J. Math. Soc. Japan
Vol. 55, No. 1, 2003

On the global rigidity of split Anosov R^n-actions

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(Received Jun. 11, 2001)

Abstract. We show that a so called split Anosov $\boldsymbol{R}^n$-action on a closed oriented $(2n+1)$-dimensional manifold is C^∞ conjugate to the suspension of a split hyperbolic affine representation of $\boldsymbol{Z}^n$ on the $(n+1)$-dimensional torus.

1. Introduction.

Let M be a closed oriented $2n+1$ dimensional C^∞ manifold, equipped with a C^∞ Riemannian metric, and let

$$\psi : M \times \boldsymbol{R}^n \to M$$

be a locally free $\boldsymbol{R}^n$-action on M. We frequently use the notation

$$\psi^\xi(x) = \psi(x, \xi)$$

for $x \in M$ and $\xi \in \boldsymbol{R}^n$.

DEFINITION 1.1. The action ψ is called a *split Anosov $\boldsymbol{R}^n$-action* if there is a continuous splitting of the tangent bundle TM of M

$$TM = T\boldsymbol{R}^n \oplus E_1 \oplus \cdots \oplus E_{n+1},$$

with the following properties, where $T\boldsymbol{R}^n$ denotes the tangent bundle of the orbit foliation of the action ψ.

1. Each line bundle E_i is invariant by the derivative ψ_*^ξ for any $\xi \in \boldsymbol{R}^n$.
2. There are elements $\xi_i \in \boldsymbol{R}^n$ $(i = 1, 2, \dots, n+1)$, and constants $\lambda > 0$ and $C > 0$ such that

$$\|\psi^{t\xi_i}(v)\| \geq Ce^{\lambda t}\|v\|, \quad \forall v \in E_i, \ \forall t > 0,$$

$$\|\psi^{t\xi_i}(v)\| \leq C^{-1}e^{-\lambda t}\|v\|, \quad \forall v \in E_j, \ (j \neq i) \ \forall t > 0.$$

2000 *Mathematics Subject Classification*. Primary 37D20; Secondary 57R30.

Key Words and Phrases. split Anosov $\boldsymbol{R}^n$-action, orbit foliation of a locally free Lie group action, codimension one Anosov diffeomorphism, hyperbolic toral automorphism, (un)stable foliation.

This research was partially supported by Grant-in-Aid for Scientific Research (A) (No. 13304005), Japan Society for the Promotion of Science.

The set of split Anosov actions forms an open subset in the space of $\boldsymbol{R}^n$-actions with the Whitney C^1 topology. Henceforth we assume $n > 1$. The purpose of this paper is to show a classification result of such actions up to C^∞ conjugacy (Theorem 3.1). Our method strongly depends on an argument of A. Katok and J. Lewis in [**KL**] for the local rigidity, and the topological classification of codimension one Anosov diffeomorphisms due to J. Franks and S. E. Newhouse.

2. Split Anosov $\boldsymbol{Z}^n$-actions.

In this section we prepare a global rigidity of so called split Anosov $\boldsymbol{Z}^n$-actions, which is a slight generalization of Theorem 4.12 of [**KL**].

Let us begin with exposing a well known fact about maximal rank abelian subgroups of $SL(n+1, \boldsymbol{Z})$ consisting of hyperbolic matrices.

PROPOSITION 2.1. *The following statements concerning a representation* $\rho : \boldsymbol{Z}^n \to SL(n+1, \boldsymbol{Z})$ *are equivalent.*

(1) *The representation* ρ *is faithful and its image* $\rho(\boldsymbol{Z}^n)$ *is generated by hyperbolic matrices.*
(2) *For any nonzero* $b \in \boldsymbol{Z}^n$, $\rho(b)$ *is hyperbolic.*
(3) *There exist elements* $a_1, a_2, \ldots, a_{n+1}$ *of* $\boldsymbol{Z}^n$ *and a direct sum decomposition*

$$\boldsymbol{R}^{n+1} = V_1 \oplus \cdots \oplus V_{n+1}$$

such that each one dimensional subspace V_i *is invariant by* $\rho(\boldsymbol{Z}^n)$ *and that* $\rho(a_i)$ *is expanding along* V_i *and contracting along* V_j $(j \neq i)$.

The proof that (1) implies (3) can be found in [**KL**], while it is an easy exercise in linear algebra to show that (3) implies (2).

