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Complex analysis of elastic symbols and construction of plane wave solutions in the half-space

Dedicated to Professor Mitsuru IKAWA on his 60th birthday

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Abstract. We examine plane waves of the elastic reduced wave equation in the half-space, and by linear combinations of them we construct the solution coinciding with any plane wave on the boundary. In the (dual) variable in the normal direction to the boundary, we apply the complex analysis to the inverse matrix of the elastic symbol.

1. Introduction.

Plane wave solutions of reduced wave equations play an important role in a variety of problems, e.g., in the representation of spectral families (cf. Dermenjian and Guillot [**1**], etc.), and in inverse problems (cf. Nakamura and Uhlmann [**4**], Wang [**6**], etc.). In this paper we consider plane wave solutions for the following general reduced wave equation in the half-space $\boldsymbol{R}^n_+ = \{x = (x_1, \dots, x_n) = (x', x_n); x_n > 0\}$:

$$\left(\sigma^2 I + \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}\right)u(x) = 0 \quad \text{in } \boldsymbol{R}^n_+, \tag{1.1}$$

where σ is an arbitrary positive (fixed) parameter. We wish to construct plane wave solutions of (1.1) which satisfy the Dirichlet condition on $\{x_n = 0\}$.

We assume that the coefficients a_{ij} $(i, j = 1, \dots, n)$ are constant real $n \times n$-matrices satisfying

(A.1) $a_{ij} = {}^t a_{ji}$, $i, j = 1, 2, \dots, n$,

(A.2) $L(\xi) \equiv \sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ is positive definite for any $\xi = {}^t(\xi_1, \dots, \xi_n) \in \boldsymbol{R}^n \backslash \{0\}$.

Thus all the eigenvalues of $L(\xi)$ are positive. We assume further that

(A.3) the multiplicity of each eigenvalue of $L(\xi)$ is independent of ξ (for $\xi \neq 0$).

In view of (A.3) we can denote the eigenvalues of $L(\xi)$ by $\lambda_j(\xi)$ $(j = 1, \dots, d;$ $0 < \lambda_1(\xi) < \cdots < \lambda_d(\xi))$.

The most frequently encountered example of a wave equation satisfying (A.1), (A.2) and (A.3) is the system of equations of isotropic elasticity. For that reason we will refer to (1.1) as an "elastic" wave equation.

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In the case of the whole space, $\boldsymbol{R}^n$, the plane waves are of the form

$$e^{i\sigma\eta x}v, \quad \text{where } \lambda_j(\eta) = 1 \text{ and } v \in \operatorname{Ker}[I - L(\eta)]$$

for some eigenvalue λ_j. In the half-space one needs to add other waves to $e^{i\sigma\eta x}v$ so that the sum satisfies the Dirichlet boundary condition. One can think of these added waves as reflections of the original plane wave by the boundary.

Our purpose is to construct the reflected waves. Namely, we construct bounded solutions satisfying the equation (1.1) and

$$u|_{x_n=0} = e^{i\sigma\eta' x'}v \quad \text{on } \boldsymbol{R}^{n-1}, \tag{1.2}$$

where v is any given vector in $\boldsymbol{C}^n$. Taking a root $\tilde{z}$ of the equation (in z)

$$\det(I - L(\eta', z)) = 0 \tag{1.3}$$

and a vector $\tilde{v}$ belonging to $\operatorname{Ker}[I - L(\eta', \tilde{z})]$, we can always make a solution of the form $e^{i\sigma(\eta' x' + \tilde{z} x_n)}\tilde{v}$, but this will only satisfy (1.2) when $\tilde{v} = v$. Thus to construct nontrivial reflected plane wave solutions we will need to use linear combinations of waves arising from the roots of (1.3), and it will be essential to determine the span of the corresponding $\tilde{v}$'s.

We make the following assumption on the eigenvalues $\{\lambda_j\}$:

(A.4) Every slowness surface $\Sigma_j = \{\xi : \lambda_j(\xi) = 1\}$ is strictly convex, and its Gaussian curvature does not vanish.

