$. We will denote by $d(g)$ [respectively $r(g)$] the *domain* [respectively, the *range*] of the morphism $g : d(g) \to r(g)$. We thus obtain functions

$$(1) \qquad\qquad d, r : \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$$

that will play an important role. The multiplication $\mu : (g, h) \to \mu(g, h) = gh$ is defined on the set $\mathscr{G}^{(2)}$ of composable pairs of arrows:

$$(2) \qquad \mu : \mathscr{G}^{(2)} = \mathscr{G}^{(1)} \times_M \mathscr{G}^{(1)} := \{(g, h) : d(g) = r(h)\} \to \mathscr{G}^{(1)}.$$

The inversion operation is a bijection $\iota(g) = g^{-1}$ of $\mathscr{G}^{(1)}$. Denoting by $u(x)$ the identity morphism of the object $x \in \mathscr{G}^{(0)}$, we obtain an inclusion $u : \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$. We see that a groupoid $\mathscr{G}$ is completely determined by the spaces $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$ and by the structural morphisms d, r, μ, u, and ι. We sometimes write

$$\mathscr{G} = (\mathscr{G}^{(0)}, \mathscr{G}^{(1)}, d, r, \mu, u, \iota).$$

The structural maps satisfy the following properties:

(i) $r(gh) = r(g)$, $d(gh) = d(h)$ for any pair $(g, h) \in \mathscr{G}^{(2)}$, and the partially defined multiplication μ is associative.

(ii) $d(u(x)) = r(u(x)) = x$, $\forall x \in \mathscr{G}^{(0)}$, $u(r(g))g = g$, and $gu(d(g)) = g$, $\forall g \in \mathscr{G}^{(1)}$, and $u : \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$ is injective.

(iii) $r(g^{-1}) = d(g)$, $d(g^{-1}) = r(g)$, $gg^{-1} = u(r(g))$, and $g^{-1}g = u(d(g))$.

By definition, a *differentiable groupoid* is a groupoid such that $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$ are smooth manifolds with corners, $\mathscr{G}^{(0)}$ is smooth, all structural morphisms are differentiable, and d, the domain map, is a submersion.

We observe that ι is a diffeomorphism and hence d is a submersion if, and only if, $r = d \circ \iota$ is a submersion. Also, it follows from the definition of submersions of manifolds with corners that each fiber $\mathscr{G}_x = d^{-1}(x) \subset \mathscr{G}^{(1)}$ is a smooth manifold without corners whose dimension n is constant on each connected component of $\mathscr{G}^{(0)}$. The étale groupoids considered in [**2**] are extreme examples of differentiable groupoids (corresponding to $\dim \mathscr{G}_x = 0$). Note that we allow $\mathscr{G}^{(1)}$ to be non-Hausdorff. If we want to make more precise the space of units $\mathscr{G}^{(0)}$ of $\mathscr{G}$, we say that "$\mathscr{G}$ is a differentiable groupoid *on* $\mathscr{G}^{(0)}$."

We now recall the definition of a Lie algebroid [**15**]. See also [**18**].

DEFINITION 1. *A Lie algebroid over a manifold M is a vector bundle A over M together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A, and a bundle map $q : A \to TM$, extended to a map between sections of these bundles, such that:*

(i) $q([X, Y]) = [q(X), q(Y)]$, *and*

(ii) $[X, fY] = f[X, Y] + (q(X)f)Y$,

for all smooth sections X and Y of A and all smooth function f on M.

The morphism q is called the *anchor map*. Note that we allow the base M in the definition above to be a manifold with corners. This is necessary in the most interesting examples related to singular spaces. We did not include in the definition the condition that $q(\Gamma(A))$ consist of vector fields tangent to each face of M, but it will be satisfied in all the cases we consider.

To a differential groupoid $\mathscr{G}$ there is naturally associated a Lie algebroid $A(\mathscr{G})$ as follows [**8**], [**15**], [**16**]. Consider first the vertical tangent bundle of $\mathscr{G}$ along the fibers of the domain map d:

$$T_d \mathscr{G} = \ker(d_*) = \bigcup_{x \in \mathscr{G}^{(0)}} T\mathscr{G}_x \subset T\mathscr{G}^{(1)}.$$

By definition, $A(\mathscr{G})$ is the restriction $T_d\mathscr{G}|_{\mathscr{G}^{(0)}}$ of $T_d\mathscr{G}$ to the set of units of $\mathscr{G}$. The space of d-vertical vector fields invariant with respect to right translations is closed with respect to the Lie bracket and identifies canonically with $\Gamma(A)$. Thus we obtain a Lie algebra structure on $\Gamma(A)$. The action on functions on the base is obtained by lifting a function on $\mathscr{G}^{(0)}$ to a function on $\mathscr{G}^{(1)}$ via r. Consequently, the anchor map $q : A(\mathscr{G}) \to T\mathscr{G}^{(0)}$ is obtained by restricting the differential of r to $A(\mathscr{G})$.

If A is a Lie algebroid and $\mathscr{G}$ is a smooth groupoid such that $A(\mathscr{G}) \simeq A$, then we say that $\mathscr{G}$ *integrates* A. Not every Lie algebroid is integrable (see [**1**] for an example).

Nevertheless, it is important to provide examples of general methods to integrate Lie algebroids.

A differentiable groupoid is called *d-connected* [respectively, *d-simply connected*] if, and only if, each set $\mathcal{G}_x$ is path connected [respectively, path connected and simply connected]. As for Lie groups, all *d*-simply connected differential groupoids with isomorphic Lie algebroids are isomorphic.

Recall [8] that, if $\mathcal{G}$ is a differentiable groupoid, then there exists a *d*-simply connected groupoid $\mathscr{P}\mathcal{G}$, uniquely determined up to isomorphism, with the same Lie algebroid as $\mathcal{G}$; it is called the *path groupoid* associated to $\mathcal{G}$. As a set, $\mathscr{P}\mathcal{G}$ consists of fixed end-point homotopy classes of paths $\gamma : [0,1] \to \mathcal{G}$ such that $d(\gamma(t))$ is constant and $\gamma(0)$ is a unit. This result is due to Moerdijk.

2. A glueying theorem.

Let M be a manifold with corners. In this paper, by *stratified manifold* we mean a smooth manifold M, possibly with corners, together with a disjoint union decomposition $M = \bigcup S$ of M by a locally finite family of submanifolds S without corners, called *open strata*, such that the closure (in M) of each stratum S is a submanifold with corners and each S is contained in a unique open face of M. (This notion is slightly stronger than that of a stratified space which is also a manifold with corners. Most of our results however hold in this greater generality.)

Consider a differentiable Lie algebroid A with anchor map $q : A \to TM$ on a manifold M. A stratification $M = \bigcup S$ of M is called *A-invariant* if, and only if, for each point $x \in S$, the range of $A_x \to T_x M$ (from the fiber of A at the point x to the tangent space to M at x) is contained in $T_x S$ (*i.e.*, $q(A_x) \subset T_x S$). The condition that

$$(3) \qquad\qquad q(A_x) \subset T_x S,$$

for all $x \in S$, is equivalent to the condition that each local diffeomorphism of the form $\exp(q(X))$, for some smooth section X of A, preserve the strata of M, or, to the condition that the restriction A_S of A to S be a Lie algebroid on S, for each S.

Define, for any subset $S \subset M$ of the set of units of $\mathcal{G}$, the groupoid

$$\mathcal{G}_S := d^{-1}(S) \cap r^{-1}(S),$$

called *the reduction of $\mathcal{G}$ to S*. If $M = \bigcup S$ is an *A*-invariant stratification of M and, moreover, $\mathcal{G}$ is a differentiable groupoid on M that integrates A, then $d^{-1}(S) = r^{-1}(S)$ and $\mathcal{G}_S = d^{-1}(S)$ satisfies $A_S \simeq A(\mathcal{G}_S)$. This shows that, in order to integrate A, we need to integrate each restriction A_S. The main point of this section is that this is also enough (Theorem 3).

