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Removable singularities of holomorphic solutions of linear partial differential equations

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Abstract. In a complex domain $V \subset \boldsymbol{C}^n$, let P be a linear holomorphic partial differential operator and K be its characteristic hypersurface. When the localization of P at K is a Fuchsian operator having a non-negative integral characteristic index, it is proved, under some conditions, that every holomorphic solution to $Pu = 0$ in $V \backslash K$ has a holomorphic extension in V. Besides, it is applied to the propagation of singularities for equations with non-involutive double characteristics.

1. Introduction.

We employ the following notation in this paper. $z = (z_1, \dots, z_n) \in \boldsymbol{C}^n$, $z' = (z_2, \dots, z_n)$, $z'' = (z_2, \dots, z_{n-1})$, $D_j = \partial/\partial z_j$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, D'' and α'' are same as z'', $\boldsymbol{N} = \{0, 1, 2, \dots\}$.

Let $n \geq 2$, $m \geq r \geq 1$ and $0 \leq \sigma \leq 1$. For simplicity, we call $P(z, D)$ a *locally Fuchsian operator* of class (m, r, σ) and write $P \in \mathscr{L}^{m,r,\sigma}$ if it is written in the form

$$P(z, D) = \left(\sum_{s=0}^{r} a_s(z)(z_1 D_1)^{r-s} \right) D_n^{m-r} + \sum_{0 \leq s \leq m,\, \alpha \in A(s)} z_1^{\ell(\alpha)} a_\alpha(z) D^\alpha \tag{1.1}$$

where $A(s) = \{\alpha; |\alpha| = m - s, \alpha_n \leq m - r, \alpha \neq (r - s, 0, \dots, 0, m - r)\}$, coefficients are all holomorphic in $V = \{z; |z_i| < r_i, \forall i\}$ $(r_i > 0)$, $a_0(0) \neq 0$ and $\ell(\alpha)$ are non-negative integers satisfying

$$\sigma \ell(\alpha) + (1 - \sigma)\alpha_1 \geq r - s \quad \text{for } |\alpha| = m - s. \tag{1.2}$$

We note that the hyperplane $z_n = h$ is characteristic for P for every h. Set

$$Q_j(z, D_1, D'') = \sum_{\alpha = (\alpha_1, \alpha'', m-j)} z_1^{\ell(\alpha)} a_\alpha(z) D^{(\alpha_1, \alpha'', 0)}. \tag{1.3}$$

By $I(z, z_1 D_1) D_n^{m-r}$ denote the first term of the right hand side of (1.1), then one can write $P = M D_n^{m-r} + \sum_{j=r+1}^{m} Q_j D_n^{m-j}$, where

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$$M(z, D_1, D'') = I(z, z_1 D_1) + Q_r(z, D_1, D''). \tag{1.4}$$

$M(z_1, z'', h, D_1, D'')$ is called the localization of P at $z_n = h$ and, by the condition (1.2), it is easy to see that $M(z_1, z'', h, D_1, D'')$ is a Fuchsian partial differential operator on the characteristic hyperplane $z_n = h$ with respect to z_1 of weight 0 for every $|h| < r_n$. (This is why we call P a locally Fuchsian operator.) As we saw in [**6**], this class is closely related with operators having non-involutive multiple characteristics.

Denote $K = \{z; z_n = 0\}$ and define the indicial polynomial by

$$I(z, \lambda) = \sum_{s=0}^{r} a_s(z)\lambda^{r-s} \tag{1.5}$$

then the following theorem holds.

Theorem 1.1 ([**4**]). *Let $P(z, D)$ be a locally Fuchsian operator of class (m, r, σ). If*

$$I(0, k) \neq 0 \quad \forall k \in \boldsymbol{N} \tag{1.6}$$

then every holomorphic solution of $Pu = 0$ on the universal covering of $V \backslash K$ has a unique holomorphic extension in V.

Our purpose in this paper is to investigate the case where the indicial polynomial $I(0, \lambda)$ has a non-negative integral root. We assume

(A.1) there exists $\nu \in \boldsymbol{N}$ such that $I(z, \nu) = 0$ on $z_1 = 0$ but $I(0, k) \neq 0$ for all $k \in \boldsymbol{N} \backslash \{\nu\}$,

(A.2) $\sigma = 1/p$ with some $p \in \{1, 2, \ldots, r+1\}$,

(A.3) $\sigma\ell(\alpha) + (1-\sigma)\alpha_1 > r - s$ when $|\alpha| = m - s$ and $r - s \geq \alpha_1$.

Let π denote the restriction of functions onto $z_1 = 0$, namely

$$\pi f(z') = f(0, z')$$

then the following proposition holds.

Proposition 1.2. *Let M be defined by* (1.4) *and suppose* (A.1). *Then*

1) *there exists a unique operator $L(z', D_1, D'') = \sum_{j=0}^{\nu} L_j(z', D'')D_1^j$ with $L_\nu = 1$ such that $\pi LM = 0$,*
2) *there exists a unique operator $R(z_1, z', D'') = \sum_{j \geq \nu} R_j(z', D'')z_1^j/j!$ with $R_\nu = 1$ such that $MR\pi = 0$,*
3) *if* (A.3) *is also supposed, the operator $\pi LQ_{r+1}R\pi$ is free from D'', namely a function of z'.*

Set

$$q(z') = \pi LQ_{r+1}R\pi. \tag{1.7}$$

Then the main result of this paper is the following theorem.

Theorem 1.3. *Let $P(z, D)$ be a locally Fuchsian operator of class (m, r, σ) and suppose* (A.1), (A.2) *and* (A.3). *Then, if*

$$q(0) \neq 0 \tag{1.8}$$

every holomorphic solution $u(z)$ to $Pu = 0$ on the universal covering of $V \backslash K$ has a unique holomorphic extension in V.

This theorem will be applied to the propagation of singularities for equations with non-involutive double characteristics in §6. The definition (1.7) looks implicit, however, an algorithm to obtain $q(z')$ will be given in §2. The following example explains the role of the condition (1.8).

Example 1.4. Let $b, c \in \boldsymbol{C}$ and consider

$$P = D_1^2 + z_1 D_1 D_n - b D_n + c.$$

$K = \{z; z_n = 0\}$ is characteristic. Since $P = (z_1 D_1 - b) D_n + D_1^2 + c$, it is easy to see that P is a locally Fuchsian operator of class $(2, 1, 1/2)$ and $I(\lambda) = \lambda - b$. Therefore Theorem 1.1 applies when $b \notin \boldsymbol{N}$.

When $b \in \boldsymbol{N}$, the conditions (A.1), (A.2) and (A.3) hold. Since $M = z_1 D_1 - b$, one can easily see that $L = D_1^b$, $R = z_1^b / b!$ and $q = c$. Therefore Theorem 1.3 applies if $c \neq 0$.

When $c = 0$, the equation $Pu = 0$ has two solutions

$$u_1 = z_n^{-(b+1)/2} (z_n - z_1^2/2)^{b/2}, \quad u_2 = z_1 z_n^{-(b+2)/2} (z_n - z_1^2/2)^{(b-1)/2}.$$

If $b \in \boldsymbol{N}$, one of u_1, u_2 is singular only on $z_n = 0$, namely the conclusion of Theorem 1.3 is not true when $b \in \boldsymbol{N}$ and $c = 0$.

Remarks. 1) Let $\alpha < \beta$ and denote $V_{\alpha\beta} = \{z \in V; \alpha < \arg z_n < \beta\}$. Under the same assumptions as in Theorem 1.3, one can prove that every holomorphic solution to $Pu = 0$ in $V_{\alpha\beta}$ has a holomorphic extension to a neighborhood of $z = 0$.

2) Let P be a linear holomorphic partial differential operator in V for which K is characteristic. The existence of a singular solution, namely a solution to $Pu = 0$ holomorphic in $V \backslash K$ but singular at K, has been studied by several authors. (See [**8**], [**10**] and their references. Cf. [**2**], [**7**], [**12**] as well.) Especially, there exist singular solutions when the localization of the principal part of P at K is a non-degenerate partial differential operator on K of order ≥ 1. (See S. Ouchi [**8**].) In our case, the localization $M(z_1, z'', 0, D_1, D'')$ of whole P at K degenerates at $z_1 = 0$ to be a Fuchsian operator, and the Theorems 1.1 and 1.3 imply the non-existence of singular solutions.

3) In $\boldsymbol{C}^2$, let $P = D_1^2 + z_1 D_1 D_2 - b(z) D_2 + c(z)$. In [**5**], the corresponding result was obtained when $b(0, z_2) = 0, 1, 2, \dots$. When $b(0) = 0, 1, 2, \dots$ but $D_2 b(0) \neq 0$, the existence of a singular solution was also verified, namely Theorem 1.3 does not hold in general by assuming only $I(0, v) = 0$ instead of (A.1).

4) In $\boldsymbol{C}^3$, let $P = z_1 D_1 D_3 + z_1^2 D_2 D_3 + D_2 + c$. Then P is a locally Fuchsian operator whose characteristic index is 0 but does not satisfy the condition (A.3). Since $u = (1/z_3) \exp[-c(z_2 - z_1^2/2)]$ is a solution to the equation $Pu = 0$, we see Theorem 1.3 is not true in general without assuming (A.3).

The theorem will be proved as follows. Take $0 < |h| < r_n$ and a branch of $u(z)$ near $z_n = h$ arbitrarily. We will prove that if h is sufficiently small, there exists a unique holomorphic solution in some neighborhood W of $z = (0, \dots, 0, h)$ containing the origin $z = 0$ to the characteristic Cauchy problem

$$Pv = 0, \quad D_n^j v = D_n^j u \quad \text{on } z_n = h, \ j = 0, 1, \dots, m - r - 1. \tag{1.9}$$

(Theorem 3.1.) Since $u(z)$ is a solution of $Pu = 0$, the uniqueness means $v(z)$ gives a holomorphic extension of u up to the origin. Then it is well known that $u(z)$ has a unique holomorphic extension in V. (Cf. [**6**, Proposition 3.3].)