Let us denote by $\mathrm{Aff}_+(T^{n+1})$ the group of the orientation preserving affine transformations of T^{n+1}. An affine representation

$$R : \boldsymbol{Z}^n \to \mathrm{Aff}_+(T^{n+1})$$

is called *split hyperbolic* if its linear part

$$R_* : \boldsymbol{Z}^n \to SL(n+1, \boldsymbol{Z})$$

satisfies the conditions of Proposition 2.1.

Let N be a closed oriented $n+1$ dimensional C^∞ manifold and let us denote by $\mathrm{Diff}_+^\infty(N)$ the group of the orientation preserving C^∞ diffeomorphisms of N.

DEFINITION 2.2. A homomorphism

$$S : \boldsymbol{Z}^n \to \mathrm{Diff}^{\infty}_{+}(N)$$

is called a *split Anosov $\boldsymbol{Z}^n$-action* if there are a continuous splitting

$$TN = E_1 \oplus \cdots \oplus E_{n+1}$$

invariant by the derivative $S(\boldsymbol{Z}^n)_*$, and elements $a_1, \dots, a_{n+1}$ of $\boldsymbol{Z}^n$ such that, for any i $(1 \leq i \leq n+1)$, $S(a_i)$ is an Anosov diffeomorphism, expanding along E_i and contracting along E_j $(j \neq i)$.

THEOREM 2.3. *Let S be a split Anosov $\boldsymbol{Z}^n$-action on an $n+1$ dimensional closed oriented manifold N. Then there exist a C^∞ diffeomorphism $h : N \to T^{n+1}$ and a split hyperbolic affine representation*

$$R : \boldsymbol{Z}^n \to \mathrm{Aff}_+(T^{n+1})$$

such that $h \circ S(b) = R(b) \circ h$ for any $b \in \boldsymbol{Z}^n$.

The rest of this section is devoted to the proof of this theorem. First of all extending the result of J. Franks [**F**], S. Newhouse [**N**] obtained the following topological classification of codimension one Anosov diffeomorphisms.

THEOREM 2.4. *Any codimension one Anosov diffeomorphism is topologically conjugate to a hyperbolic toral automorphism.*

We also need the following lemma.

LEMMA 2.5. *Let g be a homeomorphism of T^{n+1}, homotopic to the identity, which commutes with a hyperbolic toral automorphism A. Then g is a rational translation.*

PROOF. The point 0, as well as $g(0)$, is a fixed point of A. An easy computation shows that the translation by $g(0)$ commutes with A. Thus one needs only to show that g is the identity assuming $g(0) = 0$. Let $\mathscr{F}^s$ (resp. $\mathscr{F}^u$) denotes the stable (resp. unstable) foliation of f. The homomorphism g leaves these foliations invariant. As is well known, all the leaves of $\mathscr{F}^s$ and $\mathscr{F}^u$ are dense in the torus. Let $\tilde{\mathscr{F}}^s$ (resp. $\tilde{\mathscr{F}}^u$) be the lift of $\mathscr{F}^s$ (resp. $\mathscr{F}^u$) to the universal covering space $\boldsymbol{R}^{n+1}$. Then the action of $\pi_1(T^{n+1})$ on the quotient space $\boldsymbol{R}^{n+1}/\tilde{\mathscr{F}}^s$ (resp. $\boldsymbol{R}^{n+1}/\tilde{\mathscr{F}}^u$) has the property that all the orbits are dense. Now since g is homotopic to the identity, a lift $\tilde{g}$ of g satisfying $\tilde{g}(0) = 0$ commutes with all the deck transformations. Therefore the transformations on the quotient spaces $\boldsymbol{R}^{n+1}/\tilde{\mathscr{F}}^s$ and $\boldsymbol{R}^{n+1}/\tilde{\mathscr{F}}^u$ induced from $\tilde{g}$ leaves the points of the $\pi_1(T^{n+1})$-orbit of 0 fixed, and hence is the identity. This shows that $\tilde{g}$ is the identity. □

PROOF OF THEOREM 2.3. Let S be a split Anosov $\boldsymbol{Z}^n$-action on N, with $a_1, \dots, a_{n+1} \in \boldsymbol{Z}^n$ chosen as in the definition. Then by Theorem 2.4, there are a homeomorphism $h : N \to T^{n+1}$ and a hyperbolic toral automorphism A_1 such that $h \circ S(a_1) = A_1 \circ h$.

