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Residues for non-transversality of a holomorphic map to a codimension one holomorphic foliation

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Abstract. For a holomorphic map $F : X \longrightarrow Y$ and a non-singular foliation on Y, we give a residue formula which measures singularities of the lifted foliation on X by applying the Baum-Bott residue formula.

1. Introduction.

In this paper we study residues for non-transversality of a holomorphic map to a non-singular foliation. Let $F : X^n \to Y^m$ be a holomorphic map between n and m-dimensional compact complex manifolds, respectively, and assume that a non-singular codimension one foliation on Y is given. In general, the pull-back of the foliation via F is singular along the locus where F does not satisfy the transversality condition to leaves in Y. This is one of the typical way of finding singular foliations, for instance, the simplest case is the foliation given by level sets of a holomorphic function (the case where Y is a curve with the point foliation). Usually the singularity of holomorphic foliation $\mathscr{F}$ is analyzed in terms of *the normal sheaf* $\mathscr{N}_{\mathscr{F}}$ (cf. Baum-Bott [**BB**], Suwa [**S2**]), while *the conormal sheaf* $\mathscr{G}$ attracts less attention so far. However, the conomal sheaf behaves better than the normal sheaf with respect to the pull back operation. Thus, we focus on the relation between $\mathscr{N}_{\mathscr{F}}$ and $\mathscr{G}$ to define a certain residue for *the non-transversality of* F, precisely, the Baum-Bott residue for the lifted singular foliation on X. We then give a simple residue formula for the top Chern class: for instance, when the singular locus consists of isolated points p_i's, our residue is expressed by the sum of the Grothendieck residue for local defining equations around p_i (see Theorem 3.3, 3.4 below),

$$\int_X c_n(\Omega_X \otimes \mathscr{G}^\vee) = \sum_i \mathrm{Res}_{p_i} \begin{bmatrix} df_1^{(i)} \wedge \cdots \wedge df_n^{(i)} \\ f_1^{(i)}, \ldots, f_n^{(i)} \end{bmatrix}.$$

This is actually a natural extension of multiplicity formulas for singularities of holomorphic functions (over a non-singular source space) studied in [**IS**]. The proof follows directly from our definition and formal computation of the Chern character and the Baum-Bott formula for the normal sheaf $\mathscr{N}_{\mathscr{F}}$ [**BB**]. Another proof without using the Baum-Bott formula for $\mathscr{N}_{\mathscr{F}}$ is discussed in [**I**].

In Section 2, we state some basic notions of singular foliations. In Section 3, we discuss residue formulas in terms of normal and conormal sheaves. The main formula of

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this paper is in this section. In the last two sections, a few applications and an example are given.

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2. Preliminaries.

2.1. Singular holomorphic foliations.

In this subsection, we shall give some general notation and definitions on singular holomorphic foliations. For details we refer to [**BB**] and [**S2**]. Let X be a complex manifold and $\mathscr{O}_X$ the structure sheaf of X. We define a singular holomorphic foliation $\mathscr{F}$ on X to be a coherent subsheaf of the tangent sheaf Θ_X which is an involutive differential system. We call $\mathscr{F}$ *the tangent sheaf* of the foliation. We say $\mathscr{F}$ is of dimension p if a generic stalk of $\mathscr{F}$ is a rank-p free $\mathscr{O}_{X,x}$-module. We also define *the normal sheaf* $\mathscr{N}_{\mathscr{F}}$ of $\mathscr{F}$ by the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \Theta_X \longrightarrow \mathscr{N}_{\mathscr{F}} \longrightarrow 0.$$

The singular set $S(\mathscr{F})$ of $\mathscr{F}$ is defined by $S(\mathscr{F}) = \{x \in X : \mathscr{N}_{\mathscr{F},x} \text{ is not } \mathscr{O}_{X,x}\text{-free}\}$.

We can also give a definition of a singular holomorphic foliation $\mathscr{G}$ on X to be a coherent subsheaf of the cotangent sheaf Ω_X which satisfies the Frobenius integrability condition. We call $\mathscr{G}$ *the conormal sheaf* of the foliation. We also say $\mathscr{G}$ is of codimension q if the generic rank is q. We denote by $\Omega_{\mathscr{G}}$ the quotient sheaf given by

$$0 \longrightarrow \mathscr{G} \longrightarrow \Omega_X \longrightarrow \Omega_{\mathscr{G}} \longrightarrow 0.$$

The singular set $S(\mathscr{G})$ of $\mathscr{G}$ is also defined by $S(\mathscr{G}) = \{x \in X : \Omega_{\mathscr{G},x} \text{ is not } \mathscr{O}_{X,x}\text{-free}\}$.

