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Global smoothing of singular weak Fano 3-folds

By Tatsuhiro MINAGAWA

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Abstract. In this paper, we will study smoothability of a weak Fano 3-fold with only canonical singularities which is obtained as an image of a crepant primitive birational contraction from a smooth weak Fano 3-fold. Main part is on a contraction of type III.

0. Introduction.

We will work over $\boldsymbol{C}$ in this paper.

DEFINITION 0.1. Let X be a 3-dimensional normal Gorenstein projective variety which has only canonical singularities.

(i) We call X a weak Fano 3-fold when $-K_X$ is nef and big.

(ii) We call X a Fano 3-fold when $-K_X$ is ample.

DEFINITION 0.2. Let X be a 3-dimensional normal Gorenstein projective variety which has only canonical singularities, $(\Delta, 0)$ a germ of the 1-dimensional disk, and $\mathfrak{f} : \mathscr{X} \to (\Delta, 0)$ be a small deformation of X over $(\Delta, 0)$. We call $\mathfrak{f}$ a smoothing of X when the fiber $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$ is smooth for any $s \in (\Delta, 0) \backslash \{0\}$.

Let X be a smooth weak Fano 3-fold and $\phi : X \to \bar{X}$ a birational projective contraction to a weak Fano 3-fold with only canonical singularities. If $\bar{X}$ has a smoothing, then we have another smooth weak Fano 3-fold $\bar{\mathscr{X}_s}$. We want to connect weak Fano 3-folds by deformations and birational contractions as above. We call this problem Reid's fantasy for weak Fano 3-folds.

REMARK. "Original" Reid's fantasy is for Calabi-Yau 3-folds.

Thus we consider the following problem.

PROBLEM. *Let X be a weak Fano 3-folds with only canonical singularities. When does X have a smoothing?*

Known results on Problem are as follows.

1. (Namikawa, Mukai Cf. [**Na 3**] and [**Mu**])
   Let X be a Fano 3-fold with only terminal singularities. Then X has a smoothing.
2. (Namikawa, Takagi Cf. [**Na 3**], [**Ta**] and [**Mi**])
   Let X be a weak Fano 3-fold with only terminal singularities. Assume that there exists a birational proper morphism $\phi : X \to \bar{X}$ to a Fano 3-fold with only

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canonical singularities such that $\dim \phi^{-1}(x) \leq 1$ for any $x \in \bar{X}$. Then X has a smoothing.

3. ([**Mi**])
Let X be a weak Fano 3-fold with only terminal singularities. Then there exists a small deformation of X over $(\Delta, 0)$ $\mathfrak{f} : \mathscr{X} \to (\Delta, 0)$ such that the fiber $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$ has only ordinary double points for any $s \in (\Delta, 0) \backslash \{0\}$.
4. ([**Mi**])
Let X be a weak Fano 3-fold with only terminal singularities. If X is $\boldsymbol{Q}$-factorial, then X has a smoothing.
If the condition "$\boldsymbol{Q}$-factorial" is dropped, then there exists an example which does not have a smoothing.

Extending the method in Section 3 of [**Mi**], we will show the following theorems in Section 1 and 2 of this paper.

THEOREM 0.3. *Let X be a weak Fano 3-fold with only terminal singularities, $\{p_1, p_2, \dots, p_l\} \subset \mathrm{Sing}(X)$ the ordinary double points on X, and $f : Z \to X$ a small partial resolution of X such that $C_i =: f^{-1}(P_i) \cong \boldsymbol{P}^1$ and that f is an isomorphism over $X \backslash \{p_1, p_2, \dots, p_l\}$. If there is a relation in $H_2(Z, \boldsymbol{C}) : \sum_{i=1}^{l} \alpha_i [C_i] = 0$ with $\alpha_i \neq 0$ for all i, then X has a smoothing.*

THEOREM 0.4. *Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that*
(i) *X is $\boldsymbol{Q}$-factorial,*
(ii) *for any $p \in \mathrm{Sing}(X)$, the Kuranishi space of (X, p) is smooth, and*
(iii) *for any $p \in \mathrm{Sing}(X)$, (X, p) has a smoothing.*

Then X has a smoothing.

REMARK (Cf. [**Na 1**], [**Na 2**], [**Na 4**] and [**Na-St**]).
Namikawa proved the same statements of Theorem 0.3 and Theorem 0.4 for Calabi-Yau 3-fold. But the condition in Theorem 0.3 is a necessary and sufficient condition of smoothability in the case of Calabi-Yau 3-fold.

In order to consider Reid's fantasy for weak Fano 3-folds, we study "Smoothing problem" of a weak Fano 3-fold with only canonical singularities obtained as an image of a crepant primitive birational contraction from a smooth weak Fano 3-fold.

DEFINITION 0.5. Let X be a smooth weak Fano 3-fold, and $\phi : X \to \bar{X}$ a crepant birational projective morphism. We call ϕ primitive when its relative Picard number $\rho(X/\bar{X}) = 1$. Moreover, letting E be the exceptional locus of ϕ, we will define as follows.
(i) ϕ is a crepant primitive birational contraction of type I when $\dim(E) = 1$.
(ii) ϕ is a crepant primitive birational contraction of type II when $\dim(E) = 2$ and $\dim \phi(E) = 0$.
(iii) ϕ is a crepant primitive birational contraction of type III when $\dim(E) = 2$ and $\dim \phi(E) = 1$.

We treat a crepant primitive birational contraction of type III from a smooth weak Fano 3-fold in Section 3, which is the main part of this paper. On a crepant primitive birational contraction of type III from a smooth weak Fano 3-fold, we have the following theorem.

THEOREM 0.6. *Let X be a smooth weak Fano* 3*-fold and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type* III *contracting a divisor E to a curve C. Then*

(i) *C is smooth.*

(ii) *$\phi|_E : E \to C$ is a conic bundle, and each fiber is a non-singular conic, a union of two lines meeting at a point, or a double line.*

(iii) *If the general fiber of $\phi|_E$ is a non-singular curve, then E is normal and E has only rational double points.*

(iv) *If the general fiber of $\phi|_E$ is two lines meeting at a point, then singularities of E on the double line are pinch point singularities* (*of the form $x^2 + tz^2 = 0$ in* $(\boldsymbol{C}^3, 0)$).

PROOF. We can show this by the same method in [**Wi 1**], [**Wi 2**] and Section 3 of [**Wi 3**]. □

DEFINITION 0.7. Let X be a smooth weak Fano 3-fold and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type III contracting a divisor E to a curve C. We call $p \in C$ is a dissident point if the fiber of $\phi|_E : E \to C$ over p is not isomorphic to general fiber. We call the fiber over a dissident point the dissident fiber.

Theorem 0.6 enables us to define the following.

DEFINITION 0.8. Let X be a smooth weak Fano 3-fold and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type III contracting a divisor E to a curve C.

(i) The case E is normal: We call ϕ a contraction of (g, d)-type when $g = g(C)$ and $d = -K_{\overline{X}} \cdot C$. Moreover we call ϕ without dissident fibers when $\phi|_E : E \to C$ is a $\boldsymbol{P}^1$-bundle and ϕ with dissident fibers when $\phi|_E$ is not a $\boldsymbol{P}^1$-bundle.