For any element $b \in \boldsymbol{Z}^n$, define $B \in SL(n+1, \boldsymbol{Z})$ by

$$B = (h \circ S(b) \circ h^{-1})_* : H_1(T^{n+1}; \boldsymbol{Z}) \to H_1(T^{n+1}; \boldsymbol{Z}).$$

Then $B^{-1} \circ h \circ S(b) \circ h^{-1}$ is a homeomorphism, homotopic to the identity, which commutes with A_1. By virtue of Lemma 2.5, $h \circ S(b) \circ h^{-1}$ is an affine transformation. That is, we obtain a homomorphism

$$R : \boldsymbol{Z}^n \to \mathrm{Aff}_+(T^{n+1})$$

by $R(b) = h \circ S(b) \circ h^{-1}$ $(b \in \boldsymbol{Z}^n)$. Clearly R is a split hyperbolic affine representation. Now an argument in the proof of Theorem 4.2 of [**KL**] shows that h is a C^∞ diffeomorphism. The proof of Theorem 2.3 is complete. □

3. Main theorem.

Let $R : \boldsymbol{Z}^n \to \mathrm{Aff}_+(T^{n+1})$ be a split hyperbolic affine representation. There is defined a suspension manifold

$$M_R = T^{n+1} \times \boldsymbol{R}^n / (x, \xi) \sim (R(a)x, \xi - a),$$

where $a \in \boldsymbol{Z}^n$. An $\boldsymbol{R}^n$-action $(x, \xi) \cdot \xi' = (x, \xi + \xi')$ on $T^{n+1} \times \boldsymbol{R}^n$ induces an $\boldsymbol{R}^n$-action ψ_R on M_R.

Now let us state the main theorem of this paper.

THEOREM 3.1. *Any split Anosov $\boldsymbol{R}^n$-action ψ on a closed oriented $2n+1$ dimensional manifold M is C^∞ conjugate to the action ψ_R on M_R for some split hyperbolic affine representation R, i.e. there are a C^∞ diffeomorphism $h : M \to M_R$ and an automorphism Φ of $\boldsymbol{R}^n$ such that $h \circ \psi^\xi = \psi_R^{\Phi(\xi)} \circ h$ for any $\xi \in \boldsymbol{R}^n$.*

We will use the notations in Definition 1.1. Let us define four foliations associated with the flow $\psi^{\xi_i t}$.

: The strong unstable foliation $\mathscr{V}_i^u$, tangent to E_i.
: The weak unstable foliation $\mathscr{W}_i^u$, tangent to $T\boldsymbol{R}^n \oplus E^i$.
: The strong stable foliation $\mathscr{V}_i^s$, tangent to $E_1 \oplus \cdots \oplus E_{i-1} \oplus E_{i+1} \oplus \cdots \oplus E_{n+1}$.
: The weak stable foliation $\mathscr{W}_i^s$, tangent to $T\boldsymbol{R}^n \oplus E_1 \oplus \cdots \oplus E_{i-1} \oplus E_{i+1} \oplus \cdots \oplus E_{n+1}$.

REMARK 3.2. All the above foliations are continuous foliations by C^∞ leaves and the C^∞ structures of leaves vary continuously along the transverse direction. The tangential distributions of these foliations are Hölder continuous.

For the first statement, see [**HPS**]. The last statement can be shown using an argument in Section 19.1 of [**KH**]. Notice also the action ψ^ξ preserves all the above foliations.

LEMMA 3.3. *There is a C^∞ foliation $\mathscr{F}$ tangent to $E_1 \oplus \cdots \oplus E_{n+1}$ and invariant by ψ^ξ $(\forall \xi \in \boldsymbol{R}^n)$.*

PROOF. First of all by an argument analogous to that for Anosov flows one can verify that two points x and y lie on the same leaf of $\mathscr{V}_i^s$ if and only if $d(\psi^{\xi_i t}(x), \psi^{\xi_i t}(y))$ tends to 0 as t tends to ∞. This shows that $\mathscr{V}_i^s$ is the unique foliation tangent to $E_1 \oplus \cdots \oplus E_{i-1} \oplus E_{i+1} \oplus \cdots \oplus E_{n+1}$, and therefore that $\mathscr{V}_j^u$ is a subfoliation of $\mathscr{V}_i^s$ if $i \neq j$.