If $\mathscr{F}$ is reduced (or full) (see [**S2**] or [**BB**]), the tangent sheaf $\mathscr{F}$ and its annihilator $\mathscr{G}$ define equivalent foliations such that $S(\mathscr{F}) = S(\mathscr{G})$.

From now on, we concentrate ourselves on codimension one locally free singular foliations. (i.e., $\mathscr{G}$ is of rank one.) We may express $\mathscr{G}$ as follows. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of X and a collection ω of 1-forms ω_a on U_α satisfying the integrability conditions $\omega_a \wedge d\omega_\alpha = 0$. The forms ω_a are patched by the transition relations $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on $U_\alpha \cap U_\beta$. Then, $\mathscr{G}$ is defined as the locally free subsheaf of Ω_X given by the cocycle $(g_{\alpha\beta})$. We sometimes identify the collection ω of 1-forms with the conormal sheaf $\mathscr{G}$ of ω as a codimension one foliation.

2.2. Transversality condition.

Let $F : X^n \longrightarrow Y^m$ be a holomorphic map, $\tilde{\mathscr{G}}$ (the conormal sheaf of) a codimension one non-singular foliation on Y. Put $\mathscr{G} = F^{-1}\tilde{\mathscr{G}}$, the inverse image of $\tilde{\mathscr{G}}$, which defines a possibly singular foliation on X. In this case the transversality condition is described as follows. By using the Frobenius integrability condition, non-singular foliation $\tilde{\mathscr{G}}$ on Y is locally given by a Pfaffian equation $dy_m = 0$ for some local coordinates $(y_1, \ldots, y_m)$ of Y. For a local coordinate $(x_1, \ldots, x_n)$ of X, if we write F as a system of m functions

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} F_1(x_1, \ldots, x_n) \\ \vdots \\ F_m(x_1, \ldots, x_n) \end{pmatrix},$$

then the lifted foliation $\mathscr{G}$ on X is given by the collection of 1 forms $\omega = (F^* dy_m) = (dF_m)$. The non-transversal loci of F to $\tilde{\mathscr{G}}$ is nothing but the singular set of $\mathscr{G}$:

$$S(\mathscr{G}) = \{x \in X : dF_m(x) = 0\}.$$

3. Residue formula.

3.1. Locally free resolution of $\mathscr{N}_{\mathscr{F}} \otimes \mathscr{G}$.

Let $\omega = (\omega_\alpha)$ be a codimension one locally free singular holomorphic foliation and $\mathscr{G}$ the conormal sheaf of ω. We denote by $\mathscr{F} = \{v \in \Theta_X : \langle v, \omega \rangle = 0\}$ the annihilator of $\mathscr{G}$. To compute the Chern character of the coherent sheaf $\mathscr{N}_{\mathscr{F}}$, we construct a locally free resolution of $\mathscr{N}_{\mathscr{F}}$. Since ω can be regarded as a homomorphism $\iota_\omega : \mathscr{G} \longrightarrow \Omega_X$, it defines a global section $\iota_\omega \in H^0(X, \mathscr{H}om(\mathscr{G}, \Omega_X)) \simeq H^0(X, \Omega_X \otimes \mathscr{G}^\vee)$. Here $\mathscr{G}^\vee = \mathscr{H}om(\mathscr{G}, \mathscr{O}_X)$ is the dual sheaf of $\mathscr{G}$. Let $(x_1, \ldots, x_n)$ be a local coordinate of X and let $s_\alpha^\vee$ be a local frame for $\mathscr{G}^\vee$ on U_α. Then the restriction ι_{ω_α} of ι_ω to U_α is written as $\iota_{\omega_\alpha} = \sum f_i(dx_i \otimes s_\alpha^\vee)$ for some functions $f_1, \ldots, f_n$. We refer $(f_1, \ldots, f_n)$ as the local coefficients of ω (or also of ι_ω). We also regard ι_ω as the "contraction operator" $\iota_\omega : \Theta_X \otimes \mathscr{G} \longrightarrow \mathscr{O}_X$. We denote by $\mathscr{I}_\omega$ the ideal sheaf defined by the image $\mathrm{Im}(\iota_\omega : \Theta_X \otimes \mathscr{G} \longrightarrow \mathscr{O}_X)$. Let us relate the above $\mathscr{I}_\omega$ and the normal sheaf $\mathscr{N}_{\mathscr{F}}$. Since $\mathscr{G}$ is locally free, by applying $\otimes \mathscr{G}$ to the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \Theta_X \longrightarrow \mathscr{N}_{\mathscr{F}} \longrightarrow 0,$$

the following sequence

$$0 \longrightarrow \mathscr{F} \otimes \mathscr{G} \longrightarrow \Theta_X \otimes \mathscr{G} \longrightarrow \mathscr{N}_{\mathscr{F}} \otimes \mathscr{G} \longrightarrow 0$$

is also exact. Since the kernel of $\iota_\omega : \Theta_X \otimes \mathscr{G} \longrightarrow \mathscr{O}_X$ is equal to $\mathscr{F} \otimes \mathscr{G}$, we have

$$\mathscr{I}_\omega \simeq (\Theta_X \otimes \mathscr{G})/(\mathscr{F} \otimes \mathscr{G}) \simeq \mathscr{N}_{\mathscr{F}} \otimes \mathscr{G}.$$

