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On framed cobordism classes of classical Lie groups

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Abstract. It is known that any compact connected Lie group with its left invariant framing is framed null-cobordant in the p-component for any prime $p \neq 2, 3$. In this paper we will prove that the 3-components of $SO(2n+1)$ and $Sp(n)$ are zero for $n \geq 3$, $n \neq 5, 7, 11$. Combining this with the previously known results on $SO(2n)$ and $SU(n)$ consequently we see that any classical group has at most only the 2-component with some exceptions.

1. Introduction.

Let (G, L) be the pair consisting of a compact connected Lie group of dimension d and its left invariant framing. In this paper we are concerned with whether or not (G, L) is framed cobordant to the boundary of a stably parallelizable manifold. This problem is first raised by Gershenson in [**4**]. Let $[G, L]$ denote the framed cobordism class represented by (G, L) in the stable homotopy group π_d^S of spheres and let $[G, L]_{(p)}$ denote the p-component of $[G, L]$. Then this problem is referred to that of estimating the order of $[G, L]$ or $[G, L]_{(p)}$ for all primes p.

By observing its filtration associated with the Adams spectral sequence for BP, Knapp [**8**] gave the result of general nature such that $[G, L]_{(p)}$ is zero for any prime $p \geq 7$. This result was later improved by Ossa in [**14**] as follows: $72[G, L] = 0$ and in particular if G is a classical group then

$$24[G, L] = 0.$$

As to the known results of its 3-component we have the following. From the work of Becker and Schultz [**3**] (cf. [**11**]) we know that $[SO(2n), L]_{(3)} = 0$ for $n \geq 1$ and furthermore we have partial stronger results like $[SO(4), L] = 0$, $[SO(6), L] = 0$, $[SO(7), L] = 0$, $[SO(8), L] = 0$ and $[SO(9), L] = 0$ ([**3**], [**7**]). But these results were recently strengthened in [**6**] as follows: $[SO(2n), L] = 0$ for $n \geq 2$. Besides it is known [**12**] that $[SU(n), L]_{(3)} = 0$ for $n \geq 3$. For the exceptional Lie groups in addition to the classical ones we know that $[F_4, L]_{(3)} = 0$ ([**7**]), $[E_6, L]_{(3)} = 0$ ([**13**]) and $[G_2, L]_{(3)} = 0$ ([**18**], [**10**]), to be exact, $[G_2, L] = \kappa \in \pi_{14}^S$ using Toda's notation in [**17**].

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *$[SO(2n+1), L]_{(3)} = 0$ and $[Sp(n), L]_{(3)} = 0$ for any $n \geq 3$ except $n = 5, 7, 11$.*

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Putting together the result of this theorem and the facts stated above preceding to this theorem we have

THEOREM 1.2. *Let G denote the group $SO(2n)$ $(n \geq 1)$, $SO(2n+1)$ $(n \geq 3)$, $SU(n)$ $(n \geq 3)$ or $Sp(n)$ $(n \geq 3)$. Then the 3-component of (G, L) vanishes except for $G = SO(2n+1)$ and $Sp(n)$ with $n = 5, 7, 11$.*

As for the groups with rank 1 and 2 excluded from the above theorem it is proved that $[SU(2), L] = \nu$, $[SO(3), L] = 2\nu \in \pi_3^S$, $[SO(5), L] = -\beta_1 \in \pi_{10}^S$ and $[Sp(2), L] = \beta_1 \in \pi_{10}^S$ ([**1**], [**3**], [**7**]) using the notations of Toda [**17**]. We note here that $SO(n)$ and $Spin(n)$ have the same nullity in the 3-components since $[SO(n), L]_{(3)} = 2[Spin(n), L]_{(3)}$ by Lemma 7.14 of [**3**].

As is shown in Lemma 2.6 below $[SO(2n+1), L]_{(3)}$ and $[Sp(n), L]_{(3)}$ have the same order. So we only have to prove Theorem 1.1 for $SO(2n+1)$. Our proof is mainly based on the results of [**3**] and the method of [**14**]. Another key point in the proof is the use of the solution of Adams conjecture of [**15**] together with the fact stated in Lemma 4 of [**16**]. Furthermore we use the result of [**2**] for the proof of Theorem 1.1 in the six cases where $n = 6, 8, 9, 10, 12, 13$. Although we can prove $[SO(7), L]_{(3)} = 0$ and $[SO(9), L]_{(3)} = 0$ in the same way as in these six cases, we cite the stronger result of [**3**] and [**7**] as mentioned above. Unfortunately the only three cases $n = 5, 7, 11$ remains unknown. It seems that some other method is required to deal with these cases.

In section 2 we summarize several fundamental facts which we need later. In section 3 we establish a method of our computation and in the last section using this we give a proof of Theorem 1.1.

Finally I would like to thank the referee for suggesting some improvements of this paper.

2. The J-map.

We begin with recalling the construction of the J-map from [**16**], p. 314. Consider an element x of $\widetilde{KO}^{-1}(X)$ where X is a finite CW-complex with base point. We know that x can be represented by a map $f : X \to SO(N)$. In this situation, we write $\beta(f)$ for x. Let $\Omega_1^N S^N$ and $\Omega_0^N S^N$ be the spaces of all base point preserving maps of dgree 1 and 0 respectively. Then $SO(N)$ can be embedded into $\Omega_1^N S^N$ in the canonical way. By u we denote this embedding. Subtracting the identity map of S^N into itself from the composite $uf : X \to \Omega_1^N S^N$ we have a map $X \to \Omega_0^N S^N$ and then taking the adjoint of this map we get a map

$$\tilde{J} : \widetilde{KO}^{-1}(X) \to \pi_S^0(X)$$

where $\pi_S^0(X)$ is the reduced stable cohomotopy group of X in dimension 0. By definition we can check that there holds the relation

$$\tilde{J}(x+y) = \tilde{J}(x) + \tilde{J}(y) + \tilde{J}(x)\tilde{J}(y).$$

Let us set $J(x) = 1 + \tilde{J}(x) \in \pi_S^0(X^+) \cong \boldsymbol{Z} \oplus \pi_S^0(X)$ where X^+ denotes the disjoint union of X and a single point which is viewed as its base point. In fact, J is obtained by taking the adjoint of uf itself without subtracting the identity map of S^N. Then J becomes multiplicative, namely, it satisfies

$$J(x+y) = J(x)J(y) \tag{2.1}$$

and $J(0) = 1$ since $\tilde{J}(0) = 0$ by definition.

Let ρ denote the standard non-trivial n-dimensional real representation of $SO(n)$ (which is also called briefly the identity representation of $SO(n)$) and $\lambda^2\rho$ the second exterior power of ρ. We need here the expression of only $\psi^2(\beta(\rho))$ in terms of $\beta(\rho)$ and $\beta(\lambda^2\rho)$ where ψ^2 denotes the second Adams operation.

Lemma 2.2. $\psi^2(\beta(\rho)) = 2n\beta(\rho) - 2\beta(\lambda^2(\rho))$.

Proof. Let $E_\rho, E_{\rho\otimes\rho}$ and $E_{\lambda^2\rho}$ be the n-, n^2- and $\binom{n}{2}$-dimensional vector bundles associated with ρ, $\rho\otimes\rho$ and $\lambda^2\rho$ respectively over a suspension space $S^1 \wedge SO(n)^+$. Then we can write as $\beta(\rho) = [E_\rho] - n$ and $\beta(\lambda^2\rho) = [E_{\lambda^2\rho}] - \binom{n}{2}$ under the natural identification $\widetilde{KO}(S^1 \wedge SO(n)^+) \cong \widetilde{KO}^{-1}(SO(n)) \oplus \boldsymbol{Z}_2$ and $[E_{\rho\otimes\rho}] = [E_\rho]^2$ since obviously $E_{\rho\otimes\rho} \cong E_\rho \otimes E_\rho$ where $[E]$ denotes the isomorphism class of E. Now we have

$$\psi^2([E_\rho]) = [E_{\rho\otimes\rho}] - 2[E_{\lambda^2\rho}]$$

([**5**], Chapter 12, Proposition 2.5). Substituting $[E_{\rho\otimes\rho}] = [E_\rho]^2$, $[E_\rho] = \beta(\rho) + n$ and $[E_{\lambda^2\rho}] = \beta(\lambda^2\rho) + \binom{n}{2}$ into this formula we obtain the result since the products of elements of $\widetilde{KO}(S^1 \wedge SO(n)^+)$ are zero. $\square$

Let $j : SO(2n+1) \to S^d$ be an obvious collapsing map to a top cell of $SO(2n+1)$ where $d = n(2n+1)$ and let us consider the homomorphism $j^* : \pi_d^S = \pi_S^0(S^d) \to \pi_S^0(SO(2n+1)^+)$. From now on we denote by ρ the identity representation of $SO(2n+1)$ and set

$$\mu = \tilde{J}(\beta(\rho)).$$

Then we know

Lemma 2.3 ([**3**], Theorem 5.3). *$j^*([SO(2n+1), L]) = \mu^n(2-\mu)^n$ and $\mu^{n+1}(2-\mu)^n = 0$.*

Note that right multiplication is used in [**3**] to define the left invariant framing L of $SO(2n+1)$. So we will conform ourself to this manner. We also find that the map j^* is injective. As is seen below this holds for any compact connected Lie group G with framing L. We now explain this for later use. In fact the left invariant framing of the tangent bundle $L : \tau(G) \to G \times \boldsymbol{R}^d$ is given by the linear map $R_{g^{-1}*} : \tau_g(G) \to \tau_e(G)$ induced by right multiplication by g^{-1} where $\tau_g(G)$ denotes the tangent space at g and in particular the tangent space τ_e at the identity element is identified with $\boldsymbol{R}^d$.