(ii) The case E is non-normal: Let $\tilde{E}$ be the normalization of E, and $\tilde{E} \to \tilde{C} \to C$ the Stein factorization. We call ϕ a contraction of $(g, \tilde{g}, d)$-type when $g = g(C)$, $\tilde{g} = g(\tilde{C})$, and $d = -K_{\overline{X}} \cdot C$.

We will prove the following theorem on deformations of $\overline{X}$ and ϕ.

THEOREM 0.9. *Let X be a smooth weak Fano* 3*-fold and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type* III *contracting a divisor E to a curve C.*

(i) *$\overline{X}$ has a smoothing unless ϕ is of* $(0,0)$, $(0,1)$, $(0,\tilde{g},0)$, *or* $(0,\tilde{g},1)$*-type, or* $(0,3)$*-type without dissident fibers, or* $(1,1)$*-type without dissident fibers.*

(ii) *Let $\mathscr{X} \to Def(X)$ be the Kuranishi family of X. Assume that ϕ is a contraction of* $(0,0)$, $(0,1)$, $(0,\tilde{g},0)$ *or* $(0,\tilde{g},1)$*-type. Then E will deform in the family.*

(iii) *Assume that ϕ is a contraction of* $(0,3)$*-type without dissident fibers, or* $(1,1)$*-type without dissident fibers. Then there exists a small deformation of ϕ over* $(\varDelta, 0)$

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Phi"] \arrow[dr, "\mathfrak{f}"'] & \overline{\mathscr{X}} \arrow[d, "\bar{\mathfrak{f}}"] \\ & (\varDelta, 0) \end{tikzcd}$$

such that, for any $t \in (\Delta, 0)\backslash\{0\}$,

$$\Phi_t : \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive birational contraction of type I *which is a contraction of a single* $\boldsymbol{P}^1$.

Key ideas of the proof of (i) of this theorem are

(1) We will find a small deformation of ϕ over $(\Delta, 0)$

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Phi"] \arrow[dr, "\mathfrak{f}"'] & \overline{\mathscr{X}} \arrow[d, "\bar{\mathfrak{f}}"] \\ & (\Delta, 0) \end{tikzcd}$$

such that, for any $t \in (\Delta, 0)\backslash\{0\}$,

$$\Phi_t : \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive birational contraction of type I, and

(2) We will count the number of curves which are contracted by such Φ_t. It depends on not only g or $\tilde{g}$ but also d. We remark that there are similar results for Calabi-Yau 3-folds (Cf. [**Wi 1**], [**Gr 1**], [**Gr 2**]). But there are differences in the case of type III with $d \neq 0$.

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NOTATIONS.

(1) In this paper, $(\Delta, 0)$ means a germ of the 1-dimensional disk.

(2) $Def(\bullet)$ means the Kuranishi space of $\bullet$ and $T^1_\bullet$ means its tangent space at 0. We also use this notation in the case that $\bullet$ is a morphism.

(3) Let $G \cong \boldsymbol{Z}/2\boldsymbol{Z}$ acting on a $\boldsymbol{C}$-vector space F (resp. a coherent sheaf $\mathscr{F}$ on a scheme X over $\boldsymbol{C}$. In this case G acts on X trivially.). Let σ be the generator of G. We set $F^G = \{s \in F \mid \sigma s = s\}$ (resp. we define $\mathscr{F}^G$ by $\mathscr{F}^G(U) = \{s \in \mathscr{F}(U) \mid \sigma s = s\}$ for an open set U of X).

(4) Let $G \cong \boldsymbol{Z}/2\boldsymbol{Z}$ acting on a $\boldsymbol{C}$-vector space F (resp. a coherent sheaf $\mathscr{F}$ on a scheme X over $\boldsymbol{C}$. In this case G acts on X trivially.). Let σ be the generator of G. We set $F^{[-1]} = \{s \in F \mid \sigma s = -s\}$ (resp. we define $\mathscr{F}^{[-1]}$ by $\mathscr{F}^{[-1]}(U) = \{s \in \mathscr{F}(U) \mid \sigma s = -s\}$ for an open set U of X).

1. On a contraction of Type I.

We will prove Theorem 0.3 in this section, and we have a theorem on a contraction of type I as a corollary of Theorem 0.3. We prove the following theorem first.

Theorem 1.1. *Let X be a weak Fano* 3*-fold with only ordinary double points, $\{p_1, p_2, \dots, p_l\} = \mathrm{Sing}(X)$, and $\nu : Z \to X$ be a small resolution of X such that ν is an isomorphism over $X \backslash \mathrm{Sing}(X)$. Let $C_i = \nu^{-1}(P_i)$. Assume that there exists a relation in $H_2(Z, \boldsymbol{C}) : \sum_{i=1}^{l} \alpha_i [C_i] = 0$ with $\alpha_i \neq 0$ for all i, then X has a smoothing.*

Proof. Let $U = X \backslash \mathrm{Sing}(X)$, X_i a sufficiently small neighborhood of p_i, and $U_i = X_i \backslash \{p_i\}$. Under these setting, we consider the following commutative diagram:

$$\begin{tikzcd}
H^1(U, \Theta_U) \arrow[r, "\alpha"] \arrow[d, "\alpha'"] & \oplus_i H^2_{C_i}(Z, \Theta_Z) \arrow[r, "\beta"] & H^2(Z, \Theta_Z) \\
\oplus_i H^1(U_i, \Theta_{U_i}) \arrow[r, "\oplus \delta'_i"] \arrow[ur, "\oplus \delta_i"] & \oplus_i H^2_{C_i}(Z, \Omega^2_Z) \arrow[r, "\beta'"] \arrow[u, "\oplus \gamma_i"] & H^2(Z, \Omega^2_Z). \arrow[u, "\gamma"]
\end{tikzcd}$$

We remark that γ and γ_i's are defined by a section of $H^0(Z, \omega_Z^{-1})$, and the upper horizontal sequence is exact. By the assumption, there exist elements $(\eta'_i \,|\, i = 1, 2, \dots, l) \in \bigoplus_{i=1}^{l} H^2_{C_i}(Z, \Omega^2_Z)$ such that $\eta'_i \neq 0$ and $\beta'((\eta'_i \,|\, i = 1, 2, \dots, l)) = 0$. Thus there exists $\eta \in H^1(U, \Theta_U)$ such that $\alpha(\eta)_i = \gamma_i(\eta'_i)$ for $i = 1, 2, \dots, l$.

Case 1. (The case that there exists a smooth member $S \in |-K_X|$.)

We may assume that γ and γ_i are defined by $\nu^* S$. In this case, γ_i is an isomorphism for any i. Thus $\gamma_i(\eta'_i) \neq 0$. Since $\mathit{Def}(X)$ is smooth as in [**Mi**], η can be realized as a smoothing of X.

Case 2. (The case that $|-K_X|$ does not have a smooth member.)

Let $\phi_{ac} : X \to X_{ac}$ be a multi-anti-canonical morphism. In this case, as in Section 3 of [**Mi**], we may assume that $\mathit{Bs}|-K_X| = \{p_1\}$, ϕ_{ac} is an isomorphism near p_1, X_{ac} is isomorphic to $X_{2,6} \subset \boldsymbol{P}(1,1,1,1,2,3)$ which is a weighted complete intersection of multi-degree $\{2, 6\}$, and its defining homogeneous equation of degree 2 of $X_{2,6}$ in $\boldsymbol{P}(1,1,1,1,2,3)$ is given by $X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$. By the structure of X_{ac}, there exists an element $\zeta' \in T^1_{X_{ac}}$ such that ζ is locally a non-trivial deformation at $\phi_{ac}(p_1)$ and is locally the trivial deformation at any other singularities.