Let us note that if $\tilde{z}$ is a real root of (1.3), then $\lambda_l(\eta', \tilde{z}) = 1$ for some l. We say that η' is non-glancing if for each eigenvalue λ_j

$$\partial_{\xi_n}\lambda_j(\eta', z) \neq 0 \quad \text{when } z \text{ is real and } \lambda_j(\eta', z) = 1.$$

If η' is non-glancing, then (A.4) implies that there are either no solutions of $\lambda_j(\xi) = 1$ on the line $\xi = (\eta', z)$, $z \in \boldsymbol{R}$, or exactly two solutions corresponding to opposite signs for $\partial_{\xi_n}\lambda_j$. Obviously, the non-real roots of (1.3) are complex conjugates of each other. Therefore, the roots $\{z_\pm^j(\eta')\}_{j=1,\dots,d'}$ of (1.3) can be labelled in the following way if η' is non-glancing:

(i) $z_\pm^j(\eta')$ $(j = 1, \dots, k)$ are real and satisfy

$$\lambda_j(\eta', z_\pm^j(\eta')) = 1 \quad \text{and} \quad \pm\partial_{\xi_n}\lambda_j(\eta', z_\pm^j(\eta')) > 0, \tag{1.4}$$

(ii) $z_\pm^j(\eta')$ $(j = k+1, \dots, d')$ are non-real and satisfy

$$\pm \operatorname{Im} z_\pm^j(\eta') > 0.$$

Furthermore, the multiplicities of the real roots $\{z_\pm^j(\eta')\}_{j=1,\dots,k}$ coincide with those of the eigenvalues $\lambda_j(\eta', z_\pm^j(\eta'))$ (cf. Lemma 2.1 of Soga [**5**]).

The multiplicities of the non-real roots $\{z_\pm^j(\eta')\}_{j=k+1,\dots,d'}$ can exceed the dimension of the corresponding kernel of $I - L(\eta', z_\pm(\eta'))$, and this implies that the required solution may not be obtained by linear combination of the functions $e^{i\sigma(\eta' x' + z_+^j(\eta') x_n)}v^j$ with v^j chosen in $\operatorname{Ker}[I - L(\eta', z_+^j(\eta'))]$. Hence, for the non-real roots we employ solutions of another type:

$$\int_{c_+} e^{i\sigma(\eta' x' + z x_n)} z^j (I - L(\eta', z))^{-1}\, dz\, \tilde{v}, \quad (\tilde{v} \in \boldsymbol{C}^n, j = 0, 1), \tag{1.5}$$

where c_+ is a path surrounding only the roots $\{z_+^j(\eta')\}_{j=k+1,\dots,d'}$ in the complex domain. Let us note that $(I - L(\eta', z))^{-1}$ may have poles only at $\{z_\pm^j(\eta')\}_{j=1,\dots,d'}$, and that (1.5) is sum of the functions of the form $x_n^l e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}\tilde{v}$ $(j = k+1, \dots, d')$ not necessarily satisfying $\tilde{v} \in \mathrm{Ker}[I - L(\eta', z_+^j(\eta'))]$. If one includes these solutions of the form (1.5), then the boundary value problem for $\sigma^2 I - L(-i\partial_x)$ can be solved for all boundary data of the form $e^{i\sigma\eta' x'}v$, $v \in \boldsymbol{C}^n$. Our objective here is to give a short proof of this result, i.e. to prove

THEOREM 1.1. *Let* (A.1), …, (A.4) *be satisfied and let* η' *be non-glancing. Then, linear combinations of the solutions* $e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}v^j$ *for* $j = 1, \dots, k$ *and the solutions in* (1.5) *span the set of functions* $e^{i\sigma\eta' x'}v$ *for all* $v \in \boldsymbol{C}^n$ *on the boundary* $\boldsymbol{R}^{n-1}$.

Let us note that dependency of the solution (1.5) in the variable η' is C^∞ smooth in a neighborhood of any fixed η'.

In §2 we discuss an equivalent formulation of Theorem 1.1 and related results. Then we prove the reformulated version in §3. The proofs are based on considering the matrix $(I - L(\eta', z))^{-1}$ as a meromorphic function in the variable z. One can use residue calculations to show that the boundary data have the linear span given in the theorem.

