J. Math. Soc. Japan
Vol. 55, No. 4, 2003

On the structure of the group of Lipschitz homeomorphisms and its subgroups, II

Dedicated to Professor Hideki Imanishi on his 60th birthday

By Kōjun Abe and Kazuhiko Fukui†

(Received May 2, 2002)

Abstract. We study the structure of the group of Lipschitz homeomorphisms of $\boldsymbol{R}^n$ leaving the origin fixed and the group of equivariant Lipschitz homeomorphisms of $\boldsymbol{R}^n$, and show that they are perfect. Next we apply these results for the groups of Lipschitz homeomorphisms of orbifolds and the groups of foliation preserving Lipschitz homeomorphisms for compact Hausdorff C^1-foliations.

1. Introduction.

In the previous papers ([**A-F**], [**F-I2**]), we investigated the structure of the group of Lipschitz homeomorphisms of a Lipschitz manifold and its subgroups. In this paper we shall study the first homology of the group of Lipschitz homeomorphisms of $\boldsymbol{R}^n$ leaving the origin fixed and its subgroups. The first homology group of a group is given by the quotient group of the group by its commutator subgroup, and the group is said to be perfect if it coincides with its commutator subgroup.

It is known that the identity components of the groups of homeomorphisms and diffeomorphisms of $\boldsymbol{R}^n$ with compact support are perfect ([**Ma**], [**Th1**]). The results are related to the topology of the classifying spaces of foliations. There are many analogous results on the first homology of the group of automorphisms. Fukui [**F2**] proved that the identity component of the group of homeomorphisms of $\boldsymbol{R}^n$ leaving the origin fixed with compact support is perfect. This result is essentially due to D. McDuff [**Mc**]. In [**F1**], we showed that the first homology group of the identity component of the group of C^∞-diffeomorphisms of $\boldsymbol{R}^n$ leaving the origin fixed with compact support is isomorphic to $\boldsymbol{R}$.

Let $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ denote the group of Lipschitz homeomorphisms of $\boldsymbol{R}^n$ leaving the origin fixed which are isotopic to the identity through Lipschitz homeomorphisms with compact support with respect to the compact open Lipschitz topology (cf. [**A-F**]). In this paper, we would like to show the perfectness of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ and its groups.

In §2, we study the commutator subgroup of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ and prove that $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ is perfect. The idea of the proof is to decompose each element of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ into a

---

2000 *Mathematics Subject Classification*. Primary 58D05.
Key Words and Phrases. Lipschitz homeomorphisms, commutator, perfect, finite group, compact Hausdorff foliation.
† This research was partially supported by Grant-in-Aid for Scientific Research (No. 12640094), Ministry of Education, Culture, Sports, Science and Technology, Japan.

product of Lipschitz homeomorphisms supported in the disjoint unions of small balls accumulating to the origin.

In §3 we consider the representation space $\boldsymbol{R}^n$ of a finite subgroup G of the orthogonal group. Let $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ denote the identity component of the group of equivariant Lipschitz homeomorphisms of $\boldsymbol{R}^n$ with compact support. Then we show by using the same argument as that in §2 that $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ is perfect. The above results are applied to the groups of Lipschitz homeomorphisms of orbifolds.

Furthermore in §4 we apply these results to compact Hausdorff foliations. Let $(M,\mathscr{F})$ be a compact foliated manifold. Let $\mathscr{H}_{\mathrm{LIP}}(M,\mathscr{F})$ denote the identity component of the subgroup of $\mathscr{H}_{\mathrm{LIP}}(M)$ which consists of foliation preserving Lipschitz homeomorphisms. Then we obtain that $\mathscr{H}_{\mathrm{LIP}}(M,\mathscr{F})$ is perfect for compact Hausdorff C^1-foliations (cf. [**F2**]).

The authors would like to thank Takashi Tsuboi for his valuable advises.

2. Commutators of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$.

Let M be an n-dimensional Lipschitz manifold. By $\mathscr{H}_{\mathrm{LIP}}(M)$ we denote the identity component of the group of all Lipschitz homeomorphisms with compact support with respect to the compact open Lipschitz topology (see [**A-F**]). Let (M,N) be a Lipschitz manifold pair, where N is a proper Lipschitz submanifold of M. Let $\mathscr{H}_{\mathrm{LIP}}(M,N)$ denote the identity component of the subgroup of $\mathscr{H}_{\mathrm{LIP}}(M)$ consisting of Lipschitz homeomorphisms which are invariant on N.

In this section, we study commutators of $\mathscr{H}_{\mathrm{LIP}}(M,N)$. First we consider the case $\dim N = 0$, especially $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$. Then we have the following.

Theorem 2.1. *$\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ is perfect.*

First we shall prove a fragmentation lemma for $f \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ close to the identity using the function which T. Tsuboi constructed in [**Ts**].