Let $U' = X_{ac} \backslash \mathrm{Sing}(X_{ac})$, X'_j a sufficiently small neighborhood of each connected component of $\mathrm{Sing}(X)$, $U'_j = X'_j \backslash (\mathrm{Sing}(X_{ac}) \cap X'_j)$, and $E = \phi_{ac}^{-1}(\mathrm{Sing}(X_{ac})) \cap U$. We may assume that $U_1 \cong U'_1$. Under these setting, we consider the following diagram:

$$\begin{tikzcd}
H^1(X, \Theta_U) \arrow[r] & H^1(U', \Theta_{U'}) \arrow[r, "\tau"] \arrow[dr] & H^2_E(U, \Theta_U) \\
 & T^1_{X_{ac}} \arrow[r, "r"] \arrow[u] & \oplus_j H^1(U'_j, \Theta_{U'_j}). \arrow[u, "\oplus \tau_i"]
\end{tikzcd}$$

In this diagram, the upper horizontal sequence is exact. By the choice of ζ', we have that $r(\zeta')_j = 0$ for $j \neq 1$. Since ϕ_{ac} is an isomorphism near p_1, we have that $\tau_i(r(\zeta')_1) = 0$. Since $\tau(\zeta'|_{U'}) = 0$, there exists an element ζ such that $\zeta|_{U'} = \zeta'|_{U'}$. Thus we have that $\alpha(\zeta)_1 \neq 0$.

Suppose that γ and γ_i are defined by $S \in |-K_X|$ such that $S \cap \mathrm{Sing}(X) = \{p_1\}$, We know that γ_i is an isomorphism for $i \neq 1$. Thus $\alpha(\eta)_i \neq 0$ for $i \neq 1$. Thus there exists a complex number ε such that $\alpha(\eta + \varepsilon\zeta)_i \neq 0$ for all i. Since $\mathit{Def}(X)$ is smooth as in [**Mi**], there exists a realization of $\eta + \varepsilon\zeta$ which is a smoothing of X. $\square$

PROOF OF THEOREM 0.3. By Theorem 1.1 and its proof, it is enough to show that all singularities of X which are not ordinary double points are smoothed by a suitable deformation of X. There is a deformation of X to a 3-fold with only ordinary double points by (2) of Main Theorem of [**Mi**]. Considering Section 3 of [**Mi**] (refined in Section 2 of this paper), it follows from the method in the first part of the proof of Theorem 2.5 (3) $\Rightarrow$ (2) of [**Na 2**]. □

We can show the following theorem as in Theorem 5.1 of [**Gr 1**].

THEOREM 1.2. *Let X be a smooth weak Fano 3-fold, and $\phi : X \to \bar{X}$ a crepant primitive birational contraction of type* I. *Then $\bar{X}$ has a smoothing unless ϕ is a contraction of a single $\boldsymbol{P}^1$ to an ordinary double point.*

2. On a contraction of Type II.

In this section, we prove Theorem 0.4. To prove this theorem we need the following which we can prove by a little refinement of the method in Section 3 of [**Mi**].

THEOREM 2.1. *Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that, for any singularity p, the Kuranishi space of (X, p) is smooth. Then the Kuranishi space $Def(X)$ of X is smooth.*

PROPOSITION 2.2. *Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that X is $\boldsymbol{Q}$-factorial. Then there exists a smooth member $S \in |-K_X|$.*

We prove now Theorem 0.4.

PROOF OF THEOREM 0.4. Let $\{p_1, p_2, \dots, p_n\} = \mathrm{Sing}(X)$. Let $\nu : \tilde{X} \to X$ be a resolution of X such that ν is an isomorphism over $U := X \backslash \mathrm{Sing}(X)$ and its exceptional divisors $E_i := \nu^{-1}(p_i)$ have simple normal crossings. Let X_i be a sufficiently neighborhood of p_i, $U_i = X_i \backslash \{p_i\}$. We know the following proposition.

PROPOSITION 2.3 (Cf. The proof of Proposition 4 of [**Na 4**]).
If (X, p_i) is not a rigid singularity, then the homomorphism

$$\iota_i : H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}})$$

is not injective.

By Proposition 2.2, there exists a smooth member $S \in |-K_X|$, then we have the following commutative diagram defined by $\nu^* S$:

$$\begin{array}{ccc} H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}} \otimes \nu^*\omega_X^{-1}) & \xrightarrow{\iota_i'} & H^2(\tilde{X}, \Omega^2_{\tilde{X}} \otimes \nu^*\omega_X^{-1}) \\ \simeq \uparrow & & \uparrow \\ H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}) & \xrightarrow{\iota_i} & H^2(\tilde{X}, \Omega^2_{\tilde{X}}). \end{array}$$

By this diagram and Proposition 2.3, ι_i' is not injective for any i. We consider the following commutative diagram:

$$\begin{tikzcd} H^1(U,\Theta_U) \arrow[r, "\alpha'"] \arrow[d, equal] & \oplus_i H^2_{E_i}(\tilde{X},\Omega^2_{\tilde{X}}\otimes \nu^*\omega_X^{-1}) \arrow[r, "\oplus\iota_i'"] & H^2(\tilde{X},\Omega^2_{\tilde{X}}\otimes \nu^*\omega_X^{-1}) \\ H^1(U,\Theta_U) \arrow[r, "\alpha"] & \oplus_i H^1(U_i,\Theta_{U_i}). \arrow[u, "\oplus\delta_i"'] & \end{tikzcd}$$

We remark that the upper horizontal sequence is exact, and the homomorphism δ_i is factorized as follows:

$$H^1(U_i,\Theta_{U_i}) \to H^2_{E_i}(\tilde{X},\Theta_{\tilde{X}}) \to H^2_{E_i}(\tilde{X},\Omega^2_{\tilde{X}}\otimes \nu^*\omega_X^{-1}).$$

By the exactness, there exists an element $\eta \in H^1(U,\Theta_U)$ such that $\alpha'(\eta)_i \neq 0$ for any i. By Theorem 2.1, we can prove Theorem 0.4 by the Namikawa's stratification method (cf. the proof of Theorem 5 of [**Na 4**]). □

3. On a contraction of type III.

We use the following theorem of Takagi in this section,

Theorem 3.1 (Takagi) (Cf. [**Ta**]). *Let X be a weak Fano 3-fold with only canonical singularities. The complete linear system $|-2K_X|$ is base-point free.*

Proposition 3.2. *Let X be a smooth weak Fano 3-fold, and $\phi: X \to \bar{X}$ a crepant primitive birational contraction of type* III *contracting a divisor E to a curve C. Let $\tilde{E}$ be the normalization of E (when E is normal $\tilde{E} = E$), $\tilde{E} \to \tilde{C} \to C$ the Stein factorization, and $f: \tilde{E} \to X$ the induced map. Then the image of the natural map $Def(f) \to Def(X)$ has codimension*

