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On a resolvent estimate of the Stokes equation on an infinite layer

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Abstract. This paper is concerned with the standard L_p estimate of solutions to the resolvent problem for the Stokes operator on an infinite layer.

§1. Introduction.

Let $\Omega \subset \boldsymbol{R}^n$ $(n \geqq 2)$ be an infinite layer, i.e.,

$$\Omega = \{x = (x', x_n) \in \boldsymbol{R}^n \mid x' = (x_1, \ldots, x_{n-1}) \in \boldsymbol{R}^{n-1}, 0 < x_n < 1\}.$$

This paper is concerned with the resolvent problem of the Stokes operator on Ω with Dirichlet zero boundary condition:

$$(1.1) \qquad \begin{cases} (\lambda - \Delta)\boldsymbol{u} + \nabla \mathfrak{p} = \boldsymbol{f}, \ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}|_{x_n=0} = \boldsymbol{u}|_{x_n=1} = 0, \end{cases}$$

where $\boldsymbol{u} = \boldsymbol{u}(x) = (u_1(x), \ldots, u_n(x))$ and $\mathfrak{p} = \mathfrak{p}(x)$ denote the unknown velocity vector and pressure at $x \in \Omega$, respectively, and the resolvent parameter λ is contained in the sector Σ_ε which is defined as follows:

$$\Sigma_\varepsilon = \{z \in \boldsymbol{C} \backslash \{0\} \mid |\arg z| < \pi - \varepsilon\}, \quad 0 < \varepsilon < \pi/2.$$

Here and hereafter, for the differentiation, we use the following notation:

$$\Delta v = \sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k^2}, \quad \nabla = (\partial_1, \ldots, \partial_n), \quad \partial_j = \partial/\partial x_j, \quad \nabla^k v = (\partial_x^\alpha v \mid |\alpha| = k),$$

$$\Delta \boldsymbol{u} = (\Delta u_1, \ldots, \Delta u_n), \quad \nabla \cdot \boldsymbol{u} = \sum_{j=1}^{n} \partial_j u_j, \quad \nabla^k \boldsymbol{u} = (\partial_x^\alpha u_j \mid |\alpha| = k, j = 1, \ldots, n).$$

Let $\hat{W}_p^1(\Omega)$ be a function space for the pressure, which is defined as follows:

$$\hat{W}_p^1(\Omega) = \{v \in L_{p,\mathrm{loc}}(\Omega) \mid \nabla v \in L_p(\Omega)^n \text{ and}$$
$$\exists \{v_j\} \subset C_{(0)}^\infty(\bar{\Omega}) \text{ such that } \|\nabla(v_j - v)\|_{L_p(\Omega)} \to 0 \text{ as } j \to \infty\},$$

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where

$$C^{\infty}_{(0)}(\bar{\Omega}) = \{u \in C^{\infty}(\bar{\Omega}) \mid \exists R > 0 \text{ such that } u(x) = 0 \text{ for } |x'| > R\}.$$

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *For any* $\lambda \in \Sigma_\varepsilon$ *and* $\boldsymbol{f} = (f_1, \dots, f_n) \in L_p(\Omega)^n$, *there exists a unique* $\boldsymbol{u} \in W^2_p(\Omega)^n$ *which together with some* $\mathfrak{p} \in \hat{W}^1_p(\Omega)$ *solves* (1.1); $\mathfrak{p}$ *is unique up to an additive constant. Moreover, for any* $\lambda_0 > 0$ *there exists a constant* $C = C_{\lambda_0, p, \varepsilon}$ *depending only on* λ_0, p *and* ε *such that there holds the following estimate*:

$$|\lambda|\, \|\boldsymbol{u}\|_{L_p(\Omega)} + |\lambda|^{1/2} \|\nabla \boldsymbol{u}\|_{L_p(\Omega)} + \|\nabla^2 \boldsymbol{u}\|_{L_p(\Omega)} + \|\nabla \pi\|_{L_p(\Omega)} \leqq C \|\boldsymbol{f}\|_{L_p(\Omega)} \tag{1.2}$$

provided that $\lambda \in \Sigma_\varepsilon$ *and* $|\lambda| \geqq \lambda_0$.

So many results of the mathematical analysis for the incompressible viscous fluids in the whole space and in the exterior domain have been obtained when the domain is unbounded. The case where domains have noncompact boundaries have been studied in recent years as well. However, the special attention has given to problems having cylindrical and conical outlets to infinity. But, the infinite layer case has been less studied. Nazarov and Pileckas [**6**] proved the weak solvability of the Stokes and Navier-Stokes problems in the layer-like domain in weighted L_2-framework. Moreover, in [**7**] they obtained weighted a priori estimates and the asymptotic representation of the solution to the Stokes problem. We are interested in the study of the same problem as in [**7**] in the L_p-framework. Our approach is the following. Using the partial Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$ variable, we transform (1.1) to the two point boundary value problem for a system of the ordinary differential equations with parameter (ξ', λ) where $\xi' \in \boldsymbol{R}^{n-1}$ is a dual variable of x'. We solve this system exactly and apply the Fourier multiplier theorem [**4**], [**9**], [**10**] and a lemma concerning the estimate of the singular integral operator due to Agmon-Douglis-Nirenberg [**2**] to the exact formula of solutions to obtain the estimates stated in Theorem 1.1. Our method essentially follows Farwig and Sohr [**3**]. Since our domain is bounded in x_n direction, we can prove that $\lambda = 0$ is also in the resolvent set. This is one of the outstanding features of our result. Our main result is the following theorem which was already announced and proved roughly in Abe and Shibata [**1**].

Theorem 1.2. *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *Then, there exists a* $\sigma > 0$ *such that for any* $\lambda \in \Sigma_\varepsilon \cup \{z \in \boldsymbol{C} \mid |z| < \sigma\}$ *and any* $\boldsymbol{f} \in L_p(\Omega)^n$ (1.1) *admits a unique solution* $\boldsymbol{u} \in W^2_p(\Omega)^n$ *together with some* $\mathfrak{p} \in \hat{W}^1_p(\Omega)$, *where* $\mathfrak{p}$ *is unique up to an additive constant. Moreover, there holds the following resolvent estimate*:

$$|\lambda|\, \|\boldsymbol{u}\|_{L_p(\Omega)} + |\lambda|^{1/2} \|\nabla \boldsymbol{u}\|_{L_p(\Omega)} + \|\boldsymbol{u}\|_{W^2_p(\Omega)} + \|\nabla \mathfrak{p}\|_{L_p(\Omega)} \leqq C_{p,\varepsilon} \|\boldsymbol{f}\|_{L_p(\Omega)}.$$

Since our proof is rather long, we decided to divide the paper into two parts. And, this paper is concerned with the case where $\lambda \in \Sigma_\varepsilon$. The forthcoming paper will be devoted to the analysis when $\lambda = 0$.

Throughout the paper, to denote various constants we use the same letter C. By $C_{A,B,C,\dots}$ we denote a constant depending on the quantities $A, B, C, \dots$.

§2. Representation formula of solutions.

First of all, by using the solution in $\boldsymbol{R}^n$ we reduce (1.1) to the case where $\boldsymbol{f} = 0$ with inhomogeneous boundary data. In order to do this, we have to define the extension of $\boldsymbol{f}$. Let $\varphi(x_n)$ be a function in C^∞ such that $\varphi(x_n) = 1$ for $x_n \leqq 1/3$ and $\varphi(x_n) = 0$ for $x_n \geqq 2/3$. For a function $f(x)$ defined on Ω, put

$$f(x) = \varphi(x_n)f(x) + (1 - \varphi(x_n))f(x) = f_0(x) + f_1(x).$$

Let us define the even and odd extensions of f_0 and f_1 as follows:

$$f_0^e(x) = \begin{cases} \varphi(x_n)f(x', x_n) & x_n > 0, \\ \varphi(-x_n)f(x', -x_n) & x_n < 0, \end{cases} \tag{2.1}$$

$$f_1^e(x) = \begin{cases} (1 - \varphi(x_n))f(x', x_n) & x_n < 1, \\ (1 - \varphi(2 - x_n))f(x', 2 - x_n) & x_n > 1, \end{cases}$$

$$f_0^o(x) = \begin{cases} \varphi(x_n)f(x', x_n) & x_n > 0, \\ -\varphi(-x_n)f(x', -x_n) & x_n < 0, \end{cases}$$

$$f_1^o(x) = \begin{cases} (1 - \varphi(x_n))f(x', x_n) & x_n < 1, \\ -(1 - \varphi(2 - x_n))f(x', 2 - x_n) & x_n > 1. \end{cases}$$

And therefore, the even extension f^e of f and odd extension f^o of f are defined by the formulae:

$$f^e = f_0^e + f_1^e, \quad f^o = f_0^o + f_1^o. \tag{2.2}$$

Now, let us put $\boldsymbol{F} = (f_1^e, \dots, f_{n-1}^e, f_n^o)$ for given $\boldsymbol{f} = (f_1, \dots, f_n)$ in (1.1) and consider the following Stokes resolvent problem in $\boldsymbol{R}^n$:

$$(\lambda - \Delta)\boldsymbol{U} + \nabla\Phi = \boldsymbol{F}, \ \ \nabla \cdot \boldsymbol{U} = 0 \quad \text{in } \boldsymbol{R}^n. \tag{2.3}$$

Applying $\nabla\cdot$ to (2.3), we have $\Delta\Phi = \nabla \cdot \boldsymbol{F}$, and therefore $\boldsymbol{U} = (\lambda - \Delta)^{-1}(\boldsymbol{F} - \Delta^{-1}\nabla\nabla \cdot \boldsymbol{F})$. Namely, we have the following formulae for $\boldsymbol{U}$ and Φ:

$$U_j(x) = \mathscr{F}_\xi^{-1}\left[\frac{|\xi|^2\widehat{f_j^e}(\xi) - \sum_{k=1}^{n-1}\xi_j\xi_k\widehat{f_k^e}(\xi) - \xi_j\xi_n\widehat{f_n^o}(\xi)}{(\lambda + |\xi|^2)|\xi|^2}\right](x), \tag{2.4}$$

$$U_n(x) = \mathscr{F}_\xi^{-1}\left[\frac{|\xi'|^2\widehat{f_n^o}(\xi) - \sum_{k=1}^{n-1}\xi_k\xi_n\widehat{f_k^e}(\xi)}{(\lambda + |\xi|^2)|\xi|^2}\right](x), \tag{2.5}$$

$$\Phi(x) = \mathscr{F}_\xi^{-1}\left[\frac{\sum_{k=1}^{n-1}\xi_k\widehat{f_k^e}(\xi) + \xi_n\widehat{f_n^o}(\xi)}{i|\xi|^2}\right](x), \tag{2.6}$$

where $j = 1, \dots, n-1$; $i = \sqrt{-1}$; $\xi' = (\xi_1, \dots, \xi_{n-1})$; $\xi = (\xi', \xi_n) = (\xi_1, \dots, \xi_n)$; and $\hat{v}(\xi)$ and $\mathscr{F}_\xi^{-1}[w(\xi)](x)$ mean the Fourier transform of v and the Fourier inverse transform of w, respectively, namely

$$\hat{v}(\xi) = \mathscr{F}_\xi[v](\xi) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}v(x)\,dx, \quad \mathscr{F}_\xi^{-1}[w(\xi)](x) = \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{ix\cdot\xi}w(\xi)\,d\xi.$$

In order to obtain the L_p estimate for $\boldsymbol{U}$ and Φ, we begin with the following inequalities.

Lemma 2.1. *Let* $0 < \varepsilon < \pi/2$. *Then, for any* $\lambda \in \Sigma_\varepsilon$, $\xi \in \boldsymbol{R}^n$ *and* $\xi' \in \boldsymbol{R}^{n-1}$ *we have the following inequalities*:

$$|\lambda + |\xi|^2| \geqq \left(\sin\frac{\varepsilon}{2}\right)(|\lambda| + |\xi|^2), \tag{2.7}$$

$$\operatorname{Re}\sqrt{\lambda + |\xi'|^2} \geqq c_\varepsilon \sqrt{|\lambda| + |\xi'|^2}, \tag{2.8}$$

where we have chosen a branch in such a way that $\operatorname{Re}\sqrt{\lambda + |\xi|^2} > 0$ *and we have put* $c_\varepsilon = (\sin \varepsilon/2)^{3/2}$.

To obtain the L_p estimate, the following Fourier multiplier theorem (cf. [**9**], [**10**]) is the main tool in our argument.

Theorem 2.2. *For every positive number n, put* $U = \boldsymbol{R}^n \backslash \{\xi \in \boldsymbol{R}^n \mid \xi_j = 0$ *for some* $j = 1, \dots, n\}$. *Then, for every* $p \in (1, \infty)$, *there exists a positive constant* C_p *such that, for every* $P(\xi) \in C^n(U)$ *satisfying the estimate*

$$\sup_{\xi \in U, \alpha \in \{0,1\}^n} \left| \xi^\alpha \frac{\partial^\alpha P}{\partial \xi^\alpha}(\xi) \right| \leqq A$$

the operator $f(x) \mapsto \mathscr{F}_\xi^{-1}[P(\xi)\hat{f}(\xi)](x)$ *is extended to a bounded linear operator on* $L_p(\boldsymbol{R}^n)$ *with the estimate*

$$\|\mathscr{F}_\xi^{-1}[P(\xi)\hat{f}(\xi)]\|_{L_p(\boldsymbol{R}^n)} \leqq C_p A \|f\|_{L_p(\boldsymbol{R}^n)}.$$

By Lemma 2.1 and Theorem 2.2, we have the following lemma.

Lemma 2.3. *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *Then, for* $\boldsymbol{U}$ *and* Φ *defined in* (2.4)–(2.6) *we have* $\boldsymbol{U} \in W_p^2(\boldsymbol{R}^n)^n$ *and* $\Phi \in \hat{W}_p^1(\boldsymbol{R}^n) = \{\Phi \in L_{p,\mathrm{loc}}(\boldsymbol{R}^n) \mid \nabla\Phi \in L_p(\boldsymbol{R}^n)^n\}$, *and moreover for any* $\lambda \in \Sigma_\varepsilon$ *we have the estimate*:

$$|\lambda| \, \|\boldsymbol{U}\|_{L_p(\boldsymbol{R}^n)} + |\lambda|^{1/2} \|\nabla \boldsymbol{U}\|_{L_p(\boldsymbol{R}^n)} + \|\nabla^2 \boldsymbol{U}\|_{L_p(\boldsymbol{R}^n)} + \|\nabla\Phi\|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\varepsilon} \|\boldsymbol{f}\|_{L_p(\Omega)}.$$

Remark 2.4. (1) In order to obtain Lemma 2.3 only, it is not necessary to use the special extension of $\boldsymbol{f}$ like (2.2). But, a property of U_n stated in Lemma 4.2 below will be necessary to prove our main result. The special extension of $\boldsymbol{f}$ in (2.2) guarantees such property.

(2) As is well-known (cf. Farwig-Sohr [**3**], Galdi [**4**]), for any $\Phi \in \hat{W}_p^1(\boldsymbol{R}^n)$, there exists a sequence $\{\Phi_j\} \subset C_0^\infty(\boldsymbol{R}^n)$ such that $\|\nabla(\Phi - \Phi_j)\|_{L_p(\boldsymbol{R}^n)} \to 0$ as $j \to \infty$, where $C_0^\infty(\boldsymbol{R}^n)$ means the set of all functions in $C^\infty(\boldsymbol{R}^n)$ with compact support. Therefore, the restriction of Φ to Ω belongs to $\hat{W}_p^1(\Omega)$.