Let η be a C^∞ monotone increasing function on $[0,1]$ such that $\eta(x)=0$ for $x\in[0,1/2]$, $\eta(x)=1$ for $x\in[3/4,1]$ and $\eta'(x)\le 8$ for $x\in[0,1]$. Let v be the C^∞ function on the half line $(0,\infty)$ such that $v(x)=\eta(2^{2k}x)$ for $x\in[2^{-2k-1},2^{-2k}]$ ($k\ge 0$, $k\in\boldsymbol{Z}$), $v(x)=1-\eta(2^{2k+1}x)$ for $x\in[2^{-2k-2},2^{-2k-1}]$ ($k\ge -1, k\in\boldsymbol{Z}$) and $v(x)=0$ for $x\in[2,\infty)$. Then the support of v is contained in $\bigcup_{k=0}^{\infty}[2^{-2k-1},2^{-2k-1}3]$ and that of $1-v$ is contained in $\bigcup_{k=0}^{\infty}[2^{-2k-2},2^{-2k-2}3]\cup[1,\infty)$. Furthermore we note that $v=0$ on $\bigcup_{k=-1}^{\infty}[2^{-2k-3}3,2^{-2k-1}]\cup[2,\infty)$ and $v=1$ on $\bigcup_{k=-1}^{\infty}[2^{-2k-4}3,2^{-2k-2}]$. Since the absolute value of dv/dx on $[2^{-k-1},2^{-k}]$ is estimated by 2^{k+3}, we have the estimate $|dv/dx(x)| \le 2^3/x$.

Let $\lambda:\boldsymbol{R}^n-0\to[0,1]$ be the C^∞ function defined by $\lambda(x)=v(|x|)$ for $x\in\boldsymbol{R}^n-0$. Then we have the estimate $|\partial\lambda/\partial x_j(x)|\le 8/|x|$ ($j=1,2,\dots,n$). Take $f\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ which is ε-close to the identity, that is, $|f(x)-x|<\varepsilon$ and $|f(x)-x-(f(y)-y)|<\varepsilon|x-y|$ for distinct $x,y\in\boldsymbol{R}^n$. Put $f(x)=x+\varphi(x)$. We define a map $g:\boldsymbol{R}^n\to\boldsymbol{R}^n$ by

$$g(x)=\begin{cases} x+\lambda(x)\cdot\varphi(x) & (x\ne 0)\\ 0 & (x=0).\end{cases}$$

Then we have the following.

Lemma 2.2. *$g \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ for a sufficiently small $\varepsilon > 0$.*

Proof. For distinct x, y satisfying $0 \neq |x| \leq |y|$, we have

$$\begin{aligned}|\lambda(x) \cdot \varphi(x) - \lambda(y) \cdot \varphi(y)| &= |(\lambda(x) - \lambda(y)) \cdot \varphi(x) + \lambda(y) \cdot (\varphi(x) - \varphi(y))| \\ &\leq |\lambda(x) - \lambda(y)| \cdot |\varphi(x)| + |\lambda(y)| \cdot |\varphi(x) - \varphi(y)| \\ &< (16\sqrt{n}/|x|)|x - y| \cdot \varepsilon|x| + \varepsilon|x - y| \\ &= (16\sqrt{n} + 1)\varepsilon \cdot |x - y|.\end{aligned}$$

For $0 \neq |x|$, we have

$$\begin{aligned}|\lambda(x) \cdot \varphi(x) - 0| &\leq |\varphi(x) - 0| \\ &< \varepsilon \cdot |x - 0|.\end{aligned}$$

Therefore we have $g \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ for $(16\sqrt{n} + 1)\varepsilon < 1$ by Proposition 4.2 of [**A-F**]. This completes the proof. □

Corollary 2.3 (decomposition lemma). *For any $f \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ which is sufficiently close to the identity, there exist $g, h \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ such that*

(1) *g and h are close to the identity,*

(2) *$f = g \circ h$ and*

(3) *the support of g is contained in $\bigcup_{k=0}^{\infty} A(2^{-2k-1}, 2^{-2k-1}3)$ and that of h is contained in $\bigcup_{k=0}^{\infty} A(2^{-2k-2}, 2^{-2k-2}3) \cup A(1, \infty)$, where $A(s, t) = \{x \in \boldsymbol{R}^n \mid s \leq |x| \leq t\}$.*

Proof. Take g in Lemma 2.2 for any $f \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ which is sufficiently close to the identity and put $h = g^{-1} \circ f$. Then g and h satisfy the required condition. This completes the proof. □

Let $\{\mu_i\}_{i=1}^{s}$ be a partition of unity on the unit sphere S^{n-1}. For g and h as in Corollary 2.3, we put $g(x) = x + \varphi_g(x)$ and $h(x) = x + \varphi_h(x)$. We define maps $g_i, h_i : \boldsymbol{R}^n \to \boldsymbol{R}^n$ $(i = 1, 2, \ldots, s)$ by

$$g_i(x) = \begin{cases} x + \mu_i(x/|x|) \cdot \varphi_g(x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

and

$$h_i(x) = \begin{cases} x + \mu_i(x/|x|) \cdot \varphi_h(x) & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Then we have the following.