In Soga [**5**] Theorem 1.1 was proven in the case that the poles of $(I - L(\eta', z))^{-1}$ are simple. In this paper, however, the proof is fairly different. The present methods are similar to those used by Kostyuchenko and Shakalikov's [**3**] for operator pencils. We modify their general theory for use in the restricted (matrix-valued) case. There are also similarities in the constructions here to those used in the construction of Poisson operators for elliptic boundary problems.

In Kawashita and Soga [**2**] we have announced the main result (Theorem 1.1) together with outline of the proof. In the present paper we give a precise description of the proof, and add more improved arguments. (i.e., Proposition 2.4, Lemma 3.1, Remarks 3.2 and 3.3, etc.)

2. Discussion of the main theorem.

In this section we assume that (A.1), …, (A.4) in §1 are satisfied, and explain the main results. For the roots $z_+^j(\eta')$ $(j = 1, \dots, d')$ of (1.3), we have the following solutions of the equation (1.1):

$$e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}v^j, \quad v^j \in \mathrm{Ker}[I - L(\eta', z_+^j(\eta'))],\ j = 1, \dots, d'. \tag{2.1}$$

If every $\dim \mathrm{Ker}[I - L(\eta', z_+^j(\eta'))]$ equals the multiplicity of $z_+^j(\eta')$ as a root of (1.3) $(j = 1, \dots, d')$, we see that the solutions of the form (2.1) span the set of all the solutions. In general, however, for the non-real root $\tilde{z}$ of the equation (1.3) we have only

$$\dim \mathrm{Ker}[I - L(\eta', \tilde{z})] \le \text{multiplicity of } \tilde{z}. \tag{2.2}$$

For proof of this, see Remark 2.4 in Soga [**5**]. Thus in the general case we need to suspect that the set of the solutions is expanded only by the functions of the form (2.1). Moreover, the dependency in the variable η' is not necessarily C^∞ smooth (for the non-real roots). And so we employ solutions of another type for the non-real roots.

Let c_+ be a closed path in $\boldsymbol{C}$ surrounding only the non-real roots $\{z_+^j(\eta')\}_{j=k+1,\dots,d'}$, and put

$$(2.3) \qquad Q_+^j(x_n;\eta') = (2\pi i)^{-1}\int_{c_+} e^{i\sigma z x_n} z^{j-1}(I - L(\eta',z))^{-1}\,dz, \quad j = 1,2.$$

Note that the integral in (2.3) is the form encountered in construction of the Poisson operators for elliptic operators. For any $v \in \boldsymbol{C}^n$ the function $e^{i\sigma\eta' x'}Q_+^j(x_n;\eta')v$ is clearly an exponentially decreasing solution of (1.1). If one evaluates the integral in (2.3) by residues, one can verify that $e^{i\eta' x'}Q_+^j(x_n;\eta')v$ is linear combination of the functions of the form $x_n^l e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}v$ $(j = k+1,\dots,d')$ (but each of them may not satisfy (1.1)). We remark that dependency of $Q_+^j(x_n;\eta')$ in η' is C^∞ smooth (in a neighborhood of any fixed η').

We employ the following class of the solutions:

$$\begin{aligned} E_+ = \{&e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}v^j : v^j \in \operatorname{Ker}[I - L(\eta', z_+^j(\eta'))], j = 1,\dots,k\} \\ &\cup \{e^{i\sigma\eta' x'}Q_+^j(x_n;\eta')v : v \in \boldsymbol{C}^n, j = 1,2\}. \end{aligned}$$

Then, linear combinations of the solutions in E_+ span the boundary data, and we reformulate Theorem 1.1 as

Theorem 2.1. *Let η' be non-glancing. Then we have*

$$\sum_{j=1}^{k} \operatorname{Ker}[I - L(\eta', z_+^j(\eta'))] + \sum_{j=1}^{2} Q_+^j(0;\eta')\boldsymbol{C}^n = \boldsymbol{C}^n,$$

i.e., the linear span of E_+ restricted to $\{x_n = 0\}$ contains $e^{i\sigma\eta' x'}v$ for all $v \in \boldsymbol{C}^n$.