In §2, the operator M will be studied. The characteristic Cauchy problem (1.9) will be reduced to a pair of integro-differential equations in §3. We will introduce two Banach spaces $\mathscr{B}$ and $\mathscr{B}'$ of holomorphic functions in §4 and prove Theorem 3.1 by means of contraction principle in §5.

2. Operator M.

2.1. Proof of Proposition 1.2.

First, in this subsection, we prove Proposition 1.2. When $\alpha_n = m - r$ and $\alpha'' \neq 0$, it follows from (A.3) that

$$\sigma(\ell - \alpha_1) > r - s - \alpha_1 = |\alpha''| > 0 \tag{2.1}$$

and therefore $\ell > \alpha_1$. (It means M is a Fuchsian operator.)

Since $z_1^\ell D_1^{\alpha_1} = z_1^{\ell - \alpha_1} z_1^{\alpha_1} D_1^{\alpha_1}$ and $z_1^j D_1^j = \sum_{k=1}^j c_{j,k} (z_1 D_1)^k$ with certain constants $c_{j,k}$ $(c_{j,j} = 1)$, one can write

$$M(z, D_1, D'') = \sum_{s=0}^{r} M_s(z, D'')(z_1 D_1)^{r-s}$$

with $M_s(z, D'') = \sum_{|\alpha''| \leq s} z_1^{\tilde{\ell}(s, \alpha'')} b_{s, \alpha''}(z) D^{\alpha''}$, where $b_{s,\alpha''}$ are all holomorphic in V_r, $M_s(z, 0) = a_s(z)$ and $|\alpha''| < \sigma\tilde{\ell}$ for $\alpha'' \neq 0$ because of (2.1).

Denote

$$M_{s,i}(z', \zeta'') = \frac{\partial^i M_s}{\partial z_1^i}(0, z', \zeta'').$$

If $i < \tilde{\ell}$, then $\partial^i (z_1^{\tilde{\ell}} b \zeta^{\alpha''}) / \partial z_1^i = 0$ on $z_1 = 0$ for any function b. Hence $\sigma i \geq \sigma\tilde{\ell} > |\alpha''|$ for $\alpha'' \neq 0$. It means

$$\operatorname{order}_{D''} M_{s,i} < \sigma i \quad \text{for } i > 0.$$

Using $D_1^j (z_1 D_1)^{r-s} = (z_1 D_1 + j)^{r-s} D_1^j$, we have

$$\pi L M f = \sum_{i=0}^{v} \sum_{s=0}^{r} \sum_{j=0}^{i} L_i \binom{i}{j} M_{s, i-j} j^{r-s} \pi D_1^j f.$$

Set

$$\mu_{ij}(z', D'') = \sum_{s=0}^{r} \binom{i}{j} M_{s, i-j}(z', D'') j^{r-s} \tag{2.2}$$

then we have $\pi L M = 0$ if and only if

$$\sum_{i=j}^{v} L_i(z', D'')\mu_{ij}(z', D'') = 0, \quad j = 0, \dots, v. \tag{2.3}$$

Note that $\mu_{j,j} = I(0, z', j)$. Since $I(0, z', v) = 0$ and $I(0, 0, j) \neq 0$ for $j \neq v$ by the assumption (A.1), we see L_j are uniquely determined by these relations with $L_v = 1$. Besides, because $\mathrm{order}_{D''}\, \mu_{ij} < \sigma(i - j)$ when $i > j$, one can easily see that

$$\mathrm{order}_{D''}\, L_j < \sigma(v - j) \quad \text{for } j < v. \tag{2.4}$$

Thus the first part 1) of Proposition 1.2 has been proved.

Since

$$\pi D_1^i MR\pi f = \sum_{j=v}^{i} \sum_{s=0}^{r} \binom{i}{j} M_{s,i-j} j^{r-s} R_j \pi f = \sum_{j=v}^{i} \mu_{ij} R_j \pi f$$

we have $MR\pi = 0$ if and only if

$$\sum_{j=v}^{i} \mu_{ij}(z', D'') R_j(z', D'') = 0 \quad \text{for } i = v, v+1, v+2, \dots. \tag{2.5}$$

Therefore R_i $(i > v)$ are uniquely determined with $R_v = 1$ and

$$\mathrm{order}_{D''}\, R_i < \sigma(i - v) \quad \text{for } i > v. \tag{2.6}$$

Thus the second part 2) has been proved, too.

The operator Q_{r+1} is the sum of $z_1^{\ell(\alpha)} a_\alpha D^{(\alpha_1, \alpha'', 0)}$ with $\alpha = (\alpha_1, \alpha'', m - r - 1)$, $\alpha_1 + |\alpha''| = r + 1 - s$ and $0 \leq s \leq r + 1$. When $\alpha'' \neq 0$, we have $r - s - \alpha_1 = |\alpha''| - 1 \geq 0$. It then follows from the assumption (A.3) that $\sigma(\ell - \alpha_1) > r - s - \alpha_1 \geq 0$, and hence $\ell > \alpha_1$. Because $i \leq v \leq j$, the power of z_1 in the term

$$L_i D_1^i z_1^\ell a_\alpha D^{(\alpha_1, \alpha'', 0)} \frac{z_1^j}{j!} R_j$$

is at least $\ell + j - i - \alpha_1 \geq \ell - \alpha_1 > 0$, and therefore we have

$$\pi L z_1^\ell a_\alpha D^{(\alpha_1, \alpha'', 0)} R\pi = 0.$$

When $\alpha'' = 0$, we have $r - s - \alpha_1 = -1$. From the assumption (1.2) it follows that $\sigma(\ell - \alpha_1) \geq r - s - \alpha_1 = -1$, and therefore $\sigma(\alpha_1 - \ell) \leq 1$. If $v - i > 0$ or $\alpha_1 + j > v$, because of (2.3) and (2.4), we have

$$\mathrm{order}_{D''}\left\{ L_i \binom{i}{j} D_1^{i-j} (z_1^\ell a_\alpha) R_{j+\alpha_1} \right\} < \sigma(\alpha_1 + j - i).$$

Since the terms with $i - j < \ell$ all vanish on $z_1 = 0$, one may suppose $i - j \geq \ell$. Then

$$\sigma(\alpha_1 + j - i) \leq \sigma(\alpha_1 - \ell) \leq 1.$$

It means the order of D'' is zero. If $v - i = 0$ and $\alpha_1 + j = v$, the term is evidently free from D''. Thus the last part 3) has been proved.

2.2. Function $q(z')$.

Next, we explain how to calculate $q(z')$ defined by (1.7). Since $M_s(z,0) = a_s(z)$, we have

$$\mu_{ij}(z',0) = \sum_{s=0}^{r} \binom{i}{j} \frac{\partial^{i-j} a_s}{\partial z_1^{i-j}}(0,z') j^{r-s} = \binom{i}{j} \frac{\partial^{i-j} I}{\partial z_1^{i-j}}(0,z',j).$$

Next, set

$$\ell_i(z') = L_i(z',0), \quad r_i(z') = R_i(z',0).$$

Then ℓ_i $(v-p \le i \le v)$ and r_j $(v \le j \le v+p)$ are given by the relations

$$\sum_{i=j}^{v} \ell_i(z')\mu_{ij}(z',0) = 0, \quad j = v-1,\dots,v-p$$

$$\sum_{j=v}^{i} \mu_{ij}(z',0) r_j(z') = 0, \quad i = v+1,\dots,v+p$$

with $\ell_v = r_v = 1$.

When $\alpha = (k,0,m-r-1)$, as seen above, $\ell - k \ge -1/\sigma = -p$. Therefore one can write

$$Q_{r+1}(z,D_1,0) = \sum_{k=0}^{r+1} c_k(z) D_1^k, \quad c_k = O(z_1^{(k-p)^+}) \quad \text{as } z_1 \to 0$$

where $(k-p)^+ = \max\{k-p,0\}$. Hence

$$\begin{aligned} q(z') &= \pi \sum_{i,j,k} L_i(z',0) D_1^i c_k(z) D_1^k R_j(z',0) z_1^j/j! \\ &= \sum_{i,j,k} L_i(z',0) \binom{i}{j} c_{k,i-j}(z') R_{j+k}(z',0) \end{aligned}$$

where $c_{k,i}(z') = (\partial^i c_k/\partial z_1^i)(0,z')$ and the sum is taken for

$$0 \le k \le r+1, \quad i \le v, \quad j+k \ge v, \quad i-j \ge (k-p)^+.$$

The last inequality is equal to that $j+k-i \le \min\{k,p\}$. Especially we have $j+k \le v+p$ and $i \ge v-p$ and therefore $q(z')$ is given by

$$q(z') = \sum_{i,j,k} \ell_i(z') \binom{i}{j} c_{k,i-j}(z') r_{j+k}(z'). \tag{2.7}$$

2.3. Equation $M^h v = g$.

We next consider the problem

$$M^h v(z) = g(z), \quad \pi D_1^v v = \varphi(z') \tag{2.8}$$

near $z = (0, \dots, 0, h)$, where $M^h(z_1, z'', D_1, D'') = M(z_1, z'', h, D_1, D'')$. So are L^h, R^h, q^h and μ_{ij}^h.

PROPOSITION 2.1. *Let $f(z)$ and $\varphi(z')$ be holomorphic functions near $(0, \dots, 0, h)$. Then the problem* (2.8) *has a unique holomorphic solution $v(z)$ if and only if*

$$\pi L^h g = 0. \tag{2.9}$$

If we denote the solution to the equation $M^h v = g$ satisfying $\pi D_1^\nu v = 0$ by $(M^h)^{-1} g$, then the solution to (2.8) *is given by*

$$v = (M^h)^{-1} g + R^h \varphi. \tag{2.10}$$

The proof will be given in §5. For its preparation, we here explain the role of the condition (2.9). Since $\pi L^h M^h = 0$, we have $\pi L^h g = \pi L^h M^h v = 0$. Hence the condition (2.9) is necessary.