Recall from [8] that an *admissible section* of a differential groupoid $\mathcal{G}$ on M is a differentiable map

$$\sigma : M \to \mathcal{G}$$

such that $d(\sigma(x)) = x$ and the map $M \ni x \to r(\sigma(x)) \in M$ is a diffeomorphism. Then σ also defines a diffeomorphism

$$(4) \qquad\qquad \mathcal{G} \ni g \to \sigma g := \sigma(r(g))g \in \mathcal{G}.$$

The main example of an admissible section is

$$\sigma(x) = \exp(X_m) \cdots \exp(X_1)x,$$

for suitable, smooth sections $X_1, \ldots, X_m$ of A. We will discuss this type of admissible sections in more detail below.

A *differentiable family* of admissible sections is a family $\sigma_s : M \to \mathscr{G}$, $s \in [0,1]$, of maps such that each σ_s, $s \in [0,1]$, is an admissible section and the induced map $[0,1] \times M \ni (s,x) \to \sigma_s(x) \in \mathscr{G}$ is differentiable.

LEMMA 1. *Let $\mathscr{G}$ be a differentiable groupoid on M and let $\sigma_s : M \to \mathscr{G}$ be a differentiable family of local admissible sections. Then there exists a section X of A such that*

$$(5) \qquad\qquad \partial_s f(\sigma_s g)|_{s=0} = X f(\sigma_0 g), \quad g \in \mathscr{G}.$$

PROOF. Since the map $[0,1] \ni s \to \sigma_s g := \sigma_s(r(g))g$ is differentiable for all g, there exists a vector field X on $\mathscr{G}$ satisfying (5). We need to check that X is d-vertical and right invariant. We have that

$$d(\sigma_s g) = d(g),$$

which proves that X is d-vertical, by definition. Also,

$$\sigma_s(gh) = \sigma_s(r(gh))gh = \sigma_s(r(g))gh = (\sigma_s g)h,$$

which proves that X is also right invariant. $\qquad\square$

In the following, the section $X \in \Gamma(A(\mathscr{G}))$, defined in the above lemma, will be denoted $\partial_s \sigma_s|_{s=0}$. We define in the same way $\partial_s \sigma_s$ for all values of s.

We shall repeatedly use the exponential map, and hence we shall to consider diffeomorphisms obtained by integrating vector fields. Recall that a smooth vector field X on a manifold M is called *complete* if, and only if, there exists a differentiable map $\phi : \mathbf{R} \times M \to M$ such that

$$X(\phi(t,m)) = \partial_t(\phi(t,m)),$$

for all $(t,m) \in \mathbf{R} \times M$. We then define

$$\exp(X)m := \phi(1,m).$$

Note that a complete vector field on a manifold with corners M is necessarily tangent to each face of M.

As for manifolds without corners, it follows from the definition and basic results on ordinary differential equations that, if X is complete, then tX is also complete, for all $t \in \mathbf{R}$, and

$$\exp((t+s)X) = \exp(tX)\exp(sX),$$

for all s and t. Consequently, $\exp(X)$ is a diffeomorphism.

LEMMA 2. *Let $\mathscr{G}$ be a differentiable groupoid on M with Lie algebroid $A = A(\mathscr{G})$. If $X \in \Gamma(A)$ is a section such that $q(X)$ is complete, then X, regarded as a d-vertical vector field on $\mathscr{G}$, is also complete.*

PROOF.   For manifolds without corners, this is a result from Kumpera and Spencer [**7**, Appendix].   For manifolds with corners the proof is the same.   □

PROPOSITION 1.   *Let* $\pi : A \to M$ *be a Lie algebroid with anchor map* $q : A \to TM$, *and let* $X \in \Gamma(A)$ *be a section such that* $q(X)$ *is complete.   Then there exists a uniquely determined isomorphism* $E_X : A \to A$ *of Lie algebroids satisfying* (i) *and* (ii):

  (i) $\pi \circ E_X = \exp(q(X)) \circ \pi$;

  (ii) $\partial_t E_{tX}(Y)|_{t=0} = [X, Y]$.

  (iii)   *If, moreover,* $\mathcal{G}$ *is a differentiable groupoid integrating* A *and* $X, Y \in \Gamma(A)$ *are both complete, then* E_X *also satisfies*

$$\exp(X) \exp(Y) = \exp(E_X(Y)) \exp(X),$$

as admissible sections.

PROOF.   If A is integrable, then (i) is Proposition 4.1.(v), (ii) is Proposition 4.11.(iii), and (iii) is Proposition 4.11.(ii) of [**8**].   The proof of (i) and (ii) in the general case are the same.   □

A family $Y_1, \ldots, Y_n$ of smooth sections of A is a *local basis* of A at $y \in M$ if $Y_1(y), \ldots, Y_n(y)$ is a basis of A_y.   If $t = (t_1, \ldots, t_n) \in \boldsymbol{R}^n$ and $Y_1, \ldots, Y_n$ are sections of A, then we denote

$$\mathrm{Exp}(t, Y) := \exp(t_1 Y_1) \exp(t_2 Y_2) \cdots \exp(t_n Y_n).$$

Also, let

$$B_\varepsilon = \left\{ t = (t_1, \ldots, t_n) \in \boldsymbol{R}^n, \ \sum_{i=1}^n |t_i| < \varepsilon \right\}.$$

Recall that a differentiable local groupoid $\mathcal{U}$ on M satisfies all the axioms of a differential groupoid, except that the multiplication gg' is not defined for all pairs (g, g') such that $d(g) = r(g')$, but only in a neighborhood $\mathcal{U}_2$ of the diagonal in the set $\{(g, g'), d(g) = r(g')\}$.   See [**19**], [**16**] for details.

LEMMA 3.   *Let* $\mathcal{G}$ *be a local differential groupoid on* M, *and let* $\sigma : M \to \mathcal{G}$ *be an admissible section with* $r(\sigma(x_0)) = y_0$.   *Also, let* $Y_1, \ldots, Y_n$ *be a local base of* A *at* y_0. *Then, for some small* $\varepsilon > 0$ *and a relatively compact open neighborhood* U *of* x_0 *in* M, *the map*

(6) $\psi_Y^\sigma : \boldsymbol{R}^n \times M \ni (t, x) \to \mathrm{Exp}(t, Y)\sigma(x) \in \mathcal{G}$

is a diffeomorphism from $B_\varepsilon \times U$ *to a neighborhood of* $\sigma(x_0)$ *in* $\mathcal{G}$.

  If $\sigma = \exp(X_m) \cdots \exp(X_1)$, *for some integrable* $X_1, \ldots, X_m \in \Gamma(A)$, *then we denote* $\psi_Y^\sigma = \psi_Y^X$.

PROOF.   If $\mathcal{G}$ is a differentiable groupoid (not just a local one), this is Proposition 4.12 in [**8**].   For local groupoids the proof is the same.   □

Let $\mathcal{G}$ be a differentiable groupoid, and let $X_1, \ldots, X_m$ be smooth complete sections of $A(\mathcal{G})$.   Also, let Y_i be smooth, complete sections of $A(\mathcal{G})$ that form a local basis at

some point $x_0 \in M$, as above. For simplicity, we shall sometimes assume that the Y_i are compactly supported, which is a stronger assumption than completeness. Since the admissible section

$$\sigma(x) = \exp(X_m) \cdots \exp(X_1)x$$

defines a diffeomorphism $\sigma : \mathcal{G} \to \mathcal{G}$, equation (4), we obtain that the map

$$(7) \qquad \phi_Y^X : \boldsymbol{R}^n \times M \ni (t, x) \to \exp(X_m) \cdots \exp(X_1) \operatorname{Exp}(t, Y)x \in \mathcal{G}$$

is also a diffeomorphism from a set of the form $B_\varepsilon \times U$ to its image, for some small $\varepsilon > 0$ and some small open subset $U \subset M$. The maps ϕ_Y^X are slightly more convenient to work with, in what follows, than the maps ψ of the previous lemma.