Let $G \subset \boldsymbol{R}^{d+s}$ be an embedding with normal bundle ν. Using radial reduction we may regard as $\nu \subset \boldsymbol{R}^{d+s}$. We also see that this yields an embedding $\tau(G) = G \times \boldsymbol{R}^d \subset \boldsymbol{R}^{d+s} \times \boldsymbol{R}^d = \boldsymbol{R}^{2d+s}$ identifying the bundle isomorphism $L : \tau(G) \cong G \times \boldsymbol{R}^d$. Thus we obtain an embedding $\nu \oplus \tau(G) \subset \boldsymbol{R}^{2d+s}$. From this we get an obvious collapsing map $c : S^{2d+s} \to T(\nu \oplus \tau(G))$ where $T(\eta)$ denotes the Thom space of a vector bundle η. Moreover we have a bundle isomorphism $\nu \oplus \tau(G) \cong G \times \boldsymbol{R}^{d+s}$ induced by parallel translation. This induces a homeomorhism $h : T(\nu \oplus \tau(G)) \approx S^{d+s} \wedge G^+$. Then the homotopy fundamental class $\sigma(G, L)$ of the framed manifold (G, L), defined in $\pi_d^S(G^+)$, is represented as a stable map by the composition $hc : S^{2d+s} \to S^{d+s} \wedge G^+$.

Define a homomorphism $\kappa : \pi_S^0(G^+) \to \pi_S^0(S^d) = \pi_d^S$ by

$$\kappa(x) = \langle x, \sigma(G, L)\rangle_\pi \quad \text{for } x \in \pi_S^0(G^+)$$

where $\langle\, ,\rangle_\pi$ denotes the Kronecker product in the stable homotopy theory. Then by definition it is easily seen that κ becomes the left-inverse map of $j^* : \pi_d^S = \pi_S^0(S^d) \to \pi_S^0(G^+)$ so that j^* is a monomorphism. In addition we see $[G, L] = \kappa(1)$ by definition. This can be generalized as follows.

Given a map $f : G \to SO(N)$ the twisted framing $L^f : \tau(G) \oplus (G \times \boldsymbol{R}^N) \to G \times \boldsymbol{R}^{d+N}$ of L by f is defined by the formula $L^f(u, (g, v)) = (L(u), (g, f(g)^{-1}v))$ where $u \in \tau_g(G)$ and $v \in \boldsymbol{R}^N$. Then we see by construction that there holds $[G, L^f] = \langle 1, \sigma(G, L^f)\rangle_\pi = \langle J(\beta(f)), \sigma(G, L)\rangle_\pi$. For convenience we write $[G, L^f] = [G, \beta(f)]$. Then

$$[G, \beta(f)] = \kappa(J(\beta(f))).$$

Let Ad denote the adjoint representation of G. Then for any real representation γ of G we have by Lemma 4 of [**16**]

$$[G, \beta(\mathrm{Ad}) - \varepsilon\beta(\gamma)] = (-1)^d[G, \varepsilon\beta(\gamma)]$$

where $\varepsilon = \pm 1$. We give here an outline of its proof. To simplify the notation we set $\varphi = L^{\mathrm{Ad}-\varepsilon\gamma}$ and $\psi = L^{\varepsilon\gamma}$. Then $\varphi, \psi : \tau(G) \oplus \boldsymbol{R}^\ell \cong G \times (\boldsymbol{R}^d \oplus \boldsymbol{R}^\ell)$ are given as follows: $\varphi(u, (g, v)) = (g, (\mathrm{Ad}(g^{-1})_* R_{g^{-1}*}(u), \gamma(g)^\varepsilon(v)))$ and $\psi(u, (g, v)) = (g, (R_{g^{-1}*}(u), \gamma(g)^{-\varepsilon}(v)))$ where $u \in \tau_g(G)$ and $v \in \boldsymbol{R}^\ell$. By t we denote the diffeomorphism of G given by $t(g) = g^{-1}$ for $g \in G$. Then clearly $t_*\mathrm{Ad}_* R_{g^{-1}*}(u) = R_{g*}t_*(u)$ and t changes the orientation of G by the degree $(-1)^d$. Therefore it can be easily checked that these two framed manifolds (G, φ) and (G, ψ) become isomorphic through t with their orientations having a difference by the sign $(-1)^d$. Thus the above result follows.

Applying this formula to the case $G = SO(2n+1)$ we have

$$[SO(2n+1), \beta(\lambda^2\rho) - \varepsilon\beta(\gamma)] = (-1)^n[SO(2n+1), \varepsilon\beta(\gamma)]$$

because $\lambda^2\rho$ is precisely the adjoint representation of G. So using the homomorphism κ we have

$$\kappa(J(\beta(\lambda^2\rho) - \varepsilon\beta(\gamma))) = (-1)^n\kappa(J(\varepsilon\beta(\gamma))). \tag{2.4}$$

Now let us return to Lemma 2.3. Denote by $M_{(3)}$ and $x_{(3)}$'s the localizations of a group M and its elements x's at the prime 3. And identify $x \in \pi_d^S$ with $j^*(x)$ for brevity since j^* is injective. Then we have

LEMMA 2.5. *$[SO(2n+1), L]_{(3)} = (2\mu)_{(3)}^n$ and $\mu_{(3)}^{n+1} = 0$.*

PROOF. Since $SO(2n+1)$ is covered clearly with finite contractible closed subspaces, there holds $\mu^{n+s} = 0$ for some $s \geq 1$. Hence from the second formula of Lemma 2.3 it follows by induction on s that $\mu_{(3)}^{n+1} = 0$, so that the first assertion follows immediately from the first of Lemma 2.3. □

LEMMA 2.6. *$[SO(2n+1), L]_{(3)}$ and $[Sp(n), L]_{(3)}$ are of the same order.*

PROOF. By Lemma 2.5, $[SO(2n+1), L]_{(3)} = (2\tilde{J}(\beta(\rho)))^n_{(3)}$. Also for $Sp(n)$ we have a similar result $[Sp(n), L]_{(3)} = \tilde{J}(\beta(\bar{\rho}))^n_{(3)}$ by Theorem 5.3 of [**3**] where $\bar{\rho}$ denotes the obvious inclusion of $Sp(n)$ into $SO(4n)$. By inspecting the correspondence Ψ : $\pi_S^0(Sp(n))_{(3)} \cong \pi_S^0(SO(2n+1))_{(3)}$ given in [**9**], we can easily verify that $\Psi(\tilde{J}(\beta(\bar{\rho}))) = \tilde{J}(2\beta(\rho))$. Using this together with (2.1) we have $\Psi([Sp(n), L]_{(3)}) = [SO(2n+1), L]_{(3)}$. Hence the result is immediate. $\square$

3. The orders of $\kappa(\mu^k)$.

Let G be a compact connected Lie group and suppose that $\pi_S^0(G)$ is localized at the prime 3. But we will omit the index (3) attached to elements hereafter (with the exception of (3.5)). The following fact is useful for our computation and we use it below without references. Given $x \in \widetilde{KO}^{-1}(G)$ satisfying $\tilde{J}(\ell^e x) = 0$ for some non-negative two integers ℓ such that $(\ell, 3) = 1$ and e, then since $\tilde{J}(x)$ is a nilpotent element of $\pi_S^0(G)$ we can show $\tilde{J}(x) = 0$ by the inductive method using (2.1).