If the poles surrounded by c_+ (i.e. $\{z_+^j(\eta')\}_{j=k+1,\dots,d'}$) are all simple, then the solution $e^{i\sigma\eta' x'}Q_+^j(x_n;\eta')v$ is represented by a sum of the solutions of the form (2.1). Namely, we have

Proposition 2.2. *We assume that η' is non-glancing. Let $\tilde{z}$ be a root of* (1.3) *and let $\tilde{c}$ be a small circle surrounding $\tilde{z}$. If $\tilde{z}$ is the simple pole of $(I - L(\eta',z))^{-1}$, we have*

(i) $\operatorname{Ker}[I - L(\eta',\tilde{z})] = \displaystyle\int_{\tilde{c}} z^j(I - L(\eta',z))^{-1}\,dz\,\boldsymbol{C}^n, \quad j = 0,1,$

(ii) $\displaystyle\frac{1}{2\pi i}\int_{\tilde{c}} e^{i\sigma z x_n} z^j (I - L(\eta',z))^{-1}\,dz\,v = e^{i\sigma\tilde{z}x_n}\tilde{z}^j \operatorname{Res}_{z=\tilde{z}}(I - L(\eta',z))^{-1}v, \quad j = 0,1.$

For proof of (i) of this proposition, see the proof of Lemma 2.5 in Soga [**5**]. (ii) of the proposition is obvious from the definition of the residue. From Proposition 2.2 it follows that Theorem 2.1 has the corollary.

Corollary 2.3. *Let η' be non-glancing, and assume that the poles of $(I - L(\eta', z))^{-1}$ at $\{z_+^j(\eta')\}_{j=k+1,\dots,d'}$ are simple. Then the linear span of the following solutions restricted to the boundary $\{x_n = 0\}$ contains $e^{i\sigma\eta' x'}v$ for all $v \in \boldsymbol{C}^n$*:

$$\{e^{i\sigma(\eta' x' + z_+^j(\eta')x_n)}v^j; v^j \in \mathrm{Ker}[I - L(\eta', z_+^j(\eta'))], j = 1, \dots, d'\}.$$

Furthermore, we can obtain an information on when the poles of $(I - L(\eta', z))^{-1}$ will be simple:

Proposition 2.4. *Let η' be non-glancing, and let $\tilde{z}$ be a root of* (1.3). *Then $\tilde{z}$ is a simple pole of $(I - L(\eta', z))^{-1}$ if and only if the equality holds in* (2.2). *Furthermore, this equality is always satisfied when $\tilde{z}$ is real.*

Theorem 2.1 and Proposition 2.4 are proved in the next §.

3. Proofs and remarks.

Corollary 2.3 was proven in Soga [**5**]. In this section, however, we prove it (and Theorem 2.1) in a different way. The present proof uses methods similar to the ones in Kostyuchenko and Shkalikov [**3**]. Kostyuchenko and Shkalikov deal with more general cases, i.e., operator pencils. We adapt their methods to our matrix case, and obtain precise properties for our use. Throughout this section we assume that (A.1), ..., (A.4) are satisfied and that η' is non-glancing.

Let $(v, w) = \sum_{i=1}^n v_i \overline{w_i}$, and for $v \in \boldsymbol{C}^n$ put

$$F_v^l(z) = (z^{l-1}(I - L(\eta', z))^{-1}v, v), \quad l = 1, 2. \tag{3.1}$$

Then $F_v^l(z)$ becomes a meromorphic function and may have poles only at

$$\{z_\pm^j(\eta'), \infty\}_{j=1,\dots,d'}.$$

But, the presence or absence of these poles depends on v (cf. Remark 3.3).