If we put $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v}$ and $\mathfrak{p} = \Phi + \Psi$, then the problem (1.1) is reduced to the following problem for $\boldsymbol{v}$ and Ψ:

$$(\lambda - \Delta)\boldsymbol{v} + \nabla\Psi = 0, \;\; \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega, \;\; \boldsymbol{v}|_{x_n=0} = -\boldsymbol{U}|_{x_n=0}, \;\; \boldsymbol{v}|_{x_n=1} = -\boldsymbol{U}|_{x_n=1}. \tag{2.9}$$

We shall introduce the equivalent problem to (2.9). Applying $\nabla\cdot$ to (2.9) implies that

$$\Delta\Psi = 0 \quad \text{in } \Omega. \tag{2.10}$$

Applying Δ to the n-th component of the first equation of (2.9) and using (2.10), we have $(\lambda - \Delta)\Delta v_n = 0$ in Ω. Since $\nabla \cdot \boldsymbol{v} = 0$ in Ω, by the boundary condition of (2.9) we have

$$\partial_n v_n|_{x_n=a} = -\sum_{j=1}^{n-1} \partial_j v_j\Big|_{x_n=a} = \sum_{j=1}^{n-1} \partial_j U_j(x', a)$$

for $a = 0, 1$. Therefore, we have the following equation for v_n:

$$(2.11) \qquad (\lambda - \Delta)\Delta v_n = 0 \quad \text{in } \Omega, \; v_n|_{x_n=a} = -U_n(x', a), \; \partial_n v_n|_{x_n=a} = \sum_{j=1}^{n-1} \partial_j U_j(x', a)$$

for $a = 0$ and 1. And then, (2.10) and the n-th component of the first equation of (2.9) imply the equation for Ψ as follows:

$$(2.12) \qquad \Delta\Psi = 0 \quad \text{in } \Omega, \; \partial_n \Psi|_{x_n=a} = -(\lambda - \Delta)v_n|_{x_n=a}$$

for $a = 0$ and 1. Finally, the equation for $v_1, \dots, v_{n-1}$ are the following:

$$(2.13) \qquad (\lambda - \Delta)v_j = -\partial_j \Psi \quad \text{in } \Omega, \; v_j|_{x_n=a} = -U_j|_{x_n=a}$$

for $a = 0$ and 1.

First, we shall solve (2.11). In what follows, we shall write the partial Fourier transform with respect to $x' = (x_1, \dots, x_{n-1}) \in \boldsymbol{R}^{n-1}$ and its inverse transform as follows:

$$\hat{v}(\xi', x_n) = \int_{\boldsymbol{R}^{n-1}} e^{-ix'\cdot\xi'} v(x', x_n)\, dx',$$

$$\mathscr{F}_{\xi'}^{-1}[w(\xi', x_n)](x') = \frac{1}{(2\pi)^{n-1}} \int_{\boldsymbol{R}^{n-1}} e^{ix'\cdot\xi'} w(\xi', x_n)\, d\xi'.$$

For the notational simplicity, we write $A = |\xi'|$ and $B = \sqrt{\lambda + |\xi'|^2}$, below. Applying the partial Fourier transform to (2.11), we have

$$(2.14) \qquad (\partial_n^2 - A^2)(\partial_n^2 - B^2)\hat{v}_n(\lambda, \xi', x_n) = 0 \quad \text{for } 0 < x_n < 1,$$

$$\hat{v}_n|_{x_n=a} = -\hat{U}_n(\xi', a), \; \partial_n \hat{v}_n|_{x_n=a} = i\xi' \cdot \hat{U}'(\xi', a)$$

for $a = 0$ and 1, where $\hat{U}' = (\hat{U}_1, \dots, \hat{U}_{n-1})$. Since the fundamental solutions to (2.14) are $e^{-A(1-x_n)}$, e^{-Ax_n}, $e^{-B(1-x_n)}$ and e^{-Bx_n}, we look for the solution to (2.14) in the form $\hat{v}_n = a_1 e^{-A(1-x_n)} + a_2 e^{-Ax_n} + a_3 e^{-B(1-x_n)} + a_4 e^{-Bx_n}$. By the boundary condition, (a_1, a_2, a_3, a_4) satisfies the following simultaneous linear equation:

$$(2.15) \qquad L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ h_0 \\ h_1 \end{pmatrix}, \quad \text{where } L = \begin{bmatrix} e^{-A} & 1 & e^{-B} & 1 \\ 1 & e^{-A} & 1 & e^{-B} \\ Ae^{-A} & -A & Be^{-B} & -B \\ A & -Ae^{-A} & B & -Be^{-B} \end{bmatrix}$$

and

$$(2.16) \qquad g_a = -\hat{U}_n(\xi', a), \quad h_a = i\xi' \cdot \hat{U}'(\xi', a)$$

for $a = 0$ and 1. Concerning the Lopatinski determinant $\det L$, we have the following lemma.

Lemma 2.5. *If* $\lambda \in \boldsymbol{C}\backslash(-\infty, 0]$ *and* $\xi' \neq 0$, *then* $\det L \neq 0$.

Proof. If $\det L = 0$, then there exists a $(a, b, c, d) \neq (0, 0, 0, 0)$ such that $v(x_n) = ae^{-A(1-x_n)} + be^{-Ax_n} + ce^{-B(1-x_n)} + de^{-Bx_n}$ satisfies the homogeneous equation:

$$(\partial_n^2 - A^2)(\partial_n^2 - B^2)v(x_n) = 0 \quad 0 < x_n < 1,$$
$$v(0) = v(1) = \partial_n v(0) = \partial_n v(1) = 0.$$

Multiplying the equation by $\overline{v(x_n)}$, integrating the resultant formula over $(0, 1)$ and using the boundary condition, by integration by parts we have

$$(\lambda + |\xi'|^2)|\xi'|^2 \int_0^1 |v(x_n)|^2\, dx_n + (\lambda + 2|\xi'|^2) \int_0^1 |\partial_n v(x_n)|^2\, dx_n + \int_0^1 |\partial_n^2 v(x_n)|^2\, dx_n = 0.$$

When $\operatorname{Im}\lambda \neq 0$, taking the imaginary part implies that $v(x_n) = 0$. When $\operatorname{Im}\lambda = 0$ and $\operatorname{Re}\lambda > 0$, taking the real part implies that $v(x_n) = 0$. This is contradictory to $(a, b, c, d) \neq (0, 0, 0, 0)$, which completes the proof of the lemma. □

In view of Lemma 2.5, if $\lambda \in \boldsymbol{C}\backslash(-\infty, 0]$ and $\xi' \neq 0$, then the solution $\hat{v}_n$ to (2.14) and the solution v_n to (2.11) are represented as

$$\begin{aligned}
\hat{v}_n(\lambda, \xi', x_n) = \sum_{j=1}^{2}\sum_{k=1}^{2}\Bigg\{ &\left(\frac{L_{j,k}e^{-A\tau_k(x_n)}}{\det L} + \frac{L_{j,k+2}e^{-B\tau_k(x_n)}}{\det L}\right) g_{j-1} \\
&+ \left(\frac{L_{j+2,k}e^{-A\tau_k(x_n)}}{\det L} + \frac{L_{j+2,k+2}e^{-B\tau_k(x_n)}}{\det L}\right) h_{j-1} \Bigg\},
\end{aligned} \tag{2.17}$$

$$v_n(x) = \mathscr{F}_{\xi'}^{-1}[\hat{v}_n(\lambda, \xi', x_n)](x'),$$

where $L_{j,k}$ denotes the (j, k) cofactor of L. Here and hereafter, for the notational simplicity, we put

$$\tau_1(x_n) = 1 - x_n, \quad \tau_2(x_n) = x_n.$$

The $\det L$ and $L_{j,k}$ are given by the following formulae:

$$\begin{aligned}
\det L = &-(1 - e^{-2A})(1 - e^{-2B})(A^2 + B^2) \\
&+ 2AB(1 + e^{-2A})(1 + e^{-2B}) - 8ABe^{-A}B^{-B},
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
L_{1,1} = L_{2,2} &= (AB - B^2)e^{-A}e^{-2B} - 2ABe^{-B} + (AB + B^2)e^{-A}, \\
L_{1,2} = L_{2,1} &= AB - B^2 - 2ABe^{-A}e^{-B} + (AB + B^2)e^{-2B}, \\
L_{1,3} = L_{2,4} &= -(A^2 - AB)e^{-2A}e^{-B} - 2ABe^{-A} + (A^2 + AB)e^{-B}, \\
L_{1,4} = L_{2,3} &= -(A^2 - AB) - 2ABe^{-A}e^{-B} + (A^2 + AB)e^{-2A}, \\
L_{3,1} = -L_{4,2} &= -(A - B)e^{-A}e^{-2B} - 2Be^{-B} + (A + B)e^{-A}, \\
L_{3,2} = -L_{4,1} &= (A - B) + 2Be^{-A}e^{-B} - (A + B)e^{-2B}, \\
L_{3,3} = -L_{4,4} &= (A - B)e^{-2A}e^{-B} - 2Ae^{-A} + (A + B)e^{-B}, \\
L_{3,4} = -L_{4,3} &= -(A - B) + 2Ae^{-A}e^{-B} - (A + B)e^{-2A}.
\end{aligned}$$

Now, we shall consider (2.12). Taking the partial Fourier transform of (2.12), we have

$$(2.19)\qquad (\partial_n^2 - A^2)\hat{\Psi} = 0 \quad \text{for } 0 < x_n < 1, \ \ \partial_n \hat{\Psi}|_{x_n=a} = (\partial_n^2 - B^2)\hat{v}_n|_{x_n=a}$$

for $a = 0$ and 1. Since

$$(\partial_n^2 - B^2)\hat{v}_n|_{x_n=0} = -\lambda e^{-A} K_1 - \lambda K_2, \quad (\partial_n^2 - B^2)\hat{v}_n|_{x_n=1} = -\lambda K_1 - \lambda e^{-A} K_2$$

as follows from (2.17) where $K_j = (\det L)^{-1}\{L_{1,j} g_0 + L_{2,j} g_1 + L_{3,j} h_0 + L_{4,j} h_1\}$ for $j = 1, 2$, putting $\hat{\Psi}(\lambda, \xi', x_n) = a e^{-A(1-x_n)} + b e^{-A x_n}$ and inserting this formula into the boundary condition, we have

$$(2.20)\qquad \hat{\Psi}(\lambda, \xi', x_n) = \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^k e^{-A\tau_k(x_n)} \frac{\lambda}{A} \left\{ \frac{L_{j,k}}{\det L} g_{j-1} + \frac{L_{j+2,k}}{\det L} h_{j-1} \right\}.$$

Finally, we shall solve (2.13). Put

$$(2.21)\qquad F_j = \begin{cases} -\partial_j \Psi, & 0 \leqq x_n \leqq 1, \\ 0, & x_n \notin [0,1], \end{cases} \quad V_j(x) = \mathscr{F}_\xi^{-1}\left[\frac{\widehat{F_j}(\xi)}{\lambda + |\xi|^2}\right](x).$$

If we put $v_j(x) = V_j(x) + w_j(x)$, then (2.13) is reduced to the equation:

$$(2.22)\qquad (\lambda - \Delta) w_j = 0 \quad \text{in } \Omega, \ \ w_j|_{x_n=a} = -(U_j + V_j)|_{x_n=a}$$

for $a = 0$ and 1. Applying the partial Fourier transform and putting $\hat{w}_j(\lambda, \xi', x_n) = a_1 e^{-B(1-x_n)} + a_2 e^{-B x_n}$, we have

$$\begin{aligned} \hat{w}_j(\lambda, \xi', x_n) = {} & \frac{e^{-B(1+x_n)} - e^{-B(1-x_n)}}{1 - e^{-2B}} (\hat{U}_j(\xi', 1) + \hat{V}_j(\xi', 1)) \\ & + \frac{e^{-B(2-x_n)} - e^{-B x_n}}{1 - e^{-2B}} (\hat{U}_j(\xi', 0) + \hat{V}_j(\xi', 0)). \end{aligned}$$

And therefore, we have

$$(2.23)\qquad v_j(x) = V_j(x) + \mathscr{F}_{\xi'}^{-1}[\hat{w}_j(\lambda, \xi', x_n)](x'),$$

for $j = 1, \ldots, n-1$.

Finally, we shall show the uniqueness of solutions to (1.1), assuming that the existence of solution to (1.1) holds. Let $\boldsymbol{u} \in W_p^2(\Omega)^n$ and $\mathfrak{p} \in \hat{W}_p^1(\Omega)$ satisfy the homogeneous equation:

$$(2.24)\qquad (\lambda - \Delta)\boldsymbol{u} + \nabla \mathfrak{p} = 0, \ \ \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u}|_{x_n=0} = \boldsymbol{u}|_{x_n=1} = 0.$$

Let $\boldsymbol{f} = (f_1, \ldots, f_n) \in C_0^\infty(\Omega)^n$ and let $(\boldsymbol{v}, \mathfrak{q}) \in W_{p'}^2(\Omega)^n \times \hat{W}_{p'}^1(\Omega)$ be a solution to the equation:

$$(2.25)\qquad (\bar{\lambda} - \Delta)\boldsymbol{v} + \nabla \mathfrak{q} = \boldsymbol{f}, \ \ \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{v}|_{x_n=0} = \boldsymbol{v}|_{x_n=1} = 0,$$

where $p' = p/(p-1)$. Since $\mathfrak{p} \in \hat{W}_p^1(\Omega)$, there exists a sequence $\{\theta_j\} \subset C_{(0)}^\infty(\Omega)$ such that $\lim_{j\to\infty} \|\nabla(\theta_j - \mathfrak{p})\|_{L_p(\Omega)} = 0$. Since $\nabla \cdot \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v}|_{x_n=0} = \boldsymbol{v}|_{x_n=1} = 0$, we have

$$(\nabla \mathfrak{p}, \boldsymbol{v}) = \lim_{j\to\infty} (\nabla \theta_j, \boldsymbol{v}) = \lim_{j\to\infty} (\theta_j, \nabla \cdot \boldsymbol{v}) = 0$$

where we have put

$$(v, w) = \int_\Omega v(x)\overline{w(x)}\, dx, \quad (\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^n (v_j, w_j)$$

for scalar functions v, w and vector valued functions $\boldsymbol{v} = (v_1, \dots, v_n)$ and $\boldsymbol{w} = (w_1, \dots, w_n)$. Since $\mathfrak{q} \in \hat{W}^1_{p'}(\Omega)$, $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}|_{x_n=0} = \boldsymbol{u}|_{x_n=1} = 0$, in the same manner we have $(\boldsymbol{u}, \nabla \mathfrak{q}) = 0$. Therefore, by (2.24) and (2.25) we have

$$0 = ((\lambda - \Delta)\boldsymbol{u} + \nabla \mathfrak{p}, \boldsymbol{v}) = (\boldsymbol{u}, (\bar{\lambda} - \Delta)\boldsymbol{v}) = (\boldsymbol{u}, (\bar{\lambda} - \Delta)\boldsymbol{v} + \nabla \mathfrak{q}) = (\boldsymbol{u}, \boldsymbol{f}).$$

The arbitrariness of choice of $\boldsymbol{f}$ implies that $\boldsymbol{u} = 0$, which combined with (2.24) implies also that $\nabla \mathfrak{p} = 0$. Therefore, $\mathfrak{p}$ is a constant. This completes the proof of the uniqueness of the solution to (1.1).