So the solution of Adams conjecture [**15**] together with this implies that there holds $J(\psi^2(x) - x) = 1$ for any $x \in \widetilde{KO}^{-1}(G)$, namely

$$J(\psi^2(x)) = J(x).$$

Let ρ denote the identity representation of $SO(2n+1)$. Then by using Lemma 2.2 and applying this formula to $\beta(\rho), \beta(\lambda^2(\lambda^2\rho))$ and $\beta(\lambda^2(\lambda^2(\lambda^2\rho)))$ we have

$$\text{(3.1)} \qquad \text{(i)} \quad J(2\beta(\lambda^2\rho)) = J((4n+1)\beta(\rho)),$$

$$\text{(ii)} \quad J(2\beta(\lambda^2(\lambda^2\rho))) = J((4n^2+2n-1)\beta(\lambda^2\rho)),$$

$$\text{(iii)} \quad J(2\beta(\lambda^2(\lambda^2(\lambda^2\rho)))) = J((4n^4+4n^3-n^2-n-1)\beta(\lambda^2(\lambda^2\rho)))$$

which are used for the proof of the following lemma.

Before we mention the next assertion we recall the theorem of [**14**] and state briefly the idea how to prove it. Let $S \subset G$ be a circle subgroup of G and let ξ denote a canonical complex line bundle associated with the principal S-bundle $G \to G/S$. Then we find that G/S becomes a stably parallelizable (not parallelizable) manifold with the natural framing induced by L on G. We denote this framing by the same symbol L. Let $f : G \to SO(N)$ be a map such that it is written as a composite of the canonical projection $\pi : G \to G/S$ with a map $\tilde{f} : G/S \to SO(N)$. Then it is proved that there holds

$$[G, L^f] = -\langle \tilde{J}(b\xi), \sigma(G/S, L^{\tilde{f}}) \rangle_\pi$$

where $b \in \tilde{K}(S^2)$ denotes the Bott element. Moreover it is shown that $9\tilde{J}(b\xi) = 0$ and especially $3\tilde{J}(b\xi) = 0$ in the classical cases. Thus we see that there holds

$$3\kappa(J((\beta(f)))) = 0$$

for any map $f : SO(2n+1) \to SO(N)$ which factors through $SO(2n+1)/S$ for some circle subgroup $S \subset SO(2n+1)$.

We make use of this method for the proof of the lemma below. In fact this formula is often applied to a map given in the following form. Let ρ_1, ρ_2 be real representations of $SO(2n+1)$ which agree on a given circle subgroup $S \subset SO(2n+1)$, up to trivial representations. Then we write simply as

$$\rho_1|_S = \rho_2|_S.$$

Now we may assume that they have the same dimension N, if necessary, by adding trivial representations adequately and then we can define a map $f : SO(2n+1) \to SO(N)$ by $f(g) = \rho_1(g)\rho_2(g)^{-1}$ for $g \in SO(2n+1)$. Consequently, because this map obviously factors through $SO(2n+1)/S$, we have

$$3\kappa(J(s\beta(\rho_1) - s\beta(\rho_2))) = 0 \tag{3.2}$$

for any integer s. Such an argument is used repeatedly below. So to simplify our argument we will abbreviate the description about a map like f.

For convenience we introduce further an additional notation. For two real representations ρ_1, ρ_2 of $SO(2n+1)$ we write as

$$\rho_1|_S \equiv \rho_2|_S$$

if there exists another representation ρ_3 of $SO(2n+1)$ satisfying $\rho_1|_S = (\rho_2 + 3\rho_3)|_S$.

LEMMA 3.3. *$3\kappa(\mu^k) = 0$ for all $k \geq 0$.*

PROOF. Taking account of the observation given at the begining of this section we see that it is enough to prove $3\kappa(J(s\beta(\rho))) = 0$ for any integer s. We break up the proof into the three cases $n \equiv 0, 1, 2 \bmod 3$. To begin with we consider the first two cases. Take S to be the circle subgroup $SO(2) \times I_{2n-1}$ of $SO(2n+1)$ where I_{2n-1} is the identity element of degree $2n-1$. Let η be the identity representation of S. Then the restrictions of ρ and $\lambda^2\rho$ to S become as follows:

$$\rho|_S = \eta + (2n-1) \quad \text{and} \quad \lambda^2\rho|_S = (2n-1)\eta + (2n^2 - 3n + 2).$$

So we have $\lambda^2\rho|_S = (2n-1)\rho|_S$ and hence using (3.2) it is deduced that

$$3\kappa(J(2s\beta(\lambda^2\rho) - (4n-2)s\beta(\rho))) = 0.$$

Now, using the property (2.1) of J, the equality (3.1), (i) allows us to exchange $2s\beta(\lambda^2\rho)$ for $(4n+1)s\beta(\rho)$ in this bracket. Thus we obtain

$$3\kappa(J(3s\beta(\rho))) = 0 \tag{3.4}$$

so that it follows that

$$3\kappa(J(3s\beta(\lambda^2\rho))) = 0.$$

Making use of (3.4) or, to be exact, a map like f mentioned above from which (3.4) is derived enables us to carry out our computation under the modulus 3 reduction. What this means is simply that if there holds $3\kappa(J((3s+k)\beta(\rho))) = 0$, then by using (3.4) this can be reduced to $3\kappa(J(k(\beta(\rho)))) = 0$. Under this assumption we proceed to our proof.

We now consider to cut down 3 in the bracket of (3.4). In case of $n \equiv 0 \bmod 3$ we find $(\lambda^2(\lambda^2\rho) + \rho^2)|_S \equiv (\rho\lambda^2\rho + \rho)|_S$. Since $3\kappa(J(3s\beta(\lambda^2\rho))) = 0$ it follows from (3.1), (ii) that

$$3\kappa(J(2s\beta(\lambda^2(\lambda^2\rho)) + s\beta(\lambda^2\rho))) = 0$$

so that we have $3\kappa(J(3s\beta(\lambda^2(\lambda^2\rho)))) = 0$. Taking account of this it follows from the above congruence expression that

$$3\kappa(J(2s\beta(\lambda^2(\lambda^2\rho)) - 2s\beta(\lambda^2\rho) + 2s\beta(\rho))) = 0.$$

Substituting the above equality into this one we have

$$3\kappa(J(-3s\beta(\lambda^2\rho) + 2s\beta(\rho))) = 0.$$

And further, by substituting $3\kappa(J(3s\beta(\lambda^2\rho))) = 0$ we get

$$3\kappa(J(2s\beta(\rho))) = 0$$

and so by using (3.4) we have

$$3\kappa(J(s\beta(\rho))) = 0.$$

We consider next the case where $n \equiv 1 \bmod 3$. Then there hold $\lambda^2\rho|_S \equiv \rho|_S$, $\lambda^2\rho|_S \equiv \lambda^2(\lambda^2\rho)|_S$ and $\lambda^2(\lambda^2(\lambda^2\rho))|_S \equiv 0$. Using (3.4) we argue as in the preceding case. From the first two congruence expressions we have

$$3\kappa(J(s\beta(\lambda^2\rho) - s\beta(\rho))) = 0 \quad \text{and} \quad 3\kappa(J(s\beta(\lambda^2(\lambda^2\rho)) - s\beta(\lambda^2\rho))) = 0.$$

Furthermore from (3.1), (iii) we have

$$3\kappa(J(2s\beta(\lambda^2(\lambda^2(\lambda^2\rho))) + s\beta(\lambda^2(\lambda^2\rho)))) = 0$$

so that $3\kappa(J(3s\beta(\lambda^2(\lambda^2(\lambda^2\rho))))) = 0$ follows. Using this it is deduced from the third congruence expression that

$$3\kappa(J(s\beta(\lambda^2(\lambda^2(\lambda^2\rho))))) = 0.$$

Consequently by combining these four equalities we have $3\kappa(J(s\beta(\rho))) = 0$ as desired.

