J. Math. Soc. Japan
Vol. 53, No. 2, 2001

Higher torsion in the Morava K-theory of $SO(m)$ and $Spin(m)$

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(Received Jan. 20, 1999)
(Revised Nov. 12, 1999)

Abstract. We determine the module structure of Morava K-theory for the special orthogonal groups and the spinor groups at the prime 2, using the Atiyah-Hirzebruch spectral sequence.

1. Introduction.

This paper is a continuation of [**HMNS**], in which $K(n)^*(G)$ for $G = G_2$, F_4, E_6, E_7, E_8, PE_7 and $PSp(m)$ at the prime 2, PE_6 at the prime 3 and $PU(m)$ for all primes, are calculated. In this paper we compute the Morava K-theory of $SO(m)$ and $Spin(m)$ at the prime 2 using the Atiyah-Hirzebruch spectral sequence

$$H^*(G; K(n)^*) \Rightarrow K(n)^*(G).$$

The relation between the Atiyah-Hirzebruch spectral sequence and v_n torsion is described briefly as follows: if it collapses, then the connective Morava K-theory $k(n)^*(G)$ has no v_n torsion, and if $E_{2r(p^n-1)+2}^{**} = E_\infty^{**}$ for some r, then $k(n)^*(G)$ has at most v_n^r torsion.

In [**Ho**], Hodgkin notes that the Atiyah-Hirzebruch spectral sequence of mod 2 K-theory for $Spin(m)$ has only d_3 as a non zero differential for $m \geq 7$, which means that $k(1)^*(Spin(m))$ has no higher v_1 torsion. He also shows that $k(1)^*(E_7)$ and $k(1)^*(E_8)$ have higher v_1 torsion at the prime 2. Our calculation of the Atiyah-Hirzebruch spectral sequence induces that $k(n)^*(SO(m))$ and $k(n)^*(Spin(m))$ have no higher v_n torsion for any m and n. Therefore, for a simple, simply connected Lie groups G, the connective Morava K-theory $k(n)^*(G)$ has no higher v_n torsion at the prime 2 except $k(1)^*(E_7)$ and $k(1)^*(E_8)$.

For $SO(m)$, K-theory is computed by Held and Suter in [**HS**]. Rao [**Rao**] studied the Rothenberg-Steenrod spectral sequence converging to $K(n)_*(SO(2l+1))$. We compute the Atiyah-Hirzebruch spectral sequence for $K(n)^*(SO(m))$ in Section 2. When $m \leq 2^{n+1}$, the spectral sequence collapses (Theorem 2.4). Rao's result implies that, if m is odd and $m > 2^{n+1}$, then

$$\operatorname{rank}_{K(n)^*} K(n)^*(SO(m)) = 2^f,$$

2000 *Mathematics Subject Classification*. Primary 57T10; Secondary 55T25, 55N20.
Key Words and Phrases. Morava K-theory. Lie groups. Atiyah-Hirzebruch spectral sequence.

where $f = (m-1)/2 + 2^n - 1$, and we use it to see when $\mathrm{rank}_{K(n)^*} E_r^{**}$ equals 2^f. We show that there is only one non zero differential $d_{2^{n+1}-1}$ if m is odd and $m > 2^{n+1}$. If m is even and $m > 2^{n+1}$, we see by using the induced homomorphism i^* that there is also only one non zero differential $d_{2^{n+1}-1}$, where $i : SO(m) \to SO(m+1)$ is the natural inclusion.

We compute the Atiyah-Hirzebruch spectral sequence for $K(n)^*(Spin(m))$ in Section 3. When $m \le 2^{n+1}$, the spectral sequence collapses (Theorem 3.2), and when $m > 2^{n+1}$, using the induced homomorphism π^*, we see that there is only one non zero differential $d_{2^{n+1}-1}$, where $\pi : Spin(m) \to SO(m)$ is the natural projection.

The $K(n)^*$-module structures for $K(n)^*(SO(m))$ and $K(n)^*(Spin(m))$ are determined in Theorems 2.4, 2.10, 3.2 and 3.6.