Lemma 2.4. *$g_i, h_i \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ for a sufficiently small $\varepsilon > 0$.*

Proof. We prove that $g_1 \in \mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$. For distinct x, y satisfying $0 \neq |x| \leq |y|$, we have

$$
\begin{aligned}
&|\mu_1(x/|x|)\cdot\varphi_g(x)-\mu_1(y/|y|)\cdot\varphi_g(y)| \\
&\quad = |(\mu_1(x/|x|)-\mu_1(y/|y|))\cdot\varphi_g(x)+\mu_1(y/|y|)\cdot(\varphi_g(x)-\varphi_g(y))| \\
&\quad \le |\mu_1(x/|x|)-\mu_1(y/|y|)|\cdot|\varphi_g(x)|+|\mu_1(y/|y|)|\cdot|\varphi_g(x)-\varphi_g(y)| \\
&\quad < (K_1/|x|)|x-y|\cdot\varepsilon|x|+\varepsilon|x-y| \\
&\quad = (1+K_1)\varepsilon\cdot|x-y|,
\end{aligned}
$$

where K_1 denotes the Lipschitz constant of μ_1.

For $0\neq|x|$, we have

$$
\begin{aligned}
|\mu_1(x/|x|)\cdot\varphi_g(x)-0| &\le |\varphi_g(x)-0| \\
&< \varepsilon\cdot|x-0|.
\end{aligned}
$$

Therefore by using Proposition 4.2 of [**A-F**], we have $g_1\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ if $(1+K_1)\varepsilon<1$. It is similarly proved for h_1 and other g_i,h_i. This completes the proof. $\square$

The following corollary is proved by using Lemma 2.4 and the partition of unity $\{\mu_i\}_{i=1}^s$.

Corollary 2.5 (fragmentation lemma). *For any $f\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ which is sufficiently close to the identity, there exist g_i and $h_i\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ $(i=1,2,\dots,s)$ such that*

(1) *g_i and h_i are close to the identity,*

(2) *$f=g_1\circ\cdots\circ g_s\circ h_1\circ\cdots\circ h_s$ and*

(3) *the support of g_i is contained in the intersection of $\bigcup_{k=0}^{\infty}A(2^{-2k-1},2^{-2k-1}3)$ and the conic neighborhood $\overline{\{x\in\boldsymbol{R}^n\mid\mu_i(x/|x|)\neq 0\}}$ and that of h_i is contained in the intersection of $\bigcup_{k=0}^{\infty}A(2^{-2k-2},2^{-2k-2}3)\cup A(1,\infty)$ and $\overline{\{x\in\boldsymbol{R}^n\mid\mu_i(x/|x|)\neq 0\}}$ for each i.*

Proof of Theorem 2.1. Take $f\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$. We may assume that f is sufficiently close to the identity and the support of f is contained in $D^n(1)$, where $D^n(r)=\{x\in\boldsymbol{R}^n\mid|x|<r\}$. From Corollary 2.5, there exist g_i and $h_i\in\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n,0)$ $(i=1,2,\dots,s)$ such that (1) g_i and h_i are close to the identity, (2) $f=g_1\circ\cdots\circ g_s\circ h_1\circ\cdots\circ h_s$ and (3) the support of g_i is contained in the intersection of $\bigcup_{k=1}^{\infty}A(2^{-2k-1},2^{-2k-1}3)\cup A(1/2,1)$ and the conic neighborhood $\overline{\{x\in\boldsymbol{R}^n\mid\mu_i(x/|x|)\neq 0\}}$ for each i, and (4) the support of h_i is contained in the intersection of $\bigcup_{k=0}^{\infty}A(2^{-2k-2},2^{-2k-2}3)$ and $\overline{\{x\in\boldsymbol{R}^n\mid\mu_i(x/|x|)\neq 0\}}$ for each i.

We construct a Lipschitz homeomorphism $\psi:(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^n,0)$ in the following. Put $\delta=2^{-2k-9}$, $a=2^{-2k-6}7+\delta$, $b=2^{-2k-3}-\delta$ and $c=2^{-2k-4}7-\delta$. Let $\xi:[0,\infty)\to[0,\infty)$ be a homeomorphism satisfying that (1) $\xi(x)=x$ for $x\in[a-\delta,a]\cup[c,c+\delta]\cup[2^{-2}7,\infty)$ $(k\ge -1,k\in\boldsymbol{Z})$, (2) $\xi(b)=3b+4\delta$ $(k\ge -1,k\in\boldsymbol{Z})$, (3) $\xi(0)=0$ and (4) for each k, $\xi|_{[a-\delta,c+\delta]}$ is a piecewise linear homeomorphism of $[a-\delta,c+\delta]$. We note that the slope of ξ on the interval $[a,b]$ (resp. $[b,c]$) is $68/3$ (resp. $3/16$) (see Figure 1). Let $\psi:(D^n(1),0)\to(\boldsymbol{R}^n,0)$ be a Lipschitz map defined by

$$
\psi(x)=\begin{cases}(\xi(x_1),3x_2/16,\dots,3x_n/16) & (x=(x_1,\dots,x_n)\in D^n(1),x_1\ge 0)\\ (x_1,3x_2/16,\dots,3x_n/16) & (x=(x_1,\dots,x_n)\in D^n(1),x_1<0).\end{cases}
$$

We can extend ψ to the outside of $D^n(1)$ satisfying $\psi(x) = x$ for $x \notin D^n(2)$ such that ψ is a Lipschitz homeomorphism of $(\boldsymbol{R}^n, 0)$. We note that the Lipschitz constants of ψ and ψ^{-1} are estimated by $68/3$.