§3. Analysis of $\hat{v}_n(\lambda, \xi', x_n)$.

In this section, we shall estimate the coefficients of g_j and h_j of $\hat{v}_n$ defined by (2.17) studying the three cases.

3.1 Case 1.

We shall consider the case where $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfy the condition: $|\lambda| \geqq \alpha$ and $|\xi'| \leqq \gamma_{\alpha,\varepsilon}$. Here, α is an arbitrary positive number and $\gamma_{\alpha,\varepsilon}$ is a sufficiently small positive number depending only on α and ε, which will be given in Lemma 3.1 below. If we put

$$(3.1) \qquad l_1(A, B) = \frac{1 - e^{-2A}}{A}(1 - e^{-2B})(A^2 + B^2) - 2B(1 + e^{-2A})(1 + e^{-2B}) + 8Be^{-A}e^{-B},$$

by (2.18) we have

$$(3.2) \qquad \det L = -Al_1(A, B).$$

Here and hereafter, we put $A = |\xi'|$ and $B = \sqrt{\lambda + |\xi'|^2}$ like in §2.

Lemma 3.1. *Let $0 < \varepsilon < \pi/2$ and $\alpha > 0$. Then, there exist constants $\gamma_{\alpha,\varepsilon}$, $0 < \gamma_{\alpha,\varepsilon} \leqq 1$, and $c_{\alpha,\varepsilon} > 0$ depending only on α and ε such that there holds the estimate*:

$$|l_1(A, B)| \geqq c_{\alpha,\varepsilon}(1 + |B|^2)$$

provided that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfy the condition: $|\lambda| \geqq \alpha$ *and* $|\xi'| \leqq \gamma_{\alpha,\varepsilon}$.

Proof. First, we shall show that there exist constants $A_{\alpha,\varepsilon} > 0$ and $d_{\alpha,\varepsilon} > 0$ such that

$$(3.3) \qquad |l_1(A, B)| \geqq A_{\alpha,\varepsilon}|B|^2$$

provided that $0 \leqq A \leqq 1$, $\lambda \in \Sigma_\varepsilon$ and $|\lambda|^{1/2} \geqq d_{\alpha,\varepsilon}$. In fact, since

$$\frac{1 - e^{-2A}}{A} = 2\int_0^1 e^{-2A\theta}\, d\theta \geqq 2e^{-2}$$

when $0 < A \leqq 1$, we have

$$|l_1(A,B)| \geqq |B|^2\{2e^{-2}(1-e^{-2\operatorname{Re} B})(1-|A/B|^2) - 2|B|^{-1}(1+e^{-2A})(1+e^{-2\operatorname{Re} B}) - 8|B|^{-1}e^{-A}e^{-\operatorname{Re} B}\}. \tag{3.4}$$

By (2.8), we have $|B| \geqq \operatorname{Re} B \geqq c_\varepsilon|\lambda|^{1/2} \geqq c_\varepsilon d_{\alpha,\varepsilon}$ when $|\lambda|^{1/2} \geqq d_{\alpha,\varepsilon}$, and therefore

$$\begin{gathered}(1-e^{-2\operatorname{Re} B})(1-|A/B|^2) \geqq (1-e^{-2c_\varepsilon\sqrt{\alpha}})(1-(c_\varepsilon d_{\alpha,\varepsilon})^{-1}),\\ 2|B|^{-1}(1+e^{-2A})(1+e^{-2\operatorname{Re} B}) + 8|B|^{-1}e^{-A}e^{-\operatorname{Re} B} \leqq 16(c_\varepsilon d_{\alpha,\varepsilon})^{-1}.\end{gathered} \tag{3.5}$$

Put $A_{\alpha,\varepsilon} = e^{-2}(1-e^{-2c_\varepsilon\sqrt{\alpha}})/2$ and choose $d_{\alpha,\varepsilon} > 0$ so large that $(c_\varepsilon d_{\alpha,\varepsilon})^{-1} < 1/2$ and $16(c_\varepsilon d_{\alpha,\varepsilon})^{-1} \leqq A_{\alpha,\varepsilon}$, and then (3.4) and (3.5) imply (3.3).

Next, we consider the case where $\lambda \in \Sigma_\varepsilon$ and $\alpha \leqq |\lambda| \leqq d_{\alpha,\varepsilon}^2$. When $\alpha \leqq |\lambda| \leqq d_{\alpha,\varepsilon}^2$ and $0 \leqq A \leqq 1$, we have $|B| \leqq \sqrt{|\lambda| + |\xi'|^2} \leqq \sqrt{1+d_{\alpha,\varepsilon}^2}$. When $|\lambda| \geqq \alpha$ and $\lambda \in \Sigma_\varepsilon$, we have $\operatorname{Re} B \geqq c_\varepsilon|\lambda|^{1/2} \geqq c_\varepsilon\sqrt{\alpha}$. From these observations, if we put

$$K = \{z \in \boldsymbol{C} \mid |z| \leqq \sqrt{1+d_{\alpha,\varepsilon}^2}, \operatorname{Re} z \geqq c_\varepsilon\sqrt{\alpha}\},$$

then $B \in K$ when $\alpha \leqq |\lambda| \leqq d_{\alpha,\varepsilon}^2$, $\lambda \in \Sigma_\varepsilon$ and $0 \leqq A \leqq 1$. Therefore, to complete the proof, it is sufficient to prove that there exist constants $\gamma_{\alpha,\varepsilon}$, $0 < \gamma_{\alpha,\varepsilon} \leqq 1$, and $C_{\alpha,\varepsilon} > 0$ such that

$$|l_1(A,B)| \geqq C_{\alpha,\varepsilon} \quad \text{when } 0 \leqq A \leqq \gamma_{\alpha,\varepsilon},\ B \in K. \tag{3.6}$$

Since K is compact and $l_1(A,B)$ is a continuous function of (A,B), to prove (3.6) it is sufficient to prove that

$$l_1(0,B) \neq 0 \quad \text{when } \operatorname{Re} B > 0. \tag{3.7}$$

As an auxiliary problem, we consider the following ordinary differential equation:

$$\begin{gathered}\left(\frac{d}{dt}\right)^2\left(\left(\frac{d}{dt}\right)^2 - B^2\right)u(t) = 0 \quad 0 < t < 1,\\ u(0) = g_0,\ u(1) = g_1,\ u'(0) = h_0,\ u'(1) = h_1\end{gathered} \tag{3.8}$$

which is the equation corresponding to (2.14) with $A = 0$. If we put $u(t) = a_1 + a_2t + a_3e^{-Bt} + a_4e^{-B(1-t)}$, from the boundary condition we see that (a_1,a_2,a_3,a_4) satisfies the following simultaneous linear equation:

$$M\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix} = \begin{pmatrix}g_0\\g_1\\h_0\\h_1\end{pmatrix}, \quad \text{where } M = \begin{bmatrix}1 & 0 & 1 & e^{-B}\\ 1 & 1 & e^{-B} & 1\\ 0 & 1 & -B & Be^{-B}\\ 0 & 1 & -Be^{-B} & B\end{bmatrix}.$$

Since $\det M = -l_1(0,B)/2$, if $l_1(0,B) = 0$ for some B with $\operatorname{Re} B > 0$, then there exists a $(a_1,a_2,a_3,a_4) \neq (0,0,0,0)$ such that $u(t) = a_1 + a_2t + a_3e^{-Bt} + a_4e^{-B(1-t)}$ satisfies (3.8) with $g_0 = g_1 = h_0 = h_1 = 0$. Multiplying (3.8) by $\overline{u(t)}$ and integrating the resultant formula over $(0,1)$, by integration by parts we have

$$0 = \int_0^1 \left|\frac{d^2u}{dt^2}(t)\right|^2 dt + B^2 \int_0^1 \left|\frac{du}{dt}(t)\right|^2 dt.$$

Taking the imaginary part implies that

$$2(\operatorname{Re} B)(\operatorname{Im} B) \int_0^1 \left|\frac{du}{dt}(t)\right|^2 dt = 0,$$

and therefore $u = 0$ when $\operatorname{Im} B \neq 0$ and $\operatorname{Re} B \neq 0$. When $\operatorname{Im} B = 0$ and $\operatorname{Re} B \neq 0$, we have also $u = 0$. This is contradictory to $(a_1, a_2, a_3, a_4) \neq (0,0,0,0)$, which completes the proof. □

In Case 1, we transform (2.17) into the following formula:

$$\begin{aligned}(3.9)\qquad \hat{v}_n(\lambda, \xi', x_n) = \sum_{j=1}^{2} \Bigg\{ & \frac{L_{j,1} + L_{j,2}}{\det L} e^{-A\tau_1(x_n)} + \frac{L_{j,2}A}{\det L} D(A, x_n) \\ & + \sum_{m=1}^{2} \frac{L_{j,m+2}}{\det L} e^{-B\tau_m(x_n)} \Bigg\} \hat{U}_n(\cdot\,, j-1) \\ + \sum_{j=1}^{2} \Bigg\{ & \frac{L_{j+2,1} + L_{j+2,2}}{\det L} e^{-A\tau_1(x_n)} + \frac{L_{j+2,2}A}{\det L} D(A, x_n) \\ & + \sum_{m=1}^{2} \frac{L_{j+2,m+2}}{\det L} e^{-B\tau_m(x_n)} \Bigg\} i\xi' \cdot \widehat{U'}(\cdot\,, j-1),\end{aligned}$$

where we have put

$$(3.10)\qquad D(A, x_n) = \frac{e^{-Ax_n} - e^{-A(1-x_n)}}{A} = (1 - 2x_n) \int_0^1 e^{-A((1-x_n)+\theta(2x_n-1))}\, d\theta.$$

To represent the coefficients in (3.9), we also put

$$(3.11)\qquad d_k(A) = \frac{1 - e^{-kA}}{A} = k \int_0^1 e^{-\theta k A}\, d\theta.$$

Then, we have the following formulae concerning the coefficients in (3.9):

$$\begin{aligned}(3.12)\qquad & \frac{L_{1,1} + L_{1,2}}{\det L} = \frac{L_{2,1} + L_{2,2}}{\det L} = \frac{B^2(d_1(A)(1 - e^{-2B}) - B^{-1}(1 + e^{-A})(1 - e^{-B})^2)}{l_1(A, B)}, \\ & \frac{L_{3,1} + L_{3,2}}{\det L} = -\frac{L_{4,1} + L_{4,2}}{\det L} = \frac{B(d_1(A)(1 + e^{-B})^2 - B^{-1}(1 + e^{-A})(1 - e^{-2B}))}{l_1(A, B)}, \\ & \frac{L_{1,2}A}{\det L} = \frac{B^2((1 - B^{-1}A) + 2B^{-1}Ae^{-A}e^{-B} - (1 + B^{-1}A)e^{-2B})}{l_1(A, B)}, \\ & \frac{L_{2,2}A}{\det L} = \frac{B^2((1 - B^{-1}A)e^{-A}e^{-2B} + 2B^{-1}Ae^{-B} - (1 + B^{-1}A)e^{-A})}{l_1(A, B)},\end{aligned}$$

$$\frac{L_{3,2}A}{\det L} = \frac{B(1 - B^{-1}A - 2e^{-A}e^{-B} + (1 + B^{-1}A)e^{-2B})}{l_1(A,B)},$$

$$\frac{L_{4,2}A}{\det L} = \frac{B((1 - B^{-1}A)e^{-A}e^{-2B} - 2e^{-B} + (1 + B^{-1}A)e^{-A})}{l_1(A,B)},$$

$$\frac{L_{1,3}}{\det L} = \frac{L_{2,4}}{\det L} = \frac{B(-(1 - B^{-1}A)e^{-2A}e^{-B} + 2e^{-A} - (1 + B^{-1}A)e^{-B})}{l_1(A,B)},$$

$$\frac{L_{1,4}}{\det L} = \frac{L_{2,3}}{\det L} = \frac{B(-(1 - B^{-1}A) + 2e^{-A}e^{-B} - (1 + B^{-1}A)e^{-2A})}{l_1(A,B)},$$

$$\frac{L_{3,3}}{\det L} = -\frac{L_{4,4}}{\det L} = \frac{B(-d_2(A)e^{-B} - B^{-1}e^{-2A}e^{-B} + 2B^{-1}e^{-A} - B^{-1}e^{-B})}{l_1(A,B)},$$

$$\frac{L_{3,4}}{\det L} = -\frac{L_{4,3}}{\det L} = \frac{B(-d_2(A) + B^{-1} - 2B^{-1}e^{-A}e^{-B} + B^{-1}e^{-2A})}{l_1(A,B)}.$$

In order to estimate the coefficients of $\hat{v}_n$, we shall use the following lemma.

Lemma 3.2. *Let $0 < \varepsilon < \pi/2$, $\xi' \in \boldsymbol{R}^{n-1}$ and $\lambda \in \Sigma_\varepsilon$. Then, for any multi-index $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ we have the following estimates*:

$$|\partial_{\xi'}^{\alpha'} A^k| \leqq C_{\alpha',k}|\xi'|^{k-|\alpha'|} \qquad \forall k \in \boldsymbol{R},$$

$$|\partial_{\xi'}^{\alpha'} B^k| \leqq C_{\alpha',\varepsilon,k}(|\lambda| + |\xi'|^2)^{k/2}|\xi'|^{-|\alpha'|} \qquad \forall k \in \boldsymbol{R},$$

$$|\partial_{\xi'}^{\alpha'} e^{-mA}| \leqq C_{\alpha',m}|\xi'|^{-|\alpha'|}e^{-(m/2)|\xi'|} \qquad \forall m > 0,$$

$$|\partial_{\xi'}^{\alpha'} e^{mA}| \leqq C_{\alpha',m}|\xi'|^{-|\alpha'|}e^{2m|\xi'|} \qquad \forall m > 0,$$

$$|\partial_{\xi'}^{\alpha'} e^{-mB}| \leqq C_{\alpha',m}|\xi'|^{-|\alpha'|}e^{-(m/2)c_\varepsilon\sqrt{|\lambda|+|\xi'|^2}} \qquad \forall m > 0,$$

where c_ε is the same constant as in Lemma 2.1. *Here and hereafter*; $C_{A,B,C,\dots}$ *means the constant depending only on the subscripts* $A, B, C, \dots$; *and*

$$\partial_{\xi'}^{\alpha'} v(\xi') = \frac{\partial^{|\alpha'|} v}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_{n-1}^{\alpha_{n-1}}}, \quad |\alpha'| = \alpha_1 + \cdots + \alpha_{n-1}, \ \ \alpha' = (\alpha_1, \dots, \alpha_{n-1}).$$

To prove Lemma 3.2, we use the following known formula for the derivatives of the composite function, which are easily proved by induction:

$$\partial_{\xi'}^{\alpha'} w(\varphi(\xi')) = \sum_{v=1}^{|\alpha'|} \frac{d^v w(z)}{dz^v}\bigg|_{z=\varphi(\xi')} \left[\sum_{\substack{\alpha_1' + \cdots + \alpha_v' = \alpha' \\ |\alpha_i'| \geqq 1}} \partial_{\xi'}^{\alpha_1'} \varphi(\xi') \cdots \partial_{\xi'}^{\alpha_v'} \varphi(\xi') \right]. \tag{3.13}$$

If we combine (3.13) with Lemma 2.1 to treat B and the fact that there holds the relation: $|\partial_{\xi'}^{\alpha'}|\xi'|^2| \leqq 2|\xi'|^{2-|\alpha'|}$ for any multi-index α', we can show Lemma 3.2. The argument is rather standard, so that we may omit the proof (cf. Simader [**8**, Appendix 1]). By Lemma 3.1, Lemma 3.2 and Leibniz's rule, we have

$$
\begin{aligned}
(3.14)\qquad & |\partial_{\xi'}^{\alpha'}[B^k l_1(A,B)^{-1}]| \leqq C_{\alpha,\varepsilon}|\lambda|^{(1/2)(k-2)}|\xi'|^{-|\alpha'|}, \\
& |\partial_{\xi'}^{\alpha'} d_k(A)| \leqq C_{\alpha'}|\xi'|^{-|\alpha'|} \quad k=1,2, \\
& |\partial_{\xi'}^{\alpha'}\partial_n^k D(A,x_n)| \leqq C_{\alpha'}|\xi'|^{-|\alpha'|} \quad \forall k \geqq 0, \forall x_n \in [0,1],
\end{aligned}
$$

for any α' when $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geqq \alpha$ and $|\xi'| \leqq \gamma_{\alpha,\varepsilon}$ ($\leqq 1$), where we have used the fact that $c_\varepsilon|\lambda|^{1/2} \leqq |B| \leqq \sqrt{1+|\lambda|} \leqq \sqrt{1+\alpha^{-1}}|\lambda|^{1/2}$ to get the first relation. Therefore, applying (3.14), Lemma 3.1 and Leibniz's rule to the formulas in (3.12), we have the following lemma.