(i) $\geq \max\{g, g+d-1\}$ *when ϕ is a contraction of (g,d)-type.*
(ii) $\geq \max\{\tilde{g}, \tilde{g}+2d-1\}$ *when ϕ is a contraction of $(g,\tilde{g},d)$-type.*

Proof. We first show codimension $\geq g$ when ϕ is (g,d)-type (resp. $\geq \tilde{g}$ when ϕ is $(g,\tilde{g},d)$-type). (This was proved in Proposition 4.2 of [**Pa**] in the case that E is $\boldsymbol{P}^1$-bundle or any fiber of $\phi|_E$ is union of two lines meeting at a point, and this proof is a modification of it.) To show this, we need the following lemma:

Lemma 3.3. *Let $\tilde{\Omega}^2_{\bar{X}}$ be the double dual of $\Omega^2_{\bar{X}}$. We have that $H^0(\bar{X},\tilde{\Omega}^2_{\bar{X}}) = 0$.*

Proof of Lemma. Let $U = \bar{X}\backslash C$, which is a smooth locus of $\bar{X}$. By Theorem 3.1, $|-2K_{\bar{X}}|$ is base-point free. Since $\bar{X}$ has generically cA_1 or cA_2 singularities, there exists a member $D \in |-2K_{\bar{X}}|$ such that D is smooth except $D \cap C$ and D has an A_1 or A_2 singularity at each point of $D \cap C$. Let $\pi: Y = \mathrm{Spec}(\mathcal{O}_{\bar{X}} \oplus \mathcal{O}_{\bar{X}}(K_{\bar{X}})) \to \bar{X}$ be the double cover of $\bar{X}$ ramified along D, then Y is a Calabi-Yau 3-fold with only canonical singularities. Let $V = \pi^{-1}(U)$ and $G = \boldsymbol{Z}/2\boldsymbol{Z}$. We have that $(\pi_*\Omega^2_V)^G = \Omega^2_U$. So we have $H^0(V,\pi_*\Omega^2_V) = H^0(V,\Omega^2_V) = H^0(V,\Theta_V) = H^0(Y,\Theta_Y) = 0$ by the result of Kawamata [**Ka**]. Thus $H^0(\bar{X},\tilde{\Omega}^2_{\bar{X}}) = H^0(U,\Omega^2_U) = H^0(V,\pi_*\Omega^2_V)^G = 0$. □

Step 1. *When ϕ is (g,d)-type, the codimension $\geq g$.*

Proof. This is a modification of the argument from the proof of Proposition 6.5 of [**Na 1**]. Let T^1_f be the tangent space of $Def(f)$. We have an exact sequence by [**Ra**]

$$T^1_f \to T^1_X \oplus T^1_E \to H^1(X, f^*\Theta_X).$$

This induces an exact sequence

$$T_f^1 \xrightarrow{\alpha} T_X^1 \xrightarrow{\beta} H^1(E, N_{E/X}).$$

We know that $Def(X)$ is smooth by [**Pa**] and [**Mi**], thus it is enough to show that $\mathrm{rank}(\beta) \geq g$. Using the identification $\Theta_X \cong \Omega_X^2 \otimes \omega_X^{-1}$ and $N_{E/X} \cong \omega_E \otimes \omega_X^{-1}$, we may view β as a map $\beta : H^1(X, \Omega_X^2 \otimes \omega_X^{-1}) \to H^1(E, \omega_E \otimes \omega_X^{-1})$. While we can choose $S \in |-K_X|$ such that $F = S \cap E = \sum_{i=1}^d F_i$, where $F_i \cong \boldsymbol{P}^1$ is a fiber of E over C for each i, and $F_i \cap F_j = \varnothing$ if $i \neq j$. In fact there exists a member $S \in |-K_X|$ which has only rational double points by [**Re**], so C is not a base locus of $|-K_X|$. If there is a base point p on C, we consider the anti-canonical model X_{ac} of X:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\phi_{ac}"'] & \bar{X} \arrow[d, "\bar{\phi}_{ac}"] \\ & X_{ac}. \end{tikzcd}$$

As in Section 3 in [**Mi**], $Bs|-K_{X_{ac}}| = \{\bar{\phi}_{ac}(p)\}$, and general member $S_{ac} \in |-K_{X_{ac}}|$ has a single ordinary double point only at $\bar{\phi}_{ac}(p)$. By Proposition 2.1 of [**Pa**], for a general member $S \in |-K_X|$, $S \to S_{ac}$ is the minimal resolution of the ordinary double point. Thus we can choose a member of $|-K_X|$ as desired. We consider the following commutative diagram whose vertical arrows are defined by an $S \in |-K_X|$:

$$\begin{tikzcd} H^1(X, \Omega_X^2) \arrow[r, "\tilde{\beta}"] \arrow[d, "b"] & H^1(E, \omega_E) \arrow[d, "a"] \\ H^1(X, \Omega_X^2 \otimes \omega_X^{-1}) \arrow[r, "\beta"] & H^1(E, \omega_E \otimes \omega_X^{-1}). \end{tikzcd}$$

By the exact sequence

$$0 \to \omega_E \to \omega_E \otimes \omega_X^{-1} \to \omega_E \otimes \omega_X^{-1}|_F \to 0,$$

and $H^0(\omega_E \otimes \omega_X^{-1}|_F) = H^0(\omega_F) = 0$, we have that a is injective. Thus we have that

$$\mathrm{rank}(\tilde{\beta}) \leq \mathrm{rank}(a \circ \tilde{\beta}) = \mathrm{rank}(\beta \circ b) \leq \mathrm{rank}(\beta).$$

So it is enough to show that $g(C) \leq \mathrm{rank}(\tilde{\beta})$.

Let $\nu : X' \to X$ be a embedded resolution of the pair (X, E). Let $\mu = \phi \circ \nu$, E' the proper transform of E by ν. There is a commutative diagram

$$\begin{tikzcd} & H^1(E, \mathcal{O}_E) \arrow[r, "\delta"] \arrow[d, "\simeq"] & H^2(X, \Omega_X^1) \arrow[d] \\ H^1(C, \mathcal{O}_C) \arrow[r, "\simeq"] & H^1(E', \mathcal{O}_{E'}) \arrow[r, "\delta'"] & H^2(X', \Omega_{X'}^1). \end{tikzcd}$$

The vertical arrow of left-hand side is an isomorphism because E has only rational double points by Theorem 0.6, and the first horizontal arrow at the bottom is an isomorphism because a general fiber of $\phi|_E : E \to C$ is isomorhpic to $\boldsymbol{P}^1$. We remark that δ is the dual map of $\tilde{\beta}$. If we can show that δ' is injective, we have that

$$\mathrm{rank}(\tilde{\beta}) = \mathrm{rank}(\delta) \geq \mathrm{rank}(\delta') \geq g(C).$$

Thus it is enough to show that $H^1(X', \Omega^2_{X'}) \to H^1(E', \omega_{E'})$ is surjective. By the Hodge symmetry it is enough to show that $H^2(X', \Omega^1_{X'}) \to H^2(E', \Omega^1_{E'})$ is surjective. By the following 2 exact sequences

$$0 \to \mathcal{O}_{E'}(-E') \to \Omega^1_{X'}|_{E'} \to \Omega^1_{E'} \to 0$$

$$0 \to \Omega^1_{X'}(-E') \to \Omega^1_{X'} \to \Omega^1_{X'}|_{E'} \to 0,$$

it is enough to show that $H^3(X', \Omega^1_{X'}(-E')) = 0$. By the Serre duality

$$H^3(X', \Omega^1_{X'}(-E')) \cong H^0(X', \Theta_{X'}(K_{X'} + E')) \cong H^0(X', \Omega^2_{X'}(E')).$$