To prove the remaining case where $n \equiv 2 \bmod 3$ we take the circle subgroup S to be the diagonal subgroup such that $SO(2) \times I_{2n-3} \subset SO(2) \times SO(2) \times I_{2n-3} \subset SO(2n+1)$. Then we have

$$\rho|_S = 2\eta + (2n-3) \quad \text{and} \quad \lambda^2\rho|_S = \eta^2 + (4n-6)\eta + (2n^2 - 7n + 8)$$

where η is the one similar to the above. From these it follows that $(\rho^2 + (4n-6)\rho)|_S = 4\lambda^2\rho|_S$. Hence we have

$$3\kappa(J((8n-4)s\beta(\rho) - 4s\beta(\lambda^2\rho))) = 0.$$

This together with (3.1), (i) yields $3\kappa(J(3s\beta(\rho))) = 0$, so that it follows that

$$3\kappa(J(s\beta(\lambda^2\rho))) = 0.$$

By substituting this into (3.1), (ii) we have

$$3\kappa(J(s\beta(\lambda^2(\lambda^2\rho)))) = 0.$$

Moreover it holds that $(\lambda^2(\lambda^2\rho) - \rho\lambda^2\rho + 2\lambda^2\rho - \rho)|_S \equiv 0$ and so we have

$$3\kappa(J(s\beta(\lambda^2(\lambda^2\rho)) - 3s\beta(\lambda^2\rho) - 2s\beta(\rho))) = 0.$$

Clearly combining these equalities gives $3\kappa(J(s\beta(\rho))) = 0$ and completes the proof of the lemma. □

Applying Lemma 3.3 to Lemma 2.5 we see that

$$[SO(2n+1), L]_{(3)} = \kappa(1) = (-1)^n\kappa(\mu^n) \quad \text{and} \quad \mu^{n+1} = 0. \tag{3.5}$$

We now formulate a method computing $\kappa(\mu^n) = 0$ in (3.5). First we give a remark about a extraction of square root of $1+\mu$ in $\pi_S^0(SO(2n+1)^+)$. Since μ is nilpotent, $(1+\mu)^{1/2}$ can be expanded into a finite series of μ. Using Lemma 3.3 we see that there holds $(1+\mu)^{3^N} = 1$ for some large N. Furthermore since $1+3^N$ is even we see that $((1+3^N)/2)\beta(\rho)$ becomes an element of $\widetilde{KO}^{-1}(SO(2n+1)^+)$. A simple computation shows that $J(((1+3^N)/2)\beta(\rho)) = (1+\mu)^{1/2}$. This means that $(1+\mu)^{1/2}$ can be recognaized geometrically as an element of $\pi_S^0(SO(2n+1)^+)$.

By (3.1), (i) using (2.1) we have

$$J(\beta(\lambda^2\rho) - \varepsilon k\beta(\rho)) = J((2n - \varepsilon k)\beta(\rho))J(\beta(\rho))^{1/2}$$

for $k \geq 0$ and $\varepsilon = \pm 1$. Therefore from (2.4) it follows that

$$(-1)^n\kappa((1+\mu)^{\varepsilon k}) = \kappa((1+\mu)^{2n-\varepsilon k+1/2}) \tag{3.6}$$

where k and ε are as above. Here we set

$$R = \sum_{i=1}^{n}(-1)^i\mu^i \quad \text{and} \quad 1+\alpha = (1+\mu)^{2n+1/2}.$$

Using these notations we can rewrite (3.6) as

$$(-1)^n\kappa((1+\mu)^k) = \kappa((1+R)^k(1+\alpha))$$

and since $(1+\mu)^{-1} = 1+R$

$$(-1)^n\kappa((1+R)^k) = \kappa((1+\mu)^k(1+\alpha))$$

for $k \geq 0$ according as $\varepsilon = 1$ or -1. Applying induction on k to these formulas we can get the following:

$$\text{(i)} \quad (-1)^n\kappa(\mu^k) = \kappa(R^k(1+\alpha)), \tag{3.7}$$

$$\text{(ii)} \quad (-1)^n\kappa(R^k) = \kappa(\mu^k(1+\alpha))$$

for $k \geq 0$. In particular, from either of (3.7) with $k = 0$ it follows that $\kappa(\alpha) = (-1)^n\kappa(1)$, so that we see

$$\kappa(\alpha) = \kappa(1) \text{ if } n \text{ is odd} \quad \text{and} \quad \kappa(\alpha) = 0 \text{ if } n \text{ is even}. \tag{3.8}$$

The formula (3.7) is a main tool for our computation, but we need another formula to deal with the six cases as noted in the introduction. By Proposition 5.2 and the formula on p. 906 of [**2**] we have $[SO(2n+1), 2n\beta(\rho)] = 0$ and $[SO(2n+1), (2n-2)\beta(\rho)] = 0$. These imply

(3.9) (i) $\kappa((1+\mu)^{-2n}) = 0,$

(ii) $\kappa((1+\mu)^{-2n+2}) = 0.$

Apply this to (3.6). Then we also have

(3.10) (i) $\kappa((1+\alpha)(1+\mu)^{2n}) = 0,$

(ii) $\kappa((1+\alpha)(1+\mu)^{2n-2}) = 0.$

4. Proof of Theorem 1.1.

To prove Theorem 1.1 it suffices by virtue of Lemma 2.6 to show that $[SO(2n+1), L]_{(3)} = 0$. So we will prove that $\kappa(\mu^n) = 0$ or equivalently $\kappa(1) = 0$ in (3.5). (Here we continue assuming that all the elements are localized at the prime 3.) For conveience we divide n into the nine types such that $n = 9s + r$ $(0 \leq r \leq 8)$ and analyze the formulas of (3.7) with appropriate integers k's to each case. In addition we consider those of (3.9) and (3.10) for the six cases noted in the introduction. To do this we need to expand $1+\alpha$, namely $(1+\mu)^{2n+1/2}$, and R^i $(1 \leq i \leq 18)$ into series of μ up to at least the 18th degree. Since $R = -\mu(1+R)$ holds, we can get those of R^k for $n - i \leq k \leq n$ from the series expansions of such R^i's. Lemma 3.3 shows that the coefficients of these series of μ may be assumed to belong to $\boldsymbol{Z}_3$. So we will carry out our calculation under this assumption.

Now we have

$$(1+\mu)^{1/2} = 1 - \mu + \mu^2 + \mu^3 - \mu^4 + \mu^5 + \mu^9 - \mu^{10} + \mu^{11} + \mu^{12} - \mu^{13} + \mu^{14} + h_{19}$$

in $\boldsymbol{Z}_3[\mu]$ where h_i denotes the sum of the higher terms with degrees above i. Furthermore we have in $\boldsymbol{Z}_3[\mu]$

$$\begin{aligned}
R &= -\mu + \mu^2 - \mu^3 + \mu^4 - \mu^5 + \mu^6 - \mu^7 + \mu^8 - \mu^9 + \mu^{10} - \mu^{11} + \mu^{12} - \mu^{13} + \mu^{14} \\
&\quad - \mu^{15} + \mu^{16} - \mu^{17} + \mu^{18} + h_{19}, \\
R^2 &= \mu^2 + \mu^3 - \mu^5 - \mu^6 + \mu^8 + \mu^9 - \mu^{11} - \mu^{12} + \mu^{14} + \mu^{15} - \mu^{17} - \mu^{18} + h_{19}, \\
R^3 &= -\mu^3 + \mu^6 - \mu^9 + \mu^{12} - \mu^{15} + \mu^{18} + h_{19}, \\
R^4 &= \mu^4 - \mu^5 + \mu^6 + \mu^7 - \mu^8 + \mu^9 - \mu^{13} + \mu^{14} - \mu^{15} - \mu^{16} + \mu^{17} - \mu^{18} + h_{19}, \\
R^5 &= -\mu^5 - \mu^6 - \mu^8 - \mu^9 + \mu^{14} + \mu^{15} + \mu^{17} + \mu^{18} + h_{19}, \\
R^6 &= \mu^6 + \mu^9 - \mu^{15} - \mu^{18} + h_{19}, \\
R^7 &= -\mu^7 + \mu^8 - \mu^9 + \mu^{16} - \mu^{17} + \mu^{18} + h_{19}, \\
R^8 &= \mu^8 + \mu^9 - \mu^{17} - \mu^{18} + h_{19}, \\
R^9 &= -\mu^9 + \mu^{18} + h_{19},
\end{aligned}$$

$$R^{10} = \mu^{10} - \mu^{11} + \mu^{12} - \mu^{13} + \mu^{14} - \mu^{15} + \mu^{16} - \mu^{17} + \mu^{18} + h_{19},$$

$$R^{11} = -\mu^{11} - \mu^{12} + \mu^{14} + \mu^{15} - \mu^{17} - \mu^{18} + h_{19},$$

$$R^{12} = \mu^{12} - \mu^{15} + h_{19},$$

$$R^{13} = -\mu^{13} + \mu^{14} - \mu^{15} - \mu^{16} + \mu^{17} - \mu^{18} + h_{19},$$

$$R^{14} = \mu^{14} + \mu^{15} + \mu^{17} + \mu^{18} + h_{19},$$

$$R^{15} = -\mu^{15} - \mu^{18} + h_{19},$$

$$R^{16} = \mu^{16} - \mu^{17} + \mu^{18} + h_{19},$$

$$R^{17} = -\mu^{17} - \mu^{18} + h_{19},$$

$$R^{18} = \mu^{18} + h_{19}.$$

Together with these a similar series expansion of $1 + \alpha$ is required. But this varies according to the number n modulo 9. We will give this expansion at the beginning of the proof of each case and use freely it together with the relations of Lemma 3.3 and (3.5) such that $3\kappa(\mu^k) = 0$ $(k \geq 0)$ and $\kappa(\mu^n) = (-1)^n \kappa(1)$.