Throughout the rest of this section, M will be a smooth manifold and $A \to M$ will be a Lie algebroid. Moreover, $M = \bigcup S$ is an A-invariant stratification of M, $\mathcal{G}_S$ is a differentiable groupoid integrating A_S, and $\mathcal{G} = \bigcup \mathcal{G}_S$ (disjoint union).

On $\mathcal{G} = \bigcup \mathcal{G}_S$ we can then define uniquely a natural groupoid structure on $\mathcal{G}$ such that the structural morphisms of each $\mathcal{G}_S$ are obtained from those of $\mathcal{G}$ by restriction. In particular, if two arrows $g \in \mathcal{G}_S$ and $g' \in \mathcal{G}_{S'}$ are composable, then $S = S'$, also, each $\mathcal{G}_S$ is a subgroupoid of $\mathcal{G}$.

A crucial observation is that we need not use the full differentiable structure on $\mathcal{G}$ to define the maps ψ_Y^X of Lemma 3, or the maps ϕ_Y^X of equation (7), for that matter. To define ϕ_Y^X, it is enough to use only the smooth structure on each $\mathcal{G}_S$. This will allow to extend the definition of ϕ_Y^X to our case, when the groupoid $\mathcal{G}$ is obtained by glueying differential groupoids. This can be done as follows.

Let $M = \bigcup S$ be an A-invariant stratification of M. Also, let for each S $\mathcal{G}_S$ be a differentiable groupoid integrating $A_S = A|_S$ and $\mathcal{G} = \bigcup \mathcal{G}_S$, as above. If X_i and Y_j are sections of A on $M = \mathcal{G}^{(0)}$ such that the vector fields $q(X_i)$ and $q(Y_j)$ are integrable, then the restriction of these vector fields to each strata is again integrable, and hence the restriction of X_i and Y_j to each strata are integrable as vertical vector fields on $\mathcal{G}_S$. Then the map ϕ_Y^X is defined on each $\boldsymbol{R}^n \times S$ (with values in $\mathcal{G}_S$), by glueying these maps, we obtain the desired definition of $\phi_Y^X : \boldsymbol{R}^n \times M \to \mathcal{G}$, for $\mathcal{G} = \bigcup \mathcal{G}_S$.

Since the maps ϕ_Y^X and ψ_Y^X play an important role in what follows, we now spell out their properties in more detail.

LEMMA 4. (i) *The maps ϕ_Y^X and ψ_Y^X are related by*

$$\phi_Y^X = \psi_{Y'}^X,$$

where $Y_j' = E_{X_m} \circ E_{X_{m-1}} \circ \cdots \circ E_{X_1}(Y_j)$.
(ii) *Let $Y_j'' = -Y_{n+1-j}$ and $X_i'' = -X_{m+1-i}$. Then*

$$\phi_Y^X(x)^{-1} = \psi_{Y''}^{X''}(\alpha(x)),$$

where α is the diffeomorphism $\exp(q(X_m)) \cdots \exp(q(X_1))$.

PROOF. Because each $\mathcal{G}_S$ is a differential groupoid, (i) follows from Proposition 1 on each $\mathcal{G}_S$. From this we obtain the desired relation everywhere on $\mathcal{G}$.

(ii) follows from the relation

$$(\exp(X)x)^{-1} = \exp(-X)(\exp(q(X))x),$$

valid on each $\mathscr{G}_S$, and hence everywhere on $\mathscr{G}$.   □

We let as above $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be integrable sections of A. We shall also use the following lemma.

LEMMA 5. (i) *Fix* $t \in \mathbf{R}^n$ *and* $x_0 \in M$. *Then we can find* $\varepsilon > 0$, *a neighborhood* U *of x in M, and a differentiable map*

$$\tau : B_\varepsilon \times U \to \mathbf{R}^n \times M, \quad \tau(0, x_0) = (0, x_0),$$

which is a diffeomorphism onto its image, such that

$$\phi_Y^X(t + s, x) = \phi_Y^{X'}(\tau(s, x)),$$

on $B_\varepsilon \times U$, where $X_j' = X_j$, for $j = 1, \ldots, m$, and $X_{j+m}' = t_j Y_j$, for $j = 1, \ldots, n$.

(ii) *For all* $Y_1, \ldots Y_n$, $Y_1', \ldots, Y_n'$, *and* $x_0 \in M$, *there exist* $\delta > \varepsilon > 0$, *an open neighborhood U of x_0, and a differentiable map*

$$m_1 : B_\varepsilon \times B_\varepsilon \times U \to B_\delta \times M, \quad m_1(0, 0, x) = (0, x),$$

such that

$$\mathrm{Exp}(t, Y)x(\mathrm{Exp}(t', Y')x)^{-1} = \mathrm{Exp}(m(t, t', x)).$$

PROOF. We begin by writing $\exp((t_i + s_i)X_i) = \exp(t_i X_i)\exp(s_i X_i)$. Then, using Proposition 1, we can find sections Y_i^s of A, $s \in \mathbf{R}$, $Y_i^0 = Y_i$, depending smoothly on s, such that

$$\phi_Y^X(t + s, x) = \phi_Y^{X'}(\mathrm{Exp}(s, Y^s)x).$$

To complete (i), we now use a local groupoid $\mathscr{U}$ integrating A. The existence of such a $\mathscr{U}$ is ensured by [**16**]. By replacing $\mathscr{U}$ with an open neighborhood of M, if necessary, we may identify $\mathscr{U}$ with a subset of $\mathscr{G}$, using the exponential map. Then, for small ε and a relatively compact neighborhood U of x_0, the maps

$$B_\varepsilon \times U \ni (x, s) \to f_1(x, s) := \mathrm{Exp}(s, Y^s)x$$

and

$$B_\varepsilon \times U \ni (x, s) \to f_2(x, s) := \mathrm{Exp}(s, Y)x$$

are diffeomorphisms onto neighborhoods of x_0 in $\mathscr{U}$, by Lemma 3. The desired map τ is obtained from $f_2^{-1} \circ f_1$.

The proof of (ii) is similar, using the differentiability of the multiplication in a local groupoid.   □

The above result suggest to introduce the following family of maps.

THE FAMILY Φ: Let $f : V_0 \to V$ be a diffeomorphism (*i.e.*, coordinate chart) from an open subset of $\mathbf{R}^l$ to an open, relatively compact subset of $\mathscr{G}^{(0)} = M$. Because the

partition $M = \bigcup S$ is locally finite, we can find, using Lemma 3, an $\varepsilon > 0$ and sections $X_i, Y_j \in \Gamma(A)$, for which ϕ_Y^X defines a diffeomorphism from $B_\varepsilon \times (V \cap S) \to \mathcal{G}_S$, for all S such that the intersection $S \cap V$ is not empty. The family Φ then consists of all maps of the form

$$(8) \qquad \varphi(t, y) = \phi_Y^X(t, f(y)) : B_\varepsilon \times V_0 \to \mathcal{G}.$$

Recall that a *differentiable atlas* on a set M_0 is a family of injective maps

$$\varphi : V_\varphi \to M_0,$$

defined on an open subset of $\mathbf{R}^l$, for some fixed l, such that $\varphi(V_\varphi)$ is a covering of M_0, $\varphi^{-1}(\varphi_1(V_{\varphi_1})$ is an open subset V of V_φ, and the map $\varphi_1^{-1}\varphi$ is differentiable on V.

We are ready now to prove the following theorem.