There is an injection

$$\mu_* \Omega^2_{X'}(E') \hookrightarrow \tilde{\Omega}^2_{\overline{X}}$$

because both sheaves are isomorphic to each other outside a subset of codimension ≥ 2 and $\tilde{\Omega}^2_{\overline{X}}$ is a reflexible sheaf. Then we have $H^0(X', \Omega^2_{X'}(E')) = 0$ because $H^0(\overline{X}, \tilde{\Omega}^2_{\overline{X}}) = 0$ by the Lemma. □

Step 2. *When the ϕ is $(g, \tilde{g}, d)$, the codimension $\geq \tilde{g}$.*

Proof. This is a modification of the argument from the proof of Proposition 1.2 of [**Gr 2**]. By Theorem 0.6, $\tilde{\phi}_{\tilde{E}} : \tilde{E} \to \tilde{C}$ is a $\boldsymbol{P}^1$-bundle over $\tilde{C}$. Define N_f by the exact sequence

$$0 \to \Theta_{\tilde{E}} \to f^* \Theta_{\tilde{X}} \to N_f \to 0.$$

We remark that N_f is torsion free, locally free away from the inverse image of pinch points of E. Thus we have that the double dual $\tilde{N}_f$ of N_f is isomorphic to $\omega_{\tilde{E}} \otimes f^* \omega_X^{-1}$. We have an exact sequence as in [**Ra**],

$$T^1_f \to T^1_X \oplus T^1_{\tilde{E}} \to H^1(X, f^* \Theta_X).$$

This induces an exact sequence

$$T^1_f \xrightarrow{\alpha} T^1_X \xrightarrow{\beta'} H^1(\tilde{E}, N_f).$$

Let β be a composition homomorphism $T^1_X \to H^1(\tilde{E}, N_f) \to H^1(\tilde{E}, \tilde{N}_f)$. We know that $Def(X)$ is smooth by [**Pa**] and [**Mi**], thus it is enough to show that $\mathrm{rank}(\beta) \geq \tilde{g}$. Using the identifications $\Theta_X \cong \Omega^2_X \otimes \omega_X^{-1}$ and $\tilde{N}_f \cong \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}$, we may view β as a map $\beta : H^1(X, \Omega^2_X \otimes \omega_X^{-1}) \to H^1(E, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1})$. We consider the following commutative diagram:

$$\begin{tikzcd}
H^1(X, \Omega^2_X) \arrow[rr, "\tilde{\beta}"] \arrow[d] & & H^1(\tilde{E}, \omega_{\tilde{E}}) \arrow[d, hook] \\
H^1(X, \Omega^2_X \otimes \omega_X^{-1}) \arrow[rr, "\beta"] \arrow[d, "\simeq"] & & H^1(E, \omega_{\tilde{E}}) \arrow[d, "\simeq"] \\
H^1(X, \Theta_X) \arrow[r] & H^1(\tilde{E}, N_f) \arrow[r] & H^1(\tilde{E}, \tilde{N}_f).
\end{tikzcd}$$

We remark that injectivity of the first vertical map of the right-hand side is because of the same reason in Step 1. Thus it is enough to show that $\operatorname{rank}(\tilde{\beta}) \geq \tilde{g}$. Since $h^1(\tilde{E}, \omega_{\tilde{E}}) = \tilde{g}$ (because $\tilde{\phi}_{\tilde{E}} : \tilde{E} \to \tilde{C}$ is a $\boldsymbol{P}^1$-bundle over $\tilde{C}$), it is enough to show that $\tilde{\beta}$ is surjective. By the Hodge symmetry, it is enough to show that $H^2(X, \Omega_X^1) \to H^2(\tilde{E}, \Omega_{\tilde{E}}^1)$ is surjective.

We consider the following 2 exact sequences,

$$0 \to \mathscr{F}_1 \to \Omega_X^1 \to f_* f^* \Omega_X^1 \to \mathscr{F}_2 \to 0$$

where $\mathscr{F}_2$ has support on the singularities of E, and

$$0 \to \mathscr{F}_3 \to f^* \Omega_X^1 \to \Omega_{\tilde{E}}^1 \to \mathscr{F}_4 \to 0$$

where $\mathscr{F}_4$ has support on the pinch point of $\tilde{E}$. By the second exact sequence, we have that the map $H^2(\tilde{E}, f^*\Omega_X^1) \to H^2(\tilde{E}, \Omega_{\tilde{E}}^1)$ is surjective. Thus it is enough to show that $H^3(X, \mathscr{F}_1) = 0$. By the Serre duality, $H^3(X, \mathscr{F}_1) \cong H^0(X, \mathscr{F}_1{}^\vee \otimes \omega_X)^\vee$. There is an injection

$$\phi_* \mathscr{F}_1{}^\vee \otimes \omega_X \hookrightarrow \tilde{\Omega}_{\overline{X}}^2$$

because both sheaves are isomorphic to each other outside a subset of codimension ≥ 2 and $\tilde{\Omega}_{\overline{X}}^2$ is a reflexible sheaf. Thus we have that $H^0(X, \mathscr{F}_1{}^\vee \otimes \omega_X) \subseteq H^0(\overline{X}, \tilde{\Omega}_{\overline{X}}^2) = 0$ by Lemma 3.3. $\square$

We next show that the codimension $\geq g + d - 1$ when ϕ is (g, d)-type (resp. $\geq \tilde{g} + 2d - 1$ when ϕ is $(g, \tilde{g}, d)$-type). If $d = 0$, then $g \geq g + d - 1$ (resp. $\tilde{g} \geq \tilde{g} + 2d - 1$). So we may assume $d \neq 0$. Define N_f by the exact sequence

$$0 \to \Theta_{\tilde{E}} \to f^* \Theta_{\tilde{X}} \to N_f \to 0$$

as in Step 2. We remark that $N_f = N_{E/X}$ when E is normal.

Lemma 3.4. *The homomorphism $H^1(X, \Theta_X) \to H^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1})$ is surjective, where the homomorphism is induced by the composition of homomorphisms*

$$\Theta_X \to f^* \Theta_X \to N_f \to \tilde{N}_f \cong \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}.$$

Proof of Lemma. By Theorem 3.1, there exists a member $\overline{D} \in |-2K_{\overline{X}}|$ such that $\overline{D} \cap C \cap \{\text{dissident points}\} = \emptyset$ and $D := \phi^* D \in |-2K_X|$ is smooth. Taking double cover of X and $\overline{X}$ branched along D and $\overline{D}$, we have the following commutative diagrams.

$$\begin{tikzcd}[row sep=small, column sep=small]
F \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd] & \\
 & Y \arrow[rr, "\psi"] \arrow[dd] & & \overline{Y} \arrow[dd] \\
E \arrow[rr, "\pi"] \arrow[dr] & & C \arrow[dr] & \\
 & X \arrow[rr, "\phi"] & & \overline{X}
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
\tilde{F} \arrow[rr] \arrow[dr] \arrow[dd] & & \tilde{C}' \arrow[dr] \arrow[dd] & \\
 & F \arrow[rr] \arrow[dd] & & C' \arrow[dd] \\
\tilde{E} \arrow[rr] \arrow[dr] & & \tilde{C} \arrow[dr] & \\
 & E \arrow[rr] & & C
\end{tikzcd}$$