If the root $\tilde{z}$ of (1.3) is real (i.e., $\lambda_m(\eta', \tilde{z}) = 1$ for some m), we can find a precise form of $(I - L(\eta', z))^{-1}$ near $\tilde{z}$:

Lemma 3.1. *Let $\tilde{z}$ be a real number satisfying $\lambda_m(\eta', \tilde{z}) = 1$ for some m $(1 \le m \le k)$. Then on a complex neighborhood U of $\tilde{z}$ the eigenvalue $\lambda_m(\eta', z)$ of $L(\eta', z)$ can be extended analytically from the real axis (i.e., $\zeta = \lambda_m(\eta', z)$ is the root of the equation $\det(\zeta I - L(\eta', z)) = 0$ with the same multiplicity α_m as $\lambda_m(\eta', \tilde{z})$). The projection $P_m(\xi', z)$ to the generalized eigenspace of $\lambda_m(\xi', z)$ can also be extended analytically from the real axis on U, and we have*

$$(I - L(\eta', z))^{-1} = \frac{1}{1 - \lambda_m(\eta', z)} P_m(\eta', z) + M(z), \tag{3.2}$$

where $M(z)$ is a matrix-valued function analytic on U.

Proof of Lemma 3.1. Since $L(\eta', z)$ is a real symmetric matrix for real z, we have

$$L(\eta', z)P_j(\eta', z) = \lambda_j(\eta', z)P_j(\eta', z), \quad I = \sum_{j=1}^d P_j(\eta', z).$$

This yields that

$$(I - L(\eta', z))^{-1} = \sum_{j=1}^{d} (1 - \lambda_j(\eta', z))^{-1} P_j(\eta', z) \quad \text{for real } z \text{ near } \tilde{z} \tag{3.3}$$

since we have $(I - L(\eta', z)) \sum_{j=1}^{d} (1 - \lambda_j(\eta', z))^{-1} P_j(\eta', z) = I$ for real z $(\neq \tilde{z})$ near $\tilde{z}$.

For small $\delta > 0$, we set

$$P_m(\eta', z) = \frac{1}{2\pi i} \int_{|\zeta - 1| = \delta} (\zeta I - L(\eta', z))^{-1} \, d\zeta.$$

If δ is chosen so that $\det(\zeta I - L(\eta', \tilde{z}))$ has its only zero in $|\zeta - 1| \leq \delta$ at $\zeta = 1$, $P_m(\eta', z)$ is analytic in a small neighborhood of $z = \tilde{z}$, and is the projection to the generalized eigenspace of the eigenvalues in $|\zeta - 1| < \delta$. We set

$$\lambda(z) = \alpha_m^{-1} \int_{|\zeta - 1| = \delta} \zeta \frac{d}{d\zeta} \log(\det(\zeta I - L(\eta', z))) \, d\zeta.$$

Then, obviously $\lambda(z)$ is analytic near $\tilde{z}$, and is equal to the eigenvalue $\lambda_m(\eta', z)$ for real z near $\tilde{z}$. Therefore, the function

$$M(z) = (I - L(\eta', z))^{-1} - \frac{1}{1 - \lambda(z)} P_m(\eta', z)$$

is meromorphic in a neighborhood of $\tilde{z}$, and is bounded on the real axis near $\tilde{z}$ (by (3.3)), which implies that $M(z)$ has no pole in neighborhood of $\tilde{z}$. Hence, we obtain Lemma 3.1 if we can verify

(3.4) $\lambda(z)$ is the only eigenvalue of $L(\eta', z)$ with multiplicity α_m in a neighborhood of 1.

For this consider

$$s_p(z) = \int_{|\zeta - 1| = \delta} \zeta^p \frac{d}{d\zeta} \log(\det(\zeta I - L(\eta', z))) \, d\zeta, \quad p = 1, 2, \ldots.$$

Then $s_p(z)$ is analytic in a neighborhood U of $\tilde{z}$, and equal to $\alpha_m \lambda(z)^p$ $(= \alpha_m \lambda_m(\eta', z)^p)$ on the real axis near $\tilde{z}$. Therefore, from analyticity we have

$$s_p(z) = \alpha_m \lambda(z)^p \quad \text{on } U. \tag{3.5}$$

Inside of the circle $|\zeta - 1| = \delta$ there are α_m roots of $\det(\zeta I - L(\eta', z))$ counted by multiplicity when z is near $\tilde{z}$. Denote them by $\zeta_j(z)$ $(j = 1, \ldots, \alpha_m)$. Then we have $s_p(z) = \sum_{j=1}^{\alpha_m} \zeta_j(z)^p$. Therefore, using (3.5) and noting that $\log(1 - \mu) = \sum_{p=1}^{\infty} p^{-1} \mu^p$ when $|\mu|$ is small enough, we obtain