Case 1: $n = 9s$ $(s \geq 1)$. In this case there holds the relation

$$(1 + \mu)^{2n} = 1 - s\mu^9 + h_{17}$$

and so it follows that

$$1 + \alpha = 1 - \mu + \mu^2 + \mu^3 - \mu^4 + \mu^5 + (1 - s)(\mu^9 - \mu^{10} + \mu^{11} + \mu^{12} - \mu^{13} + \mu^{14}) + h_{17}.$$

We begin with the case $s = 1$, i.e., $n = 9$. By (3.8) we have

$$\kappa(1 + \mu - \mu^2 - \mu^3 + \mu^4 - \mu^5) = 0$$

and also by (3.10), (ii) and (3.9), (ii) we have

$$\kappa(\mu^3) = -\kappa(1) \quad \text{and} \quad \kappa(\mu^2) = \kappa(\mu).$$

From these formulas it follows that

$$\kappa(-1 + \mu^4 - \mu^5) = 0.$$

Furthermore by (3.7), (ii) with $k = 5, 8, 3$ we get

$$\kappa(1 - \mu^6 - \mu^7) = 0, \quad \kappa(\mu^8) = 0 \quad \text{and} \quad \kappa(\mu^4 - \mu^5 + \mu^6 + \mu^7 - \mu^8 + \mu^9) = 0.$$

Clearly combining the above four equalities together with $\kappa(\mu^9) = -\kappa(1)$ leads to the required result $\kappa(1) = 0$. So we may suppose that $s \geq 2$, i.e., $n \geq 18$.

Calculate (3.7), (ii) with $k = n - 1, n - 3, n - 5, n - 7$ and (3.7), (i) with $k = n - 10$ in this order. Then we have

$$\kappa(\mu^{n-1}) = \kappa(\mu^{n-7}) = 0, \quad \kappa(\mu^{n-3} + \mu^{n-2} + \mu^n) = 0 \quad \text{and} \quad \kappa(\mu^{n-5}) = \kappa(\mu^{n-4}) = 0.$$

Taking account of these formulas we proceed to our computation. From (3.7), (ii) with $k = n - 11, n - 12$ it follows that

$$\kappa(\mu^{n-11} - \mu^{n-10} + \mu^{n-9} - \mu^{n-8} - \mu^{n-6} + (1-s)\mu^{n-2} - \mu^{n}) = 0$$

and

$$\kappa(-\mu^{n-11} + \mu^{n-10} - \mu^{n-8} + (s-1)\mu^{n-2}) = 0.$$

Adding these two equalities yields

$$\kappa(\mu^{n-9} + \mu^{n-8} - \mu^{n-6} - \mu^{n}) = 0.$$

Moreover from (3.7), (ii) with $k = n-9$ we have

$$\kappa(-\mu^{n-9} - \mu^{n-8} + \mu^{n-7} + \mu^{n-6} + (s-1)\mu^{n}) = 0$$

and so by adding the above two equalities we obtain

$$(s+1)\kappa(\mu^{n}) = 0$$

by virtue of $\kappa(\mu^{n-7}) = 0$. Hence when $s \equiv 0, 1 \mod 3$ we see that $\kappa(\mu^{n}) = 0$. Next we check the case $s \equiv 2 \mod 3$. From (3.7), (i) with $k = n-13$ we have

$$\kappa(\mu^{n-13}) = 0$$

using $\kappa(\mu^{n-4}) = 0$. Also by (3.7), (i) with $k = n-16$ we get

$$\kappa(\mu^{n-13} + \mu^{n-7} + \mu^{n-4} - \mu^{n}) = 0.$$

Substituting $\kappa(\mu^{n-13}) = 0$ and $\kappa(\mu^{n-7}) = \kappa(\mu^{n-4}) = 0$ we obtain $\kappa(\mu^{n}) = 0$ as desired.

Case 2: $n = 9s+1$ $(s \geq 1)$. In this case we have

$$(1+\mu)^{2n} = 1 + \mu - \mu^{3} - \mu^{4} + \mu^{6} + \mu^{7} + (s-1)(\mu^{9} + \mu^{10} - \mu^{12} - \mu^{13}) + s\mu^{14} + h_{15}$$

and so

$$1 + \alpha = 1 + \mu^{3} + (s+1)(\mu^{9} + \mu^{12}) + s\mu^{14} + h_{15}.$$

By calculating (3.7), (ii) with $k = n-1, n-3, n-5, n-7, n-9$ in this order we have

(4.1) $$\kappa(\mu^{n-1}) = 0, \quad \kappa(\mu^{n-3} + \mu^{n-2} - \mu^{n}) = 0, \quad \kappa(\mu^{n-5} - \mu^{n-4} + \mu^{n-2}) = 0,$$
$$\kappa(\mu^{n-7}) = 0 \quad \text{and} \quad \kappa(\mu^{n-9} - \mu^{n-6} + \mu^{n}) = 0.$$

Using these relations it follows from (3.7), (ii) with $k = n-10$ that

$$(s-1)\kappa(\mu^{n}) = 0$$

which means that if $s \equiv 0, 2 \mod 3$ then $\kappa(\mu^{n}) = 0$. We observe here the case $s = 1$, i.e., $n = 10$. By (3.9), (i) we have $\kappa((1+\mu)^{-18}) = 0$ which yields $\kappa(\mu^{9}) = -\kappa(1)$ directly. But it follows from the first equality of (4.1) that $\kappa(\mu^{9}) = 0$ which shows $\kappa(1) = 0$. Hence we suppose that $s \geq 2$, i.e., $n \geq 19$ and prove the case where $s \equiv 1 \mod 3$.

From the calculations of (3.7), (ii) with $k = n-11, n-12$ and (3.7), (i) with $k = n-12$ we have

(4.2) $$\kappa(-\mu^{n-11}+\mu^{n-10}+\mu^{n-8}-\mu^{n-2})=0,$$

(4.3) $$\kappa(\mu^{n-11}-\mu^{n-10}+\mu^{n-9}-\mu^{n-3}-\mu^{n})=0$$

and

(4.4) $$\kappa(-\mu^{n-11}+\mu^{n-10}+\mu^{n-9}-\mu^{n-8}+\mu^{n})=0.$$

Adding (4.2) and (4.3) and also (4.3) and (4.4) we have

$$\kappa(\mu^{n-9}+\mu^{n-8}+\mu^{n})=0 \quad \text{and} \quad \kappa(-\mu^{n-9}-\mu^{n-8}-\mu^{n-3})=0$$

using the second equality of (4.1). And taking the sum of these two equalities we have $\kappa(\mu^{n-3})=\kappa(\mu^{n})$. By substituting this into the second equality of (4.1) we then have

(4.5) $$\kappa(\mu^{n-2})=0$$

so that from the third equality of (4.1) it follows that

(4.6) $$\kappa(\mu^{n-5})=\kappa(\mu^{n-4}).$$

Using the first equality of (4.1) and (4.5) it follows from (3.7), (ii) with $k=n-13$ and (3.7), (i) with $k=n-14$ that

$$\kappa(\mu^{n-13}+\mu^{n-10}+\mu^{n-4})=0$$

and

$$\kappa(\mu^{n-13}-\mu^{n-11}-\mu^{n-10}+\mu^{n-8}+\mu^{n-7}-\mu^{n-5}-\mu^{n-4}+\mu^{n})=0.$$

Taking the difference of these two equalities we have

$$\kappa(-\mu^{n-11}+\mu^{n-10}+\mu^{n-8}+\mu^{n-7}-\mu^{n-5}+\mu^{n-4}+\mu^{n})=0.$$

Subtracting (4.2) from this we have

$$\kappa(\mu^{n-7}-\mu^{n-5}+\mu^{n-4}+\mu^{n-2}+\mu^{n})=0.$$

Into this equality substituting $\kappa(\mu^{n-7})=0$, $\kappa(\mu^{n-2})=0$ and $\kappa(\mu^{n-5})=\kappa(\mu^{n-4})$ of (4.1), (4.5) and (4.6) we can get $\kappa(\mu^{n})=0$ easily.