Lemma 3.3. *Let $0 < \varepsilon < \pi/2$. Let α be any positive number and $\gamma_{\alpha,\varepsilon}$ be a number given in Lemma* 3.1. *Then, when $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geqq \alpha$ and $|\xi'| \leqq \gamma_{\alpha,\varepsilon}$ there hold the following estimates for any multi-indices α', β', integer $k \geqq 0$, $j=1,2$, $m=1,2,3,4$ and $x_n \in [0,1]$:*

$$
\left|\partial_{\xi'}^{\alpha'}\left((\xi')^{\beta'}\partial_n^k\left[\frac{L_{j,1}+L_{j,2}}{\det L}e^{-A(1-x_n)}\right]\right)\right| \leqq C_{\alpha',\beta',\varepsilon}|\xi'|^{-|\alpha'|};
$$

$$
\left|\partial_{\xi'}^{\alpha'}\left((\xi')^{\beta'}\partial_n^k\left[\frac{L_{j+2,1}+L_{j+2,2}}{\det L}e^{-A(1-x_n)}\right]\right)\right| \leqq C_{\alpha',\beta',\varepsilon}|\lambda|^{-1/2}|\xi'|^{-|\alpha'|};
$$

$$
\left|\partial_{\xi'}^{\alpha'}\left((\xi')^{\beta'}\partial_n^k\left[\frac{L_{j,2}A}{\det L}D(A,x_n)\right]\right)\right| \leqq C_{\alpha',\beta',\varepsilon}|\xi'|^{-|\alpha'|};
$$

$$
\left|\partial_{\xi'}^{\alpha'}\left((\xi')^{\beta'}\partial_n^k\left[\frac{L_{j+2,2}A}{\det L}D(A,x_n)\right]\right)\right| \leqq C_{\alpha',\beta',\varepsilon}|\lambda|^{-1/2}|\xi'|^{-|\alpha'|};
$$

$$
\left|\partial_{\xi'}^{\alpha'}\left((\xi')^{\beta'}\partial_n^k\left[\frac{L_{m,j+2}}{\det L}e^{-B\tau_j(x_n)}\right]\right)\right| \leqq C_{\alpha',\beta',\varepsilon,k}|\lambda|^{k/2-1/2}|\xi'|^{-|\alpha'|}.
$$

3.2 Case 2.

We shall consider the case where $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfy the condition: $|\xi'| \geqq \gamma$ and $|\xi'|^2 \leqq \beta_{\gamma,\varepsilon}|\lambda|$. Here, γ is an arbitrary positive number and $\beta_{\gamma,\varepsilon}$ is a positive constant depending on γ and ε, which will be chosen so small that there hold the inequalities (3.17) and (3.18), below.

Put

$$
\begin{aligned}
(3.15)\qquad l_2(A,B) = {} & (1-e^{-2A})(1-e^{-2B})(1+(AB^{-1})^2) \\
& - 2AB^{-1}(1+e^{-2A})(1+e^{-2B}) + 8AB^{-1}e^{-A}e^{-B},
\end{aligned}
$$

and then recalling (2.18) we have

$$
(3.16)\qquad \det L = -B^2 l_2(A,B).
$$

In view of (2.8), noting that $A \geqq \gamma$ and $\operatorname{Re} B \geqq c_\varepsilon\sqrt{|\lambda|+|\xi'|^2} \geqq c_\varepsilon\gamma$, we have

$$
|l_2(A,B)| \geqq (1-e^{-2\gamma})(1-e^{-2c_\varepsilon\gamma})(1-|AB^{-1}|^2) - 16|AB^{-1}|,
$$

because $\operatorname{Re} B > 0$. When $|\xi'|^2 \leqq \beta_{\gamma,\varepsilon}|\lambda|$, we have

$$|AB^{-1}| \leqq \frac{|\xi'|}{c_\varepsilon\sqrt{|\lambda| + |\xi'|^2}} \leqq c_\varepsilon^{-1}\sqrt{\beta_{\gamma,\varepsilon}}.$$

If we choose $\beta_{\gamma,\varepsilon} > 0$ so small that there hold the inequalities:

$$c_\varepsilon^{-1}\sqrt{\beta_{\gamma,\varepsilon}} \leqq 1/\sqrt{2}, \tag{3.17}$$

$$16c_\varepsilon^{-1}\sqrt{\beta_{\gamma,\varepsilon}} \leqq \frac{1}{4}(1 - e^{-2\gamma})(1 - e^{-2c_\varepsilon\gamma}), \tag{3.18}$$

then we have

$$|l_2(A, B)| \geqq \frac{1}{4}(1 - e^{-2\gamma})(1 - e^{-2c_\varepsilon\gamma}). \tag{3.19}$$

In this case, by (2.16) and (2.17) we represent $\hat{v}_n(\lambda, \xi', x_n)$ as follows:

$$\begin{aligned}\hat{v}_n(\lambda, \xi', x_n) = {} & \sum_{m=1}^{2}\sum_{j=1}^{2}\left\{\frac{L_{j,m}}{\det L}e^{-A\tau_m(x_n)} + \frac{L_{j,m+2}}{\det L}e^{-B\tau_m(x_n)}\right\}\hat{U}_n(\cdot, j-1) \\ & + \sum_{m=1}^{2}\sum_{j=1}^{2}\left\{\frac{L_{j+2,m}}{\det L}e^{-A\tau_m(x_n)} + \frac{L_{j+2,m+2}}{\det L}e^{-B\tau_m(x_n)}\right\}i\xi' \cdot \widehat{U'}(\cdot, j-1).\end{aligned} \tag{3.20}$$

Then, by (2.18) and (3.16), each coefficient in (3.20) is represented as follows:

$$\begin{aligned}
\frac{L_{1,1}}{\det L} = \frac{L_{2,2}}{\det L} &= \frac{(1 - AB^{-1})e^{-A}e^{-2B} + 2AB^{-1}e^{-B} - (1 + AB^{-1})e^{-A}}{l_2(A, B)}, \\
\frac{L_{1,2}}{\det L} = \frac{L_{2,1}}{\det L} &= \frac{(1 - AB^{-1}) + 2AB^{-1}e^{-A}e^{-B} - (1 + AB^{-1})e^{-2B}}{l_2(A, B)}, \\
\frac{L_{1,3}}{\det L} = \frac{L_{2,4}}{\det L} &= \frac{((AB^{-1})^2 - AB^{-1})e^{-2A}e^{-B} + 2AB^{-1}e^{-A} - ((AB^{-1})^2 + AB^{-1})e^{-B}}{l_2(A, B)}, \\
\frac{L_{1,4}}{\det L} = \frac{L_{2,3}}{\det L} &= \frac{((AB^{-1})^2 - AB^{-1}) + 2AB^{-1}e^{-A}e^{-B} - ((AB^{-1})^2 + AB^{-1})e^{-2A}}{l_2(A, B)}, \\
\frac{L_{3,1}}{\det L} = -\frac{L_{4,2}}{\det L} &= \frac{-(1 - AB^{-1})e^{-A}e^{-2B} + 2e^{-B} - (1 + AB^{-1})e^{-A}}{Bl_2(A, B)}, \\
\frac{L_{3,2}}{\det L} = -\frac{L_{4,1}}{\det L} &= \frac{1 - AB^{-1} - 2e^{-A}e^{-B} + (1 + AB^{-1})e^{-2B}}{Bl_2(A, B)}, \\
\frac{L_{3,3}}{\det L} = -\frac{L_{4,4}}{\det L} &= \frac{(1 - AB^{-1})e^{-2A}e^{-B} + 2AB^{-1}e^{-A} - (1 + AB^{-1})e^{-B}}{Bl_2(A, B)}, \\
\frac{L_{3,4}}{\det L} = -\frac{L_{4,3}}{\det L} &= \frac{-(1 - AB^{-1}) - 2AB^{-1}e^{-A}e^{-B} + (1 + AB^{-1})e^{-2A}}{Bl_2(A, B)}.
\end{aligned} \tag{3.21}$$

By Lemma 3.2 and Leibniz's formula, we have

$$|\partial_{\xi'}^{\alpha'} AB^{-1}| \leqq C_{\alpha',\varepsilon}(|\lambda| + |\xi'|^2)^{-1/2}|\xi'|^{1-|\alpha'|}, \tag{3.22}$$

and therefore by Lemma 3.2, Leibniz's formula, (3.13) and (3.19), we have

$$|\partial_{\xi'}^{\alpha'}(l_2(A,B)B^k)^{-1}| \leqq C_{\alpha',\varepsilon,k}(|\lambda| + |\xi'|^2)^{-k/2}|\xi'|^{-|\alpha'|}, \tag{3.23}$$

when $|\xi'| \geqq \gamma$, $\lambda \in \Sigma_\varepsilon$ and $|\xi'|^2 \leqq \beta_{\gamma,\varepsilon}|\lambda|$. Applying Lemma 3.2, (3.22), (3.23) and Leibniz's formula to (3.21), we have the following lemma.

Lemma 3.4. *Let $0 < \varepsilon < \pi/2$. Let γ be any positive number and $\beta_{\gamma,\varepsilon}$ be a constant for which* (3.17) *and* (3.18) *hold. Then, when $|\xi'| \geqq \gamma$, $\lambda \in \Sigma_\varepsilon$ and $|\xi'|^2 \leqq \beta_{\gamma,\varepsilon}|\lambda|$ we have the following estimates for any multi-index α'*:

$$\left|\partial_{\xi'}^{\alpha'}\frac{L_{j,k}}{\det L}\right| \leqq C_{\alpha',\varepsilon,\gamma}|\xi'|^{-|\alpha'|}, \qquad j = 1,2,\ k = 1,2,3,4,$$

$$\left|\partial_{\xi'}^{\alpha'}\frac{L_{j+2,k}}{\det L}\right| \leqq C_{\alpha',\varepsilon,\gamma}|\lambda|^{-1/2}|\xi'|^{-|\alpha'|}, \quad j = 1,2,\ k = 1,2,3,4.$$

3.3. Case 3.

We shall consider the case where $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfy the condition: $\beta|\lambda| \leqq |\xi'|^2$ and $|\xi'| \geqq R_{\beta,\varepsilon}$. Here, β is an arbitrary positive number and $R_{\beta,\varepsilon} > 0$ is a positive constant depending on β and ε, which will be chosen so large that there holds (3.29) below.

Since

$$\det L = -(A-B)^2 + (A-B)^2(e^{-2A} + e^{-2B}) - (A-B)^2e^{-2A}e^{-2B} + 4AB(e^{-A} - e^{-B})^2, \tag{3.24}$$

if we put

$$\frac{e^{-Ay} - e^{-By}}{A-B} = -d(A,B,y), \quad d(A,B,y) = y\int_0^1 e^{-(\theta A + (1-\theta)B)y}\,d\theta, \tag{3.25}$$

$$l_3(A,B) = 1 - (e^{-2A} + e^{-2B}) + e^{-2A}e^{-2B} - 4ABd(A,B,1)^2, \tag{3.26}$$

we have

$$\det L = -(A-B)^2 l_3(A,B). \tag{3.27}$$

Note that $\mathrm{Re}(\theta A + (1-\theta)B) \geqq c_\varepsilon A$ for $0 \leqq \theta \leqq 1$, which follows from (2.8). By (2.8) and Lemma 3.2 we have

$$|d(A,B,y)| \leqq e^{-c_\varepsilon Ay}, \quad |e^{-2B}| \leqq e^{-2c_\varepsilon A}. \tag{3.28}$$

When $\beta|\lambda| \leqq |\xi'|^2$, by (3.28) we have

$$|4ABd(A,B,1)^2| \leqq 4\sqrt{1+\beta^{-1}}A^2e^{-2c_\varepsilon A} \leqq 8\sqrt{1+\beta^{-1}}(c_\varepsilon)^{-2}e^{-c_\varepsilon A},$$

and therefore when $\beta|\lambda| \leqq |\xi'|^2$ and $|\xi'| \geqq R$ by (3.28) we have

$$|l_3(A,B)| \geqq 1 - (e^{-2R} + e^{-2c_\varepsilon R} + e^{-2(1+c_\varepsilon)R} + 8\sqrt{1+\beta^{-1}}(c_\varepsilon)^{-2}e^{-c_\varepsilon R}).$$

If we choose $R = R_{\beta,\varepsilon}$ so large that

$$e^{-2R} + e^{-2c_\varepsilon R} + e^{-2(1+c_\varepsilon)R} + 8\sqrt{1+\beta^{-1}}(c_\varepsilon)^{-2}e^{-c_\varepsilon R} \leqq 1/2,$$

we have

$$|l_3(A,B)| \geqq 1/2 \quad \text{when } \beta|\lambda| \leqq |\xi'|^2, |\xi'| \geqq R_{\beta,\varepsilon}. \tag{3.29}$$

In Case 3, we transform (2.17) into the following formula:

$$\begin{aligned}\hat{v}_n(\lambda,\xi',x_n) = \sum_{j=1}^{2}\sum_{m=1}^{2}\Bigg\{&\frac{L_{j,m}+L_{j,m+2}}{\det L}e^{-A\tau_m(x_n)} \\ &+\frac{L_{j,m+2}(A-B)}{A\det L}Ad(A,B,\tau_m(x_n))\Bigg\}\hat{U}_n(\cdot\,,j-1) \\ +\sum_{j=1}^{2}\sum_{m=1}^{2}\Bigg\{&\frac{L_{j+2,m}+L_{j+2,m+2}}{\det L}e^{-A\tau_m(x_n)} \\ &+\frac{L_{j+2,m+2}(A-B)}{A\det L}Ad(A,B,\tau_m(x_n))\Bigg\}i\xi'\cdot\widehat{U'}(\cdot\,,j-1).\end{aligned} \tag{3.30}$$

By (2.18) and (3.27), each coefficient in (3.30) is represented as follows:

$$\begin{aligned}\frac{L_{1,1}+L_{1,3}}{\det L} &= \frac{L_{2,2}+L_{2,4}}{\det L} = \frac{-(A+Be^{-A}e^{-B})d(A,B,1)+e^{-2A}e^{-B}-e^{-A}}{l_3(A,B)}, \\ \frac{L_{1,2}+L_{1,4}}{\det L} &= \frac{L_{2,1}+L_{2,3}}{\det L} = \frac{1-(e^{-A}-Bd(A,B,1))^2-ABd(A,B,1)^2}{l_3(A,B)}, \\ \frac{L_{3,1}+L_{3,3}}{\det L} &= -\frac{L_{4,2}+L_{4,4}}{\det L} = \frac{-(1-e^{-A}e^{-B})d(A,B,1)}{l_3(A,B)}, \\ \frac{L_{3,2}+L_{3,4}}{\det L} &= -\frac{L_{4,1}+L_{4,3}}{\det L} = \frac{(A+B)d(A,B,1)^2}{l_3(A,B)}, \\ \frac{L_{1,3}(A-B)}{A\det L} &= \frac{L_{2,4}(A-B)}{A\det L} = \frac{-e^{-B}(1-e^{-2A})-2Bd(A,B,1)}{l_3(A,B)}, \\ \frac{L_{1,4}(A-B)}{A\det L} &= \frac{L_{2,3}(A-B)}{A\det L} = \frac{1-e^{-2A}+2Be^{-A}d(A,B,1)}{l_3(A,B)}, \\ \frac{L_{3,3}(A-B)}{A\det L} &= -\frac{L_{4,4}(A-B)}{A\det L} = \frac{e^{-B}(1-e^{-2A})-2Ad(A,B,1)}{Al_3(A,B)}, \\ \frac{L_{3,4}(A-B)}{A\det L} &= -\frac{L_{4,3}(A-B)}{A\det L} = \frac{(1-e^{-2A})-2Ae^{-A}d(A,B,1)}{Al_3(A,B)}.\end{aligned} \tag{3.31}$$

To estimate these coefficients, we use the following lemma.