THEOREM 2. *Let $M = \bigcup S$ be an A-invariant stratification of M and, for each S, let $\mathcal{G}_S$ be a d-connected differentiable groupoid on M integrating $A_S = A|_S$. Then $\mathcal{G}$ has a differentiable structure making it a differentiable groupoid with $A(\mathcal{G}) \simeq A$ if, and only if, the family Φ, consisting of the maps in equation (8), is a differentiable atlas.*

PROOF. Suppose first that the family Φ is a differentiable atlas. We begin by showing that the groupoid structure on $\mathcal{G}$, induced from the groupoids $\mathcal{G}_S$, is compatible with the differentiable structure defined by Φ. That is, we need to check that the structural morphisms are differentiable. We now check this.

First, the domain map is differentiable and submersive because $d(\phi_Y^X(t, x)) = x$, for all X, Y, and $x \in M$. Next, it is enough to check that the map $(g', g) \to g'g^{-1}$, defined on

$$\{(g', g), d(g') = d(g)\},$$

is differentiable. Let (g', g) be such that $d(g) = d(g') = x_0$, and let $\phi_Y^X : \mathbf{R}^n \times M \to \mathcal{G}$ and $\phi_{Y'}^{X'} : \mathbf{R}^n \times M \to \mathcal{G}$ be two maps such that $\phi_Y^X(t, x_0) = g$ and $\phi_{Y'}^{X'}(t', x_0) = g'$, and such that their restrictions to some small set of the form $B_\varepsilon \times U$, $t, t' \in B_\varepsilon$, are in the family Φ. We need to show that the induced map

$$(9) \qquad \mu_1 : B_\varepsilon \times B_\varepsilon \times U \ni (t, t', x) \to \phi_{Y'}^{X'}(t', x)(\phi_Y^X(t, x))^{-1} \in \mathcal{G},$$

is differentiable. It is enough to prove this in a small neighborhood of x_0. Because Φ forms an atlas, using Lemma 5(i), we see that we may assume $t = t' = 0$, eventually by changing X, X', Y, or Y'. The differentiability of μ_1 then follows by combining Lemma 5(ii), and Lemma 4(i). This is enough to conclude that $\mathcal{G}$ is a differentiable groupoid whenever Φ is an atlas.

Let $A(\mathcal{G})$ be the Lie algebroid of $\mathcal{G}$ corresponding to the differentiable structure defined by Φ. We need to check that $A(\mathcal{G}) \simeq A$. We have that $A(\mathcal{G})|_S \simeq A_S$, by construction, and hence we can identify as a set $A(\mathcal{G})$ with the disjoint union of the restrictions A_S. The two differentiable structures on $\bigcup A_S$ (the first induced from A and the second induced from $A(\mathcal{G})$) are the same because the differential of the map ϕ_Y^σ of Lemma 3, $\sigma = id$, canonically identifies $A|_U$ and $A(\mathcal{G})|_U$, if U is as in that lemma. The Lie algebra structures on $\Gamma(A)$ and $\Gamma(A(\mathcal{G}))$ also coincide because the two possible

brackets of two vector fields coincide on each strata S, and hence they coincide everywhere. We have thus proved that, if Φ forms an atlas, then $\mathscr{G}$ is a differentiable groupoid with $A(\mathscr{G}) \simeq A$.

Conversely, suppose now that $\mathscr{G}$ is endowed with a differentiable structure, and let $g \in \mathscr{G}_S$. Since $\mathscr{G}_S$ is d-connected, we can choose vector fields $X_1, \ldots, X_m$ such that

$$g = \exp(X_m) \cdots \exp(X_1) x_0,$$

for some $x_0 \in S$. If $Y_1, \ldots, Y_n$ are chosen to form a basis of A_{x_0}, as in Lemma 3, then the map ϕ_Y^X must be a diffeomorphism of a set of the form $B_\varepsilon \times U$ onto an open neighborhood of g, for some open subset $U \subset M$. We obtain that if $\mathscr{G} = \bigcup \mathscr{G}_S$ is a differential groupoid such that $A(\mathscr{G}) \simeq A$, then the family Φ is an atlas. $\qquad\square$

In particular, we immediately obtain from the above theorem and its proof that the differentiable structure on M and the groupoid structure on $\mathscr{G}$ uniquely determine the differentiable structure on $\mathscr{G}$ satisfying $A(\mathscr{G}) \simeq A$. Indeed, the differentiable structure on $\mathscr{G}$ is determined by the family Φ.

We now turn to the main theorem of this paper, Theorem 3, which shows that the assumptions of Theorem 2 are satisfied provided that the groupoids $\mathscr{G}_S$ are d-simply connected. We shall need an extension of the concept of differentiable family of sections to our case, $\mathscr{G} = \bigcup \mathscr{G}_S$, when $\mathscr{G}_S$ are smooth groupoids but $\mathscr{G}$ is not endowed with any differentiable structure.

A *differentiable family* of admissible sections is a family $\sigma_s : M \to \mathscr{G}$, $s \in [0,1]$, of maps such that each σ_s, $s \in [0,1]$, is an admissible section, the induced map $[0,1] \times S \ni (s,x) \to \sigma_s(x) \in \mathscr{G}_S$ is differentiable for each S, and the sections $\partial_s(\sigma_s|_S)$ of A_S can be glued to a smooth section of A on $[0,1] \times M$. This definition of a differentiable family of smooth section is thus very similar to the corresponding definition in the case when $\mathscr{G}$ has a smooth structure, except that we replace the condition on the smoothness of the map $[0,1] \times M \to \mathscr{G}$ by the existence and smoothness of the derivatives $\partial_s \sigma_s$ on $[0,1] \times M$.

The following lemma is an important technical part of the proof of Theorem 3. It achieves a continuous and smooth deformation of a local admissible section of $\mathscr{G}$ to a local identity section. We denote by $\Gamma_c(E)$ the space of compactly supported, smooth sections of a vector bundle E.

LEMMA 6. *Let $\mathscr{G} = \bigcup \mathscr{G}_S$ be a union of d-simply connected differentiable groupoids with $A(\mathscr{G}_S) = A_S$, as in the statement of Theorem 2, and let $x_0 \in M$. Suppose that $X_1, \ldots, X_m \in \Gamma_c(A)$ satisfy*

$$\exp(X_m) \cdots \exp(X_1) x_0 = x_0.$$

Then we can find a differentiable family of admissible sections $\sigma_s : M \to \mathscr{G}$ such that:
    (i) $\sigma_1(x) = \exp(X_m) \cdots \exp(X_1)x$, *on M*;
    (ii) $\sigma_0(x) = x$, *for all $x \in M$*;
and, most importantly,
    (iii) $\sigma_s(x_0) = x_0$, *for all $s \in [0,1]$*.

This lemma remains true if we replace the condition that X_i be compactly supported by the condition that they be integrable.

PROOF. Consider the curve

$$\phi(t) = \exp(t'X_k)\exp(X_{k-1})\cdots\exp(X_1)x_0,$$

where $t' = (mt - k + 1)$ and k is chosen such that $0 \le t' \le 1$. By assumption, ϕ is a closed curve on $\mathscr{G}_{x_0}$, and hence, by the assumption that $\mathscr{G}$ is d-simply connected, we can continuously deform this curve to the constant curve x_0, within $\mathscr{G}_{x_0}$, through closed curves based at x_0. More precisely, we can find

$$\eta : [0, 1] \times [0, 1] \to \mathscr{G}_{x_0}$$

such that $\eta(t, 1) = \phi(t)$ and $\eta(t, 0) = \eta(0, s) = \eta(1, s) = x_0$, for all $t, s \in [0, 1]$.