Denote $\pi D_1^i g = g_i$, then we have

$$\sum_{j=0}^{i} \binom{i}{j} \sum_{s=0}^{r} M_{s,i-j}^h j^{r-s} v_j = \sum_{j=0}^{i} \mu_{ij}^h v_j = g_i, \quad i = 0, 1, 2, \dots.$$

Note that $\mu_{ii}^h = I(0, z'', h, i)$, $\mu_{ii}^h \neq 0$ for $i \neq \nu$ and $\mu_{\nu\nu}^h = 0$. Then we see that, by setting $v_\nu = \varphi$, v_j $(j \neq \nu)$ are uniquely determined by these relations if and only if

$$\begin{cases} \displaystyle\sum_{j=0}^{i} \mu_{ij}^h v_j = g_i, \quad i = 0, 1, 2, \dots, \nu - 1 \\ \displaystyle\sum_{j=0}^{\nu-1} \mu_{\nu j}^h v_j = g_\nu \end{cases}.$$

The first ν relations give one to one correspondence between $(v_0, \dots, v_{\nu-1})$ and $(g_0, \dots, g_{\nu-1})$. Under these correspondence, the last relation is equivalent to $\pi L^h g = 0$. In fact,

$$\begin{aligned} g_\nu &= -\sum_{i=0}^{\nu-1} L_i^h g_i = -\sum_{i=0}^{\nu-1} \sum_{j=0}^{i} L_i^h \mu_{ij}^h v_j \\ &= -\sum_{j=0}^{\nu-1} \left(\sum_{i=j}^{\nu-1} L_i^h \mu_{ij}^h \right) v_j = \sum_{j=0}^{\nu-1} L_\nu^h \mu_{\nu j}^h v_j = \sum_{j=0}^{\nu-1} \mu_{\nu j}^h v_j \end{aligned}$$

where we have used $\pi L^h g = \sum_{i=0}^{\nu} L_i^h g_i$ and $\sum_{i=j}^{\nu} L_i^h \mu_{ij} = 0$.

Lastly consider the case where $g = 0$ and $\pi D_1^\nu v = \varphi$. In this case, $v_j = 0$ for $j < \nu$, $v_\nu = \varphi$ and v_j $(j > \nu)$ are determined uniquely by

$$\sum_{j=\nu}^{i} \mu_{ij}^h v_j = 0.$$

Because $\sum_{j=\nu}^{i} \mu_{ij}^h R_j^h = 0$ by the definition of R_j^h, we see $v_j = R_j^h \varphi$ and hence $v = R^h \varphi$.

3. A characteristic Cauchy problem.

Let $h \in \boldsymbol{C}$ be a parameter and consider the Cauchy problem with characteristic initial hyperplane $z_n = h$.

$$\begin{cases} Pu = f \\ u = O\{(z_n - h)^{m-r}\} \quad \text{as } z_n \to h \end{cases} \tag{3.1}$$

Recall that

$$Pu = M^h D_n^{m-r} u + (M - M^h) D_n^{m-r} u + \sum_{j=r+1}^{m} Q_j D_n^{m-j} u$$

where $M = I + Q_r$. Since $\pi L^h M^h = 0$, $\pi L^h (M - M^h) = 0$ on $z_n = h$ and the third term vanishes on $z_n = h$ by the initial condition, it is necessary for f to satisfy the compatibility condition

$$\pi L^h f = 0 \quad \text{on } z_n = h. \tag{3.2}$$

Theorem 3.1. *Let P be a locally Fuchsian operator of class (m, r, σ) and assume* (A.1), (A.2) *and* (A.3). *Then there exists a constant $\delta > 0$ such that, for any $|h| < \delta$ and any holomorphic function $f(z)$ in V_r satisfying the compatibility condition* (3.2), *there exists a unique holomorphic solution $u(z)$ to the Cauchy problem* (3.1) *in $W_{h,\delta} = \{z; |z_i| < \delta \ (i < n), |z_n - h| < \delta\}$.*

Here we explain how to apply the above theorem to the Cauchy problem (1.9). Set

$$u^* = \sum_{j=0}^{m-r-1} D_n^j u(z_1, z'', h) \frac{(z_n - h)^j}{j!}$$

and $v = u^* - w$. Then (1.9) is equivalent to

$$Pw = Pu^*, \quad w = O\{(z_n - h)^{m-r}\}.$$

Now that $\pi L^h M^h = 0$ and $\pi L^h (M - M^h) = 0$ on $z_n = h$, we see

$$\pi L^h P u^* = \sum_{j=r+1}^{m} \pi L^h Q_j D_n^{m-j} u^* = \sum_{j=r+1}^{m} \pi L^h Q_j D_n^{m-j} u = \pi L^h P u = 0$$

on $z_n = h$. Thus the compatibility condition (3.2) is fulfilled and therefore the Theorem 3.1 is applicable.

Theorem 3.1 will be proved by using the contraction principle. For that purpose we rewrite the Cauchy problem (3.1). Denote

$$\begin{cases} \tilde{u} = D_n^{m-r} u \\ D_n^{-1} f(z) = \displaystyle\int_h^{z_n} f(z_1, z'', z_n)\, dz_n \\ Q = I + Q_r - M^h + Q_{r+1} D_n^{-1} + \cdots + Q_m D^{-m+r}. \end{cases} \tag{3.3}$$

Then the Cauchy problem (3.1) is written

$$M^h\tilde{u} + Q\tilde{u} = f. \tag{3.1$'$}$$

Set

$$\psi = \pi D_1^\nu \tilde{u}, \quad v = \tilde{u} - R^h\psi. \tag{3.4}$$

Then, because $M^hR^h\psi = 0$, we have

$$M^hv + Qv + QR^h\psi = f.$$

Operating πL^h and noting $\pi L^h M^h = 0$, we have

$$\pi L^h Qv + \pi L^h QR^h\psi = \pi L^h f.$$

It is easy to see that $\pi L^h IR^h\psi = I(0, z'', z_n, \nu)\psi = 0$. Since each term of Q_r has the form $z_1^\ell aD^{(\alpha_1, \alpha'', 0)}$ with $\alpha_1 + |\alpha''| = r - s$ and $\alpha'' \neq 0$, it follows from the condition (A.3) that $\ell - \alpha_1 \geq \sigma(\ell - \alpha_1) > r - s - \alpha_1 = |\alpha''| > 0$ and therefore we see $\pi L^h Q_r R^h\psi = 0$. Because R^h and D_n^{-1} are commutative, we have

$$\pi L^h Q_{r+1}^h D_n^{-1} R^h \psi = q^h D_n^{-1}\psi$$

where $Q_{r+1}^h = Q_{r+1}(z_1, z'', h, D_1, D'')$. Thus, if we denote

$$\begin{cases} \varphi = D_n^{-1}\psi, \quad w = M^h v \\ \tilde{Q} = (Q_{r+1} - Q_{r+1}^h)D_n^{-1} + Q_{r+2}D_n^{-2} + \cdots + Q_m D^{-m+r} \end{cases} \tag{3.5}$$

then the Cauchy problem (3.1) is reduced to

$$\begin{cases} w + Q(M^h)^{-1}w + QD_nR^h\varphi = f \\ \varphi + \tilde{q}^h\pi L^h Q(M^h)^{-1}w + \tilde{q}^h\pi L^h\tilde{Q}D_nR^h\varphi = \tilde{q}^h\pi L^h f \end{cases} \tag{3.6}$$

where $\tilde{q}^h = 1/q^h$.

4. Banach spaces $\mathscr{B}$ and $\mathscr{B}'$.

To consider the equations (3.6), Banach spaces are introduced and some of their fundamental properties are proved in this section.

4.1. A lemma.

The following lemma will play an important role.

Lemma 4.1. *Let $r \geq 1$, $p \geq 1$ and $\nu \geq 0$ be integers. Then there exists a correspondence $i \to i^*$ from $\mathbf{N}$ to $\mathbf{N}$ such that the following* 1)–6) *hold with some $i_0 \in \mathbf{N}$.*

1) $i_1^* \leq i_2^*$ *if* $i_1 < i_2$

2) $(\nu - p)^* = \nu^*$

3) $\nu^* + p = (\nu + 1)^*$

4) $(i + p)^* - p \leq i^*$

5) $(i - kp - 1)^* + kp \leq i^*$ *if* $1 \leq k \leq r + 1$ *or* $i \leq \nu$

6) $i^* \geq i \quad \forall i$ *and* $i^* = i$ *for* $i \geq i_0$

PROOF. Define i^* by

- $v^* = v$;
- $(v+pj+\ell)^* = \begin{cases} v+pj+p & \text{if } (r+1)\ell > j \\ v+pj+\ell & \text{if } (r+1)\ell \le j \end{cases}$

  for $j = 0,1,2,\dots$ and $\ell = 1,2,\dots,p$;
- $(v-1-pj+\ell)^* = v-pj+p$

  for $j = 1,2,\dots$ and $\ell = 1,2,\dots,p$.

Then it is easy to show 1), 2), 3) and the first half of 6).

Set $A = (i+p)^* - i^*$. When $i > v$, write $i = v+pj+\ell$ $(j \ge 0, 1 \le \ell \le p)$. Since $(r+1)\ell \le j$ implies $(r+1)\ell \le j+1$, we have

$$A = (v+jp+p+\ell)^* - (v+jp+\ell)^* \le p + \max\{0, \ell - p\} = p.$$

When $i+p < v$, write $i+p = v-1-jp+\ell$ $(j \ge 1, 1 \le \ell \le p)$. Then

$$A = (v-1-jp+\ell)^* - (v-1-jp-p+\ell)^* = p.$$

When $i \le v \le i+p$, note $i+p \le v+p$ and $i \ge v-p$. Then

$$A = (i+p)^* - v^* \le (v+p)^* - v^* = p.$$

Thus 4) has been proved.

Denote $B = i^* - (i-kp-1)^*$. When $i-kp-1 > v$, write $i-kp-1 = v+jp+\ell$ $(j \ge 0, 1 \le \ell \le p)$. Then

$$B = (v+(j+k)p+\ell+1)^* - (v+jp+\ell)^*.$$

If $\ell = p$, we have

$$B \ge v+(j+k+1)p + \min\{1,p\} - (v+jp+p) \ge kp.$$

If $\ell < p$, we have

$$B \ge v+(j+k)p - (v+jp) + \min\{1, 0, p-\ell\} \ge kp.$$

Here we have used that $(r+1)\ell \le j$ when $(r+1)(\ell+1) \le j+k$. When $i-kp-1 = v$, we have

$$B = (v+kp+1)^* - v^* \ge v+kp+1-v > kp.$$

When $i-kp-1 < v < i$, write $i-kp-1 = v-1-jp+\ell$ $(j \ge 1, 1 \le \ell \le p)$. Since $i = v+(k-j)p+\ell > v$, we have $k \ge j$ and therefore $(r+1)\ell \ge r+1 \ge k > k-j \ge 0$. Hence

$$\begin{aligned} B &= (v+(k-j)p+\ell)^* - (v-1-jp+\ell)^* \\ &= v+(k-j)p+p-(v-jp+p) = kp. \end{aligned}$$

When $i = v$,

$$B = v^* - (v - kp - 1)^* = v - (v - kp) = kp$$

for all $k \geq 0$.