By the results of Wilson, Namikawa, and Gross of Calabi-Yau 3-folds (Cf. [**Wi 1**], [**Wi 2**], [**Na 1**] and [**Gr 2**]), we have that the homomorphism

$$H^1(Y, \Omega_Y^2) \to H^1(\tilde{F}, \omega_{\tilde{F}})$$

is surjective. We have that

$$\gamma : \omega_{\tilde{F}} \xrightarrow{\simeq} \pi^*(\omega_{\tilde{E}} \otimes f^*\omega_X^{-1}).$$

Let σ be the involution on Y. We remark that $\sigma \circ \gamma = -\gamma$. We have that $(\pi_*\omega_{\tilde{F}})^{[-1]} \cong [\pi_*\pi^*(\omega_{\tilde{E}} \otimes f^*\omega_X^{-1})]^G \cong [(\omega_{\tilde{E}} \otimes f^*\omega_X^{-1}) \otimes (\mathcal{O}_{\tilde{E}} \oplus f^*\omega_X)]^G \cong \omega_{\tilde{E}} \otimes f^*\omega_X^{-1}$. We consider the following commutative diagram.

$$\begin{tikzcd}
{[\pi_*\Omega_Y^2]^{[-1]}} \arrow[d, "\simeq"] \arrow[dr] & \\
{[\pi_*\Theta_Y]^G} \arrow[r] \arrow[d] & {[\pi_*\omega_{\tilde{F}}]^{[-1]}} \arrow[d, "\simeq"] \\
\Theta_X \arrow[r] & \omega_{\tilde{E}} \otimes f^*\omega_X^{-1}
\end{tikzcd}$$

This induces a commutative diagram

$$\begin{tikzcd}
H^1(Y, \Omega_Y^2)^{[-1]} \arrow[r] \arrow[d] & H^1(\tilde{F}, \omega_{\tilde{F}})^{[-1]} \arrow[d, "\simeq"] \\
H^1(X, \Theta_X) \arrow[r] & H^1(\omega_{\tilde{E}} \otimes f^*\omega_X^{-1}).
\end{tikzcd}$$

Since the upper horizontal homomorphism is surjective, so is the horizontal homomorphism at the bottom. □

As in Step 1 and 2, we have a commutative diagram with an exact row

$$\begin{tikzcd}
T_f^1 \arrow[r, "\alpha"] & T_X^1 \arrow[r, "\beta'"] \arrow[dr, "\beta"] & H^1(\tilde{E}, N_f) \arrow[d] \\
 & & H^1(\tilde{E}, \tilde{N}_f),
\end{tikzcd}$$

and we know that $Def(X)$ is smooth. By Lemma 3.4, we know that $\operatorname{Im}\beta = h^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^*\omega_X^{-1})$. Thus it is enough to show that $h^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^*\omega_X^{-1}) \geq g + d - 1$ (resp. $\tilde{g} + 2d - 1$).

As in Step 1, we can show that there exists a member $S \in |-K_X|$ such that $F = f^*S = \sum_{i=1}^d F_i$ (resp. $\sum_{i=1}^{2d} F_i$), where $F_i \cong \boldsymbol{P}^1$ is a fiber of $\tilde{E}$ over $\tilde{C}$ for each i, and $F_i \cap F_j = \emptyset$ if $i \neq j$. Since $\omega_{\tilde{E}} \otimes f^*\omega_X^{-1}|_{F_i} \cong \omega_{F_i}$, we have an exact sequence induced by the choice of S

$$0 \to \omega_{\tilde{E}} \to \omega_{\tilde{E}} \otimes f^*\omega_X^{-1} \to \bigoplus_{i=1}^{d} \omega_{F_i} \to 0.$$

$$\left(\text{resp. } 0 \to \omega_{\tilde{E}} \to \omega_{\tilde{E}} \otimes f^*\omega_X^{-1} \to \bigoplus_{i=1}^{2d} \omega_{F_i} \to 0.\right)$$

This induces the following exact sequence:

$$\bigoplus_{i=1}^{d} H^0(F_i, \omega_{F_i}) \to H^1(\tilde{E}, \omega_{\tilde{E}}) \to H^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^*\omega_X^{-1})$$

$$\to \bigoplus_{i=1}^{d} H^1(F_i, \omega_{F_i}) \to H^2(\tilde{E}, \omega_{\tilde{E}}).$$

$$\left(\text{resp. } \bigoplus_{i=1}^{2d} H^0(F_i, \omega_{F_i}) \to H^1(\tilde{E}, \omega_{\tilde{E}}) \to H^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^*\omega_X^{-1}) \right.$$

$$\left. \to \bigoplus_{i=1}^{2d} H^1(F_i, \omega_{F_i}) \to H^2(\tilde{E}, \omega_{\tilde{E}}).\right)$$

Since $h^0(F_i, \omega_{F_i}) = 0$, $h^1(F_i, \omega_{F_i}) = 1$, $h^1(\tilde{E}, \omega_{\tilde{E}}) = g$ (resp. $=\tilde{g}$), and $h^2(\tilde{E}, \omega_{\tilde{E}}) = 1$, Proposition 3.2 follows from this exact sequence. □

By Proposition 3.2, E will not deform under a generic deformation of X, unless ϕ is a contraction of $(0,0)$, $(0,1)$, $(0,\tilde{g},0)$, or $(0,\tilde{g},1)$-type. Thus there exists a deformation of ϕ which is a crepant primitive birational contraction of type I. We want to count the number of curves contracted by the contraction of type I in the following proposition.

PROPOSITION 3.5. *Let X be a smooth weak Fano 3-fold and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type* III. *Assume that ϕ is neither $(0,0)$, $(0,1)$, $(0,\tilde{g},0)$, nor $(0,\tilde{g},1)$-type. Then there exists a small deformation of ϕ over $(\Delta, 0)$*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Phi"] \arrow[dr, "\mathfrak{f}"'] & \overline{\mathscr{X}} \arrow[d, "\mathfrak{f}"] \\ & (\Delta, 0) \end{tikzcd}$$

such that, for any $t \in (\Delta, 0)\backslash\{0\}$,

$$\Phi_t : \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive contraction of type I *which contracts*

(i) *just $2g - 2 + d\boldsymbol{P}^1$'s when ϕ is a contraction of (g,d)-type without dissident fibers with $d \geq 2$.*
(ii) *at least $2\tilde{g} - 2 + 2d\boldsymbol{P}^1$'s when ϕ is a contraction of $(g,\tilde{g},d)$-type.*
(iii) *$l\boldsymbol{P}^1$'s where $2g - 2 \leq l \leq 2g - 1$ when ϕ is a contraction of $(g,1)$-type without dissident fibers.*
(iv) *a single $\boldsymbol{P}^1$ when ϕ is a contraction of $(1,1)$-type without dissident fibers.*

PROOF. We will divide the proof into 3 cases.

Case 1 (ϕ is of (g,d)-type without dissident fibers).