LEMMA 3.5. *Let $0 < \varepsilon < \pi/2$. Let β be any positive number and $R_{\beta,\varepsilon}$ a positive constant depending on β and ε, for which* (3.29) *holds. If $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfying the condition: $\beta|\lambda| \leqq |\xi'|^2$ and $|\xi'| \geqq R_{\beta,\varepsilon}$, then, for any multi-index α' and $y \geqq 0$ there hold the following estimates*:

$$|\partial_{\xi'}^{\alpha'}(\partial_y^j A^k B^m d(A,B,y))| \leqq C_{\alpha',\varepsilon,\beta,k}|\xi'|^{j+k+m-1-|\alpha'|}e^{-(1/4)c_\varepsilon|\xi'|y},$$

where $k \geqq 1$, $m \geqq 0$, $j = 0,1,2$, and

$$|\partial_{\xi'}^{\alpha'} l_3(A,B)^{-1}| \leqq C_{\alpha',\varepsilon,\beta}|\xi'|^{-|\alpha'|}.$$

PROOF. Since $\mathrm{Re}(\theta A + (1-\theta)B) \geqq c_\varepsilon A$ as follows from (2.8), by Lemma 3.2 and Leibniz's rule we have

$$|\partial_{\xi'}^{\alpha'} d(A,B,y)| \leqq C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|} y e^{-(c_\varepsilon/2)|\xi'|y}.$$

Then, by Lemma 3.2 and Leibniz's rule we have

$$\begin{aligned}(3.32)\qquad |\partial_{\xi'}^{\alpha'} A^k B^m d(A,B,y)| &\leqq C_{\alpha',\varepsilon}|\xi'|^{k+m-|\alpha'|} y e^{-(c_\varepsilon/2)|\xi'|y}\\ &\leqq (4/c_\varepsilon)C_{\alpha',\varepsilon}|\xi'|^{k+m-1-|\alpha'|}e^{-(1/4)c_\varepsilon|\xi'|y},\end{aligned}$$

which shows the first relation for $j = 0$, because $|B| \leqq \sqrt{1+\beta^{-1}}A$. Since

$$\partial_y d(A,B,y) = -Ad(A,B,y) + e^{-By}, \quad \partial_y^2 d(A,B,y) = A^2 d(A,B,y) - (A+B)e^{-By}$$

by Lemma 3.2, (3.32) and Leibniz's formula, we have the first relation for $j = 1$ and 2. By Lemma 3.2, (3.13) and (3.29), we have the second relation, which completes the proof of the lemma. □

Applying Lemmas 3.2 and 3.5 to (3.31), we have the following lemma concerning the estimate for the coefficient in (3.31).

LEMMA 3.6. *Let $0 < \varepsilon < \pi/2$. Let β and $R_{\beta,\varepsilon}$ be the same as in Lemma* 3.5. *If $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ satisfy the condition: $\beta|\lambda| \leqq |\xi'|^2$ and $|\xi'| \geqq R_{\beta,\varepsilon}$, then for any multi-index α', and integer $m \geqq 0$ there hold the following estimates*:

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left[\frac{L_{j,l}+L_{j,l+2}}{\det L}e^{-A\tau_l(x_n)}\right]\right| \leqq C_{\alpha',\varepsilon,\beta,m}|\xi'|^{m-|\alpha'|}e^{-(1/2)|\xi'|\tau_l(x_n)};$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left[\frac{(L_{j+2,l}+L_{j+2,l+2})i\xi_k}{\det L}e^{-A\tau_l(x_n)}\right]\right| \leqq C_{\alpha',\varepsilon,\beta,m}|\xi'|^{m-|\alpha'|}e^{-(1/2)|\xi'|\tau_l(x_n)};$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left[\frac{L_{j,l+2}(A-B)}{A\det L}Ad(A,B,\tau_l(x_n))\right]\right| \leqq C_{\alpha',\varepsilon,\beta,m}|\xi'|^{m-|\alpha'|}e^{-(1/4)c_\varepsilon|\xi'|\tau_l(x_n)};$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left[\frac{L_{j+2,l+2}(A-B)i\xi_k}{A\det L}Ad(A,B,\tau_l(x_n))\right]\right| \leqq C_{\alpha',\varepsilon,\beta,m}|\xi'|^{m-|\alpha'|}e^{-(1/4)c_\varepsilon|\xi'|\tau_l(x_n)};$$

where $j = 1,2$, $l = 1,2$, $k = 1,\dots,n-1$ and $\tau_l(x_n) \geqq 0$.

§4. Estimates for v_n.

In this section, we shall derive an estimate for v_n defined in (2.17). For the sake of notational brevity, we put

$$I_{\lambda,p}(v,D) = |\lambda|\,\|v\|_{L_p(D)} + |\lambda|^{1/2}\|\nabla v\|_{L_p(D)} + \|\nabla^2 v\|_{L_p(D)}.$$

The following theorem is a main result of this section.

THEOREM 4.1. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. Let v_n be a function defined in* (2.17). *Then, there exists a constant C depending only on p, ε and λ_0 such that there holds an estimate*:

$$I_{\lambda,p}(v_n,\Omega) \leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)} \quad \text{provided that } \lambda \in \Sigma_\varepsilon \text{ and } |\lambda| \geqq \lambda_0.$$

According to the classification in §3, we divide v_n into four parts. Let λ_0 be a given positive number. In Case 1, we take $\alpha = \lambda_0$ and put $\gamma_1 = \gamma_{\lambda_0,\varepsilon}$. Next, in Case 2, we take $\gamma = \gamma_{\lambda_0,\varepsilon}/2$ and put $\gamma_2 = \beta_{\gamma,\varepsilon}$ with $\gamma = \gamma_{\lambda_0,\varepsilon}/2$. Finally, in Case 3, we take $\beta = \gamma_2/2$ and put $R = R_{\beta,\varepsilon}$ with $\beta = \gamma_2/2$. In this situation, we have the estimates of coefficients of v_n stated in Lemma 3.3 when $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geqq \lambda_0$ and $|\xi'| \leqq \gamma_1$, those stated in Lemma 3.4 when $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geqq \lambda_0$, $|\xi'|^2 \leqq \gamma_2|\lambda|$ and $|\xi'| \geqq \gamma_1/2$, and those stated in Lemma 3.6 when $\lambda \in \Sigma_\varepsilon$, $|\xi'|^2 \geqq (\gamma_2/2)|\lambda|$ and $|\xi'| \geqq R$. Let $\varphi(\xi')$ be a function in $C_0^\infty(\boldsymbol{R}^{n-1})$ such that $\varphi(\xi') = 1$ for $|\xi'| \leqq 1/2$ and $\varphi(\xi') = 0$ for $|\xi'| \geqq 3/4$. Put

$$\begin{aligned}
&\varphi_1(\lambda,\xi') = \varphi(\xi'/\gamma_1), \quad \varphi_2(\lambda,\xi') = (1-\varphi(\xi'/\gamma_1))\varphi(\xi'/(\gamma_2|\lambda|)^{1/2}),\\
&\varphi_3(\lambda,\xi') = (1-\varphi(\xi'/\gamma_1))(1-\varphi(\xi'/(\gamma_2|\lambda|)^{1/2}))\varphi(\xi'/R),\\
&\varphi_4(\lambda,\xi') = (1-\varphi(\xi'/\gamma_1))(1-\varphi(\xi'/(\gamma_2|\lambda|)^{1/2}))(1-\varphi(\xi'/R)),\\
&V_j(\lambda,\xi',x_n) = \varphi_j(\lambda,\xi')\hat{v}_n(\lambda,\xi',x_n), \quad j = 1,2,3,4.
\end{aligned}$$

Since $\sum_{j=1}^4 \varphi_j(\lambda,\xi') = 1$, we have

$$v_n(x) = \sum_{j=1}^4 W_j(x), \quad W_j(x) = \mathscr{F}_{\xi'}^{-1}[V_j(\lambda,\xi',x_n)](x').$$

We shall prove that each of $W_j(x)$ satisfies the estimate:

$$I_{\lambda,p}(W_j,\Omega) \leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)} \quad \text{provided that } \lambda \in \Sigma_\varepsilon \text{ and } |\lambda| \geqq \lambda_0. \tag{4.1}$$

We start with the estimate for W_1. Since $\operatorname{supp}\varphi_1(\lambda,\xi') \subset \{\xi' \in \boldsymbol{R}^{n-1} \mid |\xi'| \leqq 3\gamma_1/4\}$, applying Lemma 3.3 and Theorem 2.2 to (3.9), we have

$$\begin{aligned}
(4.2)\quad &|\lambda|\,\|\partial_{x'}^{\beta'} W_1(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}\\
&\leqq \sum_{a=0}^1 C_{\lambda_0,\varepsilon,p,\beta'}\left\{|\lambda|\,\|U_n(\cdot,a)\|_{L_p(\boldsymbol{R}^{n-1})} + \sum_{k=1}^{n-1}|\lambda|^{1/2}\|U_k(\cdot,a)\|_{L_p(\boldsymbol{R}^{n-1})}\right\};
\end{aligned}$$

$$(4.3) \qquad |\lambda|^{1/2}\|\partial_n\partial_{x'}^{\beta'}W_1(\cdot\,,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}+\|\partial_n^2\partial_{x'}^{\beta'}W_1(\cdot\,,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}$$
$$\leqq \sum_{k=1}^{n}\sum_{a=0}^{1} C_{\lambda_0,\varepsilon,p,\beta'}|\lambda|^{1/2}\|U_k(\cdot\,,a)\|_{L_p(\boldsymbol{R}^{n-1})}$$

for any multi-index β' when $\lambda\in\Sigma_\varepsilon$ and $|\lambda|\geqq\lambda_0$. Inserting the estimates obtained in Lemma 4.2 below into (4.2) and (4.3), we see that W_1 satisfies (4.1).

Lemma 4.2. *Let $1<p<\infty$, $0<\varepsilon<\pi/2$ and $\lambda_0>0$. Let $U_j(x)$ be the functions defined in* (2.4) *and* (2.5). *Then, we have the following estimates*:

$$\sum_{a=0}^{1}\sum_{k=1}^{n}|\lambda|^{m/2}\|U_k(\cdot\,,a)\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}, \quad m=0,1,$$

$$\sum_{a=0}^{1}|\lambda|\,\|\mathscr{F}_{\xi'}^{-1}[A^m\hat{U}_n(\xi',a)]\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_{p,\varepsilon,m,\lambda_0}\|\boldsymbol{f}\|_{L_p(\Omega)}, \quad m=-1,0,1,\dots,$$

provided that $\lambda\in\Sigma_\varepsilon$ and $|\lambda|\geqq\lambda_0$.

Proof. By the usual trace theorem, we have

$$\sum_{a=0}^{1}\sum_{k=1}^{n}|\lambda|^{1/2}\|U_k(\cdot\,,a)\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C|\lambda|^{1/2}\|\boldsymbol{U}\|_{W_p^1(\boldsymbol{R}^n)},$$

which combined with Lemma 2.3 implies the first relation, because $|\lambda|\geqq\lambda_0$.