When $i < v$, write $i = v - 1 - jp + \ell$ $(j \geq 1, 1 \leq \ell \leq p)$. Then

$$B \geq i^* - (i - kp)^* = i^* - (v - 1 - jp - kp + \ell)^*$$
$$= v - jp + p - (v - jp - kp + p) = kp$$

for all $k \geq 0$. Thus we get 5).

For $i > v$, one can write $i = v + pj + \ell$ with $j \geq 0$ and $1 \leq \ell \leq p$. Set $i_0 = v + p^2(r+1) + 1$. Then, if $i \geq i_0$, we have $j \geq p(r+1) \geq \ell(r+1)$ and therefore the second half of 6) holds. We have thus finished the proof the lemma. □

4.2. Definition of $\mathscr{B}$ and $\mathscr{B}'$

Letting $i \to i^*$ be a correspondence satisfying this lemma, we introduce two Banach spaces $\mathscr{B}$ and $\mathscr{B}'$. Let $0 < \rho < 1$, $R_1 \geq 1$, $R_c \geq 1$, $R_n \geq 1$, and set

$$g(\beta) = \beta_1^*! \prod_{j=1}^{|\beta''|+\beta_n} (\beta_1^* + jp) \tag{4.1}$$

$$G(\beta) = g(\beta)\rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n} \tag{4.2}$$

for multi-index $\beta = (\beta_1, \dots, \beta_n)$. Denote $\beta! = \beta_1! \cdots \beta_n!$, $\hat{h} = (0, \dots, 0, h)$ and $(z - \hat{h})^\beta = z_1^{\beta_1} \cdots z_{n-1}^{\beta_{n-1}} (z_n - h)^{\beta_n}$. We say a power series

$$f(z) = \sum_\beta f_\beta (z - \hat{h})^\beta / \beta! \quad (f_\beta \in \boldsymbol{C})$$

belongs to $\mathscr{B}$ and write $f \in \mathscr{B}$ if

$$\sup_\beta \frac{|f_\beta|}{G(\beta)} < \infty. \tag{4.3}$$

Denote the left hand side by $\|f\|$, then it defines a norm, with which $\mathscr{B}$ is a Banach space.

In the same way, denoting $z' = (z_2, \dots, z_n)$, we say a power series

$$\varphi(z') = \sum_{\beta'} \varphi_{\beta'} (z' - \hat{h}')^{\beta'} / \beta'! \quad (\varphi_{\beta'} \in \boldsymbol{C})$$

belongs to $\mathscr{B}'$ and write $\varphi \in \mathscr{B}'$ if

$$\sup_{\beta'} \frac{|\varphi_{\beta'}|}{G(v, \beta')} < \infty. \tag{4.4}$$

Denote the left hand side by $\|\varphi\|'$, then it also defines a norm, with which $\mathscr{B}'$ is a Banach space.

If $f(z)$ is holomorphic in $\{z; |z_i| \leq r_i' \ (1 \leq i \leq n-1), |z_n - h| \leq r_n'\}$, then it belongs to $\mathscr{B}$ if $R_1 \geq 1/r_1'$, $R_c \geq 1/(pr_c')$ and $R_n \geq 1/(pr_n')$, where $r_c = \min\{r_2, \dots, r_{n-1}\}$. In fact

$$
\begin{aligned}
|D^\beta f(c)| &\le C\left(\frac{1}{r_1'}\right)^{\beta_1}\cdots\left(\frac{1}{r_n'}\right)^{\beta_n}\beta_1!\cdots\beta_n! \\
&\le C\beta_1^*!\prod_{k=1}^{|\beta''|+\beta_n}(\beta_1^*+kp)\rho^{\beta_1^*-\beta_1}\left(\frac{1}{r_1'}\right)^{\beta_1^*}\left(\frac{1}{pr_c'}\right)^{|\beta''|}\left(\frac{1}{pr_n'}\right)^{\beta_n}.
\end{aligned}
$$

In the same way, one can see that if $\varphi(z')$ is holomorphic in $\{z; |z_i| \le r_i'\ (2 \le i \le n-1), |z_n - h| \le r_n'\}$, then it belongs to $\mathscr{B}'$ if $R_c \ge 1/(pr_c')$ and $R_n \ge 1/(pr_n')$.

Conversely, if $f \in \mathscr{B}$, then it is holomorphic in

$$R_1|z_1| + (n-2)pR_c|z''| + pR_n|z_n - h| < 1 \tag{4.5}$$

where $|z''| = \max_{2\le i\le n-1}|z_i|$. In fact, $|\sum_\beta f_\beta(z-c)^\beta/\beta!|$ is estimated

$$
\begin{aligned}
&\le C_1\|f\|\sum_\beta\frac{\beta_1^*!}{\beta!}\prod_{k=1}^{|\beta''|+\beta_n}(\beta_1^*+kp)\rho^{\beta_1^*-\beta_1}(R_1|z_1|)^{\beta_1}(R_c|z''|)^{|\beta''|}(R_n|z_n-h|)^{\beta_n} \\
&\le C_2\|f\|\sum_{\beta_1,\beta_c,\beta_n}\frac{\beta_1^*!\prod(\beta_1^*+kp)}{\beta_1!\beta_c!\beta_n!}(R_1|z_1|)^{\beta_1}((n-2)R_c|z''|)^{\beta_c}(R_n|z_n-h|)^{\beta_n} \\
&\le C_3\|f\|\sum_{m=0}^{\infty}(R_1|z_1| + (n-2)pR_c|z''| + pR_n|z_n-h|)^m.
\end{aligned}
$$

If $\varphi(z') \in \mathscr{B}'$, in the same way, one can verify that it is holomorphic in

$$(n-2)pR_c|z''| + pR_n|z_n - h| < 1.$$

4.3. Propositions.

To apply the contraction principle to (3.6), we will need to estimate $\|Q(M_h)^{-1}w\|$, $\|QD_nR^h\varphi\|$, $\|\pi L^hf\|'$ and $\|\tilde{Q}D_nR^h\varphi\|'$. We make here some preparations, employing the notation $(f)_\beta = D^\beta f(\hat{h})$ and $(k)_\ell = k(k-1)\cdots(k-\ell+1)$.

Denote

$$
\begin{cases}
I_0(z_1D_1) = \displaystyle\sum_{s=0}^{r} a_s(0)(z_1D_1)^{r-s}, \\
I_0(\lambda) = \displaystyle\sum_{s=0}^{r} a_s(0)(\lambda)^{r-s}.
\end{cases} \tag{4.6}
$$

Then, by the assumption (A.1), there exists a positive constant c_0 such that

$$|I_0(k)| \ge c_0(k+1)^r \quad \forall k \in \boldsymbol{N}\backslash\{v\}. \tag{4.7}$$

Therefore, for any holomorphic function $g(z)$ with $\pi D_1^v g = 0$, the problem

$$I_0(z_1D_1)v(z) = g(z), \quad \pi D_1^v v = 0 \tag{4.8}$$

has a unique holomorphic solution $v(z)$, which we denote by $v = I_0^{-1}g$.

PROPOSITION 4.2. *Let* (ℓ, α) *with* $\alpha_n \le m - r$ *satisfy the conditions* (1.2), (A.2) *and* (A.3), *or let* $\ell = \alpha_1 = r - s$ *and* $\alpha_n = m - r$. *Assume* (A.1) *and* $\rho R_1 > 1$, *then there exists a positive constant* C *independent of* ρ, R_1, R_c, R_n *such that*

$$\|z_1^\ell D^\alpha D_n^{-m+r} I_0^{-1} w\| \le C \left(\frac{R_c}{R_1^p}\right)^{|\alpha''|} \left(\frac{R_1^p}{R_n}\right)^{m-r-\alpha_n} \|w\| \tag{4.9}$$

for any $w \in \mathscr{B}$.

PROOF. Since $D_1^k(z_1 D_1) = (z_1 D_1 + k) D_1^k$, we have

$$(z_1^\ell D^\alpha D_n^{-m+r} I_0^{-1} w)_\beta = \frac{(\beta_1)_\ell}{I_0(\beta_1 - \ell + \alpha_1)} w_{\beta_1 - \ell + \alpha_1, \beta'' + \alpha'', \beta_n + \alpha_n - m + r}.$$

From the definition of I_0^{-1}, one may suppose $\beta_1 - \ell + \alpha_1 \ne v$.

Set

$$A = \frac{(\beta_1)_\ell}{|I_0(\beta_1 - \ell + \alpha_1)|} g(\beta_1 - \ell + \alpha_1, \beta'' + \alpha'', \beta_n + \alpha_n - m + r)$$

and note $|\alpha''| + \alpha_n - m + r = r - s - \alpha_1$. Then we have

$$A = \frac{(\beta_1)_\ell}{|I_0(\beta_1 - \ell + \alpha_1)|} (\beta_1 - \ell + \alpha_1)^*! \prod_{k=1}^{|\beta''| + \beta_n + r - s - \alpha_1} \{(\beta_1 - \ell + \alpha_1)^* + kp\}.$$

It follows from the conditions (1.2), (A.2) and (A.3) that

$$(\beta_1 - \ell - \alpha_1)^* + (r - s - \alpha_1) p \le \beta_1^*.$$

In fact, when $r - s - \alpha_1 \ge 0$, we have $\ell - \alpha_1 > p(r - s - \alpha_1)$ by (A.2) and (A.3), and therefore $\ell - \alpha_1 \ge p(r - s - \alpha_1) + 1$ (because both sides are integers). When $r - s - \alpha_1 < 0$, we have $\ell - \alpha_1 \ge p(r - s - \alpha_1)$ by (1.2) and (A.2), namely $\alpha_1 - \ell \le p(\alpha_1 - r + s)$. In both cases, the inequality follows from 4) and 5) of Lemma 4.1.