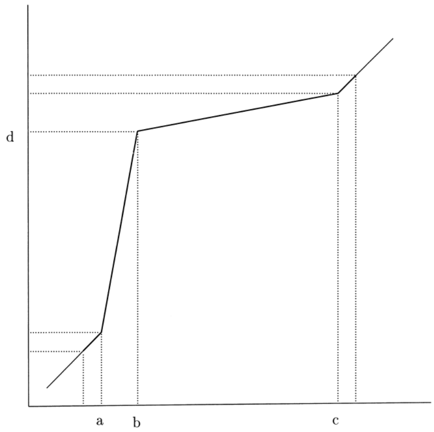

Figure 1. (where $d = 3b + 4\delta$).

We show that g_1 is written as a commutator in $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$. If we choose the function μ_1 such that the support of μ_1 is sufficiently small and $\mu_1((1, 0, \dots, 0)) = 1$, then the support of g_1 is contained in the intersection of the conic neighborhood containing the non-negative x_1-axis and the set $\{(x_1, \dots, x_n) \mid x_1 \in \bigcup_{k=0}^{\infty} [2^{-2k-1} - 2^{-2k-9}, 2^{-2k-1}3 + 2^{-2k-9}]\}$. As in the proof of Theorem 2.2 [**A-F**], we construct a Lipschitz homeomorphism ψ_0 of $\boldsymbol{R}^n$ by conjugating g_1 by ψ^j $(j = 0, 1, 2, \dots)$. Then we have $g_1 = [\psi_0, \psi]$. For other g_i, it is similarly proved by conjugating g_i by some rotation map. It is also similarly proved for h_i. Hence f is contained in the commutator subgroup of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$. This completes the proof. □

Remark 2.6. Let f be an arbitrary orientation preserving Lipschitz homeomorphism of $\boldsymbol{R}$ with compact support. Then we can show that the family $\{tf + (1-t)1_{\boldsymbol{R}}\}$ $(t \in [0, 1])$ gives a Lipschitz isotopy from the identity to f with respect to the compact open Lipschitz topology. Thus $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}, 0)$ coincides with the group of orientation preserving Lipschitz homeomorphisms of $\boldsymbol{R}$ leaving the origin fixed with compact support, which is perfect by Theorem 2.1 (cf. Theorem 3.2 of [**Ts**]).

Let $(M, N_1, \dots, N_r)$ be an $(r+1)$-tuple of Lipschitz manifolds, where N_1 is a proper Lipschitz submanifold of M and N_{i+1} is a proper Lipschitz submanifold of N_i $(i = 1, \dots, r-1)$. Let $\mathscr{H}_{\mathrm{LIP}}(M, N_1, \dots, N_r)$ denote the identity component of the subgroup of $\mathscr{H}_{\mathrm{LIP}}(M)$ consisting of Lipschitz homeomorphisms which are invariant on each N_i.

Next we study commutators of $\mathscr{H}_{\mathrm{LIP}}(M, N_1, \dots, N_r)$. Then we have the following.

Theorem 2.7. *Let $(M, N_1, \dots, N_r)$ be an $(r+1)$-tuple of Lipschitz manifolds. Then $\mathscr{H}_{\mathrm{LIP}}(M, N_1, \dots, N_r)$ is perfect.*

Proof. For $\dim N_r > 0$, it follows from Corollary 2.4 and 4.3 [**A-F**]. For

$\dim N_r = 0$, it follows by the same argument as that in the proof of Theorem 2.1. This completes the proof. $\square$

Let V_i $(i = 1, 2, \ldots, r)$ be linear subspaces of $\boldsymbol{R}^n$. Let $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, V_1, \ldots, V_r)$ denote the identity component of the subgroup of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n)$ consisting of Lipschitz homeomorphisms with compact support which are invariant on each V_i. Then we have the following.

Theorem 2.8. *$\mathscr{H}_{\mathrm{LIP}}(M, V_1, \ldots, V_r)$ is perfect.*

Proof. If $\dim(V_1 \cap \cdots \cap V_r) > 0$, then $V_1 \cap \cdots \cap V_r$ contains a line. Thus it follows from Corollary 2.4 and 4.3 [**A-F**]. If $\dim(V_1 \cap \cdots \cap V_r) = 0$, it follows by the same argument as that in the proof of Theorem 2.1. This completes the proof. $\square$

3. Commutators of $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$.

In this section we study commutators of equivariant Lipschitz homeomorphisms and Lipschitz homeomorphisms of orbifolds.