Case 3: $n=9s+2$ $(s\geq 2)$. In this case the following series expansions hold:

$$(1+\mu)^{2n}=1+\mu+\mu^{3}+\mu^{4}-s(\mu^{9}+\mu^{10}+\mu^{12}+\mu^{13})+h_{19}$$

and

$$1+\alpha=1-(s+1)\mu^{9}+(s^{2}-1)\mu^{18}+h_{19}.$$

From the computations of (3.7), (ii) with $k=n-1,n-3,n-5,n-7$ we have

(4.7) $$\kappa(\mu^{n-1}+\mu^{n})=0, \quad \kappa(\mu^{n-2})=\kappa(\mu^{n-3})=\kappa(\mu^{n-5})$$

and

$$\kappa(\mu^{n-7}+\mu^{n-6}-\mu^{n-5}-\mu^{n-4})=0.$$

Moreover let us calculate (3.7), (ii) with $k = n-9, n-11, n-13, n-16$ in this order using (4.7). Then we have

(4.8) $$\kappa(\mu^{n-9} - \mu^{n-8} + \mu^{n-6} - \mu^{n-3} + (1-s)\mu^n) = 0, \quad \kappa(\mu^{n-11} - s\mu^{n-2}) = 0,$$
$$\kappa(\mu^{n-13} + \mu^{n-12} - s(\mu^{n-4} + \mu^{n-2})) = 0$$

and

$$\kappa(\mu^{n-15} - \mu^{n-14} - \mu^{n-12} - s(\mu^{n-6} + \mu^{n-2})) = 0.$$

Similarly we have from (3.7), (ii) with $k = n-17$

(4.9) $$\kappa(-\mu^{n-17} - \mu^{n-14} + s(\mu^{n-8} + \mu^{n-2})) = 0.$$

Compute in addition (3.7), (ii) with $k = n-18$ using (4.7) and (4.8). Then we have

$$\kappa(\mu^{n-17} - \mu^{n-15} - \mu^{n-14} + \mu^{n-12} + s(-\mu^{n-8} + \mu^{n-6}) - \mu^n) = 0.$$

Subsituting the last formula of (4.8) into this equality we have

$$\kappa(\mu^{n-17} + \mu^{n-14} - \mu^n - s(\mu^{n-8} + \mu^{n-2})) = 0.$$

And adding this and (4.9) the assertion $\kappa(\mu^n) = 0$ follows immediately.

Case 4: $n = 9s + 3$ $(s \geq 0)$. We have

$$(1+\mu)^{2n} = 1 - \mu^3 + \mu^6 + s(-\mu^9 + \mu^{12} - \mu^{15} - \mu^{18}) - s^2\mu^{18} + h_{19}$$

and

$$1 + \alpha = 1 - \mu + \mu^2 + (s+1)(-\mu^9 + \mu^{10} - \mu^{11}) + (s - s^2)\mu^{18} + h_{19}.$$

Since it is known that $[SO(7), L] = 0$ ([**3**], [**7**]), we skip to the case $s = 1$, i.e., $n = 12$. From (3.9), (i) and (ii) it follows that

$$\kappa(1 + \mu^3) = 0 \quad \text{and} \quad \kappa(-\mu + \mu^2 - \mu^4 + \mu^5) = 0.$$

From (3.10), (i) we also have

(4.10) $$\kappa(-1 - \mu + \mu^2 - \mu^3 + \mu^4 - \mu^5 + \mu^6 - \mu^7 + \mu^8) = 0.$$

Calculate (3.7), (ii) with $k = 11, 9, 7, 1$. Then we have

$$\kappa(\mu^{11}) = 0, \quad \kappa(\mu^9 + \mu^{10}) = 0, \quad \kappa(\mu^7 - \mu^8 + \mu^9) = 0$$

and

$$\kappa(\mu - \mu^2 - \mu^{10} + \mu^{11}) = 0$$

so that especially from the last equality we have

$$\kappa(\mu - \mu^2 + \mu^9) = 0.$$

Substituting all the other equalities into (4.10) we find

$$\kappa(\mu^6) = 0.$$

On the other hand, from (3.10), (i) we have

$$\kappa(1-\mu^3+\mu^6)=0$$

so that we get

$$\kappa(\mu^6)=\kappa(1)$$

since $\kappa(1+\mu^3)=0$. Consequently it is immediate that $\kappa(1)=0$. Hence we proceed to our computation assuming that $s \geq 2$, i.e., $n \geq 21$.

Calculate (3.7), (ii) with $k=n-1, n-3, n-5, n-7$, (3.7), (i) with $k=n-7$ and (3.7), (ii) with $k=n-9, n-11, n-12$ in this order. Then we have

$$\kappa(\mu^{n-k})=0 \ (1\leq k\leq 3), \quad -\kappa(\mu^{n-7})=\kappa(\mu^{n-5})=\kappa(\mu^{n-4}), \quad \kappa(\mu^{n-11})=\kappa(\mu^{n-10}) \tag{4.11}$$

and

$$\kappa(\mu^{n-9}+\mu^{n-8}+\mu^{n-6}+\mu^{n-4}+(1-s)\mu^n)=0. \tag{4.12}$$

Furthermore from (3.7), (ii) with $k=n-17, n-18$ and (3.7), (i) with $k=n-18$ we get

$$\kappa(\mu^{n-17}-\mu^{n-16}+\mu^{n-15}-\mu^{n-14}+\mu^{n-13}-\mu^{n-12}-s(\mu^{n-8}-\mu^{n-7}+\mu^{n-6}))=0, \tag{4.13}$$

$$\begin{aligned}\kappa(&\mu^{n-17}-\mu^{n-16}-\mu^{n-15}+\mu^{n-12}-\mu^{n-9}-\mu^{n-8}+\mu^{n-7}-\mu^{n-6}+\mu^{n-3}\\ &-s(\mu^{n-8}-\mu^{n-7}-\mu^{n-6}+\mu^{n-3}+\mu^n)-s^2\mu^n)=0,\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\kappa(&\mu^{n-17}-\mu^{n-16}+\mu^{n-15}-\mu^{n-14}+\mu^{n-13}-\mu^{n-12}+\mu^{n-11}-\mu^{n-10}-\mu^n\\ &-s(\mu^{n-9}-\mu^{n-8}+\mu^{n-7}-\mu^{n-6}+\mu^{n-5}-\mu^{n-4}+\mu^{n-3}-\mu^{n-2}+\mu^{n-1}+\mu^n))=0.\end{aligned} \tag{4.15}$$

Take the difference of (4.13) and (4.14) using (4.11) and (4.12) and also take that of (4.14) and (4.15) similarly. Then it follows that

$$\kappa(\mu^{n-15}+\mu^{n-14}-\mu^{n-13}-\mu^{n-12}+\mu^n-s(\mu^{n-6}-\mu^n)-s^2\mu^n)=0$$

and

$$\kappa(\mu^{n-15}+\mu^{n-14}-\mu^{n-13}-\mu^{n-12}-\mu^n-s(\mu^{n-6}-\mu^n))=0.$$

From the difference of these equalities and (4.12) we obtain

$$(s^2+1)\kappa(\mu^n)=0$$

which shows $\kappa(\mu^n)=0$ clearly.

Case 5: $n=9s+4$ $(s\geq 0)$. In this case there hold the relations

$$\begin{aligned}(1+\mu)^{2n}=1-\mu+\mu^2-\mu^3+\mu^4-\mu^5+\mu^6-\mu^7+\mu^8\\ +s(-\mu^9+\mu^{10}-\mu^{11}+\mu^{12}-\mu^{13}+\mu^{14})+h_{15}\end{aligned}$$

and

$$1 + \alpha = 1 + \mu + \mu^3 + \mu^4 - (s+1)(\mu^9 + \mu^{10} + \mu^{12} + \mu^{13}) + h_{15}.$$

We begin with the case $s = 1$, i.e., $n = 13$ since $[SO(9), L]$ is known to be zero ([**3**], [**7**]). By calculating (3.9), (i) and (ii) we have

$$\kappa(\mu) = \kappa(\mu^3) = -\kappa(1).$$

Hence by calculating (3.10), (i) and (ii) we have

$$\kappa(\mu^9 + \mu^{12}) = 0 \quad \text{and} \quad \kappa(1 + \mu^9 + \mu^{10} - \mu^{12}) = 0.$$

Furthermore from (3.7), (ii) with $k = 10, 12$ it follows that

$$\kappa(\mu^{10}) = \kappa(\mu^{12}) = -\kappa(1).$$

Substituting these into the above two equalities yields the contradictory two results

$$\kappa(\mu^9) = \kappa(1) \quad \text{and} \quad \kappa(\mu^9) = -\kappa(1).$$

Therefore it is immediate that $\kappa(1) = 0$.