Let $Z \cong \boldsymbol{P}^1$ be any fiber of $\phi|_E$ over $p \in C$ and $i : Z \hookrightarrow X$ the natural closed embedding. We have that $N_{Z/X} \cong N_{Z/E} \oplus (N_{E/X}|_Z)$ where $N_{Z/E} \cong \mathcal{O}_Z$. We consider the following commutative diagram:

$$\begin{tikzcd} T_i^1 \arrow[r] & T_X^1 \arrow[r, "\iota"] \arrow[d, "\beta"] & H^1(Z, N_{Z/X}) \arrow[d, "\simeq"] \\ & H^1(E, N_{E/X}) \arrow[r, "\tau"] & H^1(Z, N_{E/X}|_Z). \end{tikzcd}$$

We remark that the upper horizontal sequence is exact as in [**Ra**]. Thus for $\eta \in T^1_X$, Z extends sideways to first order in the first order deformation corresponding to η if and only if $\tau \circ \beta(\eta) = 0$. Using the identification $N_{E/X} \cong \omega_E \otimes \omega_X^{-1}$, we may view τ as a map

$$\tau : H^1(E, \omega_E \otimes \omega_X^{-1}) \to H^1(\omega_E \otimes \omega_X^{-1}|_Z).$$

By the relative duality, we know that

$$R^1\phi_*\omega_E \cong R^1\phi_*(\omega_{E/C} \otimes \phi^*\omega_C) \cong (R^1\phi_*\omega_{E/C}) \otimes \omega_C \cong \omega_C.$$

Thus we have the isomorphisms

$$R^1\phi_*(\omega_E \otimes \omega_X^{-1}) \cong R^1\phi_*(\omega_E \otimes \phi^*(\omega_{\bar{X}}^{-1}|_C)) \cong (R^1\phi_*\omega_E) \otimes \omega_{\bar{X}}^{-1} \cong \omega_C \otimes \omega_{\bar{X}}^{-1}.$$

Considering the Leray spectral sequence, we have an identification

$$\begin{tikzcd}
H^1(E, \omega_E \otimes \omega_X^{-1}) \arrow[r, "\tau"] \arrow[d, "\simeq"] & H^1(Z, \omega_E \otimes \omega_X^{-1}|_Z) \arrow[d, "\simeq"] \\
H^0(C, \omega_C \otimes \omega_{\bar{X}}^{-1}) \arrow[r, "\bar{\tau}"] & H^0(p, \omega_C \otimes \omega_{\bar{X}}^{-1}|_p).
\end{tikzcd}$$

By the assumption, there exists an element $\xi \in H^0(C, \omega_C \otimes \omega_{\bar{X}}^{-1})$ such that $\eta \neq 0$. Then there exists an element $\eta \in T^1_X$ such that $\beta(\eta) = \xi$ by Lemma 3.4. From the above argument, the fiber over a point where $\bar{\tau}(\xi)$ vanishes extends sideways to first order in the first order deformation corresponding to η. Because $\bar{\tau}(\xi)$ vanishes at $2g - 2 + d$ points, $2g - 2 + d\boldsymbol{P}^1$'s extends sideways in a general first order deformation.

Next, we consider the obstruction of the morphism $i : Z \hookrightarrow X$. We consider the following commutative diagram:

$$\begin{tikzcd}
T^1_X \arrow[r, "\iota"] \arrow[d, "\beta"] & H^1(Z, N_{Z/X}) \arrow[r] \arrow[d, "\simeq"] & T^2_i \arrow[r] & T^2_X \\
H^0(C, \omega_C \otimes \omega_{\bar{X}}^{-1}) \arrow[r, "\bar{\tau}"] & H^0(p, \omega_C \otimes \omega_{\bar{X}}^{-1}|_p). & &
\end{tikzcd}$$

We remark that the upper horizontal sequence is exact. If $d \neq 1$, the linear system $|K_C - (K_{\bar{X}}|_C)|$ is base-point free. Thus $\bar{\tau}$ is surjective. We know that β is surjective by Lemma 3.4, thus ι is surjective. If $d = 1$, in this case $g \neq 0$ by the assumption, we have that $|K_C - (K_{\bar{X}}|_C)| = |K_C| + q$ where $q = -K_{\bar{X}} \cdot C$. Thus $\bar{\tau}$ is surjective if $p \neq q$. This completes the proof in this case except (iv).

Case 2 (ϕ is of $(g, \tilde{g}, d)$-type). Let $\tilde{E}$ be the normalization of E, and $\tilde{E} \to \tilde{C} \to C$ the Stein factorization. We consider the following commutative diagram:

$$\begin{tikzcd}
\tilde{E} \arrow[r] \arrow[d, "\tilde{\phi}_{\tilde{E}}"] \arrow[rr, bend left, "f"] & E \arrow[r, hook] \arrow[d, "\phi|_E"] & X \arrow[d, "\phi"] \\
\tilde{C} \arrow[r] \arrow[rr, bend right, "h"'] & C \arrow[r, hook] & \bar{X}.
\end{tikzcd}$$

Let $Z \cong \boldsymbol{P}^1$ be any fiber of $\tilde{\phi}_{\tilde{E}}$ over $p \in \tilde{C}$ and $i : Z \hookrightarrow X$ be the natural closed embedding. There is a natural morphism $N_{Z/X} \to \tilde{N}_f|_Z$ for any fiber Z. We remark that $N_{Z/X} \cong N_{Z/\tilde{E}} \oplus (\tilde{N}_f|_Z)$ where $N_{Z/\tilde{E}} \cong \mathcal{O}_Z$ for a general fiber Z. We consider the following commutative diagram:

$$\begin{tikzcd}
T_i^1 \arrow[r] & T_X^1 \arrow[r, "\iota"] \arrow[d, "\beta"] & H^1(Z, N_{Z/X}) \arrow[d, "\zeta"] \\
& H^1(\tilde{E}, \tilde{N}_f) \arrow[r, "\tau"] \arrow[d, "\simeq"] & H^1(Z, \tilde{N}_f|_Z) \arrow[d, "\simeq"] \\
& H^0(\tilde{C}, \omega_{\tilde{C}} \otimes h^*\omega_{\overline{X}}^{-1}) \arrow[r, "\bar{\tau}"] & H^0(p, \omega_{\tilde{C}} \otimes h^*\omega_{\overline{X}}^{-1}|_p).
\end{tikzcd}$$

We remark that ζ is an isomorphism for a general fiber Z and we can show that the lower vertical arrows are isomorphisms by the similar reason in Step 1. The above commutative diagram tells us that Z will not extend in a first order deformation corresponding to η if $\bar{\tau} \circ \beta(\eta) \neq 0$ for any fiber Z, and that Z extends sideways to first order in the first order deformation corresponding to η if $\bar{\tau} \circ \beta(\eta) = 0$ for a general fiber Z. Since the degree of $K_{\tilde{C}} + h^*(-K_{\overline{X}})$ is $2\tilde{g} - 2 + 2d$ and $d \neq 0$, $|K_{\tilde{C}} + f^*(-K_{\overline{X}})|$ is base-point free. Thus we can show this proposition by the same method in Step 1 in this case. We remark that we only considered a fiber of $\tilde{\phi}_{\tilde{E}}$ in this proof, thus we need "at least" in this statement.