2. Special orthogonal groups.

In this section and Section 3, we use two lemmas which appeared in the previous paper [**HMNS**]. Before introducing the lemmas, we recall some properties of the Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$. First, it is a spectral sequence of $K(n)^*$-algebra. Secondly, the Morava K-theories enjoy the Künneth isomorphism and so do the E_r-terms of their Atiyah-Hirzebruch spectral sequence. Therefore, if G is a Hopf space, E_r^{**} have the $K(n)^*$-Hopf algebra structures. Moreover the Hopf algebra structure on E_2^{**} is given by that on $H^*(G; K(n)^*)$. The Hopf algebra structure on E_∞^{**} is related to that on $K(n)^*(G)$. One needs to note that, if $p = 2$, the product of $K(n)^*(G)$ is not always commutative, while that of E_∞^{**} is always commutative.

The first lemma we need is:

LEMMA 2.1. *If $x \in E_r^{m,0}$ and $d_r(x') = 0$, for all $x' \in E_r^{u,0}$ with $u < m$, then $d_r(x)$ is primitive.*

If there is no primitive element in $E_r^{s,*}$ for $s \ge r+2$, the lemma implies that $d_r = 0$.

The second one is due to Yagita [**Yag**]:

LEMMA 2.2. *The first non-trivial differential in the Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$ is $d_{2(p^n-1)+1}$ and is represented by a unit multiple of Milnor's operation Q_n.*

We now compute the Atiyah-Hirzebruch spectral sequence for $K(n)^*(SO(m))$. First we recall the mod 2 ordinary cohomology for $SO(m)$. Throughout this paper, suffixes of elements represent their degree.

THEOREM 2.3. *We have*

$$H^*(SO(m); \mathbf{F}_2) \cong \Delta(x_1, x_2, \ldots, x_{m-1}),$$

where $|x_i| = i$. *The action of the* mod 2 *Steenrod operation is given by*

$$Sq^j x_i = \binom{i}{j} x_{i+j}.$$

The basis of the primitive elements are $\{x_i \mid 1 \le i < m\}$.

For simplicity we consider separately the two cases: (1) $m \le 2^{n+1}$ and (2) $m > 2^{n+1}$.

THEOREM 2.4. *The Atiyah-Hirzebruch spectral sequence for* $K(n)^*(SO(m))$ *collapses whenever* $m \le 2^{n+1}$ *and there is an isomorphism of* $K(n)^*$*-modules*

$$K(n)^*(SO(m)) \cong K(n)^* \otimes H^*(SO(m); \mathbf{F}_2).$$

PROOF. The cases where the differential may be non zero are $d_{2i(2^n-1)+1}$ where i is the positive integer. Using Lemma 2.1, one can see that all differentials are zero since there is no primitive element of degree equal or higher than 2^{n+1}. □

NOTATION 2.5.

$$f(i,r) = 2^i(4r+2) - (2^{n+1} - 1),$$
$$k(r) = n - [\log_2(2r+1)],$$
$$h(m,r) = -[-\log_2(m/(4r+2))].$$

Next, we consider case (2). We need to calculate $Q_n x_i$ for Lemma 2.2 where $Q_n = Q_{n-1} Sq^{2^n} + Sq^{2^n} Q_{n-1}$ is the Milnor operation.

LEMMA 2.6. *The following equality holds*

$$Q_n x_i = i x_{i+2^{n+1}-1}.$$

PROOF. We use induction on n. If $n = 0$, we have

$$Q_0 x_i = Sq^1 x_i = i x_{i+1}.$$

Next, suppose that the lemma is true for $n = k$. Then,

$$\begin{aligned} Q_{k+1} x_i &= Q_k Sq^{2^{k+1}} x_i + Sq^{2^{k+1}} Q_k x_i \\ &= Q_k \binom{i}{2^{k+1}} x_{i+2^{k+1}} + i Sq^{2^{k+1}} x_{i+2^{k+1}-1} \end{aligned}$$

$$= i\binom{i}{2^{k+1}}x_{i+2^{k+2}-1} + i\binom{i+2^{k+1}-1}{2^{k+1}}x_{i+2^{k+2}-1}$$

$$= ix_{i+2^{n+1}-1},$$

since $\binom{i}{2^{k+1}} + \binom{i+2^{k+1}-1}{2^{k+1}} \equiv \binom{i}{2^{k+1}} + \binom{i+2^{k+1}}{2^{k+1}} \equiv 1 \bmod 2$ if i is odd. □