Let G be a finite subgroup of $O(n)$ which acts on $\boldsymbol{R}^n$ linearly. Let $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ be the subgroup of $\mathscr{H}_{\mathrm{LIP}}(\boldsymbol{R}^n, 0)$ which consists of G-equivariant Lipschitz homeomorphisms isotopic to the identity through G-equivariant Lipschitz homeomorphisms with compact support. Then we have the following.

Theorem 3.1. *$\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ is perfect.*

For $p \in S^{n-1}$ and $r > 0$, we denote $U_p(r) = \{x \in S^{n-1} \mid |x - p| < r\}$. Then we can choose a finite points $\{p_i \in S^{n-1} \mid 1 \le i \le s\}$ and a sufficiently small positive number ε as follows.

(1) Each $U_{p_i}(2\varepsilon)$ is G_{p_i}-invariant, where G_{p_i} is the isotropy subgroup of G at p_i.
(2) If $\tau \in G$ with $\tau \ne 1$, then $U_{p_i}(2\varepsilon) \cap \tau U_{p_i}(2\varepsilon) = \emptyset$ for each i.
(3) $\mathscr{C} = \{G \cdot U_{p_i}(\varepsilon) \mid 1 \le i \le s\}$ is a G-invariant open covering of S^{n-1}.

Let $\{\mu_i\}_{i=1}^{s}$ be a G-invariant partition of unity subordinate to $\mathscr{C}$. Then we have the following proposition by using the same argument as that in §2.

Proposition 3.2. *For any $f \in \mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ which is sufficiently close to the identity, there exist $g_i, h_i \in \mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ $(i = 1, 2, \ldots, s)$ such that*
(1) *g_i and h_i are close to the identity,*
(2) *$f = g_1 \circ \cdots \circ g_s \circ h_1 \circ \cdots \circ h_s$ and*
(3) *the support of g_i is contained in the intersection of $\overline{\bigcup_{k=0}^{\infty} A(2^{-2k-1}, 2^{-2k-1}3)}$ and the conic neighborhood $\overline{\{x \in \boldsymbol{R}^n \mid \mu_i(x/|x|) \ne 0\}}$ and that of h_i is contained in the intersection of $\bigcup_{k=0}^{\infty} A(2^{-2k-2}, 2^{-2k-2}3) \cup A(1, \infty)$ and $\overline{\{x \in \boldsymbol{R}^n \mid \mu_i(x/|x|) \ne 0\}}$ for each i.*

Proof of Theorem 3.1.
(1) Case where the origin is an isolated fixed point.
We define equivariant Lipschitz functions $\bar{\mu}_i$ and $\bar{\bar{\mu}}_i$ $(i = 1, 2, \ldots, s)$ on S^{n-1} by

$$\bar{\mu}_i(x) = \begin{cases} 1 & (x \in U_{p_i}(\varepsilon)) \\ -(2/\varepsilon)|x - p_i| + 3 & (x \in U_{p_i}(3\varepsilon/2) \setminus U_{p_i}(\varepsilon)) \\ 0 & (x \notin U_{p_i}(3\varepsilon/2)) \end{cases}$$

and

$$\bar{\bar{\mu}}_i(x) = \begin{cases} 1 & (x \in U_{p_i}(3\varepsilon/2)) \\ -(2/\varepsilon)|x - p_i| + 4 & (x \in U_{p_i}(2\varepsilon) \setminus U_{p_i}(3\varepsilon/2)) \\ 0 & (x \notin U_{p_i}(2\varepsilon)). \end{cases}$$

We may assume $\mu_1((1,0,\dots,0)) = 1$. We denote by C the conic component of $\overline{\{x \in \boldsymbol{R}^n \mid \bar{\bar{\mu}}_i(x/|x|) \neq 0\}}$ containing the non-negative x_1-axis. We define a map $\bar{\psi}_C$ on C by

$$\bar{\psi}_C(x) = \begin{cases} (\bar{\mu}_1(x/|x|)\xi(x_1) + (1 - \bar{\mu}_1(x/|x|))x_1, & \\ \quad (1 - (13/16)\bar{\bar{\mu}}_1(x/|x|))x_2, \dots, (1 - (13/16)\bar{\bar{\mu}}_1(x/|x|))x_n) & (x \in C \text{ and } x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Here ξ is the map in the proof of Theorem 2.1. Then we see that $\bar{\psi}_C$ is Lipschitz. We extend $\bar{\psi}_C$ to a map $\bar{\psi} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ equivariantly as follows:

$$\bar{\psi}(x) = \begin{cases} \tau^{-1} \cdot \bar{\psi}_C(\tau \cdot x) & (\tau \cdot x \in C \text{ for some } \tau \in G) \\ x & (\tau \cdot x \notin C \text{ for any } \tau \in G). \end{cases}$$

Then we can see that $\bar{\psi}$ is well-defined and is an equivariant Lipschitz homeomorphism.