To prove the second relation, we use the formula (2.5) and our special extension of $\boldsymbol{f}$. By the definition of f_j^e and f_j^o, we have

$$\begin{aligned}\hat{U}_n(\xi',0)=\frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\lambda+|\xi|^2)|\xi|^2}&\left\{\int_0^{2/3}(e^{-iy_n\xi_n}-e^{iy_n\xi_n})\varphi(y_n)\hat{f}_n(\xi',y_n)\,dy_n\right.\\&\left.+\int_{1/3}^{1}(e^{-iy_n\xi_n}-e^{-i(2-y_n)\xi_n})(1-\varphi(y_n))\hat{f}_n(\xi',y_n)\,dy_n\right\}d\xi_n\\-\sum_{k=1}^{n-1}\frac{\xi_k}{2\pi}\int_{-\infty}^{\infty}\frac{\xi_n}{(\lambda+|\xi|^2)|\xi|^2}&\left\{\int_0^{2/3}(e^{-iy_n\xi_n}+e^{iy_n\xi_n})\varphi(y_n)\hat{f}_k(\xi',y_n)\,dy_n\right.\\&\left.+\int_{1/3}^{1}(e^{-iy_n\xi_n}+e^{-i(2-y_n)\xi_n})(1-\varphi(y_n))\hat{f}_k(\xi',y_n)\,dy_n\right\}d\xi_n.\end{aligned}$$

By the residue theorem, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ia\xi_n}}{(\lambda+|\xi|^2)|\xi|^2}\,d\xi_n=\frac{1}{2\lambda}\left(\frac{e^{-|a|A}}{A}-\frac{e^{-|a|B}}{B}\right),$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ia\xi_n}\xi_n}{(\lambda+|\xi|^2)|\xi|^2}\,d\xi_n=\frac{i}{2\lambda}\,\frac{a}{|a|}\,(e^{-|a|A}-e^{-|a|B})$$

for $a \in \boldsymbol{R} \backslash \{0\}$, which inserted into the representation formula of $\hat{U}_n(\xi', 0)$ implies that

$$(4.4)\qquad \hat{U}_n(\xi', 0) = \frac{A^2}{2\lambda} \int_{1/3}^1 \left\{ \left(\frac{e^{-y_n A}}{A} - \frac{e^{-y_n B}}{B} \right) - \left(\frac{e^{-(2-y_n)A}}{A} - \frac{e^{-(2-y_n)B}}{B} \right) \right\} (1 - \varphi(y_n)) \hat{f}_n(\xi', y_n)\, dy_n$$
$$+ \sum_{k=1}^{n-1} \frac{i\xi_k}{2\lambda} \int_{1/3}^1 \{(e^{-y_n A} - e^{-y_n B}) + (e^{-(2-y_n)A} - e^{-(2-y_n)B})\}(1 - \varphi(y_n)) \hat{f}_k(\xi', y_n)\, dy_n.$$

Employing the same argument, we have also

$$(4.5)\qquad \hat{U}_n(\xi', 1) = \frac{A^2}{2\lambda} \int_0^{2/3} \left\{ \left(\frac{e^{-(1-y_n)A}}{A} - \frac{e^{-(1-y_n)B}}{B} \right) - \left(\frac{e^{-(1+y_n)A}}{A} - \frac{e^{-(1+y_n)B}}{B} \right) \right\} \varphi(y_n) \hat{f}_n(\xi', y_n)\, dy_n$$
$$- \sum_{k=1}^{n-1} \frac{i\xi_k}{2\lambda} \int_0^{2/3} \{(e^{-(1-y_n)A} - e^{-(1-y_n)B}) + (e^{-(1+y_n)A} - e^{-(1+y_n)B})\} \varphi(y_n) \hat{f}_k(\xi', y_n)\, dy_n.$$

By Lemma 3.2 we have

$$|\partial_{\xi'}^{\alpha'}(A^k e^{-tA})| \leqq C_{\alpha',k} |\xi'|^{k-|\alpha'|} e^{-(t/2)A} \leqq C_{\alpha',k} |\xi'|^{-|\alpha'|},$$
$$|\partial_{\xi'}^{\alpha'}(A^k B^{-1} e^{-tB})| \leqq C_{\alpha',k,\lambda_0} |\xi'|^{k-|\alpha'|} e^{-(t/2)c_\varepsilon A} \leqq C_{\alpha',k,\lambda_0} |\xi'|^{-|\alpha'|},$$

when $t \geqq 1/3$. If $1/3 \leqq y_n \leqq 1$, then $y_n \geqq 1/3$ and $2 - y_n \geqq 1$ in (4.4). And also, if $0 \leqq y_n \leqq 2/3$, then $1 - y_n \geqq 1/3$ and $1 + y_n \geqq 1$ in (4.5). Therefore, applying Theorem 2.2 to (4.4) and (4.5) with respect to x' variables, we have

$$\|\mathscr{F}_{\xi'}^{-1}[\lambda A^k \hat{U}_n(\xi', a)]\|_{L_p(\boldsymbol{R}^{n-1})} \leqq C \int_0^1 \|\boldsymbol{f}(\cdot, x_n)\|_{L_p(\boldsymbol{R}^{n-1})}\, dx_n \leqq C \|\boldsymbol{f}\|_{L_p(\Omega)},$$

for any integer $k \geqq -1$ and $a = 0, 1$, which completes the proof of the lemma. □

Now, we shall estimate W_2. We divide W_2 into the following four parts:

$$W_2 = W_{1g} + W_{2g} + W_{1h} + W_{2h}$$

where

$$W_{1g}(x) = \sum_{m=1}^2 \sum_{j=1}^2 \mathscr{F}_{\xi'}^{-1} \left[\varphi_2(\lambda, \xi') \frac{L_{j,m}}{\det L} e^{-A\tau_m(x_n)} \hat{U}_n(\xi', j-1) \right](x'),$$

$$W_{2g}(x) = \sum_{m=1}^2 \sum_{j=1}^2 \mathscr{F}_{\xi'}^{-1} \left[\varphi_2(\lambda, \xi') \frac{L_{j,m+2}}{\det L} e^{-B\tau_m(x_n)} \hat{U}_n(\xi', j-1) \right](x'),$$

$$W_{1h}(x) = \sum_{m=1}^{2}\sum_{j=1}^{2} \mathscr{F}_{\xi'}^{-1}\left[\varphi_2(\lambda,\xi')\frac{L_{j+2,m}}{\det L} e^{-A\tau_m(x_n)} i\xi'\cdot \widehat{U'}(\xi', j-1)\right](x'),$$

$$W_{2h}(x) = \sum_{m=1}^{2}\sum_{j=1}^{2} \mathscr{F}_{\xi'}^{-1}\left[\varphi_2(\lambda,\xi')\frac{L_{j+2,m+2}}{\det L} e^{-B\tau_m(x_n)} i\xi'\cdot \widehat{U'}(\xi', j-1)\right](x'),$$

(cf. (3.20) and (2.16)). Since $\operatorname{supp}\varphi_2(\lambda,\xi') \subset \{\xi' \in \boldsymbol{R}^{n-1} \mid \gamma_1/2 \leqq |\xi'| \leqq 3(\gamma_2|\lambda|)^{1/2}/4\}$, by Lemma 3.2, Lemma 3.4 and Leibniz's rule, we have

$$\left|\partial_{\xi'}^{\alpha'}\left(\varphi_2(\lambda,\xi')\frac{L_{j,k}}{\det L}e^{-yA}\right)\right| \leqq C_{\alpha',\lambda_0,\varepsilon}|\xi'|^{-|\alpha'|}, \tag{4.6}$$

when $y \geqq 0$, $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geqq \lambda_0$ and $\xi' \in \boldsymbol{R}^{n-1}$. By (4.6), Theorem 2.2 and Lemma 4.2, we have

$$|\lambda|\,\|W_{1g}(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})} \leqq C_{p,\lambda_0,\varepsilon}|\lambda|\sum_{j=0}^{1}\|U_n(\cdot,j)\|_{L_p(\boldsymbol{R}^{n-1})} \leqq C_{p,\lambda_0,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{4.7}$$

In order to continue the estimate, we shall use the following lemma.

Lemma 4.3. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$ *and* $\lambda_0 > 0$. *Put* $\boldsymbol{R}_1 = (-\infty, 1]$, $\boldsymbol{R}_2 = [0,\infty)$ *and* $\boldsymbol{R}^n_m = \boldsymbol{R}^{n-1}\times \boldsymbol{R}_m$, $m = 1,2$.

(1) *Let* D *be a set in* $\boldsymbol{C}$ *and* $\Phi(\lambda,\xi',x_n)$ *be a function defined on* $D\times(\boldsymbol{R}^{n-1}\backslash\{0\})\times\boldsymbol{R}_2$. *Assume that* $\Phi(\lambda,\xi',x_n)$ *belongs to* $C^\infty((\boldsymbol{R}^{n-1}\backslash\{0\})\times\boldsymbol{R}_2)$ *for each* $\lambda\in D$ *and satisfies the estimate*:

$$|\partial_{\xi'}^{\alpha'}\partial_n^k\Phi(\lambda,\xi',x_n)| \leqq M|\xi'|^{k-|\alpha'|}e^{-\delta|\xi'|x_n}, \tag{4.8}$$

for any multi-index α' *with* $|\alpha'|\leqq n-1$, $\xi'\in\boldsymbol{R}^{n-1}\backslash\{0\}$, $\lambda\in D$, $k=0,1,2$ *and* $x_n\in\boldsymbol{R}_2$ *with some positive constants* M *and* δ. *Given* $g(x)\in W_p^2(\boldsymbol{R}^n)$ *and* $a\in\boldsymbol{R}$, *we put*

$$V_a^m(x) = \mathscr{F}_{\xi'}^{-1}[\Phi(\lambda,\xi',\tau_m(x_n))\hat{g}(\xi',a)](x'), \quad m=1,2.$$

Then, there hold the following estimates:

$$\|V_a^m(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})} \leqq C_pM\|g(\cdot,a)\|_{L_p(\boldsymbol{R}^{n-1})}, \quad x_n\in\boldsymbol{R}_m, \tag{4.9}$$

$$\|\nabla^k V_a^m\|_{L_p(\boldsymbol{R}^n_m)} \leqq C_pM\|\nabla^k g\|_{L_p(\boldsymbol{R}^n)}, \quad k=1,2. \tag{4.10}$$

(2) *Let* $\Phi(\lambda,\xi')$ *be a function defined on* $\Sigma_\varepsilon\times(\boldsymbol{R}^{n-1}\backslash\{0\})$. *Assume that* $\Phi(\lambda,\xi')$ *belongs to* $C^\infty(\boldsymbol{R}^{n-1}\backslash\{0\})$ *for each* $\lambda\in\Sigma_\varepsilon$ *and satisfies the estimate*:

$$|\partial_{\xi'}^{\alpha'}\Phi(\lambda,\xi')| \leqq M|\xi'|^{-|\alpha'|}, \quad |\alpha'|\leqq n-1 \tag{4.11}$$

for some positive constant M. *Given* $g(x)\in W_p^2(\boldsymbol{R}^n)$, *we put*

$$W_a^m(x) = \mathscr{F}_{\xi'}^{-1}[\Phi(\lambda,\xi')e^{-B\tau_m(x_n)}\hat{g}(\xi',a)](x').$$

Then, we have the following estimates:

$$I_{\lambda,p}(W_a^m,\boldsymbol{R}^n_m) \leqq C_{p,\lambda_0,\varepsilon}MI_{\lambda,p}(g,\boldsymbol{R}^n). \tag{4.12}$$

REMARK 4.4. Let $\Psi(\lambda,\xi')$ be a function defined on $D\times(\boldsymbol{R}^{n-1}\backslash\{0\})$ which belongs to $C^\infty(\boldsymbol{R}^{n-1}\backslash\{0\})$ and satisfies the estimate: $|\partial_{\xi'}^{\alpha'}\Psi(\lambda,\xi')|\leqq M$ for any α' with $|\alpha'|\leqq n-1$ and $\lambda\in D$. By Lemma 3.2 and Leibniz's rule, we see that $\Psi(\lambda,\xi')e^{-Ax_n}$ satisfies (4.8).

PROOF. (1) By Theorem 2.2 we have (4.9) immediately. In order to prove (4.10), we consider the function

$$g_\delta(x)=\mathscr{F}_{\xi'}^{-1}[e^{-(\delta/2)Ax_n}\hat{g}(\xi',a)](x')=2\int_{\boldsymbol{R}^{n-1}}\frac{\partial E}{\partial x_n}(x'-y',(\delta/2)x_n)g(y',a)\,dy'$$

where E is a fundamental solution of $-\Delta$ of the form: $E(x)=c_n|x|^{-(n-2)}$ when $n\geqq 3$ and $E(x)=c_2\log|x|$ when $n=2$ with some constant c_n depending on n. By a theorem due to Agmon-Douglis-Nirenberg [**2**, Theorem 3.3] (cf. also Galdi [**4**, Theorem 9.6]), we have

$$\|\nabla^k g_\delta\|_{L_p(\boldsymbol{R}_+^n)}\leqq C_{p,\delta}\|\nabla^k g(\cdot,a+\cdot)\|_{L_p(\boldsymbol{R}_+^n)}\leqq C_{p,\delta}\|\nabla^k g\|_{L_p(\boldsymbol{R}^n)},\quad k=1,2. \tag{4.13}$$

Put $|D'|^k h(x')=\mathscr{F}_{\xi'}^{-1}[|\xi'|^k\hat{h}(\xi')](x')$. Noting that $|\xi'|=-\sum_{j=1}^{n-1}(i\xi_j|\xi'|^{-1})i\xi_j$, by Theorem 2.2 we have

$$\|\,|D'|^k h\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_{k,p}\sum_{|\beta'|=k}\|\partial_{x'}^{\beta'}h\|_{L_p(\boldsymbol{R}^{n-1})}. \tag{4.14}$$

By (4.13) and (4.14) we have

$$\|\partial_{x'}^{\beta'}|D'|^k g_\delta\|_{L_p(\boldsymbol{R}_+^n)}\leqq C_{p,\delta,k}\|\nabla^{k+|\beta'|}g\|_{L_p(\boldsymbol{R}^n)}. \tag{4.15}$$

To prove (4.10), we write

$$\partial_n^k V_a^2(x)=\mathscr{F}_{\xi'}^{-1}[\partial_n^k\Phi(\lambda,\xi',x_n)|\xi'|^{-k}e^{(\delta/2)Ax_n}\widehat{|D'|^k g_\delta}(\xi',x_n)](x').$$

By (4.8), Lemma 3.2 and Leibniz's rule, we have

$$|\partial_{\xi'}^{\alpha'}[\partial_n^k\Phi(\lambda,\xi',x_n)|\xi'|^{-k}e^{(\delta/2)Ax_n}]|\leqq C_\delta M|\xi'|^{-|\alpha'|}$$

for any multi-index α' with $|\alpha'|\leqq n-1$ and $x_n\in\boldsymbol{R}_2$, and therefore by Theorem 2.2 we have

$$\|\partial_{x'}^{\beta'}\partial_n^k V_a^2(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_\delta M\|\partial_{x'}^{\beta'}|D'|^k g_\delta(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})},$$

which combined with (4.15) implies (4.10) for $\boldsymbol{R}_2^n$. If we use the change of variable: $x_n=1-y_n$ we have also (4.10) for $\boldsymbol{R}_1^n$.