We assume that $S(\mathscr{G})$ consists of isolated points only. Since the local coefficients $(f_1, \ldots, f_n)$ of ω_α is a regular sequence near a singular point $p_\alpha \in U_\alpha$, the Koszul complex of sheaves

$$0 \to \wedge^n(\Theta_X \otimes \mathscr{G}) \to \wedge^{n-1}(\Theta_X \otimes \mathscr{G}) \to \cdots \to \wedge^1(\Theta_X \otimes \mathscr{G}) \to \mathscr{O}_X \to 0$$

is exact except for the last term with the boundary operator

$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i-1} f_i e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_p,$$

where $\{e_1, \ldots, e_n\}$ is a local frame of $\Theta_X \otimes \mathscr{G}$. (See [**GH**] or [**FL**].) Thus by replacing the last term by $\mathscr{I}_\omega = \mathscr{N}_{\mathscr{F}} \otimes \mathscr{G}$, it gives the Koszul resolution of $\mathscr{N}_{\mathscr{F}} \otimes \mathscr{G}$ as

$$0 \to \wedge^n(\Theta_X \otimes \mathscr{G}) \to \wedge^{n-1}(\Theta_X \otimes \mathscr{G}) \to \cdots \to \wedge^1(\Theta_X \otimes \mathscr{G}) \to \mathscr{N}_{\mathscr{F}} \otimes \mathscr{G} \to 0.$$

3.2. Chern classes of $\mathscr{N}_{\mathscr{F}}$.

This subsection is devoted to a simple computation of $c_n(\mathscr{N}_{\mathscr{F}})$. From the definition of the characteristic classes of coherent sheaves, we write

$$\mathrm{ch}(\mathscr{N}_{\mathscr{F}} \otimes \mathscr{G}) = \mathrm{ch}\left(\sum_{i=1}^{n}(-1)^{i-1} \wedge^i (\Theta_X \otimes \mathscr{G})\right).$$

By using Theorem 10.1.1 of [**H**], we have

$$\begin{aligned}
\mathrm{ch}\left(\sum_{i=1}^{n}(-1)^{i-1} \wedge^i (\Theta_X \otimes \mathscr{G})\right) &= \mathrm{ch}\left(-\left(\sum_{i=0}^{n}(-1)^{i} \wedge^i (\Theta_X \otimes \mathscr{G})\right) + \mathscr{O}_X\right) \\
&= \mathrm{ch}(\mathscr{O}_X) - \mathrm{ch}\left(\sum_{i=0}^{n}(-1)^{i} \wedge^i (\Theta_X \otimes \mathscr{G})\right) \\
&= 1 - \mathrm{td}^{-1}(\Omega_X \otimes \mathscr{G}^\vee) c_n(\Omega_X \otimes \mathscr{G}^\vee) \\
&= 1 - c_n(\Omega_X \otimes \mathscr{G}^\vee),
\end{aligned}$$

where the last equality holds for dimensional reason. Now we have the Chern character of $\mathscr{N}_{\mathscr{F}}$ as

$$\mathrm{ch}(\mathscr{N}_{\mathscr{F}}) = \big(1 - c_n(\Omega_X \otimes \mathscr{G}^\vee)\big)\,\mathrm{ch}(\mathscr{G}^\vee).$$

Next, we compute the Chern classes of $\mathscr{N}_{\mathscr{F}}$. We denote by ch_i the terms of i-th degree in ch. We see from the above equality that $\mathrm{ch}_i(\mathscr{N}_{\mathscr{F}}) = \frac{1}{i!}c_1(\mathscr{G}^\vee)^i$ for $i \leq n-1$ and $\mathrm{ch}_n(\mathscr{N}_{\mathscr{F}}) = \frac{1}{n!}c_1(\mathscr{G}^\vee)^n - c_n(\Omega_X \otimes \mathscr{G}^\vee)$.