Let us suppose that $s \geq 2$, i.e., $n \geq 22$. And compute (3.7), (ii) with $k = n-1$, $n-3, n-7, n-8$, (3.7), (i) with $k = n-8$ and (3.7), (ii) with $k = n-9$ in this order. Then it follows that

$$\kappa(\mu^{n-3}) = \kappa(\mu^{n-1}) = \kappa(\mu^n), \quad \kappa(-\mu^{n-7} + \mu^{n-6} - \mu^{n-4} + \mu^n) = 0,$$
$$\kappa(\mu^{n-6} + \mu^{n-5} + \mu^{n-4}) = 0, \quad \kappa(\mu^{n-7} + \mu^{n-4} + \mu^{n-2}) = 0$$

and

$$\kappa(-\mu^{n-9} + \mu^{n-7} - \mu^{n-6} + \mu^{n+4} + (s-1)\mu^n) = 0.$$

Adding the second formula and the fifth yields

$$\kappa(\mu^{n-9} - s\mu^n) = 0.$$

Using the above equalities we have from (3.7), (i) and (ii) with $k = n-11$

$$\kappa(\mu^{n-11} + \mu^{n-10} + (s+1)\mu^{n-2} - s\mu^n) = 0 \quad \text{and} \quad \kappa(\mu^{n-11} + \mu^{n-10} + (s+1)(\mu^{n-2} + \mu^n)) = 0.$$

Subtracting we get

$$(s-1)\kappa(\mu^n) = 0.$$

Hence we see that if $s \equiv 0, 2 \bmod 3$ then $\kappa(\mu^n) = 0$. For the proof of the case $s \equiv 1 \bmod 3$ we consider the following equalities obtained from (3.7), (ii) with $k = n-13, n-14$ and (3.7), (i) with $k = n-14$.

$$\kappa(\mu^{n-13}(-1 + \mu + \mu^3 + \mu^4 + \mu^9 + \mu^{10} + \mu^{12} + \mu^{13})) = 0,$$
$$\kappa(\mu^{n-13}(\mu^2 + \mu^3 + \mu^8 + \mu^9 + \mu^{11} + \mu^{12} + \mu^{13})) = 0$$

and

$$\kappa(\mu^{n-13}(-1+\mu+\mu^2-\mu^3+\mu^4+\mu^8-\mu^9+\mu^{10}+\mu^{11}-\mu^{12}+\mu^{13}))=0.$$

Then by combining these we obtain $\kappa(\mu^n)=0$ immediately.

Case 6: $n=9s+5$ $(s\geq 1)$. In this case we have

$$(1+\mu)^{2n}=1+\mu+(1-s)(\mu^9+\mu^{10})+h_{15}$$

and

$$1+\alpha=1-\mu^3+\mu^6-(s+1)(\mu^9-\mu^{12})+h_{15}.$$

We get

$$\begin{aligned}(4.16)\qquad &\kappa(\mu^{n-1}+\mu^n)=0,\quad \kappa(\mu^{n-3}-\mu^{n-2}-\mu^n)=0,\quad \kappa(\mu^{n-5}+\mu^{n-2})=0,\\ &\kappa(\mu^{n-7}+\mu^{n-6}-\mu^{n-5}+\mu^{n-4}+\mu^n)=0,\\ &\kappa(-\mu^{n-9}+\mu^{n-8}+\mu^{n-5}+\mu^{n-4}+(s+1)\mu^n)=0\end{aligned}$$

and

$$\kappa(\mu^{n-6}-\mu^{n-5})=0$$

from the computations of (3.7), (ii) with $k=n-1,n-3,n-5,n-7,n-9$ and (3.7), (i) with $k=n-9$ in this order. Moreover using (4.16) from (3.7), (i) with $k=n-10$, $n-11$ and (3.7), (ii) $k=n-11$ we have

$$\kappa(-\mu^{n-9}+\mu^{n-8}+s\mu^n)=0, \tag{4.17}$$

$$\kappa(\mu^{n-11}-\mu^{n-8}-s\mu^n)=0 \tag{4.18}$$

and

$$\kappa(-\mu^{n-11}+\mu^{n-8}-\mu^{n-5}+(s-1)\mu^{n-2})=0. \tag{4.19}$$

By virtue of (4.16) we also have

$$\kappa(\mu^{n-9}-\mu^{n-8}-\mu^n)=0 \tag{4.20}$$

by taking the difference of the equalities

$$\begin{aligned}&\kappa(\mu^{n-13}+\mu^{n-12}-\mu^{n-11}-\mu^{n-10}+\mu^{n-9}-\mu^{n-8}-\mu^{n-3}+\mu^{n-2}-\mu^{n-1}-\mu^n\\ &\quad -s(\mu^{n-4}+\mu^{n-3}-\mu^{n-2}+\mu^n))=0\end{aligned}$$

and

$$\begin{aligned}&\kappa(\mu^{n-13}+\mu^{n-12}-\mu^{n-11}-\mu^{n-10}-\mu^{n-9}+\mu^{n-8}+\mu^{n-7}+\mu^{n-6}-\mu^{n-5}+\mu^{n-4}\\ &\quad -\mu^{n-3}+\mu^{n-2}+\mu^{n-1}+\mu^n-s(\mu^{n-4}+\mu^{n-1}))=0\end{aligned}$$

which are obtained from (3.7), (i) and (ii) with $k=n-13$. In a similar way we have

$$\kappa(\mu^{n-2}+(1-s)\mu^{n-5})=0 \tag{4.21}$$

by taking the difference of the equalities

$$\kappa(\mu^{n-11} - \mu^{n-8} - s(\mu^{n-5} + \mu^{n-2})) = 0 \quad \text{and} \quad \kappa(\mu^{n-11} - \mu^{n-8} - \mu^{n-5} - (s+1)\mu^{n-2}) = 0$$

obtained from (3.7), (i) and (ii) with $k = n - 14$. Adding (4.17) and (4.20) yields

$$(s-1)\kappa(\mu^n) = 0.$$

This implies that if $s \equiv 0, 2 \mod 3$ then $\kappa(\mu^n) = 0$. So we suppose that $s \equiv 1 \mod 3$. Then by virtue of (4.21) it follows that $\kappa(\mu^{n-2}) = 0$. Hence we have

$$\kappa(\mu^{n-5} + \mu^n) = 0$$

by adding (4.18) and (4.19). Now we have

$$\kappa(\mu^{n-5}) = 0$$

since $\kappa(\mu^{n-5} + \mu^{n-2}) = 0$ by (4.16). Consequently we have $\kappa(\mu^n) = 0$.

Case 7: $n = 9s + 6$ $(s \geq 0)$. The following series expansion is enough for the present case:

$$(1+\mu)^{2n} = 1 + \mu^3 + (1-s)(\mu^9 + \mu^{12}) + h_{13}$$

and so

$$\begin{aligned} 1 + \alpha = 1 &- \mu + \mu^2 - \mu^3 + \mu^4 - \mu^5 + \mu^6 - \mu^7 + \mu^8 \\ &- (s+1)\mu^9 - \mu^{10} + \mu^{11} + (s+1)\mu^{12} + h_{13}. \end{aligned}$$

Suppose that $s \geq 1$, i.e., $n \geq 15$ and calculate (3.7), (ii) with $k = n-1, n-3, n-5$ and (3.7), (i) with $k = n - 8$ in this order. Then it follows that

$$\kappa(\mu^{n-1}) = 0, \quad \kappa(\mu^{n-3} + \mu^{n-2} - \mu^n) = 0, \quad \kappa(\mu^{n-5} - \mu^{n-4} + \mu^{n-2}) = 0 \quad \text{and} \quad \kappa(\mu^{n-7}) = 0.$$

Moreover from (3.7), (i) with $k = n - 9$ we have

$$\kappa(\mu^{n-9} + \mu^{n-8} - (s+1)\mu^n) = 0.$$

Using these formulas we obtain

$$\kappa(\mu^{n-11} - \mu^{n-10} - \mu^{n-8} - \mu^{n-2} - \mu^n) = 0 \quad \text{and} \quad \kappa(\mu^{n-11} - \mu^{n-10} - \mu^{n-8} - \mu^{n-2}) = 0$$

from (3.7), (i) and (ii) with $k = n - 12$. By subtracting we obtain the required result $\kappa(\mu^n) = 0$.

Here we now consider the remaining case $s = 0$, i.e., $n = 6$. The first three formulas above are valid for the present case. Hence we have

$$\kappa(\mu^5) = 0, \quad \kappa(1 - \mu^3 - \mu^4) = 0 \quad \text{and} \quad \kappa(\mu - \mu^2 + \mu^4) = 0.$$

But from (3.9), (i) it follows that

$$\kappa(1 - \mu + \mu^2 - \mu^3 + \mu^4 - \mu^5 + \mu^6) = 0.$$

Substitute these formulas into this equality. Then it is immediate that $\kappa(\mu^6) = 0$ so that we have $\kappa(1) = 0$ since $\kappa(\mu^6) = \kappa(1)$.