Consider the two dimensional foliation $\mathscr{G}_{1,2}$ obtained as the transverse intersection of $\mathscr{V}_3^s, \mathscr{V}_4^s, \ldots, \mathscr{V}_{n+1}^s$. Then the two one dimensional foliations $\mathscr{V}_1^u$ and $\mathscr{V}_2^u$ are subfoliations of $\mathscr{G}_{1,2}$. Let φ_i^t be a nonsingular flow tangent to $\mathscr{V}_i^u$ such that $\|(d/dt)\varphi^t(x)\| = 1$ for any $t \in \boldsymbol{R}$ and $x \in M$. Then for any $x \in M$ and any small $s, t \in \boldsymbol{R}$, there are $s' = s'(x, s, t)$ and $t' = t'(x, s, t)$ such that

$$\varphi_1^t \varphi_2^s(x) = \varphi_2^{s'} \varphi_1^{t'}(x).$$

Of course the same thing holds for any other pair of indices.

Let $B(a)$ be a small open metric ball centered at $a \in M$. For any permutation σ of the indices $1, 2, \ldots, n+1$, define a submanifold $F(a)$ of $B(a)$ consisting of those points x which can be represented as

$$x = \varphi_{\sigma(1)}^{t_1} \varphi_{\sigma(2)}^{t_2} \cdots \varphi_{\sigma(n+1)}^{t_{n+1}}(a), \quad |t_i| < k,$$

for some $k > 0$. Choosing the ball $B(a)$ and the constant k appropriately, one gets that $F(a)$ is a connected subset of $B(a)$, which is independent of the choice of the permutation σ. This shows that $F(a)$ is tangent to $E_1 \oplus \cdots \oplus E_{n+1}$. It is clear from the construction that the family $\{F(a) \mid a \in M\}$ defines a foliation, which is denoted by $\mathscr{F}$. Notice that for any $\xi \in \boldsymbol{R}^n$, the map ψ^ξ preserves the foliation $\mathscr{F}$. For small $\varepsilon > 0$, let

$$D(a; \varepsilon) = \{y \in M \mid \psi^{-\xi}(y) \in F(a), |\xi| < \varepsilon\}.$$

Define a map $p : D(a; \varepsilon) \to \boldsymbol{R}^n$ by $p(y) = \xi$ if $\psi^{-\xi}(y) \in F(a)$.

Now according to [**J**], a continuous function on a C^∞ manifold is C^∞ if it is C^∞ along leaves of two transverse foliations of complementary dimension which satisfy the conditions of Remark 3.2.

Therefore on each leaf of $\mathscr{W}_1^u$, the function p is C^∞. On the other hand on each leaf of $\mathscr{V}_1^s$, the function p is constant. This shows that p is a C^∞ function, proving that $\mathscr{F}$ is a C^∞ foliation. □

Let $X_1, \dots, X_n$ be a linear basis of $\boldsymbol{R}^n$. Via the action ψ, they induce vector fields on M, which are denoted by the same letters by some abuse. Let Y be an arbitrary vector field tangent to the foliation $\mathscr{F}$. Then $[X_i, Y]$ is also tangent to $\mathscr{F}$, since $\mathscr{F}$ is preserved by ψ^ξ. Define 1-forms $\omega_1, \dots, \omega_n$ by

$$\omega_i|_{T\mathscr{F}} = 0, \quad \omega_i(X_j) = \delta_{ij}, \quad i, j = 1, 2, \dots, n,$$

where $T\mathscr{F}$ denotes the tangent bundle of $\mathscr{F}$. They are linearly independent closed 1-forms and the foliation $\mathscr{F}$ is characterized by

$$T\mathscr{F} = \mathrm{Ker}(\omega_1 \wedge \cdots \wedge \omega_n).$$

Let ω_i' be a nonsingular closed 1-form near ω_i in the C^1 topology which represents a rational cohomology class. Then the foliation $\mathscr{F}'$ defined by

$$T\mathscr{F}' = \mathrm{Ker}(\omega_1' \wedge \cdots \wedge \omega_n')$$

is a bundle foliation. In particular all the leaves of $\mathscr{F}'$ are compact.