First, $c_1(\mathscr{N}_{\mathscr{F}}) = c_1(\mathscr{G}^\vee)$ is obvious. Next we show that $c_2(\mathscr{N}_{\mathscr{F}}) = 0$. From the Newton formula

$$2!\,\mathrm{ch}_2(\mathscr{N}_{\mathscr{F}}) - \mathrm{ch}_1(\mathscr{N}_{\mathscr{F}})c_1(\mathscr{N}_{\mathscr{F}}) + 2c_2(\mathscr{N}_{\mathscr{F}}) = 0,$$

we see that

$$2!\,\frac{1}{2!}c_1(\mathscr{G}^\vee)^2 - c_1(\mathscr{G}^\vee)c_1(\mathscr{G}^\vee) + (-1)^2 2c_2(\mathscr{N}_{\mathscr{F}}) = 0,$$

which implies $c_2(\mathscr{N}_{\mathscr{F}}) = 0$.

In the same way, we see $c_i(\mathscr{N}_{\mathscr{F}}) = 0$ for $2 \leq i \leq n-1$. Indeed, assuming that $c_2(\mathscr{N}_{\mathscr{F}}) = \cdots = c_{i-1}(\mathscr{N}_{\mathscr{F}}) = 0$, we have

$$
\begin{aligned}
0 &= i!\,\mathrm{ch}_i(\mathscr{N}_{\mathscr{F}}) - \mathrm{ch}_{i-1}(\mathscr{N}_{\mathscr{F}})c_1(\mathscr{N}_{\mathscr{F}}) + \mathrm{ch}_{i-2}(\mathscr{N}_{\mathscr{F}})c_2(\mathscr{N}_{\mathscr{F}}) + \cdots + (-1)^i\, i\, c_i(\mathscr{N}_{\mathscr{F}}) \\
&= i!\,\frac{1}{i!}c_1(\mathscr{G}^\vee)^i - c_1(\mathscr{G}^\vee)^{i-1}c_1(\mathscr{N}_{\mathscr{F}}) + (-1)^i\, i\, c_i(\mathscr{N}_{\mathscr{F}}) \\
&= (-1)^i\, i\, c_i(\mathscr{N}_{\mathscr{F}}).
\end{aligned}
$$

Finally we also see

$$
\begin{aligned}
0 &= n!\,\mathrm{ch}_n(\mathscr{N}_{\mathscr{F}}) - \mathrm{ch}_{n-1}(\mathscr{N}_{\mathscr{F}})c_1(\mathscr{N}_{\mathscr{F}}) + \mathrm{ch}_{n-2}(\mathscr{N}_{\mathscr{F}})c_2(\mathscr{N}_{\mathscr{F}}) + \cdots + (-1)^n\, n\, c_n(\mathscr{N}_{\mathscr{F}}) \\
&= n!\left(\frac{1}{n!}c_1(\mathscr{G}^\vee)^n - c_n(\Omega_X \otimes \mathscr{G}^\vee)\right) - c_1(\mathscr{G}^\vee)^{n-1}c_1(\mathscr{N}_{\mathscr{F}}) + (-1)^n\, n\, c_n(\mathscr{N}_{\mathscr{F}}) \\
&= (-1)^n\, n\, c_n(\mathscr{N}_{\mathscr{F}}) - n!c_n(\Omega_X \otimes \mathscr{G}^\vee)
\end{aligned}
$$

which implies $c_n(\mathscr{N}_{\mathscr{F}}) = (-1)^n(n-1)!\,c_n(\Omega_X \otimes \mathscr{G}^\vee)$.

We summarize the above computations as follows.

Proposition 3.1.

$$
\begin{gathered}
c_1(\mathscr{N}_{\mathscr{F}}) = c_1(\mathscr{G}^\vee) \\
c_2(\mathscr{N}_{\mathscr{F}}) = \cdots = c_{n-1}(\mathscr{N}_{\mathscr{F}}) = 0
\end{gathered}
$$

and

$$
c_n(\mathscr{N}_{\mathscr{F}}) = (-1)^n(n-1)!\,c_n(\Omega_X \otimes \mathscr{G}^\vee).
$$

3.3. Residue formula.

As observed in Proposition 3.1, for a codimension one foliation we only have $\alpha\, c_n(\mathscr{N}_{\mathscr{F}}) + \beta\, c_1(\mathscr{N}_{\mathscr{F}})^n$ as the n-th symmetric polynomial $\varphi(\mathscr{N}_{\mathscr{F}})$. Thus by applying the Baum-Bott residue formula we have

$$
\int_X \left(\alpha\, c_n(\mathscr{N}_{\mathscr{F}}) + \beta\, c_1(\mathscr{N}_{\mathscr{F}})^n\right) = \alpha \operatorname{Res}_{c_n}(\mathscr{N}_{\mathscr{F}}, S(\mathscr{F})) + \beta \operatorname{Res}_{c_1^n}(\mathscr{N}_{\mathscr{F}}, S(\mathscr{F})),
$$

where Res_{c_n} and $\operatorname{Res}_{c_1^n}$ are the Baum-Bott residues.