By an approximation argument, we can assume that $\eta(k/m, s)$ depends smoothly on s, for each integer k. Moreover, after replacing m by a large multiple lm and each X_k, $k = 1, \ldots, m$, by $l^{-1}X_k$ repeated l times, we can assume that there exist compactly supported sections $X_k^s \in \Gamma(A)$, depending smoothly on s, such that $X_k^1 = X_k$, $X_k^0 = 0$, and

$$\eta\left(\frac{k}{m}, s\right) = \exp(X_k^s)\eta\left(\frac{k-1}{m}, s\right).$$

The desired deformation is obtained by letting

$$\sigma_s(x) = \exp(X_m^s)\cdots\exp(X_1^s)x.$$

To see that σ_s is a smooth family of admissible sections, we use Proposition 1 for each $\mathscr{G}_S$ and the fact that each $\exp(X_k^s)$ is a smooth admissible section. $\qquad\square$

We continue to assume that $\mathscr{G} = \bigcup \mathscr{G}_S$ is as in the statement of Theorem 3.

PROPOSITION 2. *Let σ_s, $s \in [0, 1]$, be a differentiable family of local admissible sections. Assume that $\sigma_s(x_0) = x_0$, for all $s \in [0, 1]$, and that $\sigma_0(x) = x$, for all $x \in M$. Let $Y_1, \ldots, Y_n$ be a local basis of A at x_0. Then there exist $\varepsilon > \delta > 0$, a neighborhood U of x_0 in M, and a differentiable map*

$$\tau : [0, 1] \times B_\delta \times U \to B_\varepsilon$$

such that: (i) $\tau(s, 0, x_0) = 0$, (ii) for each fixed $s \in [0, 1]$ and $x \in U$, the map $B_\delta \ni t \to \tau(s, t, x) \in B_\varepsilon$ is a diffeomorphism onto its image, and

$$\text{(iii)} \quad \sigma_s \operatorname{Exp}(t, Y)x = \operatorname{Exp}(\tau(s, t, x), Y)x,$$

for all $s, t \in [0, 1]$, and $x \in U$.

PROOF. Let $\mathscr{U}$ be a local groupoid integrating A. Choose $\varepsilon > 0$ and a neighborhood U_1 of x_0 small enough such that

$$\eta : B_{2\varepsilon} \times U_1 \to \mathscr{U}, \quad \eta(t, x) = \operatorname{Exp}(t, Y)x,$$

is a diffeomorphism onto an open neighborhood V of x_0 in $\mathscr{U}$. Let $U_0 \subset U_1$ be a compact neighborhood of x_0. Because $K := \exp(\bar{B}_\varepsilon \times U_0)$ is a compact subset of the open set V and σ is continuous, $\sigma_s(x_0) = x_0$, we can find a neighborhood U of x_0 such

that $\sigma_s k$ is defined in $\mathcal{U}$ whenever $k \in K$, and $d(k) \in U$, and such that

$$\sigma_s(U_1)K \subset V.$$

Then we can define

$$\tau(s, t, x) = \eta^{-1}(\sigma_s \operatorname{Exp}(t, Y)x).$$

Because $\sigma_s \operatorname{Exp}(t, Y)x \in V \subset \mathcal{U}$ and η is a diffeomorphism, we obtain that τ is smooth also. $\square$

THEOREM 3.   *Let A be a Lie algebroid on a manifold with corners M. Suppose that M has an A-invariant stratification $M = \bigcup S$ such that, for each stratum S, the restriction A_S is integrable, then A is integrable.*

   More precisely, let $\mathcal{G}_S$ be d-simply connected differential groupoids such that $A(\mathcal{G}_S) \simeq A_S$. Then the disjoint union $\mathcal{G} = \bigcup \mathcal{G}_S$ is naturally a differentiable groupoid such that $A(\mathcal{G}) \simeq A$.

PROOF.   By Theorem 2, we see that it is enough to show that the family Φ, defined using the maps ϕ_Y^X of equation (7), is an atlas. Let ϕ_Y^X and $\phi_{Y'}^{X'}$ be two such maps, defined on $B_\varepsilon \times U$ and, respectively, on $B_{\varepsilon'} \times U'$, such that their images intersect. Let $g \in \mathcal{G}$ be an element of this intersection. As in the proof of Theorem 2, we can arrange that $\phi_Y^X(0, x_0) = \phi_{Y'}^{X'}(0, x_0) = g$.   Then

$$\phi_Y^X(0, x_0) = \exp(X_m) \cdots \exp(X_1)x_0 = \exp(X_{m'}') \cdots \exp(X_1')x_0 = \phi_{Y'}^{X'}(0, x_0),$$

and hence

$$x_0 = \exp(-X_1') \cdots \exp(-X_{m'}')\exp(X_m) \cdots \exp(X_1)x_0,$$

so we may also assume that $X_i' = 0$ (and hence $g = x_0$). By replacing V and V' with some smaller, relatively compact neighborhoods, if necessary, we may further assume that the sections X_i are compactly supported.   Let

$$\sigma(x) = \exp(X_m) \cdots \exp(X_1)x.$$

Since each $\mathcal{G}_S$ is d-simply connected, by Lemma 6 we can find a smooth family $\sigma_s : M \to \mathcal{G}$, $s \in [0, 1]$, such that σ_0 is the identity, $\sigma_1 = \sigma$, the restriction to each groupoid $\mathcal{G}_S$ is differentiable. The smoothness of the family σ_s in this case is reduces to showing that $\partial_s \sigma_s$ is a smooth section of A over M. Let τ be the map defined in Proposition 2 using the admissible sections σ_s, and let $\tau_1(t, x) = \tau(1, t, x)$.   Then

$$\phi_Y^X(t, x) = \phi_{Y'}^{X'}(\tau_1(t, x), x),$$

in a small neighborhood of x_0. Since τ_1 is a local diffeomorphism for each fixed x, this proves the theorem. $\square$

3.   Applications to foliations.

   Recall that a Lie algebroid $\pi : A \to M$, with anchor map $q : A \to TM$, is called *regular* if, and only if, the range of q has locally constant rank. Then the sections of

$q(A)$ are the sections of the tangent bundle to a foliation $\mathscr{F}_A$ whose tangent bundle, a sub-bundle of TM, is denoted $T\mathscr{F}_A$. If, moreover, there exists a morphism $\rho : \Gamma(T\mathscr{F}_A) \to \Gamma(A)$ such that $q \circ \rho = id$, then we may assume that A is a *semi-direct product*.

Also, if A is regular, the kernel of q is a bundle of Lie algebras. See also [**6**], [**20**]. Recall that a result of Douady and Lazard from [**5**] states that every bundle of Lie algebras is integrable. In particular the kernel $\ker(q)$ is integrable. (I am greatful to Alan Weinstein for pointing out this reference to me.)

We say that A is (isomorphic to) the semi-direct product $\ker(q) \rtimes q(A)$ if there exists a morphism of Lie algebras $\rho : \Gamma(q(A)) \to \Gamma(A)$ that is a right inverse to q.

PROPOSITION 3. *Let A be a regular algebroid with anchor map $q : A \to TM$. Assume that A is the semi-direct product $\ker(q) \rtimes q(A)$. Then A is integrable.*

Before proceeding to the proof, we first introduce some terminology and make some comments. If $E \to M$ is a vector bundle on a foliated manifold M, then a leafwise connection on E is a linear map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*\mathscr{F}),$$

satisfying the Leibnitz identity. Here $\mathscr{F}$ is the foliation of M, $T\mathscr{F}$ is the tangent bundle to this foliation, and $T^*\mathscr{F}$ is the dual of this bundle. The equivalent definitions of a connection in terms of parallel transport or equivariant splittings of tangent spaces of principal bundles extend to the "leafwise" setting also.