When $r - s - \alpha_1 > 0$, it holds that

$$\frac{(\beta_1 - \ell + \alpha_1)_{\alpha_1}}{|I_0(\beta_1 - \ell + \alpha_1)|} \prod_{k=1}^{r-s-\alpha_1} \{(\beta_1 - \ell + \alpha_1)^* + kp\} \le C_1$$

with a positive constant C_1 independent of β_1. Therefore

$$\begin{aligned} A &\le C_1 (\beta_1)_{\ell - \alpha_1} (\beta_1 - \ell + \alpha_1)^*! \prod_{k=1}^{|\beta''| + \beta_n} \{(\beta_1 - \ell + \alpha_1)^* + (r - s - \alpha_1) p + kp\} \\ &\le C_2 \beta_1^*! \prod_{k=1}^{|\beta''| + \beta_n} \{(\beta_1^* + kp)\} = C_2 g(\beta) \end{aligned}$$

where we used $\beta_1^* = \beta_1$ for large β_1 (Lemma 4.1).

When $r - s - \alpha_1 \le 0$, it holds that

$$\frac{\{(\beta_1 - \ell + \alpha_1)^*\}_{\alpha_1}}{|I_0(\beta_1 - \ell + \alpha_1)| \prod_{k=r-s-\alpha_1+1}^{0} \{(\beta_1 - \ell + \alpha_1)^* + kp\}} \le C_3$$

with a positive constant C_3 independent of β_1. Therefore

$$A \le C_3(\beta_1)_\ell\{(\beta_1 - \ell + \alpha_1)^* - \alpha_1\}! \prod_{k=1}^{|\beta''|+\beta_n} \{(\beta_1 - \ell + \alpha_1)^* + (r - s - \alpha_1)p + kp\}$$

$$\le C_4\beta_1^*! \prod_{k=1}^{|\beta''|+\beta_n} \{(\beta_1^* + kp)\} = C_4 g(\beta)$$

where we used $\beta_1^* = \beta_1$ for large β_1 again.

On the other hand, if $0 < \rho < 1$ and $\rho R_1 > 1$, then

$$\rho^{(\beta_1-\ell+\alpha_1)^*-(\beta_1-\ell+\alpha_1)} R_1^{(\beta_1-\ell+\alpha_1)^*} R_c^{|\beta''+\alpha''|} R_n^{\beta_n+\alpha_n-m+r}$$

$$= \rho^{\beta_1^*-\beta_1} R_c^{|\beta''|} R_n^{\beta_n} \rho^{\ell-\alpha_1} (\rho R_1)^{(\beta_1-\ell+\alpha_1)^*-\beta_1^*} R_c^{|\alpha''|} R_n^{\alpha_n-m+r}$$

$$\le \rho^{\beta_1^*-\beta_1} R_c^{|\beta''|} R_n^{\beta_n} \left(\frac{R_c}{R_1^p}\right)^{|\alpha''|} \left(\frac{R_1^p}{R_n}\right)^{m-r-\alpha_n}$$

where we used $\ell - \alpha_1 \ge p(r - s - \alpha_1) = p(|\alpha''| + \alpha_n - m + r)$. Thus the proof has been finished. $\square$

Proposition 4.3. *Suppose $\rho R_1 > 1$. Then there exists a positive constant C independent of ρ, R_1, R_c and R_n such that*

$$\|z_1^{\gamma_1}(z'')^{\gamma''}(z_n - h)^{\gamma_n} w\| \le C\rho^{\gamma_1}\left(\frac{1}{pR_c}\right)^{|\gamma''|}\left(\frac{1}{pR_n}\right)^{\gamma_n}\|w\| \tag{4.10}$$

for any $\gamma = (\gamma_1, \gamma'', \gamma_n)$ and any $w \in \mathscr{B}$.

Proof.

$$|(z_1^{\gamma_1}(z'')^{\gamma''}(z_n - h)^{\gamma_n} w)_\beta| = (\beta_1)_{\gamma_1}(\beta'')_{\gamma''}(\beta_n)_{\gamma_n}|w_{\beta_1-\gamma_1,\beta''-\gamma'',\beta_n-\gamma_n}|$$

$$\le (\beta_1)_{\gamma_1}(\beta'')_{\gamma''}(\beta_n)_{\gamma_n} G(\beta_1 - \gamma_1, \beta'' - \gamma'', \beta_n - \gamma_n)\|w\|.$$

Since $\beta_1^* \ge \beta_1$, we have

$$\frac{(\beta_1)_{\gamma_1}(\beta_1 - \gamma_1)^*!}{\beta_1^*!} \le \frac{(\beta_1 - \gamma_1)^*!}{(\beta_1^* - \gamma_1)!} \le \frac{(\beta_1 - \gamma_1)^*!}{(\beta_1 - \gamma_1)!} \le \sup_k \frac{k^*!}{k!} = \max_{k \le i_0} \frac{k^*!}{k!}$$

and therefore

$$(\beta_1)_{\gamma_1}(\beta'')_{\gamma''}(\beta_n)_{\gamma_n} g(\beta_1 - \gamma_1, \beta'' - \gamma'', \beta_n - \gamma_n)$$

$$\le C\beta_1^*!(\beta'')_{\gamma''}(\beta_n)_{\gamma_n} \prod_{k=1}^{|\beta''-\gamma''|+\beta_n-\gamma_n} (\beta_1^* + kp)$$

$$\le Cp^{-|\gamma''|-\gamma_n}\beta_1^*! \prod_{k=1}^{|\beta''|+\beta_n} (\beta_1^* + kp)$$

where $C = \max_{k \le i_0} k^*!/k!$.

On the other hand, $\rho R_1 > 1$ implies

$$\begin{aligned}
&\rho^{(\beta_1-\gamma_1)^*-(\beta_1-\gamma_1)} R_1^{(\beta_1-\gamma_1)^*} R_c^{|\beta''-\gamma''|} R_n^{\beta_n-\gamma_n} \\
&\quad = \rho^{\gamma_1}(\rho R_1)^{(\beta_1-\gamma_1)^*-\beta_1^*} R_c^{-|\gamma''|} R_n^{-\gamma_n} \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n} \\
&\quad \le \rho^{\gamma_1} R_c^{-|\gamma''|} R_n^{-\gamma_n} \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n}.
\end{aligned}$$

Thus we get (4.10). □

Proposition 4.4. *Suppose $a(z)$ satisfies*

$$|D^\gamma a(\hat{h})| \le A_0 \gamma! R_0^\gamma \tag{4.11}$$

for all γ with positive constants A_0 and R_0. Then, if

$$\rho R_0 < \frac{1}{2}, \quad \frac{R_0}{pR_c} < \frac{1}{2}, \quad \frac{R_0}{pR_n} < \frac{1}{2}, \quad \rho R_1 > 1 \tag{4.12}$$

there exists a constant C independent of ρ, R_1, R_c, R_n such that

$$\|aw\| \le CA_0\|w\|, \quad w \in \mathscr{B}. \tag{4.13}$$

Proof. In the same way as in the proof of Proposition 4.3, we have

$$\begin{aligned}
|(aw)_\beta| &= \left|\sum_\gamma \binom{\beta}{\gamma} a_\gamma w_{\beta-\gamma}\right| \\
&\le \sum_\gamma A_0 (\beta_1)_{\gamma_1} (\beta'')_{\gamma''} (\beta_n)_{\gamma_n} R_0^{\gamma_1+|\gamma''|+\gamma_n} G(\beta_1-\gamma_1, \beta''-\gamma'', \beta_n-\gamma_n)\|w\| \\
&\le A_0 C_1 \sum_\gamma (\rho R_0)^{\gamma_1} \left(\frac{R_0}{pR_c}\right)^{|\gamma''|} \left(\frac{R_0}{pR_n}\right)^{\gamma_n} G(\beta)\|w\|.
\end{aligned}$$

Therefore, if (4.12) is fulfilled, the inequality (4.13) holds with a positive constant C independent of ρ, R_1, R_c and R_n. □

Now, one can write L^h appearing in Proposition 2.1 as

$$L^h = \sum_{\alpha_1+|\alpha''| \le \nu} b^h_{\alpha_1,\alpha''}(z'') D_1^{\alpha_1} D''^{\alpha''}$$

where $b^h_{\nu,0} = 1$, $|\alpha''| < \sigma(\nu - \alpha_1)$ if $\alpha_1 < \nu$ and $b^h_{\alpha_1,\alpha''}(z'')$ are holomorphic functions. Taking δ sufficiently small, one may suppose

$$|D^{\gamma''} b^h_{\alpha_1,\alpha''}(0)| \le A_0 \gamma''! R_0^{|\gamma''|} \tag{4.14}$$

for all γ'' with positive constants A_0, R_0 independent of $|h| < \delta$ and (α_1, α'').