Let us also compute the local residues. For $\varphi = c_n$, we can see that Res_{c_n} is given by the Grothendieck residue. First we assume that the singular set $S(\mathscr{G})$ consists only of p. Here we recall the following result in [**S1**].

Proposition 3.2 ([**S1**]). *For an $(n-1)$-dimensional locally free foliation $\mathscr{F}$, the Baum-Bott residue is given by*

$$
\operatorname{Res}_{c_n}(\mathscr{N}_{\mathscr{F}}, p) = (-1)^n(n-1)!\dim \operatorname{Ext}^1_{\mathscr{O}_p}(\Omega_{\mathscr{G},p}, \mathscr{O}_p).
$$

We can observe the dimension of $\operatorname{Ext}^1_{\mathscr{O}_p}(\Omega_{\mathscr{G},p}, \mathscr{O}_p)$ more precisely as follows. Let us consider the dual of the exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \Omega_X \longrightarrow \Omega_{\mathscr{G}} \longrightarrow 0.$$

Then we have

$$0 \longrightarrow \mathscr{F} \longrightarrow \Theta_X \longrightarrow \mathscr{G}^\vee \longrightarrow \mathscr{E}xt^1_{\mathscr{O}_X}(\Omega_{\mathscr{G}}, \mathscr{O}_X) \longrightarrow 0.$$

Thus by applying $\otimes\mathscr{G}$, we also have

$$0 \longrightarrow \mathscr{I}_\omega \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E}xt^1_{\mathscr{O}_X}(\Omega_{\mathscr{G}}, \mathscr{O}_X) \otimes \mathscr{G} \longrightarrow 0.$$

We note that the supports of the sheaves $\mathscr{O}_X/\mathscr{I}_\omega$ and $\mathscr{E}xt^1_{\mathscr{O}_X}(\Omega_{\mathscr{G}}, \mathscr{O}_X)$ are the singular set of $\mathscr{G}$. Thus in the isolated singular case, they are sky-scraper sheaves. Therefore the isomorphism $\mathscr{O}_X/\mathscr{I}_\omega \simeq \mathscr{E}xt^1_{\mathscr{O}_X}(\Omega_{\mathscr{G}}, \mathscr{O}_X) \otimes \mathscr{G}$ gives

$$\dim \operatorname{Ext}^1_{\mathscr{O}_p}(\Omega_{\mathscr{G},p}, \mathscr{O}_p) = \dim(\mathscr{O}_p/\mathscr{I}_{\omega,p}) = \operatorname{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix},$$

where $(f_1, \dots, f_n)$ is the local coefficient of ω and $\operatorname{Res}_p \left[\begin{smallmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{smallmatrix}\right]$ is the Grothendieck residue symbol (see [**O**] or Chapter 5 of [**GH**]). The Baum-Bott residue formula for $\mathscr{N}_{\mathscr{F}}$ [**BB**] is translated as follows.

Theorem 3.3 (Baum-Bott residue formula for codimension one foliations). *Let ω be a codimension one foliation with conormal sheaf $\mathscr{G}$, and $\mathscr{F}$ the annihilator of $\mathscr{G}$. We suppose that $S(\mathscr{G}) = \{p_1, \dots, p_k\}$ and we write $\iota_\omega = \sum f_i^{(j)}(dx_i \otimes s^\vee)$ as a section of $\Omega_X \otimes \mathscr{G}^\vee$ near p_j with a local frame $s^\vee$ of $\mathscr{G}^\vee$. Then we have*

$$\int_X c_n(\Omega_X \otimes \mathscr{G}^\vee) = \sum_{j=1}^k \operatorname{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \dots, f_n^{(j)} \end{bmatrix},$$

and for the Baum-Bott residue $\operatorname{Res}_{c_1^n}(\mathscr{N}_{\mathscr{F}}, S(\mathscr{F}))$,

$$\int_X c_1(\mathscr{G}^\vee)^n = \operatorname{Res}_{c_1^n}(\mathscr{N}_{\mathscr{F}}, S(\mathscr{F})).$$

Proof. This is given by

$$\begin{aligned} \int_X c_n(\Omega_X \otimes \mathscr{G}^\vee) &= \frac{(-1)^n}{(n-1)!} \int_X c_n(\mathscr{N}_{\mathscr{F}}) \\ &= \sum_{j=1}^k \dim \operatorname{Ext}^1_{\mathscr{O}_{p_j}}(\Omega_{\mathscr{G},p_j}, \mathscr{O}_{p_j}) \\ &= \sum_{j=1}^k \operatorname{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \dots, f_n^{(j)} \end{bmatrix}. \end{aligned}$$

□

Now we give the main result of this paper. Let $F : X^n \longrightarrow Y^m$, $\tilde{\mathscr{G}}$, $\mathscr{G} = F^{-1}\tilde{\mathscr{G}}$ be as in Subsection 2.2.