$$\begin{aligned} \log \prod_{j=1}^{\alpha_m} (1 - \mu \zeta_j(z)) &= \sum_{j=1}^{\alpha_m} \log(1 - \mu \zeta_j(z)) = \sum_{j=1}^{\alpha_m} \sum_{p=1}^{\infty} \zeta_j^p(z) \frac{\mu^p}{p} \\ &= \sum_{p=1}^{\infty} s_p(z) \frac{\mu^p}{p} = \sum_{p=1}^{\infty} \alpha_m \lambda(z)^p \frac{\mu^p}{p} = \alpha_m \log(1 - \mu \lambda(z)). \end{aligned}$$

Hence, it follows that $\prod_{j=1}^{\alpha_m}(1-\mu\zeta_j(z)) = (1-\mu\lambda(z))^{\alpha_m}$, which holds for all μ from the analyticity in μ. Thus, putting $\mu = \zeta^{-1}$, we have

$$\prod_{j=1}^{\alpha_m}(\zeta - \zeta_j(z)) = (\zeta - \lambda(z))^{\alpha_m}$$

when z is near $\tilde{z}$. This means that (3.4) is correct. Thus Lemma 3.1 is proved. □

Remark 3.2. If η' is non-glancing, we can rewrite (3.2) as

$$(I - L(\eta', z))^{-1} = -\frac{\{\partial_{\xi_n}\lambda_m(\eta', \tilde{z})\}^{-1}}{z - \tilde{z}} P_m(\eta', \tilde{z}) + R(z),$$

for a function $R(z)$ analytic near $\tilde{z}$, since $\partial_{\xi_n}\lambda_m(\eta', \tilde{z}) \neq 0$.

This remark gives the precise form of $(I - L(\eta', z))^{-1}$ at the real pole, and shows that Proposition 2.4 for real $\tilde{z}$ is correct. We will not require such detailed information about the complex poles.

Let us note that orthogonality of v to $\mathrm{Ker}[I - L(\eta', \bar{\tau})]$ is related with analyticity of F_v^l at τ:

Remark 3.3. Assume that $(I - L(\eta', z))^{-1}$ has a simple pole at $\tau \in \{z_{\pm}^j(\eta')\}_{j=1,\dots,d'}$. Then, $F_v^l(z)$ $(l = 1, 2)$ becomes analytic at τ if v is orthogonal to $\mathrm{Ker}[I - L(\eta', \bar{\tau})]$.

Let us check this Remark briefly. By the alternative theorem, we have $w \in \boldsymbol{C}^n$ such that

$$v = {}^t\overline{(I - L(\eta', \bar{\tau}))}w.$$

Since ${}^t\overline{L(\eta', z)} = L(\eta', \bar{z})$, there exists a function $R(z)$ analytic near τ such that

$$v = (I - L(\eta', z))w + (z - \tau)R(z)w.$$

Hence we have

$$F_v^l(z) = z^{l-1}(w, v) + (z^{l-1}(z - \tau)(I - L(\eta', z))^{-1}R(z)w, v).$$

Noting that $(z - \tau)(I - L(\eta', z))^{-1}$ is analytic near τ, we obtain the remark.

Proof of Theorem 2.1. Let $v \in \boldsymbol{C}^n$ be orthogonal to

$$\sum_{j=1}^{k} \mathrm{Ker}[I - L(\eta', z_+^j(\eta'))] + \sum_{j=1}^{2} Q_+^j(0; \eta')\boldsymbol{C}^n,$$

and insert this v into $F_v^l(z)$ in (3.1). To prove Theorem 2.1 we have only to show that $v = 0$. By calculation of the residue at $z = \infty$, for large $r > 0$ we have

$$(3.6) \qquad \frac{1}{2\pi i}\int_{|z|=r} z^{l-1}(I - L(\eta', z))^{-1}\,dz = \frac{1}{2\pi i}\int_{|\zeta|=r^{-1}} \zeta^{1-l}(\zeta^2 I - L(\zeta\eta', 1))^{-1}\,d\zeta$$