Case 8: $n = 9s + 7$ $(s \geq 1)$. In this case we have

$$(1+\mu)^{2n} = 1 - \mu + \mu^2 + \mu^3 - \mu^4 + \mu^5 + (1-s)(\mu^9 - \mu^{10} + \mu^{11} + \mu^{12}) + h_{13}$$

and

$$1 + \alpha = 1 + \mu - (s+1)\mu^9 - s\mu^{10} + h_{13}.$$

Calculate (3.7), (ii) with $k = n-1, n-3, n-5, n-7$ in this order. Then we have

$$\kappa(\mu^{n-1} - \mu^n) = 0, \quad \kappa(\mu^{n-3} + \mu^n) = 0, \quad \kappa(\mu^{n-5} + \mu^{n-4} + \mu^{n-2} + \mu^{n-1}) = 0$$

and

$$\kappa(\mu^{n-7} - \mu^{n-6}) = 0.$$

Moreover from the computations of (3.7), (i) and (ii) with $k = n-9, n-10$ we have

$$\kappa(-\mu^{n-9} + \mu^{n-3}) = 0, \quad \kappa(\mu^{n-4} - \mu^{n-3}) = 0 \tag{4.22}$$

and

$$\kappa(\mu^{n-8} - \mu^{n-6}) = 0, \quad \kappa(-\mu^{n-9} + \mu^{n-6} + (1-s)\mu^n) = 0.$$

Similar calculations of (3.7), (i) and (ii) with $k = n-11$ give

$$\kappa(-\mu^{n-11} - \mu^{n-10} + \mu^{n-9} + \mu^{n-8} + (s+1)\mu^{n-2} + \mu^n) = 0 \tag{4.23}$$

and

$$\kappa(-\mu^{n-11} - \mu^{n-10} - \mu^{n-8} + (s-1)\mu^{n-2} + s\mu^n) = 0.$$

Taking the difference of two equalities of (4.23) we have

$$\kappa(\mu^{n-9} - \mu^{n-6} - \mu^{n-2} + (1-s)\mu^n) = 0 \tag{4.24}$$

since $\kappa(\mu^{n-8}) = \kappa(\mu^{n-6})$ by (4.22). Furthermore taking the sum of the first formula of (4.23) and the equality

$$\kappa(\mu^{n-11} + \mu^{n-10} + \mu^{n-9} - \mu^{n-8} + \mu^{n-6} - \mu^n + s(\mu^{n-2} + \mu^n)) = 0$$

obtained from (3.7), (ii) with $k = n-12$ we have

$$\kappa(-\mu^{n-9} + \mu^{n-6} + (1-s)\mu^{n-2} + s\mu^n) = 0.$$

Add this and (4.24). Then we have

$$\kappa(\mu^n - s\mu^{n-2}) = 0. \tag{4.25}$$

By taking the difference of the fourth of (4.22) and the equality

$$\kappa(-\mu^{n-9} + \mu^{n-6} + \mu^{n-2} + (s-1)\mu^n) = 0$$

obtained from (3.7), (i) with $k = n-12$ we have

$$\kappa(\mu^{n-2} + (1-s)\mu^n) = 0.$$

Hence by substituting this into (4.25) we obtain

$$\kappa((s^2 - s - 1)\mu^n) = 0$$

which implies $\kappa(\mu^n) = 0$ obviously.

Case 9: $n = 9s + 8$ $(s \geq 0)$. The required expansion series are

$$(1+\mu)^{2n} = 1 + \mu - \mu^3 - \mu^4 + \mu^6 + \mu^7 + (1-s)(\mu^9 - \mu^{10} - \mu^{12}) + h_{13}$$

and

$$1 + \alpha = 1 + \mu^3 - s(\mu^9 + \mu^{12}) + h_{13}.$$

We postpone showing the case $s = 0$. Supposing that $s \geq 1$, i.e., $n \geq 17$ we calculate (3.7), (ii) with $k = n-1, n-3, n-5, n-6, n-7, n-9$ and (3.7), (i) with $k = n-7, n-9$ in this order. Then we have

$$(4.26) \qquad \kappa(\mu^{n-1} + \mu^n) = 0, \quad \kappa(\mu^{n-2} - \mu^n) = 0, \quad \kappa(\mu^{n-3}) = \kappa(\mu^{n-5}) = 0,$$
$$\kappa(\mu^{n-7} + \mu^{n-6}) = 0, \quad \kappa(\mu^{n-4} + \mu^n) = 0, \quad \kappa(\mu^{n-9} - \mu^{n-8}) = 0$$

and

$$\kappa(\mu^{n-6} + (1-s)\mu^n) = 0.$$

Taking the above equalities into account we have

(4.27)
$$\kappa(-\mu^{n-6} + s(\mu^n - \mu^{n-8})) = 0, \quad \kappa(\mu^{n-11} + \mu^{n-8} - s\mu^{n-2}) = 0 \quad \text{and} \quad \kappa(\mu^{n-11} + \mu^{n-8}) = 0$$

from (3.7), (i) with $k = n-10, n-12$ and (3.7), (ii) with $k = n-11$. Subtracting the second equality of (4.27) from the third we have

$$s\kappa(\mu^{n-2}) = 0, \quad \text{so that} \quad s\kappa(\mu^n) = 0$$

by virtue of $\kappa(\mu^{n-2}) = \kappa(\mu^n)$ by (4.26). Hence we conclude that if $s \equiv 1, 2 \mod 3$ then $\kappa(\mu^n) = 0$.

Consider the sum of the last equality of (4.26) and the first of (4.27). Then we have

$$\kappa(\mu^n - s\mu^{n-8}) = 0.$$

It is therefore seen that if $s \equiv 0 \mod 3$ then $\kappa(\mu^n) = 0$.

Finally we consider the case $s = 0$, i.e., $n = 8$. From the equalities of (4.26) together with $\kappa(\mu^8) = \kappa(1)$ it follows that

$$\kappa(\mu^7) = -\kappa(1) \quad \text{and} \quad \kappa(\mu^2) = -\kappa(\mu).$$

Furthermore from (3.9), (i) and (3.10), (i) we have

$$\kappa(\mu) = -\kappa(1) \quad \text{and} \quad \kappa(1 - \mu + \mu^2 + \mu^7 - \mu^8) = 0.$$

By substituting the first three equalities into the last one we obtain $\kappa(1) = 0$ immediately. This completes the proof of Theorem 1.1.

References

[1] M. F. Atiyah and L. Smith, Compact Lie groups and stable homotopy of spheres, Topology, **13** (1974), 135–142.

[2] L. Astey, M. A. Guest and G. Pastor, Lie groups as framed boundaries, Osaka J. Math., **25** (1988), 891–907.

[3] J. C. Becker and R. E. Schultz, Fixed point indices and left invariant framings, Lecture Notes in Math., **657**, Berlin-Heidelberg-New York, Springer, 1977, 1–31.

[4] H. H. Gershenson, A problem in compact Lie groups and framed cobordism, Pacific. J. Math., **51** (1974), 121–129.

[5] D. Husemoller, Fibre Bundles, McGraw-Hill, 1966.

[6] M. Kamata and H. Minami, The special orthogonal groups $SO(2n)$ as framed boundaries, Kyushu J. Math., **54** (2000), 147–153.

[7] K. Knapp, Rank and Adams filtration of a Lie group, Topology, **17** (1978), 41–51.

[8] K. Knapp, On odd-primary components of Lie groups, Proc. Amer. Math. Soc., **79** (1980), 147–152.

[9] A. T. Lundell, The p-equivalence of $SO(2n+1)$ and $Sp(n)$, Pacific J. Math., **110** (1984), 161–166.

[10] K. Maruyama, The stable cohomotopy ring of G_2, Publ. Res. Inst. Math. Sci., Kyoto Univ., **23** (1987), 731–736.

[11] H. Minami, A remark on odd-primary components of special unitary groups, Osaka J. Math., **21** (1984), 457–460.

[12] H. Minami, On the 3-component of $SU(n)$ as a framed manifold, Kyushu J. Math., **55** (2001), 183–187.

[13] H. Minami and M. Kamata, A remark on the framed null-cobordantness of the exceptional Lie group E_6, Kyushu J. Math., **56** (2002), 83–87.

[14] E. Ossa, Lie groups as framed manifolds, Topology, **21** (1982), 315–323.

[15] D. Quillen, The Adams conjecture, Topology, **10** (1971), 67–80.

[16] B. Steer, Orbits and the homotopy class of a compactification of a classical map, Topology, **15** (1976), 383–393.

[17] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Stud., No. 49, Princeton Univ. Press, Princeton, N.J., 1962.

[18] R. M. W. Wood, Framing the exceptional Lie group G_2, Topology, **15** (1976), 303–320.

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