Proposition 4.5. *Take δ sufficiently small and suppose* (4.12), (4.14), $(R_c/R_1^p) < 1$ *and $|h| < \delta$. Then there is a positive constant C independent of h, ρ, R_1, R_c and R_n such that*

$$\|\pi L^h f\|' \le C\|f\|. \tag{4.15}$$

PROOF. It follows from the definition of the norm $\|\cdot\|'$ that

$$\|\pi D_1^{\nu} f\|' \le \|f\|.$$

Next, consider the remainder terms.

$$|(\pi D_1^{\alpha_1} D''^{\alpha''} f)_{\beta'',\beta_n}| = |f_{\alpha_1,\alpha''+\beta'',\beta_n}| \le G(\alpha_1, \alpha''+\beta'', \beta_n)\|f\|.$$

Since $p|\alpha''| < \nu - \alpha_1$ and p is an integer, we have $\alpha_1 \le \nu - 1 - p|\alpha''|$. Therefore it follows from Lemma 4.1 that

$$\alpha_1^* \le (\nu - 1 - p|\alpha''|)^* \le \nu^* - p|\alpha''|.$$

It means $\alpha_1^* + p|\alpha''| \le \nu^* = \nu$, and therefore we have

$$\begin{aligned}\alpha_1^*! \prod_{k=1}^{|\alpha''+\beta''|+\beta_n} (\alpha_1^* + kp) &\le \alpha_1^*! \prod_{k=1}^{|\alpha''|} (\alpha_1^* + kp) \prod_{k=1}^{|\beta''|+\beta_n} (\nu^* + kp) \\ &\le \nu^*! \prod_{k=1}^{|\beta''|+\beta_n} (\nu^* + kp) = g(\nu, \beta'', \beta_n).\end{aligned}$$

On the other hand, because $\rho < 1$ and $R_1 \ge 1$,

$$\begin{aligned}\rho^{\alpha_1^* - \alpha_1} R_1^{\alpha_1^*} R_c^{|\alpha''+\beta''|} R_n^{\beta_n} &\le R_1^{\alpha_1^* - \nu^*} R_c^{|\alpha''|} \rho^{\nu^*-\nu} R_1^{\nu^*} R_c^{|\beta''|} R_n^{\beta_n} \\ &\le (R_c/R_1^p)^{|\alpha''|} \rho^{\nu^*-\nu} R_1^{\nu^*} R_c^{|\beta''|} R_n^{\beta_n} \le \rho^{\nu^*-\nu} R_1^{\nu^*} R_c^{|\beta''|} R_n^{\beta_n}.\end{aligned}$$

Using Proposition 4.4, one can complete the proof easily. □

PROPOSITION 4.6. *If ℓ, α with $\alpha_n \le m - r$ and $\alpha \ne (r - s, 0, m - r)$ satisfy the conditions* (1.2), (A.2) *and* (A.3), *then there exists a positive constant C independent of ρ, R_1, R_c and R_n such that the following inequality holds for all $\varphi \in \mathscr{B}'$.*

$$\|z_1^{\ell} D^{\alpha} D_n^{-m+r+1} \varphi(z') z_1^{\nu}/\nu!\| \le C\rho^{-p} \left(\frac{R_c}{R_1^p}\right)^{|\alpha''|} \left(\frac{R_1^p}{R_n}\right)^{m-r-1-\alpha_n} \|\varphi\|'.$$

Besides it holds that

$$\|z_1 D_n \varphi(z') z_1^{\nu}/\nu!\| \le C\rho^{1-p} \frac{R_n}{R_1^p} \|\varphi\|'.$$

PROOF.

$$\begin{aligned}|(z_1^{\ell} D^{\alpha} D_n^{-m+r+1} \varphi(z') z_1^{\nu}/\nu!)_{\beta}| &= |(\beta_1)_{\ell} \varphi_{\alpha''+\beta'', \alpha_n+\beta_n-m+r+1}| \\ &\le CG(\nu, \alpha''+\beta'', \alpha_n+\beta_n-m+r+1)\|\varphi\|'\end{aligned}$$

where $\beta_1 - \ell + \alpha_1 = \nu$ and C is a constant depending on ν.

In the same way as in the proof of Proposition 4.2, by Lemma 4.1, it follows from the conditions (1.2), (A.2) and (A.3) that

$$(\beta_1 - \ell + \alpha_1 - p)^* + p(r - s - \alpha_1 + 1) \le \beta_1^*.$$

(Note this is not true when $r - s = \alpha_1 = \ell$.) Since $v^* = (v - p)^* = (\beta_1 - \ell + \alpha_1 - p)^*$ and $r - s - \alpha_1 + 1 = |\alpha''| + \alpha_n - m + r + 1$, this inequality implies

$$v^* + p(|\alpha''| + \alpha_n - m + r + 1) \leq \beta_1^*.$$

Then it is easy to see that

$$v^*! \prod_{k=1}^{|\alpha''+\beta''|+\alpha_n+\beta_n-m+r+1} (v^* + kp) \leq g(\beta).$$

On the other hand, taking $\beta_1 - v = \ell - \alpha_1 \geq p(|\alpha''| + \alpha_n - m + r)$ into account as well, we have

$$\begin{aligned}
&\rho^{v^*-v} R_1^{v^*} R_c^{|\alpha''+\beta''|} R_n^{\alpha_n+\beta_n-m+r+1} \\
&\quad = \rho^{\beta_1-v}(\rho R_1)^{v^*-\beta_1^*} R_c^{|\alpha''|} R_n^{\alpha_n-m+r+1} \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n} \\
&\quad \leq \rho^{-p} \left(\frac{R_c}{R_1^p}\right)^{|\alpha''|} \left(\frac{R_1^p}{R_n}\right)^{m-r-1-\alpha_n} \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n}.
\end{aligned}$$

Thus we have obtained the first half of the assertion. Now let us work on the second half.

$$|(z_1 D_n \varphi(z') z_1^v / v!)_\beta| = |\beta_1 \varphi_{\beta'',\beta_n+1}| \leq CG(v, \beta'', \beta_n + 1) \|\varphi\|'$$

where $\beta_1 - 1 = v$ and C is a constant depending on v. Since $v^* + p = (v + 1)^* = \beta_1^*$, we have

$$g(v, \beta'', \beta_n + 1) \leq g(\beta).$$

Moreover, we see

$$\begin{aligned}
\rho^{v^*-v} R_1^{v^*} R_c^{|\beta''|} R_n^{\beta_n+1} &= \rho^{\beta_1-v}(\rho R_1)^{v^*-\beta_1^*} R_n \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n} \\
&= \rho(\rho R_1)^{-p} R_n \rho^{\beta_1^*-\beta_1} R_1^{\beta_1^*} R_c^{|\beta''|} R_n^{\beta_n}.
\end{aligned}$$

Therefore the second half of the Proposition has been proved. □

5. Proof of Theorem 3.1.

In this section, the Theorem 3.1 is proved by applying the contraction principle to the reduced equations (3.6). Since all the coefficients of $P, M^h, Q, \tilde{Q}, L^h$ and $\tilde{q}^h = 1/q^h$ itself are holomorphic near the origin, one may suppose there are positive constants δ_0, A_0 and R_0 such that all the coefficients and $\tilde{q}$ satisfy (4.11), (4.12) of Proposition 4.4, and $(R_c/R_1^p) \leq 1/2$, for any $|h| < \delta_0$. Then one may use all propositions prepared in the former section.

5.1. Estimate of $(M^h)^{-1}$.

First, we obtain an à priori estimate of the solution to

$$M^h v = g, \quad \pi D_1^v v = 0 \tag{5.1}$$

where $\pi L^h g = 0$ is supposed.

Since $\pi D_1^\nu v = 0$, it holds that $v = I_0^{-1} I_0 v$ (See (4.8)), and therefore the equation can be written as

$$I_0 v + (M^h - I_0) I_0^{-1} I_0 v = g$$

where

$$M^h - I_0 = \sum_{s=0}^{r} \{a_s(z_1, z'', h) - a_s(0,0,0)\}(z_1 D_1)^{r-s}$$
$$+ \sum_{s=0}^{r} \sum_{\alpha_1 + |\alpha''| = r-s,\, \alpha'' \neq 0} z_1^\ell a_{(\alpha_1, \alpha'', m-r)}(z_1, z'', h) D^{(\alpha_1, \alpha'', 0)}.$$

By Proposition 4.2, we have

$$\|z_1^\ell D^{(\alpha_1, \alpha'', 0)} I_0^{-1} I_0 v\| \le C \left(\frac{R_c}{R_1^p}\right)^{|\alpha''|} \|I_0 v\|$$
$$\|(z_1 D_1)^{r-s} I_0^{-1} I_0 v\| \le C \|I_0 v\|.$$

One can write

$$a_s(z_1, z'', h) - a_s(0,0,0) = \sum_{j=1}^{n-1} \tilde{a}_j(z_1, z'', h) z_j + h \tilde{a}_n(z_1, z'', h)$$

and suppose all $\tilde{a}_j$ and $a_{(\alpha_1, \alpha'', m-r)}$ fulfill (4.11) and (4.12) in Proposition 4.4 for any $|h| < \delta_0$, by replacing δ_0, A_0 and R_0 with other ones if necessary. Thus, by Propositions 4.3 and 4.4, we have

$$\|(M^h - I_0) I_0^{-1} I_0 v\| \le C \left\{ \frac{R_c}{R_1^p} + \rho + \frac{1}{R_c} + |h| \right\} \|I_0 v\|.$$

Therefore, there exists a positive constant δ_1 such that, if

$$\rho < \delta_1, \quad \frac{1}{R_c} < \delta_1, \quad \frac{R_c}{R_1^p} < \delta_1, \quad |h| < \delta_1 \tag{5.2}$$

then

$$\|(M^h - I_0) I_0^{-1} I_0 v\| \le \frac{1}{2} \|I_0 v\| \tag{5.3}$$

and the solution to (5.1) satisfies the inequality

$$c_0 \|v\| \le \|I_0 v\| \le 2\|g\| \tag{5.4}$$

where c_0 is the constant appearing in (4.7).

5.2. Proof of Proposition 2.1.

Next, we consider the problem (2.8), i.e.

$$M^h v(z) = g(z), \quad \pi D_1^\nu v(z') = \varphi(z')$$

supposing $\pi L^h g = 0$, and complete the proof of Proposition 2.1.

Set $w = \sum_{j=0}^{\nu} v_j(z')z_1^j/j!$ with $v_\nu = \varphi$. Recall that v_i $(i < \nu)$ are determined by

$$\mu_{ii}^h v_i = g_i - \sum_{j=0}^{i-1} \mu_{ij}^h v_j$$

where $\mu_{ii}^h(z_1, z'') \neq 0$ and $\mathrm{order}_{D''} \mu_{ij}^h(z_1, z'', D'') < \sigma(i-j)$ for $j < i$. One can write

$$\mu_{ij}^h = \sum_{p|\alpha''|<i-j} \mu_{ij\alpha''}^h(z_1, z'')D^{\alpha''}$$

and suppose $1/\mu_{ii}^h, \mu_{ij\alpha''}^h/\mu_{ii}^h$ all satisfy (4.11) and (4.12) in Proposition 4.4 for any $|h| < \delta_0$ by replacing δ_0, A_0 and R_0 with other ones if necessary.