Case 3 ((iv) of this proposition). If any fiber will not deform, then the Kähler cone of X is not locally constant at $0 \in Def(X)$. But it contradicts Page 63 of [**Pa**] and Theorem 3.1. $\square$

Proposition 3.6. *Let X be a smooth weak Fano 3-fold, and $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type* III. *Assume that ϕ is a contraction of (g, d)-type with dissident fibers and is neither $(0,0)$ nor $(0,1)$-type. Then there exists a small deformation of ϕ over $(\varDelta, 0)$*

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\Phi"] \arrow[dr, "\mathfrak{f}"'] & \overline{\mathscr{X}} \arrow[d, "\bar{\mathfrak{f}}"] \\
& (\varDelta, 0)
\end{tikzcd}$$

such that, for any $t \in (\varDelta, 0) \backslash \{0\}$,

$$\Phi_t : \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive birational contraction of type I *which is not a contraction of a single $\boldsymbol{P}^1$ to an ordinary double point.*

Proof. We can show this proposition by the same method in the proof of Theorem 1.3 of [**Gr 2**].

Proposition 3.7. *Let X be a smooth weak Fano 3-fold, $\phi : X \to \overline{X}$ a crepant primitive birational contraction of type* III, *and $\mathscr{X} \to Def(X)$ the Kuranishi family of X. Assume that ϕ is a contraction of $(0,0)$, $(0,1)$, $(0, \tilde{g}, 0)$, or $(0, \tilde{g}, 1)$-type. Then E will deforms in the family.*

Proof. The case $d = 0$ was proved by Paoletti (cf. the proof of Lemma 3.6 of [**Pa**]), thus we may assume $d = 1$.

We first treat the case ϕ is of $(0,1)$-type. When ϕ is without dissident fibers, $h^1(E, \mathcal{O}_E(E)) = h^1(E, \omega_E \otimes \omega_X^{-1}) = h^0(C, \omega_C \otimes \omega_{\bar{X}}^{-1}) = 0$. Consider the exact sequence in Step 1 of Proposition 3.6,

$$0 \to \omega_E \to \omega_E \otimes \omega_X^{-1} \to \omega_F \to 0$$

where $F \cong \boldsymbol{P}^1$. This induces a long exact sequence

$$\begin{aligned} 0 = H^1(E, \omega_E \otimes \omega_X^{-1}) &\to H^1(F, \omega_F) \to H^2(E, \omega_E) \\ &\to H^2(E, \omega_E \otimes \omega_X^{-1}) \to 0. \end{aligned}$$

Thus we can show that $h^2(E, \mathcal{O}_E(E)) = h^2(E, \omega_E \otimes \omega_X^{-1}) = 0$. Thus E will deform in the Kuranishi family of X.

When E is of $(0,1)$-type with dissident fibers. As in the case E is $(0,1)$-type without dissident fibers, we can prove that $h^2(E, \mathcal{O}_E(E)) = 0$ if $h^1(E, \mathcal{O}_E(E)) = 0$. So it is enough to show that $h^1(E, \mathcal{O}_E(E)) = 0$. We can consider the following commutative diagram:

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\mu} & E \\ \Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \phi|_E} \\ E' & \xrightarrow{\phi'} & C \cong \boldsymbol{P}^1. \end{array}$$

where $\mu : \hat{E} \to E$ is the minimal resolution of E, and $\phi' : E' \to C$ is a $\boldsymbol{P}^1$-bundle.

Since $\omega_{\hat{E}} \simeq \mu^* \omega_E$, we have that

$$H^1(\hat{E}, \omega_{\hat{E}} \otimes \mu^* \omega_X^{-1}) = H^1(\hat{E}, \mu^*(\omega_E \otimes \omega_X^{-1})).$$

Since $\mu_* \mu^*(\omega_E \otimes \omega_X^{-1}) \simeq \omega_E \otimes \omega_X^{-1}$, we have that

$$H^1(E, \mathcal{O}_E(E)) \cong H^1(E, \omega_E \otimes \omega_X^{-1}) \hookrightarrow H^1(\hat{E}, \omega_{\hat{E}} \otimes \mu^* \omega_X^{-1})$$

by the Leray spectral sequence.

Thus it is enough to show that

$$H^1(\hat{E}, \omega_{\hat{E}} \otimes \mu^* \omega_X^{-1}) \cong H^1(\hat{E}, \mu^* \omega_X) = 0.$$

Since $\phi^* \omega_{\bar{X}} \simeq \omega_X$, we have that $\nu^*(\phi')^* \omega_{\bar{X}} \simeq \mu^* \phi^* \omega_{\bar{X}} \simeq \mu^* \omega_X$, thus

$$H^1(\hat{E}, \mu^* \omega_X) = H^1(\hat{E}, \nu^*(\phi')^* \omega_{\bar{X}}).$$

Since $\nu_* \nu^*((\phi')^* \omega_{\bar{X}}) \simeq (\phi')^* \omega_{\bar{X}}$, we have that

$$H^1(E', (\phi')^* \omega_{\bar{X}}) \cong H^1(\hat{E}, \nu^*(\phi')^* \omega_{\bar{X}})$$

by the Leray spectral sequence. As is the case ϕ is a $(0,1)$-type without dissident fibers, we can show that

$$H^1(E', (\phi')^* \omega_{\bar{X}}) \cong H^1(E', \omega_{E'} \otimes (\phi')^* \omega_{\bar{X}}^{-1}) = 0.$$

Thus the rest is the case ϕ is of $(0, \tilde{g}, 1)$-type. If there exists a small deformation of ϕ over $(\Delta, 0)$

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Phi"] \arrow[dr, "f"'] & \overline{\mathscr{X}} \arrow[d, "\bar{f}"] \\ & (\Delta, 0) \end{tikzcd}$$

such that, for any $t \in (\Delta, 0) \backslash \{0\}$,

$$\Phi_t : \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive birational contraction of type I. We consider the following commutative diagram as in Case 2 of the proof of Proposition 3.6:

$$\begin{tikzcd} \tilde{E} \arrow[r] \arrow[d, "\tilde{\phi}_{\tilde{E}}"] \arrow[rr, bend left, "f"] & E \arrow[r, hook] \arrow[d, "\phi|_E"] & X \arrow[d, "\phi"] \\ \tilde{C} \arrow[r] \arrow[rr, bend right, "h"'] & C \arrow[r, hook] & \overline{X}. \end{tikzcd}$$

Then Φ_t contracts at least $2\tilde{g}\boldsymbol{P}^1$'s for $t \in (\Delta, 0) \backslash \{0\}$ which are deformations of fibers of $\tilde{\phi}_{\tilde{E}}$, and these fibers are chosen generically in fibers of $\tilde{\phi}_{\tilde{E}}$ as in Case 2 of the proof of Proposition 3.6. We remark that the morphism $\tilde{C} \to C$ is a finite morphism branched over at least 2 points on C by Hurwitz formula, and these points on C are dissident points. By [**Pa**], we know that the fiber of $\tilde{\phi}_{\tilde{E}}$ whose image by $\phi \circ f$ is a dissident point deforms in the Kuranishi family of X. It is a contradiction. $\square$

Combining these propositions and Theorem 1.2, we can prove Theorem 0.9.

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Tatsuhiro Minagawa
Department of Mathematics
Tokyo Institute of Technology
Ohokayama, Meguro
Tokyo 152-0033
Japan
E-mail: minagawa@math.titech.ac.jp