Now, we shall show (4.12). By Lemma 3.2, we have

$$|\partial_{\xi'}^{\alpha'}[\Phi(\lambda,\xi')e^{-Bx_n}]|\leqq C_{\alpha',\varepsilon}Me^{-(c_\varepsilon/2)|\lambda|^{1/2}x_n}|\xi'|^{-|\alpha'|}$$

for any α' with $|\alpha'|\leqq n-1$. By Theorem 2.2, we have

$$|\lambda|\,\|W_a^2(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_{p,\varepsilon}Me^{-(c_\varepsilon/2)|\lambda|^{1/2}x_n}\|g(\cdot,a)\|_{L_p(\boldsymbol{R}^{n-1})}$$

for $x_n \geqq 0$. Integrating the p-th power of the both sides over the interval $[0, \infty)$, we obtain

$$|\lambda|^p \| W_a^2 \|_{L_p(\boldsymbol{R}_+^n)}^p \leqq (C_{p,\varepsilon} M)^p 2 c_\varepsilon^{-1} |\lambda|^{p-1/2} \| g(\cdot, a) \|_{L_p(\boldsymbol{R}^{n-1})}^p.$$

On the other hand,

$$\| g(\cdot, a) \|_{L_p(\boldsymbol{R}^{n-1})}^p = -\int_a^\infty \frac{\partial}{\partial x_n} \left[\int_{\boldsymbol{R}^{n-1}} |g(x)|^p \, dx' \right] dx_n \leqq p \| g \|_{L_p(\boldsymbol{R}^n)}^{p-1} \| \partial_n g \|_{L_p(\boldsymbol{R}^n)}.$$

Combining these inequalities implies that

$$|\lambda| \, \| W_a^2 \|_{L_p(\Omega)} \leqq C_{p,\varepsilon} M \{ |\lambda| \, \| g \|_{L_p(\boldsymbol{R}^n)} + |\lambda|^{1/2} \| \nabla g \|_{L_p(\boldsymbol{R}^n)} \}. \tag{4.16}$$

To estimate the derivative of W_a^2, let us write

$$\begin{aligned} \partial_n^k W_a^2(x) &= |\lambda|^{k/2} \mathscr{F}_{\xi'}^{-1} [\Phi(\lambda, \xi')(-B)^k (|\lambda|^{k/2} + |\xi'|^k)^{-1} e^{-Bx_n} \hat{g}(\xi', a)](x') \\ &\quad + \mathscr{F}_{\xi'}^{-1} [\Phi(\lambda, \xi')(-B)^k (|\lambda|^{k/2} + |\xi'|^k)^{-1} e^{-Bx_n} e^{\delta A x_n} \widehat{|D'|^k g_\delta}(\xi', x_n)](x') \end{aligned}$$

where $\delta = c_\varepsilon / 4$. Since

$$|\partial_{\xi'}^{\alpha'} [\Phi(\lambda, \xi')(-B)^k (|\lambda|^{k/2} + |\xi'|^k)^{-1}]| \leqq C_{\alpha',\varepsilon} M |\xi'|^{-|\alpha'|},$$

$$|\partial_{\xi'}^{\alpha'} [\Phi(\lambda, \xi')(-B)^k (|\lambda|^{k/2} + |\xi'|^k)^{-1} e^{-Bx_n} e^{\delta A x_n}]| \leqq C_{\alpha',\varepsilon} M |\xi'|^{-|\alpha'|},$$

for any α' with $|\alpha'| \leqq n-1$ as follows from Lemma 3.2 and (4.11), by Theorem 2.2, (4.10), (4.15) and (4.16) we have (4.12) for $m = 2$. Using the change of variable: $x_n = 1 - y_n$, (4.12) with $m = 1$ follows from (4.12) with $m = 2$. This completes the proof of the lemma. □

By Lemma 3.4, (4.10), (4.12) and Lemma 2.3, we have

$$\begin{aligned} |\lambda| \, \| W_{2g} \|_{L_p(\Omega)} &+ \sum_{j=1}^{2} \{ |\lambda|^{1/2} \| \nabla W_{jg} \|_{L_p(\Omega)} + \| \nabla^2 W_{jg} \|_{L_p(\Omega)} \} \\ &\leqq C_{\lambda_0,p,\varepsilon} I_{\lambda,p}(U_n, \boldsymbol{R}^n) \leqq C_{p,\lambda_0,\varepsilon} \| \boldsymbol{f} \|_{L_p(\Omega)}. \end{aligned} \tag{4.17}$$

Since

$$\left| \partial_{\xi'}^{\alpha'} \left(\varphi_2(\lambda, \xi') \sqrt{\lambda} \frac{L_{j+2,k}}{\det L} \right) \right| \leqq C_{\alpha',\lambda_0,\varepsilon} |\xi'|^{-|\alpha'|}$$

as follows from Lemma 3.4, by (4.9), (4.10) and Lemma 2.3 we have

$$|\lambda| \, \| W_{1h} \|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon} |\lambda|^{1/2} \sum_{j=1}^{n-1} \| \nabla U_j \|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\lambda_0,\varepsilon} \| \boldsymbol{f} \|_{L_p(\Omega)}, \tag{4.18}$$

$$|\lambda|^{1/2} \| \nabla W_{1h} \|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon} \sum_{j=1}^{n-1} \| \nabla^2 U_j \|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\lambda_0,\varepsilon} \| \boldsymbol{f} \|_{L_p(\Omega)}.$$

Since

$$\left|\partial_{\xi'}^{\alpha'}\left[\varphi_2(\lambda,\xi')\xi_k\frac{L_{j+2,k}}{\det L}\right]\right| \leqq C_{\alpha',\lambda_0,\varepsilon}|\xi'|^{-|\alpha'|}$$

as follows from Lemma 3.4, by (4.10), (4.12) and Lemma 2.3, we have

$$\begin{aligned}(4.19)\qquad &\|\nabla^2 W_{1h}\|_{L_p(\Omega)} + I_{\lambda,p}(W_{2h},\Omega)\\ &\leqq C_{p,\lambda_0,\varepsilon}\sum_{j=1}^{n-1} I_{\lambda,p}(U_j,\boldsymbol{R}^n) \leqq C_{p,\lambda_0,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}.\end{aligned}$$

Combining (4.7), (4.17), (4.18) and (4.19), we see that W_2 satisfies (4.1).

Next, we shall estimate W_3. Since $\operatorname{supp}\varphi_3(\lambda,\xi') \subset \{\xi' \in \boldsymbol{R}^{n-1} \mid \gamma_1/2 \leqq |\xi'| \leqq 3R/4\} \cap \{\xi' \in \boldsymbol{R}^{n-1} \mid |\xi'/(\gamma_2|\lambda|)^{1/2}| \geqq 1/2\}$, if $(\gamma_2|\lambda|)^{1/2} > 3R/2$, then $\operatorname{supp}\varphi_3(\lambda,\xi') = \emptyset$. Namely, if $|\lambda| > (3R/2\gamma_2^{1/2})^2$, then $W_3 = 0$. Therefore, it suffices to estimate W_3 under the assumption: $\lambda_0 \leqq |\lambda| \leqq (3R/2\gamma_2^{1/2})^2$. But, by Lemma 2.5 we know that $L_{j,k}/\det L$ are C^∞ functions, and then by Theorem 2.2, Lemma 2.3 and Lemma 4.2, we see immediately that W_3 satisfies (4.1).

Finally, we shall estimate W_4. Since $|\lambda| \leqq (2/\gamma_2)|\xi'|^2$ on $\operatorname{supp}\varphi_4(\lambda,\xi')$, we have $I_{\lambda,p}(W_4,\Omega) \leqq C_{\lambda_0,p,\varepsilon}\|\nabla^2 W_4\|_{L_p(\Omega)}$. On the other hand, in view of Lemma 3.6, applying (4.10) to (3.30) and using Lemma 2.3, we have $\|\nabla^2 W_4\|_{L_p(\Omega)} \leqq C_{\lambda_0,p,\varepsilon}\|\nabla^2 U\|_{L_p(\boldsymbol{R}^n)}$. Combining these two estimates we see that W_4 satisfies (4.1) which completes the proof of Theorem 4.1.

§5. Estimates for the pressure term.

In this section, we shall prove the following theorem.

Theorem 5.1. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$ *and* $\lambda_0 > 0$. *Then, there exists a* $\Psi \in \hat{W}_p^1(\Omega)$ *such that* Ψ *solves* (2.12) *and satisfies the estimate*:

$$\|\nabla\Psi\|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}$$

provided that $\lambda \in \Sigma_\varepsilon$ *and* $|\lambda| \geqq \lambda_0$.

Let $\varphi_j(\lambda,\xi')$, $j = 1,2,3,4$, be the same functions as in §4 and let $\hat{\Psi}(\lambda,\xi',x_n)$ be a function defined in (2.20). Put

$$(5.1)\qquad \Xi_l(\lambda,\xi',x_n) = \varphi_l(\lambda,\xi')\hat{\Psi}(\lambda,\xi',x_n) = \sum_{j=1}^{2}\Xi_{lj}(\lambda,\xi',x_n),$$

where

$$\Xi_{l1}(\lambda,\xi',x_n) = \sum_{j=1}^{2}\sum_{m=1}^{2}\varphi_l(\lambda,\xi')(-1)^m\frac{L_{j,m}}{\det L}e^{-A\tau_m(x_n)}\frac{\lambda}{A}\hat{U}_n(\xi',j-1),$$

$$\Xi_{l2}(\lambda,\xi',x_n) = \sum_{j=1}^{2}\sum_{m=1}^{2}\varphi_l(\lambda,\xi')(-1)^m\frac{L_{j+2,m}}{\det L}e^{-A\tau_m(x_n)}\frac{\lambda}{A}i\xi'\cdot\widehat{U'}(\xi',j-1).$$

Since $\Xi_j \in \mathscr{S}'(\boldsymbol{R}^{n-1})$, $j = 2, 3, 4$, we can define the partial Fourier inverse transform $\mathscr{F}_{\xi'}^{-1}[\Xi_j(\lambda, \xi', x_n)](x')$. But, Ξ_1 has singularity at $\xi' = 0$, and when $n = 2$, Ξ_1 in fact is not locally integrable, and therefore we can not define $\mathscr{F}_{\xi'}^{-1}[\Xi_1(\lambda, \xi', x_n)](x')$. Therefore, we start with the following lemma which will be proved in the appendix below.

Lemma 5.2. *Let $A(\xi', x_n)$ be a function in $C^\infty((\boldsymbol{R}^{n-1}\backslash\{0\}) \times [0, 1])$ which satisfies the condition*:

$$|\partial_{\xi'}^{\alpha'} \partial_n^m A(\xi', x_n)| \leqq C_{\alpha'} |\xi'|^{m-1-|\alpha'|}$$

for any multi-index α', integer $m \geqq 0$ and $x_n \in [0, 1]$. Moreover, we assume that $A(\xi', x_n) = 0$ when $|\xi'| \geqq \gamma_0$ with some $\gamma_0 > 0$. Given $f(x') \in L_p(\boldsymbol{R}^{n-1})$, we put

$$v_j(x) = \mathscr{F}_{\xi'}^{-1}[i\xi_j A(\xi', x_n) \hat{f}(\xi')](x'), \quad j = 1, \ldots, n-1,$$

$$v_n(x) = \mathscr{F}_{\xi'}^{-1}[\partial_n A(\xi', x_n) \hat{f}(\xi')](x').$$

Then, there exists a $u \in \hat{W}_p^1(\Omega)$ such that $\partial_k u = v_k$, $k = 1, \ldots, n$ and

$$\|\partial_k u\|_{L_p(\Omega)} \leqq C_p \|f\|_{L_p(\boldsymbol{R}^{n-1})}.$$

Remark 5.3. Since

$$|\partial_{\xi'}^{\alpha'}(i\xi_j A(\xi', x_n))| \leqq C_{\alpha'} |\xi'|^{-|\alpha'|}, \quad |\partial_{\xi'}^{\alpha'}(\partial_n A(\xi', x_n))| \leqq C_{\alpha'} |\xi'|^{-|\alpha'|},$$

by Theorem 2.2 $v_j(x)$ and $v_n(x)$ are well-defined as functions in $L_p(\boldsymbol{R}^{n-1})$ with respect to $x' \in \boldsymbol{R}^{n-1}$ for any $x_n \in [0, 1]$ and

$$\|v_j\|_{L_p(\Omega)} \leqq C_p \|f\|_{L_p(\boldsymbol{R}^{n-1})}, \quad j = 1, \ldots, n.$$

Therefore, the point of the lemma is to show the existence of $u \in \hat{W}_p^1(\Omega)$.

Now, we shall apply Lemma 5.2 to $\Xi_1(\lambda, \xi', x_n)$. In view of Lemma 3.1, (3.2) and (2.18), applying Lemma 3.2 we see that

$$\left|\partial_{\xi'}^{\alpha'}\left[\varphi_1(\lambda, \xi') \frac{L_{j,m}}{\det L} e^{-Ax_n}\right]\right| \leqq C_{\alpha',\varepsilon} |\xi'|^{-1-|\alpha'|},$$

$$\left|\partial_{\xi'}^{\alpha'}\left[\varphi_1(\lambda, \xi') \frac{\sqrt{\lambda} i\xi_k}{A} \frac{L_{j+2,m}}{\det L} e^{-Ax_n}\right]\right| \leqq C_{\alpha',\varepsilon} |\xi'|^{-1-|\alpha'|},$$

for $m = 1, 2$ and $x_n \geqq 0$ provided that $|\lambda| \geqq \lambda_0$ and $\lambda \in \Sigma_\varepsilon$. By Lemma 5.2, we know that there exists a $\Psi_1(x) \in \hat{W}_p^1(\Omega)$ such that

$$\partial_j \Psi_1(x) = \mathscr{F}_{\xi'}^{-1}[i\xi_j \Xi_1(\lambda, \xi', x_n)](x'), \quad j = 1, \ldots, n-1, \tag{5.2}$$

$$\partial_n \Psi_1(x) = \mathscr{F}_{\xi'}^{-1}[\partial_n \Xi_1(\lambda, \xi', x_n)](x'),$$

$$\|\nabla \Psi_1\|_{L_p(\Omega)} \leqq C_p \sum_{j=1}^2 \left\{ \|\mathscr{F}_{\xi'}^{-1}[\lambda A^{-1} \hat{U}_n(\xi', j-1)]\|_{L_p(\boldsymbol{R}^{n-1})} + \sum_{k=1}^{n-1} \|\mathscr{F}_{\xi'}^{-1}[\sqrt{\lambda} \hat{U}_k(\xi', j-1)]\|_{L_p(\boldsymbol{R}^{n-1})} \right\}. \tag{5.3}$$

Combining (5.3) and Lemma 4.2 implies that

$$\|\nabla \Psi_1\|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{5.4}$$

For $j = 2,3,4$, we put $\Psi_j(x) = \mathscr{F}_{\xi'}^{-1}[\Xi_j(\lambda,\xi',x_n)](x')$. We shall estimate $\Psi_j(x)$, $j = 2,3,4$. First, we shall consider $\Psi_2(x)$. Put $\Psi_{2j}(x) = \mathscr{F}_{\xi'}^{-1}[\Xi_{2j}(\lambda,\xi',x_n)](x')$. Note that

$$\partial_n \Psi_{21}(x) = -\sum_{j=1}^{2}\sum_{m=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\varphi_2(\lambda,\xi')\frac{L_{j,m}}{\det L}e^{-A\tau_m(x_n)}\lambda\hat{U}_n(\cdot\,,j-1)\right](x'),$$

$$\partial_{x'}^{\beta'}\Psi_{21}(x) = \sum_{j=1}^{2}\sum_{m=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\varphi_2(\lambda,\xi')(-1)^m\frac{(i\xi')^{\beta'}}{A}\frac{L_{j,m}}{\det L}e^{-A\tau_m(x_n)}\lambda\hat{U}_n(\cdot\,,j-1)\right](x').$$

Since

$$\left|\partial_{\xi'}^{\alpha'}\left[\varphi_2(\lambda,\xi')\frac{L_{j,k}}{\det L}\right]\right| \leqq C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|},$$

$$\left|\partial_{\xi'}^{\alpha'}\left[\varphi_2(\lambda,\xi')\frac{(i\xi')^{\beta'}}{A}\frac{L_{j,k}}{\det L}\right]\right| \leqq C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|}, \quad |\beta'| = 1,$$

as follows from Lemma 3.4, by Theorem 2.2 we have

$$\begin{aligned} &\|\Psi_{21}(\cdot\,,x_n)\|_{L_p(\boldsymbol{R}^{n-1})} + \|\nabla\Psi_{21}(\cdot\,,x_n)\|_{L_p(\boldsymbol{R}^{n-1})} \\ &\quad \leqq C_{\lambda_0,p,\varepsilon}\sum_{a=0}^{1}\{|\lambda|\,\|\mathscr{F}_{\xi'}^{-1}[A^{-1}\hat{U}_n(\cdot\,,a)]\|_{L_p(\boldsymbol{R}^{n-1})} + |\lambda|\,\|\mathscr{F}_{\xi'}^{-1}[\hat{U}_n(\cdot\,,a)]\|_{L_p(\boldsymbol{R}^{n-1})}\}. \end{aligned} \tag{5.5}$$