$$= 0 \quad (\text{when } l = 1), \quad = -a_{nn}^{-1} \quad (\text{when } l = 2),$$

where a_{nn} is the matrix from (1.1). Since there are the poles $\{z_{\pm}^{j}(\eta')\}_{j=1,\dots,d'}$ inside of a large circle $|z| = r$, by (3.6) we have

$$\sum_{j=1}^{d'} \frac{1}{2\pi i} \int_{c_{+}^{j} \cup c_{-}^{j}} z^{l-1} (I - L(\eta', z))^{-1} \, dz = (1 - l) a_{nn}^{-1} \quad (l = 1, 2), \tag{3.7}$$

where $c_{\pm}^{j}$ is a small circle surrounding $z_{\pm}^{j}(\eta')$.

Let c_{+} (resp. c_{-}) be a closed path in $\boldsymbol{C}$ surrounding only the non-real roots $\{z_{+}^{j}(\eta')\}_{j=k+1,\dots,d'}$ (resp. $\{z_{-}^{j}(\eta')\}_{j=k+1,\dots,d'}$), and put

$$\tilde{Q}_{\pm}^{l} = \frac{1}{2\pi i} \int_{c_{\pm}} z^{l-1} (I - L(\eta', z))^{-1} \, dz \quad (l = 1, 2).$$

Note that $\tilde{Q}_{+}^{l} = Q_{+}^{l}(0; \eta')$. From the equality ${}^{t}\overline{(I - L(\eta', z))^{-1}} = (I - L(\eta', \bar{z}))^{-1}$ we see that

$${}^{t}\overline{\tilde{Q}_{-}^{l}} = \tilde{Q}_{+}^{l}. \tag{3.8}$$

Since v is orthogonal to each $\operatorname{Ker}[I - L(\eta', z_{+}^{j}(\eta'))]$ $(= P_{j}(\eta', z_{+}^{j}(\eta'))\boldsymbol{C}^{n})$ for $j = 1, \dots, k$, Lemma 3.1 (Remark 3.2 and Remark 3.3) implies that $\{z_{+}^{j}(\eta')\}_{j=1,\dots,k}$ are not the poles of $F_{v}^{l}(z)$ $(l = 1, 2)$. Therefore, using (3.8) and the hypothesis that v is orthogonal to $Q_{+}^{l}(0; \eta')\boldsymbol{C}^{n}$ $(l = 1, 2)$, for large $r > 0$ we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=r} F_{v}^{l}(z) \, dz &= \sum_{j=1}^{k} \left(\frac{1}{2\pi i} \int_{c_{-}^{j}} z^{l-1} (I - L(\eta', z))^{-1} \, dz \; v, v \right) + (\tilde{Q}_{+}^{l} v, v) + (v, \tilde{Q}_{+}^{l} v) \\ &= -\sum_{j=1}^{k} \{\partial_{\xi_n} \lambda_{j}(\eta', z_{-}^{j}(\eta'))\}^{-1} z_{-}^{j}(\eta')^{l-1} (P_{j}(\eta', z_{-}^{j}(\eta'))v, v) \quad (l = 1, 2). \end{aligned}$$

It follows from (3.7) that $(2\pi i)^{-1} \int_{|z|=r} F_{v}^{l}(z) \, dz = (1 - l)(a_{nn}^{-1} v, v)$ $(l = 1, 2)$ for large $r > 0$. Hence we have

(3.9)

$$-\sum_{j=1}^{k} \{\partial_{\xi_n} \lambda_{j}(\eta', z_{-}^{j}(\eta'))\}^{-1} z_{-}^{j}(\eta')^{l-1} (P_{j}(\eta', z_{-}^{j}(\eta'))v, v) = (1 - l)(a_{nn}^{-1} v, v) \quad (l = 1, 2).$$

Since $\partial_{\xi_n} \lambda_{j}(\eta', z_{-}^{j}(\eta')) < 0$ for all $j = 1, \dots, k$, we have $(P_{j}(\eta', z_{-}^{j}(\eta'))v, v) = 0$ $(j = 1, \dots, k)$ from (3.9) with $l = 1$. Therefore, by (3.9) with $l = 2$, we have $(a_{nn}^{-1} v, v) = 0$. It follows from (A.2) that a_{nn} is positive definite, and hence $v = 0$, which proves Theorem 2.1. □