The structure of regular, semi-direct product Lie algebroids is as follows. First, there exists a *leafwise flat* connection ∇, $\nabla_Z(X) = [\rho(Z), X]$, on $\ker(q)$, which preserves its Lie bundle structure, that is

$$\nabla_Z \nabla_{Z'} - \nabla_{Z'} \nabla_Z = \nabla_{[Z, Z']},$$

and

$$\nabla_Z([X, Y]) = [\nabla_Z(X), Y] + [X, \nabla_Z(Y)],$$

for all $Z, Z' \in \Gamma(T\mathscr{F}_A)$ and $X, Y \in \Gamma(\ker(q))$. Then

$$A \simeq T\mathscr{F}_A \oplus \ker(q)$$

as vector bundles, with anchor map given by the projection onto the first component, and with Lie bracket on $\Gamma(A)$ defined by

$$[(Z, X), (Z', X')] = ([Z, Z'], \nabla_Z(X') - \nabla_{Z'}(X) + [X, X']),$$

for all $Z, Z' \in \Gamma(T\mathscr{F}_A)$ and $X, X' \in \Gamma(\ker(q))$.

With these comments, we are now ready to prove Proposition 3.

PROOF. Recall that $\mathscr{P}\mathscr{F}$, the *path groupoid* of the foliation $\mathscr{F}$, consists of fixed end point homotopy classes of paths γ that are fully contained in a single leaf, with respect to homotopies *within* that leaf. As explained above, the morphism ρ defines a leafwise flat connection on $\ker(q)$ that preserves the Lie bracket. Let $\mathscr{K}$ be a d-simply connected groupoid that integrates $\ker(q)$.

We define then a groupoid $\mathscr{G}$ that integrates A as follows. As a smooth manifold,

$$\mathscr{G} = \{(g, \gamma) \in \mathscr{K} \times \mathscr{P}\mathscr{F}, d(g) = \gamma(1)\}.$$

To define the multiplication, observe first that the leafwise flat connection on $\ker(q)$ defines a parallel transport map

$$\rho(\gamma) : \ker(q)_{\gamma(0)} \to \ker(q)_{\gamma(1)},$$

which is a Lie algebra isomorphism for any path γ fully contained in a leaf. Since $\mathcal{K}_x$ is simply connected for each x, we obtain by exponentiation a group morphism

$$\rho(\gamma) : \mathcal{K}_{\gamma(0)} \to \mathcal{K}_{\gamma(1)}.$$

That is, the leafwise flat connection on $\ker(q)$ lifts to a leafwise flat connection on $\mathcal{K}$ that preserves the Lie group structure on the fibers.

We are ready now to define the groupoid structure on $\mathcal{G}$. Note first that $d(g) = r(g)$ for $g \in \mathcal{K}$. Then $d(g, \gamma) = \gamma(0)$, $r(g, \gamma) = r(g)$, and the product on $\mathcal{G}$ is given by the formula

$$(g, \gamma)(g', \gamma') = (g\rho(\gamma)(g'), \gamma\gamma'),$$

where the composition of paths is given by concatenation. The flatness of ∇ gives that $\rho(\gamma\gamma') = \rho(\gamma)\rho(\gamma')$, which guarantees the associativity of the product. $\qquad \square$

In certain cases we get integrability without assuming that A is a semi-direct product.

PROPOSITION 4. *Let A be a regular Lie algebroid with anchor map $q : A \to TM$ such that $\ker(q)$ is a bundle of* semisimple *Lie algebras. Then A is integrable.*

PROOF. We may assume that M is connected. Since semisimple Lie algebras are rigid, all fibers of $\ker(q)$ will be isomorphic Lie algebras. Fix one of these algebras and denote it by $\mathfrak{g}$. Also, let G_0 be the group of automorphisms of $\mathfrak{g}$. Then, if we define

$$P = \bigcup_x \mathrm{Iso}(\mathfrak{g}, \ker(q)_x),$$

(the fibers are the sets of Lie algebra isomorphisms $\mathfrak{g} \to \ker(q)_x$), we obtain a G_0-principal bundle on M, which acquires by pull-back a foliation $\mathscr{F}$ of the same codimension as $\mathscr{F}_A$. The path groupoid $\mathscr{PF}$ of the foliation $\mathscr{F}$ has an induced free action of G_0, and we define the groupoid $\mathcal{G}$ by $\mathcal{G} = \mathscr{PF}/G_0$. The composition of two paths in $\mathcal{G}$ is obtained by choosing composable liftings in $\mathscr{PF}$. Because the Lie algebra of G_0 is $\mathfrak{g}$, we obtain that $\mathcal{G}$ integrates A. $\qquad \square$

With the above results in mind, we now define A-invariant regular stratifications.

DEFINITION 2. Let $\pi : A \to M$ be a Lie algebroid with anchor map $q : A \to TM$ on the manifold with corners M. An invariant stratification $M = \bigcup S$ is called regular if, and only if, the restriction $A_S := A|_S$ is regular for each S.

Although almost all interesting Lie algebroids have invariant regular stratifications, this is not true in general. Consider, for example, a closed subset $B \subset \boldsymbol{R}^n$ with empty interior, which is not a manifold. Let $\phi \geq 0$ be a smooth function that vanishes exactly on B, and let $\mathscr{V}(\boldsymbol{R}^n)$ be the Lie algebra of vector fields on $\boldsymbol{R}^n$. Since $\phi\mathscr{V}(\boldsymbol{R}^n)$ is a free $\mathscr{C}^\infty(\boldsymbol{R}^n)$ module, using the Serre-Swan theorem, we can define a vector bundle A such

that $\Gamma(A) = \phi\mathcal{V}(\mathbf{R}^n)$. Moreover,

$$[\phi X, \phi Y] = \phi(X(\phi)Y - Y(\phi)X + \phi[X, Y]),$$

for all vector fields X and Y on $\mathbf{R}^n$, which shows that A is a Lie algebroid. It is not difficult to see that A has no invariant regular stratification.

Although not all Lie algebroids have regular stratifications, this notion is useful because it is easier to integrate regular algebroids than general Lie algebroids, and we know that in order to integrate a Lie algebroid, it is enough to integrate it over each strata (Theorem 3).

THEOREM 4. *Let A be a Lie algebroid with anchor map $q : A \to TM$ on a manifold M with a regular A-invariant stratification $M = \bigcup S$. Assume, for each S, that either A_S is the semi-direct product $\ker(q)_S \rtimes q(A_S)$, or that $\ker(q)|_S$ is a bundle of semisimple Lie algebras. Then A is integrable.*

PROOF. Using Proposition 3 or Proposition 4, we see that each of the algebroids A_S, obtained by restricting A to the stratum S, is integrable. By Theorem 3, the Lie algebroid A is then integrable. □

Let us consider now in greater detail a class of examples that is useful for the construction of pseudodifferential operators on complex algebraic varieties. It is possible to associate several "natural" algebroids to a complex algebraic variety, so the following constructions lead to a family of algebras of pseudodifferential operators associated to a complex algebraic variety. The details of this construction will be presented in a future paper. We do not know at this point which of the many algebras that one obtains is the "right" algebra to associate to a complex algebraic manifold, but we hope to address this question in a future paper.

Let M be a manifold with corners, such that each hyperface (*i.e.*, face of maximal dimension) $H \subset M$ is given by $H = \{x_H = 0\}$ for some function $x_H \geq 0$ with $dx_H \neq 0$ on H, (*i.e.*, x_H is a defining function of H.) If $F \subset M$ is an arbitrary face of M of codimension k, then F is an open component of the intersection of the hyperfaces containing it. The set $x_1, \ldots, x_k$ of defining functions of these hypersurfaces are called the defining functions of F; thus F is a connected component of $\{x_1 = x_2 = \cdots = x_k = 0\}$.

We first introduce the class of Lie algebroids we are interested in on a manifold with corners M. We call these algebroids "quasi-homogeneous," and we now proceed to construct them. First we need some related definitions that will make it easier to describe our settings.

DEFINITION 3. A Lie flag on a manifold M is an increasing finite sequence of sub-bundles

$$E_0 \subset E_1 \subset \cdots \subset E_l \subset E_\infty := TM$$

such that $[\Gamma(E_i), \Gamma(E_j)] \subset \Gamma(E_{i+j})$.