Let $f \in \mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$. We may assume that f is sufficiently close to the identity and the support of f is contained in $D^n(1)$. From Proposition 3.2, there exist $g_i, h_i \in \mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ $(i = 1, 2, \dots, s)$ such that (1) g_i and h_i are close to the identity, (2) $f = g_1 \circ \cdots \circ g_s \circ h_1 \circ \cdots \circ h_s$ and (3) the support of g_i is contained in the intersection of $\overline{\bigcup_{k=1}^{\infty} A(2^{-2k-1}, 2^{-2k-1}3) \cup A(1/2, 1)}$ and the conic neighborhood $\overline{\{x \in \boldsymbol{R}^n \mid \mu_i(x/|x|) \neq 0\}}$ and that of h_i is contained in the intersection of $\overline{\bigcup_{k=0}^{\infty} A(2^{-2k-2}, 2^{-2k-2}3)}$ and $\overline{\{x \in \boldsymbol{R}^n \mid \mu_i(x/|x|) \neq 0\}}$ for each i. We show that g_i and h_i are commutators. By the same argument as that in the proof of Theorem 2.1, we have that g_1 is the commutator of an element of $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ and $\bar{\psi}$. For other g_i, it is similarly proved by taking new coordinates. It is also similarly proved for h_i. Hence f is contained in the commutator subgroup of $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$. This completes the proof.

(2) Case where the origin is not an isolated fixed point.

In this case the fixed point set contains a line (one dimensional subspace of $\boldsymbol{R}^n$). We take new coordinates of $\boldsymbol{R}^n$ such that the line is the x_1-axis and define a Lipschitz homeomorphism $\phi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ by

$$\phi(x) = ((1/3)x_1 + c, (1/3)x_2, \dots, (1/3)x_n) \quad \text{for some constant } c$$

on a neighborhood of the origin (cf. Theorem 2.2 of [**A-F**]). Since ϕ is equivariant, the equivariant version of Theorem 2.7 follows. Thus any f in $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$ is contained in the commutator subgroup of $\mathscr{H}_{\mathrm{LIP},G}(\boldsymbol{R}^n)$. This completes the proof. □

Definition 3.3 (Satake [**S**] and Thurston [**Th2**]). A paracompact Hausdorff space M is called a (smooth) orbifold if there exists an open covering $\{U_i\}$ of M, closed under finite intersections, satisfying the following conditions.

(1) For each U_i, there are a finite group G_i, a (smooth) effective action of G_i on an open set $\tilde{U}_i$ of $\boldsymbol{R}^n$ and a homeomorphism $\phi_i : U_i \to \tilde{U}_i/G_i$.

(2) Whenever $U_i \subset U_j$, there is a (smooth) imbedding $\phi_{ij} : \tilde{U}_i \to \tilde{U}_j$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j \\ {\scriptstyle \phi_i^{-1}\circ\pi_i}\downarrow & & \downarrow{\scriptstyle \phi_j^{-1}\circ\pi_j} \\ U_i & \longrightarrow & U_j, \end{array}$$

where $\pi_k : \tilde{U}_k \to \tilde{U}_k/G_k$ $(k = i, j)$ are the natural projections. Each (U_i, ϕ_i) is called a local chart of M.

Let M, N be two orbifolds. A continuous map $f : M \to N$ is called a Lipschitz map if for any point p in M, there exist a local chart (U_i, ϕ_i) of M around p and a local chart $(V_\lambda, \psi_\lambda)$ of N around $f(p)$ such that $f(U_i) \subset V_\lambda$ and for $f : U_i \to V_\lambda$, there is an equivariant Lipschitz map $\tilde{f} : \tilde{U}_i \to \tilde{V}_\lambda$ satisfying $f \circ \phi_i^{-1} \circ \pi_i = \psi_\lambda^{-1} \circ \pi_i \circ \tilde{f}$. As in §4 of [**A-F**], we can introduce the compact open Lipschitz topology on the space of Lipschitz maps from M to N, $C_{\mathrm{LIP}}(M, N)$. A homeomorphism $f : M \to M$ is called a Lipschitz homeomorphism if f and f^{-1} are Lipschitz.

Let $\mathscr{H}_{\mathrm{LIP}}(M)$ denote the group of Lipschitz homeomorphisms of M which are isotopic to the identity through Lipschitz homeomorphisms with compact support. Then we have the following.