Proof of Proposition 2.4. We have seen that the statement for the real $\tilde{z}$ is correct. Let us verify the former part of the proposition. For functions $G(z)$ we denote $\partial_{z}^{j} G(z)$ by $G^{(j)}(z)$, and put

$$A(z) = I - L(\eta', z), \quad \tilde{A}(z) = \operatorname{cof}[I - L(\eta', z)].$$

If the multiplicity of the root $\tilde{z}$ (of (1.3)) equals 1, then one see that $A(z)$ has a simple pole at $\tilde{z}$ immediately from

$$A(z)^{-1} = \frac{1}{\det A(z)} \tilde{A}(z). \tag{3.10}$$

Therefore, we assume that the multiplicity of the root $\tilde{z}$ (denoted by α) is not smaller than 2 (i.e., $2 \le \alpha$).

Firstly, let the equality hold in (2.2). Then the number of linearly independent column vectors in the matrix $A(\tilde{z})$ equals $n - \alpha$. From this fact we see that

$$\tilde{A}^{(i)}(\tilde{z}) = 0, \quad i = 0, \dots, \alpha - 2, \tag{3.11}$$

as is shown in the proof of Lemma 2.3 in Soga [**5**]. Differentiating $A(z)\tilde{A}(z) = (\det A(z))I$ α-times and putting $z = \tilde{z}$, by (3.11) we have

$$\alpha A^{(1)}(\tilde{z})\tilde{A}^{(\alpha-1)}(\tilde{z}) + A(\tilde{z})\tilde{A}^{(\alpha)}(\tilde{z}) = (\partial_z^\alpha \det A)(\tilde{z})I.$$

If $\tilde{A}^{(\alpha-1)}(\tilde{z}) = 0$, then it follows from $\det A(\tilde{z}) \det \tilde{A}^{(\alpha)}(\tilde{z}) = \{(\partial_z^\alpha \det A)(\tilde{z})\}^n \neq 0$, which is not consistent with $(\det A)(\tilde{z}) = 0$. Therefore we obtain

$$\tilde{A}^{(\alpha-1)}(\tilde{z}) \neq 0. \tag{3.12}$$

(3.10), (3.11) and (3.12) mean that $\tilde{z}$ is a simple pole of $A(z)^{-1}$.

Next, let $A(z)^{-1}$ be simple at $\tilde{z}$. Then we see from (3.10) that (3.11) and (3.12) must hold for the multiplicity α of the root $\tilde{z}$. Hence, we can rewrite (3.10) in the following form:

$$A(z)^{-1} = \frac{a}{z - \tilde{z}} \tilde{A}^{(\alpha-1)}(\tilde{z}) + B(z),$$

where a is a non-zero constant and $B(z)$ is a matrix-valued function analytic at $\tilde{z}$. Taking constant matrices R and S such that $R^{-1}\tilde{A}^{(\alpha-1)}(\tilde{z})S^{-1}$ becomes a diagonal matrix D with elements equal to 1 or 0, we have $A(z)^{-1} = a(z - \tilde{z})^{-1}RDS + B(z)$. This yields that

$$\det A(z)^{-1} = (z - \tilde{z})^{-\beta} f(z),$$

where $\beta = \operatorname{rank} \tilde{A}^{(\alpha-1)}(\tilde{z})$ and $f(z)$ is a function analytic at $\tilde{z}$. The above equality implies that $\det A(z) = \{\det A(z)^{-1}\}^{-1}$ has $\tilde{z}$ as a zero point of order $\le \beta$, and therefore $\alpha \le \beta$. Hence, α is not greater than $\dim \operatorname{Ker}[A(\tilde{z})]$ since we have

$$\tilde{A}^{(\alpha-1)}(\tilde{z})\boldsymbol{C}^n = \operatorname{Ker}[A(\tilde{z})]$$

by the same methods as in the proof of Lemma 2.5 in Soga [**5**]. Let us note that the methods in Soga [**5**] can be applied if (3.11) holds. Thus the equality must hold in (2.2). □

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