We assume that $S(\mathscr{G})$ consists of isolated points $\{p_1, \ldots, p_k\}$. Let $F_m^{(j)}$ be a function with which $dF_m^{(j)}$ defines $\mathscr{G}$ near p_j as above and set $f_i^{(j)} = \frac{\partial F_m^{(j)}}{\partial x_i}$ so that $dF_m^{(j)} = f_1^{(j)} dx_1 + \cdots + f_n^{(j)} dx_n$. Then we have

$$\int_X c_n(\Omega_X \otimes \mathscr{G}^\vee) = (-1)^n \biggl(\int_X c_n(X) + \sum_{l=1}^n \int_X c_{n-l}(\Theta_X) c_1(\mathscr{G})^l \biggr)$$
$$= \sum_{j=1}^k \operatorname{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.$$

Here we have the main result.

Theorem 3.4 (residue formula for non-transversality). *Let $F : X^n \longrightarrow Y^m$ be a holomorphic map of generic rank-r and $\tilde{\mathscr{G}}$ a codimension one non-singular foliation on Y. We assume that the non-transversal points of F to $\tilde{\mathscr{G}}$ are $\{p_1, \ldots, p_k\}$, then we have*

$$\chi(X) + \sum_{l=1}^r \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathscr{G}})^l = (-1)^n \sum_{j=1}^k \operatorname{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.$$

Proof. We denote by X^* the set of generic points where the rank of F is r. By using the projection formula,

$$\int_X c_{n-l}(\Theta_X) c_1(\mathscr{G})^l = \int_{X^*} c_{n-l}(\Theta_X) F^*(c_1(\tilde{\mathscr{G}})^l) = \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathscr{G}})^l.$$

It is obvious that the above terms are zero for $r \leq l$. □

4. Application.

4.1. Multiplicity formula.

As an example, let us consider the case where $F : X^n \longrightarrow C$ is a map for a curve C and $\tilde{\mathscr{G}} = \Omega_C$. Then the above formula implies the multiplicity formula [**IS**], [**F**], [**IV**].

Theorem 4.1 (the multiplicity formula). *Let $F : X^n \longrightarrow C$ be a holomorphic map onto a compact complex curve C with the generic fiber M_F. If F has finite number of isolated critical points $\{p_1, \ldots, p_k\}$, then for $n \geq 2$, we have*

$$\chi(X) - \chi(M_F)\chi(C) = (-1)^n \sum_{j=1}^k \mu(F, p_j)$$

where $\mu(F, p_j)$ is the Milnor number of F at p_j.

Proof. It is clear that the residues which appear in the right hand side of the Theorem 3.4 are $\dim(\mathscr{O}/\mathscr{I}_{dF}) = \mu(F, p_i)$. The left hand side of the Theorem 3.4 is written by

$$\chi(X) + \big(c_{n-1}(\Theta_X)c_1(\mathscr{G}) \frown [X]\big).$$

We have

$$\begin{aligned} c_{n-1}(\Theta_X)c_1(\mathscr{G}^\vee) &= \left(c_{n-1}(\Theta_F) + \sum_{l\geq 2} c_{n-l}(\Theta_F)c_1(\mathscr{G}^\vee)^{l-1}\right)c_1(\mathscr{G}^\vee) \\ &= -c_{n-1}(\Theta_F)c_1(F^{-1}\Omega_C) + \sum_{l\geq 2}(-1)^l c_{n-l}(\Theta_F)c_1(F^{-1}\Omega_C)^l \\ &= -c_{n-1}(\Theta_F)F^*c_1(\Omega_C) + \sum_{l\geq 2}(-1)^l c_{n-l}(\Theta_F)F^*(c_1(\Omega_C)^l) \\ &= c_{n-1}(\Theta_F)F^*c_1(\Theta_C). \end{aligned}$$