Theorem 3.4. *Let M be an orbifold. Then $\mathscr{H}_{\mathrm{LIP}}(M)$ is perfect.*

Proof. Take $f \in \mathscr{H}_{\mathrm{LIP}}(M)$. Then we may assume that f is close to the identity and isotopic to the identity. Then taking an appropriate partition of unity on M, we may assume that $f = f_k \circ \cdots \circ f_1$ such that each $\mathrm{supp}(f_i)$ is contained in a local chart (U_i, ϕ_i) of M. Since each $f_i : U_i \to U_i$ is a Lipschitz homeomorphism isotopic to the identity, we have an equivariant Lipschitz homeomorphism $\tilde{f}_i : \tilde{U}_i \to \tilde{U}_i$ satisfying $f_i \circ \phi_i^{-1} \circ \pi_i = \phi_i^{-1} \circ \pi_i \circ \tilde{f}_i$. We may assume that $\tilde{f}_i$ is contained in $\mathscr{H}_{\mathrm{LIP}, G_i}(\tilde{U}_i)$, which is identified with $\mathscr{H}_{\mathrm{LIP}, G_i}(\boldsymbol{R}^n)$. Furthermore by considering an isotropy subgroup we may assume that each G_i is a subgroup of $O(n)$. From Theorem 3.1, each $\tilde{f}_i$ is a product of commutators of elements of $\mathscr{H}_{\mathrm{LIP}, G_i}(\boldsymbol{R}^n)$. Hence each f_i is expressed as a product of commutators of elements of $\mathscr{H}_{\mathrm{LIP}}(U_i)$. Therefore f is contained in the commutator subgroup of $\mathscr{H}_{\mathrm{LIP}}(M)$. This completes the proof. $\square$

4. Application to foliations.

In this section we apply the results in §2, 3 to study commutators of foliation preserving Lipschitz homeomorphisms.

Let M be a compact C^1-manifold without boundary and $\mathscr{F}$ a compact Hausdorff codimension q C^1-foliation of M, where $\mathscr{F}$ is said to be *Hausdorff* if the leaf space $M/\mathscr{F}$ is Hausdorff. Then we have a nice picture of the local behavior of $\mathscr{F}$ as follows.

Proposition 4.1 ([**E**]). *There is a generic leaf L_0 with property that there is an open dense subset of M, where the leaves have all trivial holonomy and are all diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite subgroup $G(L)$ of $O(q)$ such that $G(L)$ acts freely on L_0 on the right and $L_0/G(L) \cong L$. Let D^q be the unit disk. We foliate $L_0 \times D^q$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x, y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and*

$y \in D^q$. So we have a foliation induced on $U = L_0 \times_{G(L)} D^q$. The leaf corresponding to $y = 0$ is $L_0/G(L)$. Then there is a C^1-imbedding $\varphi : U \to M$ with $\varphi(U) = U(L)$, which preserves leaves and $\varphi(L_0/G(L)) = L$.

Let $\mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F})$ be the subgroup of $\mathscr{H}_{\mathrm{LIP}}(M)$ consisting of foliation preserving Lipschitz homeomorphisms which are isotopic to the identity through foliation preserving Lipschitz homeomorphisms (see [**F-I1**], [**F-I2**] for the definitions). Then we have the following.

Theorem 4.2. *Let M be a compact C^1-manifold without boundary and $\mathscr{F}$ a compact Hausdorff codimension q C^1-foliation of M. Then $\mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F})$ is perfect.*

Proof. Let $\mathscr{F}_0$ be the foliation on $L_0 \times D^q$ with leaves of the form $L_0 \times \{pt\}$ and let $\mathscr{H}_{\mathrm{LIP}}(L_0 \times D^q, \mathscr{F}_0)$ be the group consisting of foliation preserving Lipschitz homeomorphisms which are isotopic to the identity through foliation preserving Lipschitz homeomorphisms satisfying that the supports are contained in $\operatorname{int} L_0 \times D^q$. Take a leaf L of $\mathscr{F}$. Then L has a saturated neighborhood $U(L)$ of L as in Proposition 4.1, which is identified with $U = L_0 \times_{G(L)} D^q$. We put $\mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0) = \{f \in \mathscr{H}_{\mathrm{LIP}}(L_0 \times D^q, \mathscr{F}_0) \mid f \text{ is } G(L)\text{-equivariant}\}$. Then since $\pi : L_0 \times D^q \to L_0 \times_{G(L)} D^q = U(L)$ is a covering, we have the natural homomorphism $\pi : \mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0) \to \mathscr{H}_{\mathrm{LIP}}(U(L), \mathscr{F}|_{U(L)})$ which is surjective, where $\mathscr{H}_{\mathrm{LIP}}(U(L), \mathscr{F}|_{U(L)}) = \{f \in \mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F}) \mid \operatorname{supp}(f) \subset \operatorname{int} U(L)\}$.

Let $f \in \mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F})$ satisfying $\operatorname{supp}(f) \subset \operatorname{int} U(L)$. We may assume that f is close to the identity. Then we can take $\hat{f} \in \mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0)$ satisfying that $\pi(\hat{f}) = f$ and $\hat{f}$ is close to the identity.