It follows from the definition that, if $E_0 \subset \cdots \subset E_l \subset TM$ is a Lie flag on M, then the bundles E_0 and E_l are integrable. We do not assume the above inclusions to be strict.

We now describe the type of behavior we want for the vector fields that are sections of a quasi-homogeneous Lie algebroid. Let p_i denote the projection onto the ith component of a product.

DEFINITION 4. Let $H = \{x_H = 0\} \subset M$ be a hyperface, and let

$$\phi_H = (\pi_H, x_H) : V_H \to H \times [0, \varepsilon)$$

be a diffeomorphism defined in a neighborhood V_H of H in M. Also, let

$$E_0^H \subset \cdots \subset E_{l_H}^H \subset TH$$

be a Lie flag on H and $d_H \in N \cup \{\infty\}$. Then a vector field X on V_H is called (E_i, ϕ_H, d_H)-adapted if, and only if,

$$(10) \qquad X \in \sum_{j=0}^{l_H} x_H^j \Gamma(\pi_H^*(E_i^H)|_{V_H}) + \mathscr{C}^\infty(V_H) x_H^{d_H+1} \partial_{x_H},$$

(we agree that $x_H^\infty = 0$).

Finally, the sections of our quasi-homogeneous Lie algebroids will consist of vector fields that are "adapted" to each hyperface, but the data at each hyperface must be compatible.

DEFINITION 5. A boundary Lie datum $\mathscr{D} = (E_i^H, \phi_H, d_H)$ on a manifold with corners M, where H ranges through the set of hyperfaces of M, consists of:
(i) Lie flags $E_0^H \subset \cdots \subset E_{l_H}^H \subset TH$ such that all intersections $E_{i_1}^{H_1} \cap \cdots \cap E_{i_t}^{H_t}$, $i_j \in \{0, 1, \ldots, l_{H_j}, \infty\}$, as well as all finite sums of such intersections, have constant rank on the set where they are defined and form a distributive lattice.
(ii) Diffeomorphisms

$$\phi_H = (\pi_H, x_H) : V_H \to H \times [0, \varepsilon)$$

such that $\pi_H \pi_{H'} = \pi_{H'} \pi_H$ and $x_H \pi_{H'} = x_H$ on $V_H \cap V_{H'}$, for all hyperfaces H and H'.
(iii) Degrees $d_H \in N \cup \{\infty\}$.

If $\mathscr{D} = (E_i^H, \phi_H, d_H)$ is a boundary Lie datum on M, then the diffeomorphism ϕ_H defines a complement NH to TH in $TM|_H$.

PROPOSITION 5. *Let $\mathscr{D} = (E_i^H, \phi_H, d_H)$ be a boundary Lie datum on the manifold with corners M. Suppose that $\mathscr{F}$ is a foliation of M such that*

$$T\mathscr{F}|_H = \begin{cases} E_{l_H}^H + NH, & \text{if } d_H < \infty, \\ E_{l_H}^H, & \text{if } d_H = \infty. \end{cases}$$

If we define

$$\mathscr{A}_{\mathscr{D}} := \{X \in \Gamma(T\mathscr{F}), X \text{ is } (E_i^H, \phi_H, d_H)\text{-adapted, for each hyperface } H\},$$

then $\mathscr{A}_{\mathscr{D}}$ is a Lie algebra and a projective $\mathscr{C}^\infty(M)$-module.

PROOF. The set of points satisfying (10) at a face H is closed under the Lie bracket, by definition of a Lie flag. Since $\Gamma(T\mathscr{F})$ is also closed under the Lie bracket, it follows that $\mathscr{A}_{\mathscr{D}}$ is a Lie algebra.

Fix a point x_0 in the interior of a face F of codimension k with defining functions $x_1, \ldots, x_k$. Let $H_1, \ldots, H_k$ be the corresponding hyperfaces, $H_j = \{x_j = 0\}$, and let $d_i = d_{H_i}$. Also let

$$\pi = \pi_{H_1} \cdots \pi_{H_k} : V := \bigcap V_{H_j} \to F,$$

where the order of composition is not important because $\pi_H \pi_{H'} = \pi_{H'} \pi_H$, by the definition of boundary Lie datum.

Let $v = (v_1, \ldots, v_k)$, $0 \le v_i \le l_{H_i}$, be a multi-index and

$$E_v = \pi^*(E_{v_1}^{H_1} \cap \cdots \cap E_{v_k}^{H_k}),$$

which, we recall, is a vector bundle on $H_1 \cap \cdots \cap H_k$, again by the definition of a boundary Lie datum. Also, let

$$v^{(i)} = (v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_k)$$

and choose, arbitrarily, a complement Y_v to $\sum E_{v^{(i)}}$ in E_v.

Let $\mathscr{Z}$ be the set of vector fields $\{x_i^{d_i+1} \partial_{x_i}, d_i < \infty\}$ defined using the diffeomorphism $(\pi, x_1, \ldots, x_k) : V = \bigcap V_{H_i} \to F \times [0, \varepsilon)^k$, for some small ε. Then the restriction of $\mathscr{A}_{\mathscr{D}}$ to V is

$$(11) \qquad \mathscr{C}^\infty(V) \mathscr{A}_{\mathscr{D}}|_V = \left(\sum_{v=(v_1,\ldots,v_k)} x_1^{v_1} \cdots x_k^{v_k} \Gamma(\pi^*(Y_v)) \right) \oplus \left(\bigoplus_{Z_i \in \mathscr{Z}} \mathscr{C}^\infty(V) Z_i \right)$$

$$\simeq \Gamma\left(\bigoplus_v Y_v \oplus \mathbf{R}^{\mathscr{Z}} \right).$$

Since this is a $\mathscr{C}^\infty(V)$-projective module—the module of sections of a bundle isomorphic to the direct sum of $\bigoplus_v Y_v$ and the trivial bundle generated by the set $x_i^{d_i+1} \partial_{x_i}$ $(d_i < \infty)$, we obtain that $\mathscr{A}_{\mathscr{D}}$ is a projective $\mathscr{C}^\infty(M)$-module, as desired. □

Let us examine now a particular case of this construction. Assume that in the definition above $\mathscr{F} = M$, that is, that there exists a single leaf, and that in the boundary Lie data $d_H = 0$ and $E_1^H = TH$. The only choice then is that of the integrable bundles E_0^H, because the choice of the diffeomorphisms ϕ_H is not important. The conditions that these bundles have to satisfy are that, for each face F (contained in k distinct hyperplanes H, where k is the codimension of F), there exist $2k$ commuting surjective projections

$$p_{FH} : TM|_F \to E_0^H|_F, \quad \text{and} \quad q_{FH} : TM|_F \to TH|_F.$$

We denote by $A_{\mathscr{D}}$ the vector bundle on M with sections $\mathscr{A}_{\mathscr{D}}$, defined by the above proposition. Because $\mathscr{A}_{\mathscr{D}}$ is a Lie algebra, $A_{\mathscr{D}}$ is a Lie algebroid. A Lie algebroid A is called *quasi-homogeneous* if it is isomorphic to one of the form $A_{\mathscr{D}}$ obtained as above.

Proposition 6. *Let $A \subset TM$ be a quasi-homogeneous algebroid. Then the set of open faces of M defines an A-invariant regular stratification of M. Moreover, if $S = \mathrm{Int}(F)$ is an open face of M, then the Lie algebroid $q_S : A_S := A|_S \to TS$ is integrable.*

Proof. Let $\mathscr{F}$ and (E_i^H, ϕ_H, d_H) be the foliation and, respectively, the boundary Lie data defining A. Observe that the sections of A are vector fields that are tangent to all faces of M, and hence each open face of M is A-invariant. This means that the stratification of M by open faces is an A-invariant stratification of M.