It follows from the definition of the norm that $\|g_i(z')z_1^i/i!\| \le \|g\|$. By Proposition 4.4, we have $\|v_0\| \le C_0\|g\|$. Suppose $\|v_j z_1^j/j!\| \le C_j\|g\|$ for all $j < i < \nu$. Then

$$\left|\left(\frac{z_1^i}{i!}D^{\alpha''}v_j\right)_\beta\right| = |v_{j,\alpha''+\beta'',\beta_n}| \le C_j\|g\|G(j, \alpha''+\beta'', \beta_n)$$

where $\beta_1 = i$. Since $p|\alpha''| < i-j$, we have $j + p|\alpha''| + 1 \le i$ and therefore $j^* + p|\alpha''| \le i^*$ by Lemma 4.1. It is then easy to see that, if $\rho < 1$, $\rho R_1 > 1$ and $R_c/R_1^p < 1$, then $G(j, \alpha''+\beta'', \beta_n) \le G(\beta)$ and $\|v_i z_1^i/i!\| \le C_i\|g\|$ with some constant C_i. Thus we have

$$\|w\| \le C\|g\| + \|\varphi\|'.$$

Next we estimate $M^h w$.

$$\begin{aligned}|(z_1^\ell D^{(\alpha_1,\alpha'',0)}w)_\beta| &= |(\beta_1)_\ell w_{\beta_1-\ell+\alpha_1,\alpha''+\beta'',\beta_n}| \\ &\le CG(\beta_1-\ell+\alpha_1, \alpha''+\beta'', \beta_n)\|w\|\end{aligned}$$

where C is a constant depending on ν. Since $(\beta_1-\ell+\alpha_1)^* + p|\alpha''| \le \beta_1^*$ by Lemma 4.1 and the assumption (A.3), we see easily that, if $\rho < 1$, $\rho R_1 > 1$ and $R_c/R_1^p < 1$, then $G(\beta_1-\ell+\alpha_1, \alpha''+\beta'', \beta_n) \le G(\beta)$. Then, taking Proposition 4.4 into account, we get

$$\|M^h w\| \le C\{\|g\| + \|\varphi\|'\}$$

and therefore $M^h w \in \mathscr{B}$.

Set $\tilde{v} = v - w$ and $\tilde{g} = g - M^h w$, then we have

$$M^h\tilde{v} = I_0\tilde{v} + (M^h - I_0)I_0^{-1}I_0\tilde{v} = \tilde{g}. \tag{5.5}$$

By the consideration in §2, it follows from $\pi L^h g = 0$ that $\tilde{g} = O(z_1^{\nu+1})$. So, defining $\tilde{\mathscr{B}} = \{\tilde{f} \in \mathscr{B}; \tilde{f} = O(z_1^{\nu+1})\}$, we consider the equation (5.5) in this space.

Note $\tilde{g} = g - M^h w \in \tilde{\mathscr{B}}$ if $g \in \mathscr{B}$ and $\varphi \in \mathscr{B}'$, $(M^h - I_0)I_0^{-1}I_0\tilde{v} = O(z_1^{\nu+1})$ if $I_0\tilde{v} = O(z_1^{\nu+1})$ and $\pi L^h\tilde{f} = 0$, $\pi D_1^\nu\tilde{f} = 0$ for $\tilde{f} \in \tilde{\mathscr{B}}$. Then, by the inequality (5.3) and the contraction principle, we see the unique existence of solution $\tilde{v} \in \tilde{\mathscr{B}}$ to the equation (5.5). We have thus completed the proof of Proposition 2.1. Hereafter the constant ρ shall be fixed.

5.3. Estimate of φ.

Next, we consider the equation

$$\varphi + \tilde{q}^h \pi L^h \tilde{Q} D_n R^h \varphi = \psi \tag{5.6}$$

in the space $\mathscr{B}'$, where

$$\tilde{Q} D_n = \sum_{\alpha_n \le m-r-2} a_\alpha z_1^\ell D^\alpha D_n^{-m+r+1} + \sum_{\alpha_n = m-r-1} (z_n - h) \tilde{a}_\alpha z_1^\ell D^\alpha D_n^{-m+r+1}.$$

By replacing δ_0, A_0 and R_0 with other ones if necessary, one may suppose (4.11) and (4.12) for all $a_\alpha, \tilde{a}_\alpha$ and $|h| < \delta_0$.

Set $v_1 = \varphi z_1^v / v!$ and $v_2 = R^h \varphi - v_1$. If $R_c / R_1^p < 1$ and $R_1^p / R_n < 1$, by Propositions 4.3, 4.4 and 4.6, we have

$$\|\tilde{Q} D_n v_1\| \le C_1 \frac{R_1^p}{R_n} \|\varphi\|' + C_2 \frac{1}{R_n} \|\varphi\|' \le C_3 \frac{R_1^p}{R_n} \|\varphi\|'.$$

Here and hereafter C denotes a constant which may depend on ρ but not on R_1, R_c, R_n.

Since $M^h D_n R^h = M^h R^h D_n = 0$, v_2 fulfills

$$M^h D_n v_2 = -M^h D_n v_1, \quad \pi D_1^v v_2 = 0.$$

By Proposition 4.6, if $R_c / R_1^p < 1$, we have

$$\|z_1^\ell D^{(\alpha_1, \alpha'', 1)} v_1\| \le C \frac{R_n}{R_1^p} \|\varphi\|', \quad \|z_1 D_n v_1\| \le C \frac{R_n}{R_1^p} \|\varphi\|'$$

for $\alpha_1 + |\alpha''| = r - s$ and ℓ satisfying the condition (A.3). Note that $I^h(z_1, z'', z_1 D_1) D_n v_1 = I^h(z_1, z'', v) D_n v_1 = a z_1 D_n v_1$ with some holomorphic function a. Since one may suppose a satisfies (4.11) and (4.12) for any $|h| < \delta_0$ by replacing δ_0, A_0, R_0 if necessary, we have

$$\|M^h D_n v_1\| \le C \frac{R_n}{R_1^p} \|\varphi\|'.$$

Hence, by the inequality (5.3), we have

$$\|I_0 D_n v_2\| \le 2 \|M^h D_n v_1\| \le C \frac{R_n}{R_1^p} \|\varphi\|'. \tag{5.7}$$

Therefore, by Propositions 4.2 and 4.4, we see

$$\|\tilde{Q} D_n v_2\| = \|\tilde{Q} I_0^{-1} I_0 D_n v_2\| \le C_1 \left(\frac{R_1^p}{R_n} \right)^2 \|I_0 D_n v_2\| \le C_2 \frac{R_1^p}{R_n} \|\varphi\|'.$$

Thus, by using Propositions 4.4 and 4.5, we obtain

$$\|\tilde{q}^h \pi L^h \tilde{Q} D_n R^h \varphi\|' \le C \frac{R_1^p}{R_n} \|\varphi\|'.$$

Therefore there exists a positive constant δ_2 such that, if

$$\frac{R_1^p}{R_n} < \delta_2 \tag{5.8}$$

then $\|\tilde{q}^h L^h \tilde{Q} D_n R^h \varphi\|' \le (1/2)\|\varphi\|'$ and by the contraction principle we see the unique existence of solution to the equation (5.6), which satisfies

$$\|\varphi\|' = \|(1 + \tilde{q}^h \pi L^h \tilde{Q} D_n R^h)^{-1}\psi\|' \le 2\|\psi\|'. \tag{5.9}$$

5.4. Projection Λ.

We now consider the equation (3.6). It follows from its second equation that

$$\varphi = (1 + \tilde{q}^h \pi L^h \tilde{Q} D_n R^h)^{-1} \tilde{q}^h \pi L^h \{f - Q(M^h)^{-1} w\}. \tag{5.10}$$

With this expression, the first equation of (3.6) is written as

$$w + \Lambda Q(M^h)^{-1} w = \Lambda f \tag{5.11}$$

where

$$\Lambda = 1 - Q D_n R^h (1 + \tilde{q}^h \pi L^h \tilde{Q} D_n R^h)^{-1} \tilde{q}^h \pi L^h. \tag{5.12}$$

Denote $\hat{\mathscr{B}} = \{f \in \mathscr{B}; \pi L^h f = 0\}$. Then it is easy to see

$$\Lambda f \in \hat{\mathscr{B}} \quad \text{if } f \in \mathscr{B}, \qquad \Lambda f = f \quad \text{if } f \in \hat{\mathscr{B}}.$$

It means Λ defines a projection from $\mathscr{B}$ to $\hat{\mathscr{B}}$. We consider the equation (5.11) in the space $\hat{\mathscr{B}}$.

5.5. Contraction principle.

Recall

$$Q = \sum_{\alpha_n \le m-r-1} a_\alpha z_1^\ell D^\alpha D_n^{-m+r} + \sum_{\alpha_n = m-r} (z_n - h)\tilde{a}_\alpha z_1^\ell D^\alpha D_n^{-m+r}.$$

Then, by Propositions 4.2, 4.3, 4.4 and the inequality (5.3), we first get

$$\|Q(M^h)^{-1} w\| = \|Q I_0^{-1} I_0 (M^h)^{-1} w\| \le C_1 \frac{R_1^p}{R_n} \|I_0 (M^h)^{-1} w\| \le C_2 \frac{R_1^p}{R_n} \|w\|.$$