To calculate the $E_{2^{n+1}}$-term, we replace the expression of the algebra structure of $H^*(SO(m);\mathbf{F}_2)$ by

$$H^*(SO(m);\mathbf{F}_2) \cong \Delta(x_1, \dots, x_{2i+1}, \dots, x_{[m/2]-1}) \otimes \mathbf{F}_2[x_2, \dots, x_{4r+2}, \dots, x_{4[(m-3)/4]+2}]/(x_{4r+2}^{2^{h(m,r)}}).$$

We now define differential modules $(M_{m,r} : d)$ for $0 \le r \le [(m-3)/4]$ as follows:

1. If $0 \le r < 2^{n-1}$, then

$$M_{m,r} = \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{h(m,r)}}) \otimes \Delta(x_{f(k(r),r)}, \dots, x_{f(i,r)}, \dots, x_{f(h(m,r)-1,r)});$$

$$dx_{4r+2} = 0, \quad dx_{f(i,r)} = x_{4r+2}^{2^i}.$$

2. If $2^{n-1} \le r \le [(m-3)/4]$, then

$$M_{m,r} = \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{h(m,r)}}) \otimes \Delta(x_{f(0,r)}, \dots, x_{f(i,r)}, \dots, x_{f(h(m,r)-1,r)});$$

$$dx_{4r+2} = 0, \quad dx_{f(i,r)} = x_{4r-2}^{2^i}.$$

Note that the first generator of the simple system is $x_{f(k(r),r)}$ in case 1 but $x_{f(0,r)}$ in case 2.

Therefore, we have the isomorphism of differential modules with respect to the Milnor operation Q_n:

$$(H^*(SO(m);\mathbf{F}_2) : Q_n) \cong \bigotimes_{r=0}^{[(m-3)/4]} (M_{m,r} : d) \otimes (\Delta(x_{2i+1} \mid [(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1) : 0)$$

as differential modules.

Lemma 2.7. *If $0 \le r < 2^{n-1}$, then there exist elements $y_{f(i,r)}$ for $k(r)+1 \le i < h(m,r)$ and $y_{f(h(m,r),r)}$ such that*

$$H(M_{m,r} : d) \cong \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{k(r)}}) \otimes \Delta(y_{f(k(r)+1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)}),$$

and if $2^{n-1} \le r \le [(m-3)/4]$, then

$$H(M_{m,r} : d) \cong \Delta(y_{f(1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)}).$$

PROOF. We prove the case for $0 \le r < 2^{n-1}$. We rechoose the generators as follows:

$$y_{f(i,r)} = x_{f(i,r)} + x_{f(k(r),r)} x_{4r+2}^{2^i - 2^{k(r)}}$$

for $k(r) < i < h(m,r)$. Then

$$M_{m,r} = \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{h(m,r)}}) \otimes \Delta(x_{f(k(r),r)}, y_{f(k(r)+1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r)-1,r)}),$$

where

$$dx_{f(k(r),r)} = x_{4r+2}^{2^{k(r)}}, \quad dx_{4r+2} = dy_{f(i,r)} = 0.$$

Thus, we have

$$H(M_{m,r} : d) = \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{k(r)}}) \otimes \Delta(y_{f(k(r)+1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)}),$$

where $y_{f(h(m,r),r)} = x_{f(k(r),r)} x_{4r+2}^{2^{h(m,r)-k(r)}}$.

The case for $2^{n-1} \le r \le [(m-3)/4]$ can be proved similarly. □

In this way we get the $E_{2^{n+1}}$-term of the Atiyah-Hirzebruch spectral sequence as follows:

$$E_{2^{n+1}} = K(n)^* \otimes \bigotimes_{r=0}^{[(m-3)/4]} H(M_{m,r} : d)$$
$$\otimes \Delta(x_{2i+1} \mid [(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1).$$

We use the following result to show that $E_\infty = E_{2^{n+1}}$.