Hence

$$c_{n-1}(\Theta_X)c_1(\mathscr{G}) = -c_{n-1}(\Theta_F)F^*c_1(\Theta_C).$$

Since the regular leaves of $\mathscr{G}$ are the generic fibers ($\iota : M_F \longrightarrow X$), we see that $\iota^{-1}\Theta_F$ is the tangent sheaf of M_F. Thus by applying the projection formula for $\iota^*F : M_F \longrightarrow p$, we have

$$\int_{M_F} c_{n-1}(\Theta_F) = \chi(M_F)1_C(p)$$

for almost everywhere $p \in C$. Now we obtain

$$\begin{aligned} \int_X c_{n-1}(\Theta_F)F^*c_1(\Theta_C) &= \chi(M_F)\int_C c_1(\Theta_C) \\ &= \chi(M_F)\chi(C) \end{aligned}$$

which prove the formula. □

Remark 4.2.
1. In the above proof, arguments on the application of the projection formula is a little bit rough. We can give a concrete proof on the projection formula without measure zero sets also by applying the Čech-de Rham techniques (see [**IS**]).
2. The above arguments cannot be applied to the case that X is a Riemann surface since one dimensional foliations defined as the annihilator of codimension one foliations are always non-singular.

4.2. The surface case.

Let $F : X^2 \longrightarrow Y^m$ be a map from a compact complex surface. In this case we write down the formula in Theorem 4.1 into geometric form. We set that $y_m = F_m^{(j)}(x_1, x_2)$ to be the m-th entry of a local representation of F near p_j and write $dF_m^{(j)} = f_1^{(j)} dx_1 + f_2^{(j)} dx_2$. Then the above formula is

$$\chi(X) + F_*(c_1(X) \frown [X]) \frown c_1(\tilde{\mathscr{G}}) + F_*[X] \frown c_1(\tilde{\mathscr{G}})^2 = \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge df_2^{(j)} \\ f_1^{(j)}, f_2^{(j)} \end{bmatrix}.$$

If the generic rank of F is one, the last term in the left-hand side of the above vanishes and we have

$$\chi(X) + \chi(M_F)\big(F_*[X] \frown c_1(\tilde{\mathscr{G}})\big) = \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge df_2^{(j)} \\ f_1^{(j)}, f_2^{(j)} \end{bmatrix},$$

where M_F is the generic fiber of F.

Remark 4.3. In [**T**], Torii gave the above results by computing $c(\mathscr{N}_F) = c(\Theta_X - \mathscr{F})$ directly using the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \Theta_X \longrightarrow \mathscr{N}_{\mathscr{F}} \longrightarrow 0$$

of the annihilator $\mathscr{F}$ of $\mathscr{G} = F^{-1}\tilde{\mathscr{G}}$. Then $c_2(\mathscr{N}_F) = c_1(\Theta_X)c_1(\mathscr{F}) + c_2(\Theta_X)$ is given and by applying the adjunction formula $\mathscr{G} = \mathscr{F} \otimes K_X$ on surfaces, the formula for the surface cases are also proved.

5. Example.

Let $Y = \boldsymbol{C}P^3 \times \boldsymbol{C}P^1$. We set $([X_0 : X_1 : X_2 : X_3], [Y_0 : Y_1])$ the homogeneous coordinates of Y. We give a non-singular foliation $\tilde{\mathscr{G}} = \pi^{-1}\Omega_{\boldsymbol{C}P^1}$ as the trivial extension via the natural projection $\pi : \boldsymbol{C}P^3 \times \boldsymbol{C}P^1 \longrightarrow \boldsymbol{C}P^1$. Let

$$X = V\big({X_0}^l + {X_1}^l + {X_2}^l + {X_3}^l\big) \cap V(X_0Y_0 + X_1Y_1)$$

be the non-singular subvariety of Y. We consider the inclusion map $F = \iota : X \hookrightarrow Y$.

To see the non-transversal points, we take the inhomogeneous coordinates over $X_0 \neq 0$ and $Y_0 \neq 0$ as $(s, x, y) = \big(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \frac{X_3}{X_0}\big)$ and $z = \frac{Y_1}{Y_0}$. Then a local coordinate of X are given by (x, y) such that the local defining equations of X are $1 + s^l + x^l + y^l = 0$ and $1 + sz = 0$. For the function $z = (-1)^{-\frac{1}{l}}(1 + x^l + y^l)^{-\frac{1}{l}}$, we set that

$$\varphi = \frac{\partial z}{\partial x} = (-1)^{-\frac{1}{l}} x^{l-1}(1 + x^l + y^l)^{-\frac{l+1}{l}}$$
$$\psi = \frac{\partial z}{\partial y} = (-1)^{-\frac{1}{l}} y^{l-1}(1 + x^l + y^l)^{-\frac{l+1}{l}}.$$

Since z-axis is the transversal direction for the leaves, the non-transversal conditions are given by $\varphi = \psi = 0$ such that $(x, y) = (0, 0)$. Thus, with the defining equations, we see that the non-transversal points are

$$(s, x, y; z) = (\omega_k,\ 0,\ 0; -\omega_{l-k-1})_{k=0,\dots,l-1},$$

where we denote by ω_k the l-roots of -1. It is easily check that the non-transversal points never appear on the hyperplane $X_0 = 0$.