On the one hand, we also have the epimorphism $r : \mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0) \to \mathscr{H}_{\mathrm{LIP}, G(L)}(D^q)$ defined by the restriction. We put $\bar{f} = r(\hat{f})$. Note that $\bar{f}$ is close to the identity. By Theorem 3.1, $\bar{f}$ is expressed as a product of commutators $\prod_{i=1}^{k}[\bar{f}_{2i-1}, \bar{f}_{2i}]$, where $\bar{f}_i \in \mathscr{H}_{\mathrm{LIP}, G(L)}(D^q)$. Since r is surjective, we can lift $\bar{f}_i$ to $\hat{f}_i$ in $\mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0)$ $(i = 1, 2, \ldots, 2k)$. We may assume that $\prod_{i=1}^{k}[\hat{f}_{2i-1}, \hat{f}_{2i}]$ is close to the identity since we can take $\hat{f}_i$ satisfying $\hat{f}_i(x, y) = (x, \bar{f}_i(y))$ $(i = 1, 2, \ldots, 2k)$ for $x \in L_0$, $y \in D^q$. Then $\hat{f} \circ (\prod_{i=1}^{k}[\hat{f}_{2i-1}, \hat{f}_{2i}])^{-1}$ is contained in the kernel of r. We may assume that $\hat{f} \circ (\prod_{i=1}^{k}[\hat{f}_{2i-1}, \hat{f}_{2i}])^{-1}$ is close to the identity in the identity component of $\ker r$, $\mathscr{H}_{\mathrm{LIP}, L, G(L)}(L_0 \times D^q, \mathscr{F}_0)$. By the same argument as that in the proof of Theorem 3.4 of [**F-I2**], we see that $\hat{f} \circ (\prod_{i=1}^{k}[\hat{f}_{2i-1}, \hat{f}_{2i}])^{-1}$ is contained in the commutator subgroup of $\mathscr{H}_{\mathrm{LIP}, L, G(L)}(L_0 \times D^q, \mathscr{F}_0)$. Thus $\hat{f}$ is expressed as a product of commutators of elements in $\mathscr{H}_{\mathrm{LIP}, G(L)}(L_0 \times D^q, \mathscr{F}_0)$. Hence $f = \pi(\hat{f})$ is also expressed as a product of commutators of elements in $\mathscr{H}_{\mathrm{LIP}}(U(L), \mathscr{F}|_{U(L)})$.

Any $f \in \mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F})$ is decomposed as $f = f_1 \circ \cdots \circ f_s$, where each $\operatorname{supp}(f_i)$ is contained in a saturated neighborhood of a leaf as above. Thus we see that $\mathscr{H}_{\mathrm{LIP}}(M, \mathscr{F})$ is perfect. This completes the proof. □

References

[A-F] K. Abe and K. Fukui, On the structure of the group of Lipschitz homeomorphisms and its subgroups, J. Math. Soc. Japan, **53** (2001), 501–511.

[B] A. Banyaga, The Structure of Classical Diffeomorphism Groups, Math. Appl., Kluwer Academic Publishers, 1997.

[E] D. B. A. Epstein, Foliations with all leaves compact, Ann. Inst. Fourier (Grenoble), **26** (1976), 265–282.

[F1] K. Fukui, Homologies of the group $Diff^{\infty}(\boldsymbol{R}^n, 0)$ and its subgroups, J. Math. Kyoto Univ., **20-3** (1980), 475–487.

[F2] K. Fukui, Commutators of foliation preserving homeomorphisms for certain compact foliations, Publ. Res. Inst. Math. Sci., **34** (1998), 65–73.

[F-I1] K. Fukui and H. Imanishi, On commutators of foliation preserving homeomorphisms, J. Math. Soc. Japan, **51-1** (1999), 227–236.

[F-I2] K. Fukui and H. Imanishi, On commutators of foliation preserving Lipschitz homeomorphisms, J. Math. Kyoto Univ., **41-3** (2001), 507–515.

[Ma] J. N. Mather, The vanishing of the homology of certain groups of homeomorphisms, Topology, **10** (1971), 297–298.

[Mc] D. McDuff, The lattice of normal subgroups of the group of diffeomorphisms or homeomorphisms of an open manifold, J. London Math. Soc. (2), **18-2** (1978), 353–364.

[S] I. Satake, The Gauss-Bonnet theorem for V-manifolds, J. Math. Soc. Japan, **9-4** (1957), 462–492.

[Th1] W. Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. (N.S.), **80** (1974), 304–307.

[Th2] W. Thurston, The Geometry and Topology of 3-manifolds, Mimeographed notes, Princeton Univ.

[Ts] T. Tsuboi, On the perfectness of groups of diffeomorphisms of the interval tangent to the identity at the endpoints, Proceedings of Foliations: Geometry and Dynamics (Warsaw), edited by P. Walczak and et al, World Scientific (Singapore), 2002, 421–440.

Kōjun ABE
Department of Mathematical Sciences
Shinshu University
Matsumoto 390-8621
Japan
E-mail: kojnabe@gipac.shinshu-u.ac.jp

Kazuhiko FUKUI
Department of Mathematics
Kyoto Sangyo University
Kyoto 603-8555
Japan
E-mail: fukui@cc.kyoto-su.ac.jp