Fix a face F of codimension k with defining functions $x_1, \ldots, x_k$, such that $H_j = \{x_j = 0\}$. Let $S := \mathrm{Int}(F)$ be the interior of F, and denote $A_S := A|_S$.

For $v_0 = (0, \ldots, 0)$, the intersection

$$E_{v_0} := E_0^{H_1} \cap \cdots \cap E_0^{H_k} \subset TS$$

is an integrable sub-bundle of TS such that $q_S : A_S \to TS$ maps $\Gamma(A_S)$ surjectively onto $\Gamma(E_{v_0})$. Moreover, the map $\pi = \pi_{H_1} \circ \cdots \circ \pi_{H_k}$ defines a splitting of A_S, that is, a Lie algebra morphism $\Gamma(E_{v_0}) \to \Gamma(A_S)$. (This splitting is implicit in the description of $\mathscr{A}_{\mathscr{D}}$ given in (11).) Consequently, A_S is integrable, by Proposition 3. $\qquad\square$

Theorem 5. *Every quasi-homogeneous algebroid is integrable.*

Proof. This follows from Proposition 6 and Theorem 4. $\qquad\square$

Because the construction of the differentiable groupoid integrating a regular, semi-direct product Lie algebroid depends on the construction of a differentiable groupoid $\mathscr{K}$ integrating $\ker(q_S)$, it is useful to have an explicit construction of $\mathscr{K}$ in the above proposition. To this end, we use the notation in the proof of Proposition 5. Let $Z_0 \subset Z := \{x_i^{d_i+1} \partial_{x_i}\}$ be the set of normal vectors (with respect to the decomposition induced by $(\pi, x_1, \ldots, x_k)$) such that $d_i = d_{H_i} = 0$. Also, let A_0 be the sub-bundle of $A|_S$ generated by Z_0 and A_1 be the vector bundle generated by the complement of Z_0 in Z and by E_v, for $v \neq (0, \ldots, 0)$. We then obtain the split exact sequence

$$0 \to A_1 \to \ker(q_S) \to A_0 \to 0.$$

Now, each of the bundles A_0 and A_1 is a Lie algebroid with vanishing anchor map. In fact, each of A_0 and A_1 are bundles of commutative Lie algebras, and hence are integrable: to integrate them, we just consider each A_0 and A_1 as a bundle of commutative Lie groups. Moreover, A_1 is an ideal (in Lie algebra sense) of $\ker(q_S)$, and the sub-bundle A_1 acts by derivations on A_0 (the weights are exactly given by the exponents $v = (v_i)$). This action by derivations of each of the fibers $(A_1)_x$ on $(A_0)_x$ exponentiates to an action of the Lie group with Lie algebra $(A_1)_x$ on the Lie group with Lie algebra $(A_0)_x$. Denote the resulting semidirect product by $\mathscr{K}_x$. In this way we obtain on $\mathscr{K} = \mathscr{K}_x$ a Lie algebra bundle structure, which is isomorphic to $\ker(q_S)$, as a fiber bundle, via the exponential map.

If A is an integrable Lie algebroid, then we denote by $\mathscr{G}_A$ a d-simply-connected differential groupoid that integrates A, which is unique up to isomorphism. If A is a quasi-homogeneous algebroid and $\mathscr{G}$ is any differential groupoid on M with Lie algebroid A, then we call $\mathscr{G}$ a *quasi-homogeneous* groupoid.

We now make some elementary remarks on quasi-homogeneous algebroids and groupoids. If $A \subset M$ is a quasi-homogeneous algebroid and $F \subset M$ is a face of M, then A_F is not a quasi-homogeneous algebroid unless $F = M$. Let $N_F \subset TM$ be the normal bundle to F, with the trivialization given by the boundary Lie data used in the definition of A. Then A_F is the semidirect product of $A_F \cap TF$ and N. The same will be true of the integrating groupoids. Thus, assume that F has codimension k. Then there exists an action of $\mathbf{R}^k$ on $\mathscr{G}_{A_F \cap TF}$, which fixes F, the set of units of $\mathscr{G}_{A_F \cap TF}$, such that

$$(12) \qquad \mathscr{G}_{A_F} \simeq \mathscr{G}_{A_F \cap TF} \rtimes \mathbf{R}^k.$$

This relates the differential groupoids $\mathscr{G}_{A_F}$, associated to the faces of M, to the quasi-homogeneous differential groupoids $\mathscr{G}_{A_F \cap TF}$.

Let us now take a closer look at the simplest example of a quasi-homogeneous, non-regular algebroid, the algebroid $\mathscr{V}_b(M)$ of all vector fields tangent to the boundary $\partial M \neq \varnothing$ of a manifold with boundary M. The two strata of M are ∂M and $\mathrm{Int}(M) = M \setminus \partial M$. If X is a topological space, we denote by $\mathscr{P}_X$ its *path groupoid*. Then the d-simply-connected differential groupoids $\mathscr{G}_1$ and $\mathscr{G}_2$ that integrate $\mathscr{V}_b(M)|_{\partial M} = T(\partial M) \oplus N_{\partial M}$ and $\mathscr{V}_b(M)|_{\mathrm{Int}(M)}$ are, up to isomorphism,

$$\mathscr{G}_1 \simeq \mathscr{P}_{\partial M} \times \mathbf{R}, \quad \text{and} \quad \mathscr{G}_2 \simeq \mathscr{P}_{\mathrm{Int}(M)}.$$

By Theorem 3, or directly, the two groupoids $\mathscr{G}_1$ and $\mathscr{G}_2$ above can be smoothly glued to form a differentiable groupoid

$$\mathscr{G}_{\mathscr{V}_b(M)} = \mathscr{G}_1 \cup \mathscr{G}_2$$

that integrates $\mathscr{V}_b(M)$. This groupoid will be Hausdorff if, and only if, the morphism $\pi_1(\partial M) \to \pi_1(M)$ is injective. The domain map

$$d : \mathscr{G}_{\mathscr{V}_b(M)} \to M$$

is a fibration (that is, it has the homotopy lifting property) if, and only if, $\pi_1(\partial M) \to \pi_1(M)$ is surjective.

In general, the groupoid $\mathscr{G}_{\mathscr{V}_b(M)}$ is much larger than the "stretched product" M_b^2 considered by Melrose [**10**], which also gives a groupoid that integrates $\mathscr{V}_b(M)$ after we remove its off-diagonal faces. We get the same groupoid only if both $\pi_1(\partial M)$ and $\pi_1(M)$ are trivial. Unlike our d-simply connected groupoid $\mathscr{G}_{\mathscr{V}_b(M)}$, the groupoid obtained from M_b^2 is always Hausdorff and the domain map d is a fibration.

Let $r \geq 2$ be an integer. Then the same discussion applies to $\mathscr{V}_{b,k}(M)$, the Lie algebra of vector fields that at the boundary are of the form $x^r \partial_x + \sum_j \partial_{y_j}$, for a suitable coordinate systems $(x, y_1, \ldots, y_{n-1})$ in a neighborhood of a point of the boundary $\partial M = \{x = 0\}$ and a suitable choice of a complement to $T(\partial M)$ in $TM|_{\partial M}$.

As a final remark, we now use the algebroid $\mathscr{V}_b(M)$ to show that some conditions on the groupoids $\mathscr{G}_S$ are necessary in Theorem 3. Otherwise the glued groupoid might not be a smooth manifold. Indeed, consider the same groupoid $\mathscr{G}_2 \simeq \mathscr{P}_{\mathrm{Int}(M)}$ for the big-open strata $\mathrm{Int}(M)$, but a smaller one for the boundary:

$$\mathscr{G}_1 \simeq (\partial M \times \partial M) \times \mathbf{R}.$$

Then $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ is not a smooth manifold if $\pi_1(M)$ is non-trivial.

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