THEOREM 2.8. ([**Rao**]) *$K(n)_*(SO(2l+1))$ is isomorphic to the following modules as $K(n)_*$-modules*:

1. *for* $l \ge 2^{n+1}$,

$$K(n)_* \otimes \bigotimes_{i=2^{n-1}}^{[(l-1)/2]-2^{n-1}} \Lambda(\bar{\beta}_{2i}) \otimes \bigotimes_{i=2[(l-1)/2]-2^n+2}^{l-1} \Lambda(\bar{\beta}_i)$$
$$\otimes \bigotimes_{i=[l/2]}^{l-1} \Lambda(\bar{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{2^{n-1}-1} \Gamma_{k(i)}(\gamma_i);$$

2. *for* $2^n \le l < 2^{n+1}$,

$$K(n)_* \otimes \left(\bigotimes_{i=0}^{2^{n-1}-1} \Gamma_{k(i)+1}(\bar{\beta}_i) \otimes \bigotimes_{i=2^{n-1}}^{l-1} \Lambda(\bar{\beta}_i) \right) \Big/ (\bar{\beta}_i \mid 0 \le i \le l - 2^n)$$
$$\otimes \bigotimes_{i=2^n-1}^{l-1} \Lambda(\bar{\alpha}_{2i+1}),$$

where $\Gamma_k(x)$ *is the truncated divided power algebra of height* k *which is the dual of* $\mathbf{F}_2[x]/(x^{2^k})$.

Corollary 2.9. *If* $l > 2^n$, *then we have*

$$\mathrm{rank}_{K(n)_*} K(n)_*(SO(2l+1)) = 2^f,$$

where $f = l + \sum_{i=0}^{2^{n-1}-1} k(i) = l + 2^n - 1$.

Proof. Recall that $k(i) = n - [\log_2(2i+1)]$. The set $\{1, 2, 3, \dots, 2^n - 1\}$ is the disjoint union of

$$\{(2i+1), 2(2i+1), \dots, 2^j(2i+1), \dots, 2^{k(i)-1}(2i+1)\}$$

for $0 \le i \le 2^{n-1} - 1$. This shows that

$$\sum_{i=0}^{2^{n-1}-1} k(i) = 2^n - 1.$$

First, we consider the case 1 of Theorem 2.8:

$$\begin{aligned} \log_2(\mathrm{rank}_{K(n)_*} K(n)_*(SO(2l+1))) &= ([(l-1)/2] - 2^{n-1}) - 2^{n-1} + 1 \\ &\quad + (l-1) - (2[(l-1)/2] - 2^n + 2) + 1 \\ &\quad + (l-1) - [l/2] + 1 + \sum_{i=0}^{2^{n-1}-1} k(i) \\ &= 2l - 1 - [(l-1)/2] - [l/2] + \sum_{i=0}^{2^{n-1}-1} k(i) \\ &= l + \sum_{i=0}^{2^{n-1}-1} k(i). \end{aligned}$$

Next, we consider the case 2 of Theorem 2.8:

$$\begin{aligned} \log_2(\mathrm{rank}_{K(n)_*} K(n)_*(SO(2l+1))) &= \sum_{i=0}^{2^{n-1}-1} (k(i)+1) \\ &\quad + (l-1) - 2^{n-1} + 1 - (l - 2^n) - 1 \\ &\quad + (l-1) - (2^n - 1) + 1 \\ &= l + \sum_{i=0}^{2^{n-1}-1} k(i). \end{aligned}$$

Consequently, we get the required equation. □

If m is odd, we have $\mathrm{rank}_{K(n)^*} E_{2^{n+1}} = \mathrm{rank}_{K(n)_*} K(n)_*(SO(m))$ and hence $E_\infty = E_{2^{n+1}}$.

Let i be the natural inclusion $i : SO(2l) \to SO(2l+1)$, where $l = 2^t(2s+1)$ and consider the induced map in the $E_{2^{n+1}}$-term:

$$i^* : E_{2^{n+1}}(SO(2l+1)) \to E_{2^{n+1}}(SO(2l)).$$

The generators which are in the image of i^* are clearly permanent cycles. The generators which are not in the image of i^* are $x_{l-2^{n+1}+1}$ and the generators of $H(M_{2l,s})$ of odd degree. But there is no primitive element with even degree and higher than 2^{n+1}, and so they are permanent cycles. Hence $E_\infty = E_{2^{n+1}}$ for all m and we get the following theorem.