Define vector fields $X_1', \dots, X_n'$ tangent to $T\boldsymbol{R}^n$ by

$$\omega_i'(X_j') = \delta_{ij} \quad (i, j = 1, 2, \dots n).$$

They commute with each others and define an $\boldsymbol{R}^n$-action ψ'. The orbit foliation of ψ' is the same as that of ψ, and ψ'^ξ preserves the bundle foliation $\mathscr{F}'$. Since we have chosen ω_i' sufficiently close to ω_i, the $\boldsymbol{R}^n$-action ψ' is also split Anosov. Now let N be a leaf of $\mathscr{F}'$ and let

$$\Gamma = \{\xi \in \boldsymbol{R}^n \mid \psi'^\xi(N) = N\}.$$

Then Γ is a discrete cocompact subgroup of $\boldsymbol{R}^n$, and is isomorphic to $\boldsymbol{Z}^n$. To show that Γ is cocompact, notice that ψ' induces a locally free $\boldsymbol{R}^n$ action on the base space $M/\mathscr{F}$, which is an n dimensional manifold.

Now the restriction of ψ' to Γ defines a homomorphism

$$S : \Gamma \to \mathrm{Diff}_+^\infty(N),$$

which is split Anosov. Using Theorem 2.3, one can show that S is conjugate to a split hyperbolic affine representation on T^{n+1}. In particular the manifold M is a T^{n+1}-bundle over T^n with hyperbolic monodromies. Now succesive computations of Wang sequences show

$$\mathrm{Hom}(\pi_1(M), \boldsymbol{R}) = H^1(M; \boldsymbol{R}) \cong \boldsymbol{R}^n.$$

With this in mind, let us return to the study of the original foliation $\mathscr{F}$. Since $\mathscr{F}$ is defined by the closed 1-forms $\omega_1, \dots, \omega_n$, $\mathscr{F}$ is a Riemannian foliation, *i.e.* admits a holonomy invariant transverse Riemannian metric. Furthermore the foliation $\mathscr{F}$ is without holonomy, *i.e.* the leafwise holonomy group of any leaf is trivial. As a consequence $\mathscr{F}$ satisfies the following property. Let us denote by D^n the unit disc in $\boldsymbol{R}^n$. For any leaf L of $\mathscr{F}$, there is a C^∞ submersion.

$$H : L \times D^n \to M,$$

such that

: $H(x, 0) = x$ $(\forall x \in L)$,
: $H(L \times \{y\})$ $(\forall y \in D^n)$ is contained in a leaf of $\mathscr{F}$,
: $H(\{x\} \times D^n)$ $(\forall x \in L)$ is contained in an orbit of the action ψ and is a unit disc centered at x w.r.t. the holonomy invariant transverse metric.

Consider the lift $\tilde{\mathscr{F}}$ (resp. $\tilde{\omega}_i$) of $\mathscr{F}$ (resp. ω_i) to the universal covering space $\tilde{M}$ of M. Let $p_i : \tilde{M} \to \boldsymbol{R}$ be a primitive of $\tilde{\omega}_i$. The foliation $\tilde{\mathscr{F}}$ is defined by the submersion

$$p = (p_1, \dots, p_n) : \tilde{M} \to \boldsymbol{R}^n.$$

The above property of the Riemannian foliation $\mathscr{F}$ implies that p is a locally trivial bundle map onto $\boldsymbol{R}^n$. Consider the quotient space $\mathscr{M} = \tilde{M}/\tilde{\mathscr{F}}$ and the map $q : \mathscr{M} \to \boldsymbol{R}^n$ induced from p. Then q is a covering map and thus a homeomorphism. That is, the bundle $p : \tilde{M} \to \boldsymbol{R}^n$ has a connected fiber.

Since the deck transformation of $\pi_1(M)$ leaves the 1-forms $\tilde{\omega}_i$ invariant, there is defined a homomorphism

$$\theta : \pi_1(M) \to \boldsymbol{R}^n$$

such that

$$p(\gamma \cdot x) = p(x) + \theta(\gamma), \quad \gamma \in \pi_1(M), \ \ x \in \tilde{M}.$$

The image $\theta(\pi_1(M))$ must be cocompact, since otherwise the map p induces a continuous map from a compact set $M = \pi_1(M)\backslash\tilde{M}$ onto a noncompact set $\mathrm{Cl}(\theta(\pi_1(M)))\backslash\boldsymbol{R}^n$.

But this shows that the image $\theta(\pi_1(M))$ is discrete in $\boldsymbol{R}^n$, since we have $\mathrm{Hom}(\pi_1(M); \boldsymbol{R}) \cong \boldsymbol{R}^n$. That is, for any leaf L of $\mathscr{F}$, the image $p(\pi^{-1}(L))$ is discrete, where π is the universal covering map. This implies that the leaves of the foliation $\mathscr{F}$ are compact. Applying the previous argument once again to the action ψ, the proof of Theorem 3.1 is complete.

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