Now we compute the local residue. We see

$$\frac{d\varphi \wedge d\psi}{\varphi\psi} = \left((l-1)^2 + (l^2-1)\frac{x^l + y^l}{1 + x^l + y^l}\right)\frac{dx}{x} \wedge \frac{dy}{y},$$

which implies

$$\mathrm{Res} = \frac{1}{(2\pi\sqrt{-1})^2}\int_{|x|=|y|=\varepsilon} \frac{d\varphi \wedge d\psi}{\varphi\psi} = (l-1)^2.$$

Indeed, this number coincides the Milnor number μ_z of the function z at the origin. Since the number of non-transversal points is l, we have

$$\sum \mathrm{Res} = l(l-1)^2.$$

On the other hand, we also compute the global index. For the hyperplane bundle $[H_n]$ on $\boldsymbol{CP}^n$, we set $\mathscr{O}_n(k) = \mathscr{O}_{\boldsymbol{CP}^n}\big([H_n]^{\otimes k}\big)$. (On this notations, see Chapter 1 of [**GH**].) We recall that the associate line bundles of $V({X_0}^l + {X_1}^l + {X_2}^l + {X_3}^l)$ and $V(X_0Y_0 + X_1Y_1)$ are $\mathscr{O}_3(l)$ and $\mathscr{O}_3(1) \otimes \mathscr{O}_1(1)$ respectively. Thus $\mathscr{E} = \mathscr{O}_3(l) \oplus (\mathscr{O}_3(1) \otimes \mathscr{O}_1(1))$ is a global extension of the normal sheaf of X in Y:

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_Y|_X \longrightarrow \mathscr{E}|_X \longrightarrow 0.$$

Here we use the following interpretation:

$$\begin{aligned}\chi(X) + \int_{F_*(c_1(X)\frown[X])} c_1(\tilde{\mathscr{G}}) &= \int_X c_2\big(\Omega_X \otimes F^{-1}\pi^{-1}\tilde{\mathscr{G}}^\vee\big) \\ &= \int_M c_2(\mathscr{E})c_2\big((\Omega_Y - \mathscr{E}^\vee) \otimes \tilde{\mathscr{G}}^\vee\big).\end{aligned}$$

We remark that we omit the term of $c_1(\tilde{\mathscr{G}})^2$ in 4.2 since this vanishes by dimensional reason. Let $h = c_1([H_3])$ and $p = c_1([H_1])$ be the generators of $H^*(\boldsymbol{CP}^3 \times \boldsymbol{CP}^1)$ with the relation $h^4 = p^2 = 0$. We note that the total Chern classes of locally free sheaves appears in the above integration are $c(\Omega_M) = (1+h)^4(1+p)^2$, $c(\mathscr{E}) = (1+lh)(1+h+p)$ and $c(\tilde{\mathscr{G}}) = (1-2p)$. Then the quadratic terms of

$$c\big((\Omega_Y - \mathscr{E}^\vee) \otimes \tilde{\mathscr{G}}^\vee\big) = \frac{c(\Omega_M \otimes \tilde{\mathscr{G}}^\vee)}{c(\mathscr{E}^\vee \otimes \tilde{\mathscr{G}}^\vee)}$$

$$= \frac{1 + (6p - 4h) + (6h^2 - 16ph) + (\text{higher terms})}{(1 + (2p + lh))(1 + (p - h))}$$

are

$$c_2\big((\Omega_Y - \mathscr{E}^\vee) \otimes \tilde{\mathscr{G}}^\vee\big) = (l-2)ph + (l^2 - 3l + 3)h^2.$$

We also have

$$c_2(\mathscr{E}) = lh(p+h).$$

Therefore we see

$$c_2(\mathscr{E})c_2\big((\Omega_Y - \mathscr{E}^\vee) \otimes \tilde{\mathscr{G}}^\vee\big) = lh(p+h)\big((l-2)ph + (l^2 - 3l + 3)h^2\big)$$

$$= l(l-1)^2 h^3 p$$

such that we obtain

$$\chi(X) + \int_{F_*(c_1(X) \frown [X])} c_1(\tilde{\mathscr{G}}) = l(l-1)^2 h^3 p \frown [Y]$$

$$= l(l-1)^2.$$

This coincides with the sum of the local residues.

We note that the Euler number $\chi(X)$ of X is $l(l^2 - 4l + 7)$ and $\int_{F_*(c_1(X) \frown [X])} c_1(\tilde{\mathscr{G}}) = \sum \mathrm{Res} - \chi(X) = 2l(l-3)$. Thus the Euler number of the generic leaf on X is $l(l-3)$.

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