Theorem 2.10. *If $m > 2^{n+1}$, then $K(n)^*(SO(m))$ is isomorphic to the following module as $K(n)^*$-modules*:

$$K(n)^*(SO(m)) \cong K(n)^* \otimes \bigotimes_{r=0}^{2^{n-1}-1} \begin{pmatrix} \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{k(r)}}) \\ \otimes \\ \Lambda(y_{f(k(r)+1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)}) \end{pmatrix}$$

$$\otimes \bigotimes_{r=2^{n-1}}^{[(m-3)/4]} \Lambda(y_{f(1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)})$$

$$\otimes \Lambda(x_{2i+1} \mid [(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1).$$

3. Spinor groups.

We calculate the Atiyah-Hirzebruch spectral sequence for $K(n)^*(Spin(m))$ in this section. First we recall the mod 2 ordinary cohomology for $Spin(m)$.

Theorem 3.1. ([**IKT**]) *There is an isomorphism*

$$H^*(Spin(m); \mathbf{F}_2) \cong \Lambda(x_i, z \mid 3 \le i < m, i \ne 4, 8, \dots, 2^{t-1}),$$

where $2^{t-1} < m \le 2^t$, $\deg x_i = i$, $\deg z = 2^t - 1$. *The following equations hold*

$$Sq^j x_i = \binom{i}{j} x_{i+j},$$

$$Sq^1 z = \sum_{\substack{i+j=2^{t-1} \\ i<j}} x_{2i}x_{2j} \quad \text{and} \quad Sq^j z = 0 \quad \text{for } j > 1.$$

For simplicity we consider separately the two cases: (1) $m \le 2^{n+1}$ and (2) $m > 2^{n+1}$. Observe that the proof of the case (1) is similar to that of Theorem 2.4.

THEOREM 3.2. *The Atiyah-Hirzebruch spectral sequence for $K(n)^*(Spin(m))$ collapses whenever $m \leq 2^{n+1}$ and there is an isomorphism of $K(n)^*$-modules*

$$K(n)^*(Spin(m)) \cong K(n)^* \otimes H^*(Spin(m); \mathbf{F}_2).$$

Next we consider the case (2). It has been shown that $Q_n x_i = i x_{i+2^{n+1}-1}$. We need some lemmas to calculate $Q_n z$. The following lemma is well known.

LEMMA 3.3. *Let $a = \sum_{i=0}^{l} a_i 2^i$ and $b = \sum_{i=0}^{l} b_i 2^i$, where $0 \leq a_i, b_i \leq 1$. Then*

$$\binom{b}{a} \equiv \prod_{i=0}^{l} \binom{b_i}{a_i} \mod 2.$$

The following lemma is necessary to prove Proposition 3.5.

LEMMA 3.4. *Let j be an integer such that $1 \leq j < 2^{n+1}$ and $j \equiv 2^{k-1} \bmod 2^k$ for some $k \leq n+1$. If $n \leq t-2$, then*

$$\binom{2i}{j}\binom{2^t - 2^{n+1} - 2 - 2i}{2^{n+1} - j} \equiv 0 \mod 2.$$

PROOF. If $2i \equiv -2, -4, \ldots, -2^{k-1} \bmod 2^k$, then we have the expansions

$$2^t + 2^{n+1} - 2 - 2i = b_l 2^l + \cdots + b_k 2^k + b_{k-2} 2^{k-2} + \cdots + b_0 2^0,$$
$$2^{n+1} - j = a_l 2^l + \cdots + a_k 2^k + 2^{k-1}.$$