By Lemma 3.4, we have

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left[\varphi_2(\lambda,\xi')\frac{i\xi_k\sqrt{\lambda}}{A}\frac{L_{j+2,l}}{\det L}e^{-Ax_n}\right]\right| \leqq C_{\alpha',\varepsilon}|\xi'|^{m-|\alpha'|}$$

for $m = 0,1,2$, $j = 1,2$, $l = 1,2$ and $x_n \geqq 0$. Applying (4.9) and (4.10) to Ψ_{22}, we have

$$\begin{aligned} &\|\Psi_{22}(\cdot\,,x_n)\|_{L_p(\boldsymbol{R}^{n-1})} + \|\nabla\Psi_{22}\|_{L_p(\Omega)} \\ &\quad \leqq C_{\lambda_0,p,\varepsilon}\left\{\sum_{a=0}^{1}\sum_{k=1}^{n-1}|\lambda|^{1/2}\|U_k(\cdot\,,a)\|_{L_p(\boldsymbol{R}^{n-1})} + \sum_{k=1}^{n-1}|\lambda|^{1/2}\|\nabla U_k\|_{L_p(\boldsymbol{R}^n)}\right\}. \end{aligned} \tag{5.6}$$

Combining Lemma 2.3, Lemma 4.2, (5.5) and (5.6) implies that $\Psi_2 \in W_p^1(\Omega)$ and

$$\|\Psi_2\|_{W_p^1(\Omega)} \leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{5.7}$$

As was stated in §4, $\varphi_3(\lambda,\xi') = 0$ when $|\lambda| > (3R/2\gamma_2^{1/2})^2$ and $\xi' \in \boldsymbol{R}^{n-1}$. Therefore, in view of Lemma 2.5, we see immediately that $\Psi_3 \in W_p^1(\Omega)$ and

$$\|\Psi_3\|_{W_p^1(\Omega)} \leqq C_{p,\lambda_0,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{5.8}$$

Finally, we shall estimate Ψ_4. To do this, we rewrite Ξ_{4j} as follows:

$$\Xi_{41}(\lambda,\xi',x_n)=\sum_{j=1}^{2}\sum_{m=1}^{2}\varphi_4(\lambda,\xi')(-1)^m\frac{\lambda}{A^2}\frac{L_{j,m}}{\det L}e^{-A\tau_m(x_n)}A\hat{U}_n(\xi',j-1),$$

$$\Xi_{42}(\lambda,\xi',x_n)=\sum_{j=1}^{2}\sum_{k=1}^{n-1}\sum_{m=1}^{2}\varphi_4(\lambda,\xi')(-1)^m\frac{i\xi_k\lambda}{A^2}\frac{L_{j+2,m}}{\det L}e^{-A\tau_m(x_n)}A\hat{U}_k(\xi',j-1).$$

Note that $\lambda/(A-B)=-(A+B)$. By (3.27) and (2.18), we have

$$\frac{\lambda}{A}\frac{L_{1,1}}{\det L}=\frac{\lambda}{A}\frac{L_{2,2}}{\det L}=\frac{(A+B)B(e^{-A}e^{-2B}-2Ad(A,B,1)-e^{-A})}{Al_3(A,B)}, \tag{5.9}$$

$$\frac{\lambda}{A}\frac{L_{1,2}}{\det L}=\frac{\lambda}{A}\frac{L_{2,1}}{\det L}=\frac{(A+B)B(1+2Ad(A,B,1)e^{-B}-e^{-2B})}{Al_3(A,B)},$$

$$\frac{\lambda}{A}\frac{L_{3,1}}{\det L}=-\frac{\lambda}{A}\frac{L_{4,2}}{\det L}=\frac{-(A+B)(e^{-A}e^{-2B}+2Bd(A,B,1)-e^{-A})}{Al_3(A,B)},$$

$$\frac{\lambda}{A}\frac{L_{3,2}}{\det L}=-\frac{\lambda}{A}\frac{L_{4,1}}{\det L}=\frac{(A+B)(1-2Bd(A,B,1)e^{-B}-e^{-2B})}{Al_3(A,B)}.$$

Applying Lemma 3.5, (3.32) and Lemma 3.2 to (5.9), we have

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left(\varphi_4(\lambda,\xi')\frac{\lambda}{A^2}\frac{L_{j,l}}{\det L}e^{-Ax_n}\right)\right|\leqq C_{\alpha',\varepsilon}|\xi'|^{m-|\alpha'|},$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_n^m\left(\varphi_4(\lambda,\xi')\frac{i\xi_k\lambda}{A^2}\frac{L_{j+2,l}}{\det L}e^{-Ax_n}\right)\right|\leqq C_{\alpha',\varepsilon}|\xi'|^{m-|\alpha'|},$$

for $j,l=1,2$, $m=0,1,2$ and $x_n\geqq 0$. Therefore, applying (4.9) and (4.10) to Ψ_4, we have

$$\|\Psi_4(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C_{\lambda_0,p,\varepsilon}\sum_{k=1}^{n}\|\mathscr{F}_{\xi'}^{-1}[A\hat{U}_k(\cdot,a)]\|_{L_p(\boldsymbol{R}^{n-1})}, \tag{5.10}$$

$$\|\nabla\Psi_4\|_{L_p(\Omega)}\leqq C_{\lambda_0,p,\varepsilon}\sum_{k=1}^{n}\|\nabla\mathscr{F}_{\xi'}^{-1}[A\hat{U}_k]\|_{L_p(\Omega)}.$$

Since

$$\|\mathscr{F}^{-1}[A\hat{U}_k(\cdot,x_n)]\|_{L_p(\boldsymbol{R}^{n-1})}\leqq C\sum_{j=1}^{n-1}\|\partial_jU_k(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}$$

as follows from (4.15), by (5.10), the trace theorem and Lemma 2.3 we see that $\Psi_4\in W_p^1(\Omega)$ and

$$\|\Psi_4\|_{W_p^1(\Omega)}\leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{U}\|_{W_p^2(\boldsymbol{R}^n)}\leqq C_{\lambda_0,p,\varepsilon}\|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{5.11}$$

Since $W_p^1(\Omega)\subset\hat{W}_p^1(\Omega)$, if we put $\Psi=\Psi_1+\Psi_2+\Psi_3+\Psi_4$, then by (5.4), (5.7), (5.8) and (5.11) we see that Ψ has the required properties in Theorem 5.1.

§6. Estimates for v_j, $j = 1, \dots, n-1$.

In this section, we shall prove the following theorem.

THEOREM 6.1. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$ *and* $\lambda_0 > 0$. *Let* $v_j(x)$, $j = 1, \dots, n-1$, *be the functions defined in* (2.23). *Then, there holds the following estimate*:

$$|\lambda| \, \|v_j\|_{L_p(\Omega)} + |\lambda|^{1/2} \|\nabla v_j\|_{L_p(\Omega)} + \|\nabla^2 v_j\|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon} \|\boldsymbol{f}\|_{L_p(\Omega)}$$

provided that $\lambda \in \Sigma_\varepsilon$ *and* $|\lambda| \geqq \lambda_0$.

For the notational simplicity, we also use the notation $I_{\lambda,p}(v, D)$ in this section, which was defined in §4. Let V_j, $j = 1, \dots, n-1$, be the functions defined in (2.21). By Theorem 2.2 and Theorem 5.1, we have

$$I_{\lambda,p}(V_j, \boldsymbol{R}^n) \leqq C_{p,\varepsilon} \|F_j\|_{L_p(\boldsymbol{R}^n)} = C_{p,\varepsilon} \|\partial \Psi / \partial x_j\|_{L_p(\Omega)} \leqq C_{p,\lambda_0,\varepsilon} \|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{6.1}$$

Let $\hat{w}_j$, $j = 1, \dots, n-1$, be the functions defined in (2.22). Since $\lambda \in \Sigma_\varepsilon$ and $|\lambda| \geqq \lambda_0$, we have

$$|1 - e^{-2B}| \geqq 1 - e^{-2 \operatorname{Re} B} \geqq 1 - e^{-2c_\varepsilon (\lambda_0)^{1/2}}$$

as follows from (2.8), and therefore by Lemma 3.2

$$\left| \partial_{\xi'}^{\alpha'} \left[\frac{e^{-B}}{1 - e^{-2B}} \right] \right| \leqq C_{\alpha',\lambda_0,\varepsilon} |\xi'|^{-|\alpha'|}. \tag{6.2}$$

Applying (4.12) to (2.22), we have

$$I_{\lambda,p}(w_j, \Omega) \leqq C_{\lambda_0,p,\varepsilon} (I_{\lambda,p}(U_j, \boldsymbol{R}^n) + I_{\lambda,p}(V_j, \boldsymbol{R}^n)),$$

which combined with Lemma 2.3 and (6.1) implies that

$$I_{\lambda,p}(w_j, \Omega) \leqq C_{p,\lambda_0,\varepsilon} \|\boldsymbol{f}\|_{L_p(\Omega)}. \tag{6.3}$$

If we put $v_j = V_j + w_j$, then by (6.1) and (6.3) we have Theorem 6.1.

Appendix. A Proof of Lemma 5.2.

The essential part of our proof of Lemma 5.2 is the following.

LEMMA Ap.1. *Let* $A(\xi', x_n)$ *and* $v_j(x)$, $j = 1, \dots, n$ *be the same functions as in Lemma* 5.2. *Then, there exists a sequence* $\{u_k\} \subset C_{(0)}^\infty(\bar{\Omega})$ *such that*

$$\lim_{k \to \infty} \left\| \frac{\partial u_k}{\partial x_j} - v_j \right\|_{L_p(\Omega)} = 0, \quad j = 1, \dots, n.$$

PROOF. An idea of our proof is based on the argument in a proof of Lemma 2.4 of Kozono and Yamazaki [**5**]. Let χ be a function in $C_0^\infty(\boldsymbol{R}^{n-1})$ such that $0 \leqq \chi \leqq 1$ and $\chi(\xi') = 1$ for $|\xi'| \leqq 1$ and $\chi(\xi') = 0$ for $|\xi'| \geqq 2$. Put

$$w_k(x) = \mathscr{F}_{\xi'}^{-1} [(1 - \chi(2^k \xi')) A(\xi', x_n) \chi(\xi'/\gamma_0) \hat{f}(\xi')](x').$$

Since $(1-\chi(2^k\xi'))A(\xi',x_n)\chi(\xi'/\gamma_0)\hat{f}(\xi')$ has a compact support, $w_k \in C^\infty(\boldsymbol{R}^{n-1}\times[0,1])$. Put $\phi_0(\xi') = \chi(\xi') - \chi(2\xi')$ and $g_{0j}(x') = \mathscr{F}_{\xi'}^{-1}[\phi_0(\xi')\xi_j|\xi'|^{-2}](x')$. Since

$$\begin{aligned}&(1-\chi(2^k\xi'))A(\xi',x_n)\chi(\xi'/\gamma_0)\hat{f}(\xi')\\&\quad = -\sum_{h=-k+1}^{k}\sum_{j=1}^{n-1}[\phi_0(2^{-h}\xi')i\xi_j|\xi'|^{-2}]i\xi_jA(\xi',x_n)\hat{f}(\xi')\end{aligned}$$

when k is large enough, we have

$$\text{(Ap.1)}\qquad \|w_k\|_{L_p(\Omega)} \leqq C_0 2^k \quad \text{where } C_0 = \sum_{j=1}^{n-1}\|g_{0j}\|_{L_1(\boldsymbol{R}^{n-1})}\|v_j\|_{L_p(\Omega)}.$$

If we put $B_j(\xi',x_n) = i\xi_jA(\xi',x_n)$, $j=1,\dots,n-1$, and $B_n(\xi',x_n) = \partial_nA(\xi',x_n)$, we have

$$\begin{aligned}\frac{\partial w_k}{\partial x_j} - v_j &= -\mathscr{F}_{\xi'}^{-1}[\chi(2^k\xi')B_j(\xi',x_n)\chi(\xi'/\gamma_0)\hat{f}(\xi')]\\&= -\mathscr{F}_{\xi'}^{-1}[\chi(2^k\xi')B_j(\xi',x_n)] * \mathscr{F}_{\xi'}^{-1}[\chi(\xi'/\gamma_0)\hat{f}(\xi')],\end{aligned}$$

where $*$ means the convolution with respect to the variable x'. By Theorem 2.2, $\|\mathscr{F}_{\xi'}^{-1}[\chi(2^k\xi')B_j(\xi',x_n)]\|_{L_p(\boldsymbol{R}^{n-1})} \leqq C\|\mathscr{F}_{\xi'}^{-1}[\chi(2^k\xi')]\|_{L_p(\boldsymbol{R}^{n-1})}$ for any $x_n \in [0,1]$. Therefore, noting that $\|\mathscr{F}_{\xi'}^{-1}[\chi(2^k\xi')]\|_{L_p(\boldsymbol{R}^{n-1})} \leqq 2^{-(n-1)k/p'}\|\mathscr{F}_{\xi'}^{-1}[\chi]\|_{L_p(\boldsymbol{R}^{n-1})}$ with $1/p+1/p'=1$, by Young's inequality we have

$$\begin{aligned}&\left\|\frac{\partial w_k}{\partial x_j} - v_j\right\|_{L_p(\boldsymbol{R}^{n-1})}\\&\quad\leqq C2^{-(n-1)k/p'}\|\mathscr{F}_{\xi'}^{-1}[\chi]\|_{L_p(\boldsymbol{R}^{n-1})}\|\mathscr{F}_{\xi'}^{-1}[\chi(\xi'/\gamma_0)]\|_{L_1(\boldsymbol{R}^n)}\|f\|_{L_p(\boldsymbol{R}^{n-1})},\end{aligned}$$

which implies that

$$\text{(Ap.2)}\qquad \lim_{k\to\infty}\left\|\frac{\partial w_k}{\partial x_j} - v_j\right\|_{L_p(\Omega)} = 0, \quad j=1,\dots,n-1.$$

Put $u_k(x) = \chi(2^{-2k}x')w_k(x)$, and then $u_k \in C^\infty_{(0)}(\bar{\Omega})$. Observe that

$$\begin{aligned}\frac{\partial u_k}{\partial x_j}(x) - v_j(x) &= \chi(2^{-2k}x')\left[\frac{\partial w_k}{\partial x_j} - v_j\right]\\&\quad + 2^{-2k}\frac{\partial\chi}{\partial x_j}(2^{-2k}x')w_k(x) - [1-\chi(2^{-2k}x')]v_j(x)\end{aligned}$$

for $j=1,\dots,n-1$. By (Ap.1) and (Ap.2), we see easily that

$$\lim_{k\to\infty}\left\|\frac{\partial u_k}{\partial x_j} - v_j\right\|_{L_p(\Omega)} = 0, \quad j=1,\dots,n-1.$$

Since

$$\frac{\partial u_k}{\partial x_n} - v_n = \chi(2^{-2k}x')\left[\frac{\partial w_k}{\partial x_n} - v_n\right] - [1-\chi(2^{-2k}x')]v_n,$$

by the fact that $v_n \in L_p(\Omega)$ and (Ap.2), we have also

$$\lim_{k\to\infty}\left\|\frac{\partial u_k}{\partial x_n} - v_n\right\|_{L_p(\Omega)} = 0,$$

which completes the proof of the lemma. □

Let $\{u_k\}$ be a sequence constructed in Lemma Ap.1. Employing the same argument as in the proof of Lemma 5.1 in Galdi [**4**, II], by using Poincaré's inequality we can find a $u \in L_{p,\mathrm{loc}}(\Omega)$ such that

$$\lim_{k\to\infty}\left\|\frac{\partial u_k}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L_p(\Omega)} = 0$$

for $j = 1, 2, \dots, n$. This completes the proof of Lemma 5.2.

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