From them it follows that

$$\binom{2^t + 2^{n+1} - 2 - 2i}{2^{n+1} - j} \equiv 0 \mod 2.$$

We can similarly show that

$$\binom{2i}{j} \equiv 0 \mod 2,$$

if $2i \equiv 0, 2, 4, \ldots, 2^{k-1} - 2 \bmod 2^k$. Therefore, we have

$$\binom{2i}{j}\binom{2^t - 2^{n+1} - 2 - 2i}{2^{n+1} - j} \equiv 0 \mod 2$$

for all i. □

PROPOSITION 3.5. *For $n \geq 1$ we have*

$$Q_n z = \sum_{i=2^n+1}^{2^{t-2}+2^{n-1}-1} x_{2i} x_{2^t+2^{n+1}-2-2i}.$$

Proof. We use induction on n. If $n = 1$, we have

$$\begin{aligned}
Q_1 z &= Sq^2 Sq^1 z \\
&= Sq^2 \sum_{i=3}^{2^{t-2}-1} x_{2i} x_{2^t - 2i} \\
&= \sum_{i=3}^{2^{t-2}-1} (i x_{2i} x_{2^t+2-2i} + i x_{2i+2} x_{2^t-2i}) \\
&= \sum_{i=3}^{2^{t-2}-1} i x_{2i} x_{2^t+2-2i} + \sum_{i=4}^{2^{t-2}} (i-1) x_{2i} x_{2^t+2-2i} \\
&= \sum_{i=3}^{2^{t-2}} x_{2i} x_{2^t+2-2i}.
\end{aligned}$$

Suppose that the lemma is true for $n = k$. Then, we have

$$\begin{aligned}
Q_{k+1} z &= Sq^{2^{k+1}} Q_k z \\
&= \sum_{i=2^k+1}^{2^{t-2}+2^{k-1}-1} \sum_{j=0}^{2^{k+1}} \binom{2i}{j} \binom{2^t + 2^{k+1} - 2 - 2i}{2^{k+1} - j} x_{2i+j} x_{2^t+2^{k+2}-2-2i-j} \\
&= \sum_{i=2^k+1}^{2^{t-2}+2^{k-1}-1} \binom{2^t + 2^{k+1} - 2 - 2i}{2^{k+1}} x_{2i} x_{2^t+2^{k+2}-2-2i} \\
&\quad + \sum_{i=2^k+1}^{2^{t-2}+2^{k-1}-1} \binom{2i}{2^{k+1}} x_{2i+2^{k+1}} x_{2^t+2^{k+1}-2-2i} \\
&= \sum_{i=2^{k+1}+1}^{2^{t-2}+2^{k-1}-1} \binom{2^t + 2^{k+1} - 2 - 2i}{2^{k+1}} x_{2i} x_{2^t+2^{k+2}-2-2i} \\
&\quad + \sum_{i=2^{k+1}+1}^{2^{t-2}+2^k+2^{k-1}-1} \binom{2i - 2^{k+1}}{2^{k+1}} x_{2i} x_{2^t+2^{k+2}-2-2i},
\end{aligned}$$

since $x_{2^t+2^{k+2}-2-2i} = 0$ if $i < 2^{k+1}$, and $x_{2i} = 0$ if $i = 2^{k+1}$. If $k > t-2$, then $Q_{k+1} = 0$ since both $2^{t-2} + 2^k + 2^{k-1} - 1$ and $2^{t-2} + 2^{k-1} - 1$ are smaller than $2^{k+1} + 1$. Therefore, suppose that $k \le t-2$. We have the expansion

$$i = \sum a_l 2^l.$$

If $a_k = 0$, then

$$\binom{2^t + 2^{k+1} - 2 - 2i}{2^{k+1}} = 0 \quad \text{and} \quad \binom{2i - 2^{k+1}}{2^{k+1}} = 1,$$

while if $a_k = 1$, then

$$\binom{2^t + 2^{k+1} - 2 - 2i}{2^{k+1}} = 1 \quad \text{and} \quad \binom{2i - 2^{k+1}}{2^{k+1}} = 0.$$

Consequently, we get the required result

$$Q_n z = \sum_{i=2^n+1}^{2^{t-2}+2^{n-1}-1} x_{2i} x_{2^t+2^{n+1}-2-2i}. \qquad \square$$

Replacing the generator z by

$$w = z + \sum_{i=2^n+1}^{2^{t-2}+2^{n-1}-1} x_{2i-2^{n+1}+1} x_{2^t+2^{n+1}-2-2i},$$

we can represent the cohomology of the spinor group as follows:

$$H^*(Spin(m); \mathbf{F}_2) \cong \Delta(x_3, \dots, x_{2i+1}, \dots, x_{[m/2]-1}, w)$$
$$\otimes \mathbf{F}_2[x_6, \dots, x_{4r+2}, \dots, x_{[(m-3)/4]+2}]/(x_{4r+2}^{2^{h(m,r)}}).$$

Since $Q_n w = 0$, we have the homology with respect to the Milnor operation Q_n:

$$(H^*(Spin(m)) : Q_n) \cong \bigotimes_{r=1}^{[(m-3)/4]} (M_r : d)$$
$$\otimes (\Delta(x_{f(n+1,0)}, \dots, x_{f(i,0)}, \dots, x_{f(h(m,0)-1,0)}), 0)$$
$$\otimes (\Delta(x_{2i+1}, w \,|\, [(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1), 0).$$

Consequently, we get the $E_{2^{n+1}}$-term of the Atiyah-Hirzebruch spectral sequence

$$E_{2^{n+1}} = \left(\bigotimes_{r=1}^{[(m-3)/4]} H(M_r) \right) \otimes \Delta(x_{f(j,0)}, x_{2i+1}, w),$$

where $n + 1 \le j < h(m, 0)$ and $[(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1$.

Let π be the natural projection $\pi : Spin(m) \to SO(m)$ and consider the induced map in the $E_{2^{n+1}}$-term

$$\pi^* : E_{2^{n+1}}(SO(m)) \to E_{2^{n+1}}(Spin(m)).$$

The generators which are in the image of π^* are clearly permanent cycles. The generators which are not in the image of π^* are $x_{f(n+2,0)}$ and w. Since there is no primitive element with even degree and higher than 2^{n+1}, they are permanent cycles. Therefore, $E_\infty = E_{2^{n+1}}$ for all m and we get the following.

Theorem 3.6. *If $m > 2^{n+1}$, then $K(n)^*(Spin(m))$ is isomorphic to the following module as $K(n)^*$-module*

$$K(n)^*(Spin(m)) \cong K(n)^* \otimes \bigotimes_{r=1}^{2^{n-1}-1} \left(\begin{array}{c} \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{k(r)}}) \\ \otimes \\ \Delta(y_{f(k(r)+1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)}) \end{array} \right)$$

$$\otimes \bigotimes_{r=2^{n-1}}^{[(m-3)/4]} \Delta(y_{f(1,r)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)})$$

$$\otimes \Delta(x_{f(n+2,0)}, \dots, x_{f(i,0)}, \dots, x_{f(h(m,0)-1,0)})$$

$$\otimes \Delta(x_{2[(m-1)/2]-2^{n+1}+3}, \dots, x_{2i+1}, \dots, x_{2[m/2]-1}, w).$$

Finally, we have the following remark.

Remark 3.7. *Quite similarly one can calculate the Atiyah-Hirzebruch spectral sequence for $K(n)^*(PO(4l+2))$, where $PO(4l+2)$ is the projective orthogonal group, using the natural projection $SO(4l+2) \to PO(4l+2)$ as follows:*

1. *If $l \le 2^{n-1}$, then the Atiyah-Hirzebruch spectral sequence collapses and the $K(n)^*$-module structure is given as follows:*

$$K(n)^* \otimes H^*(PO(4l+2)).$$

2. *If $l > 2^{n-1}$, then there is only one non zero differential $d_{2^{n+1}-1}$ and the $K(n)^*$-module structure is given as follows:*

$$K(n)^* \otimes \mathbf{F}_2[x_2]/(x_2^{n+1}) \otimes \Delta(y_{f(k(0)+2,0)}, \dots, y_{f(i,0)}, \dots, y_{f(h(m,0),0)})$$

$$\otimes \bigotimes_{r=1}^{2^{n-1}-1} \left(\begin{array}{c} \mathbf{F}_2[x_{4r+2}]/(x_{4r+2}^{2^{h(m,r)}}) \\ \otimes \\ \Delta(x_{f(k(r)+1,r)}, \dots, x_{f(i,r)}, \dots, x_{f(h(m,r),r)}) \end{array} \right)$$

$$\otimes \bigotimes_{r=2^{n-1}}^{[(m-3)/4]} \Delta(y_{f(r,1)}, \dots, y_{f(i,r)}, \dots, y_{f(h(m,r),r)})$$

$$\otimes \Delta(y_1, x_{2i+1} \,|\, [(m-1)/2] - 2^n + 1 \le i \le [m/2] - 1),$$

where $m = 4l + 2$.

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