Next we estimate $\|Q D_n R^h \varphi\|$. As in the paragraph 5.3, set $R^h \varphi = v_1 + v_2$, $v_1 = \varphi z_1^\nu / \nu!$. Since, by the assumption (A.1),

$$\{I - I^h\} D_n v_1 = \{I(z, \nu) - I(z_1, z'', h, \nu)\} D_n v_1 = a z_1 (z_n - h) D_n v_1$$

with some holomorphic function a depending holomorphically on h, it follows from Propositions 4.3, 4.4 and 4.6 that $\|\{I - I^h\} D_n v_1\| \le (C/R_1^p)\|\varphi\|'$. Then it is easy to see

$$\|Q D_n v_1\| \le C\|\varphi\|'.$$

Since $\|I_0 D_n v_2\| \le C(R_n/R_1^p)\|\varphi\|'$ by (5.7), in the same way as in the estimation of $Q(M^h)^{-1} w$ one can obtain

$$\|Q D_n v_2\| = \|Q I_0^{-1} I_0 D_n v_2\| \le C_1 \frac{R_1^p}{R_n} \|I_0 D_n v_2\| \le C_2 \|\varphi\|'$$

and therefore

$$\|QD_nR^h\varphi\| \le C\|\varphi\|'.$$

Thus we have

$$\begin{aligned}\|\Lambda f\| &\le \|f\| + \|QD_nR^h(1+\tilde{q}^h\pi L^h\tilde{Q}D_nR^h)^{-1}\tilde{q}^h\pi L^hf\| \\ &\le \|f\| + C_1\|(1+\tilde{q}^h\pi L^h\tilde{Q}D_nR^h)^{-1}\tilde{q}^h\pi L^hf\|' \\ &\le \|f\| + C_2\|\tilde{q}^h\pi L^hf\|' \le C_3\|f\|\end{aligned}$$

where Proposition 4.5 and (5.9) were used. Therefore

$$\|\Lambda Q(M^h)^{-1}w\| \le C_3\|Q(M^h)^{-1}w\| \le C_4\frac{R_1^p}{R_n}\|w\|.$$

Hence, there is a constant δ_3 such that, if

$$\frac{R_1^p}{R_n} \le \delta_3 \tag{5.13}$$

then by the contraction principle we see the unique existence of the solution $w \in \hat{\mathscr{B}}$ to the equation (5.11) for any $f \in \mathscr{B}$. Let w be this solution and define φ by (5.10), then $\varphi \in \mathscr{B}'$, $\tilde{u} = (M^h)^{-1}w + R^hD_n^{-1}\varphi \in \mathscr{B}$ and it gives a solution to the equation (3.1′) which is equivalent to the Cauchy problem (3.1).

5.6. End of the proof.

If $f(z)$ is holomorphic in V_r, it belongs to $\mathscr{B}$ with

$$R_1 > \frac{1}{r_1}, \quad R_c > \frac{1}{pr_c}, \quad R_n > \frac{2}{pr_n}$$

for any $|h| < r_n/2$. Set $\delta_1^* = \min\{1/2, \delta_0, \delta_1, 1/(2R_0)\}$ and let

$$\rho < \delta_1^*, \quad \frac{1}{R_c} < \delta_1^*, \quad \frac{R_c}{R_1^p} < \delta_1^*, \quad |h| < \delta_1^*, \quad \rho R_1 > 1.$$

Fix ρ and set $\delta_3^* = \min\{r_n/2, \delta_1^*, \delta_2, \delta_3\}$. Then, if we take R_n such that

$$\frac{R_1^p}{R_n} < \delta_3^*$$

(we remark δ_2, δ_3 depend on ρ), we see (4.12), (5.2), (5.8) and (5.13) are satisfied and therefore the solution $\tilde{u}$ is holomorphic in $R_1|z_1| + (n-2)pR_c|z''| + pR_n|z_n - h| < 1$. Thus, by setting

$$\delta = \min\left\{\frac{1}{3R_1}, \frac{1}{3(n-2)pR_c}, \frac{1}{3pR_n}, \delta_3^*\right\}$$

we can finish the proof of Theorem 3.1.

6. Application to the propagation of singularities.

Theorem 1.3 as well as Theorem 1.1 can be applied to the propagation of singularities for equations with non-involutive double characteristics.

Let $S = \{z; z_1 = 0\}$, $T = \{z; z_1 = z_n = 0\}$, denote the dual variable of z by $\zeta = (\zeta_1, \dots, \zeta_n)$ and suppose that $P = P(z, D)$ is a linear partial differential operator of second order with holomorphic coefficients in V whose principal symbol $P_2(z, \zeta)$, $(z, \zeta) \in V \times \boldsymbol{C}^n$, satisfies the following condition.

Condition (N). There are holomorphic functions $\lambda^k(z, \zeta')$, $(k = 1, 2)$, in a neighborhood of $(z, \zeta') = (0, \mu')$, $\mu' = (0, \dots, 0, 1) \in \boldsymbol{C}^{n-1}$, such that, if we denote $X^k = \zeta_1 - \lambda^k(z, \zeta')$, then

(N1) $P_2(z, \zeta) = aX^1X^2$, where $a(z)$ is the coefficient of D_1^2 in P,
(N2) $(\partial P_2/\partial \zeta_1)_{\zeta_1 = \lambda_1(z, \zeta')} = 0$ on $\{z_1 = 0, \zeta' = \mu'\}$,
(N3) $\{X^1, aX^2\}_{\zeta_1 = \lambda_1(z, \zeta')} \neq 0$ at $(z, \zeta') = (0, \mu')$, where $\{X^1, aX^2\}$ denotes the Poisson bracket.

Let $\Phi^k(z)$ be the solution to the Cauchy problem

$$D_1 \Phi - \lambda^k(z, D'\Phi) = 0, \quad \Phi(0, z') = z_n \tag{6.1}$$

for each $k = 1, 2$ and denote by K^k the hypersurface $\{z; \Phi^k(z) = 0\}$, which is a characteristic hypersurface issued from T. Replacing r_1 by smaller one if necessary, one may suppose both K^1 and K^2 are connected in V.

Definition 6.1. Let Ω be an open connected set in $\boldsymbol{C}^n$ and $\partial\Omega$ its boundary. Let $u(z)$ be a holomorphic function in a neighborhood of a point $z^0 \in \Omega$ and have a holomorphic extension in the universal covering $\mathscr{R}(\Omega)$. We say $\hat{z} \in \partial\Omega$ is a point of strong (weak respectively) analytic continuation of $u(z)$ if it is analytically continued up to $\hat{z}$ along any (some respectively) path $z = z(t)$, $0 \le t \le 1$, satisfying

$$z(0) = z^0, \quad z(1) = \hat{z}, \quad z(t) \in \Omega \quad \text{for } 0 \le t < 1.$$

Set

$$A^k(z) = \frac{\partial}{\partial z_1}\left\{\frac{\partial P_2}{\partial \zeta_1}(z, D\Phi^k(z))\right\}, \quad B^k(z) = \{P_2 + P_1\}\Phi^k(z)$$

and denote $\Omega = V - K^1 \cup K^2 \cup S$. Then, if $A^1(0)k + B^1(0) \neq 0$, $\forall k \in \boldsymbol{N}$, the following property holds ([**6**]).

Property (AC). *Let $u(z)$ be a holomorphic function in a neighborhood of a point $z^0 \in \Omega$, satisfy $Pu = 0$ and have a holomorphic extension on the universal covering $\mathscr{R}(\Omega)$. If $u(z)$ has a point of strong analytic continuation $\hat{z} \in K^2$ and a point of weak analytic continuation $a \in (S - T)$, then it has a unique holomorphic extension in V.*

This section aims to consider if this property can be true when $A^1(0)\nu + B^1(0) = 0$, $\exists \nu \in \boldsymbol{N}$. We suppose

Condition (N$'$). Set

$$P_1'(z,\zeta) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 P_2}{\partial\zeta_i\partial z_i} + P_1$$

where P_1 denotes the first order term of P, then

(N4) $A^1(z)v + B^1(z) = 0$ on $z_1 = 0$ for some $v \in \boldsymbol{N}$,

(N5) $(\partial P_2/\partial\zeta_1)_{\zeta_1=\lambda^1}$, $\{X^1, aX^2\}_{\zeta_1=\lambda^1}$ and $(P_1')_{\zeta_1=(\lambda^1+\lambda^2)/2}$ are free from $\zeta'' = (\zeta_2,\dots,\zeta_{n-1})$ on $\{z_1 = 0\}$.

Then the following proposition holds.

PROPOSITION 6.2. *Define*

$$Q(z,D) = \sum_{i=1}^{n}\frac{\partial P_2}{\partial\zeta_i}(z, D\Phi^1(z))D_i + B^1(z) \tag{6.2}$$

then, under the conditions (N) *and* (N′), *it holds that*

i) *there exists a unique operator* $L(z',D) = D_1^v + \sum_{j=1}^{v} L_j(z',D')D_1^{v-j}$ *such that* $\pi LQ = 0$,

ii) *there exists a unique operator* $R(z,D') = z_1^v/v! + \sum_{j\geq v+1} R_j(z',D')z_1^j/j!$ *such that* $QR\pi = 0$,

iii) *the operator* $\pi LPR\pi$ *is free from* D', *namely a function of* z'.

Denote

$$q^1(z') = \pi LPR\pi. \tag{6.3}$$

Then, by Theorem 1.3, one can prove the following thorem.

THEOREM 6.3. *Assume Conditions* (N), (N′) *and*

$$q^1(0) \neq 0. \tag{6.4}$$

Then the property (AC) *holds true.*

REMARK. If $u(z)$ takes initial data on S holomorphic on $S\backslash T$ but singular at T, the theorem means at least weak singularities appear everywhere on K_2. (In other words, every point of K_2 is not of strong analytic continuation.) Concerning the existence of holomorphic solutions in the universal covering of $V - K_1 \cup K_2 \cup S$, see C. Wagschal [**13**], J. Persson [**11**] and S. Ouchi [**8**].

Theorem 6.3 is proved as follows. The function $u(z)$ has a holomorphic extension in $\mathscr{R}(V - K^1 \cup S)$ (see [**6**, Proposition 3.4]) and consequently so does it in $\mathscr{R}(V - K^1)$ (see [**6**, Proposition 3.5]). Therefore the proof is completed by the following proposition.

PROPOSITION 6.4. *If* $u(z)$ *is holomorphic in* $\mathscr{R}(V - K^1)$ *and satisfies* $Pu = 0$, *then it has a unique holomorphic extension in* V.

Hereafter we prove this proposition by applying Theorem 1.3.

PROPOSITION 6.5. *The conditions* (N), (N′) *and* (6.4) *are invariant under any regular change of variables*

$$w_1 = z_1, \quad w_j = f_j(z) \quad (2 \le j \le n)$$

satisfying $f_j(0, z') = z_j$.

The proof of this proposition is elementary and so we omit it.

By the condition (N1), the operator P can be written in the form

$$P = a\left\{ \left(D_1 + \sum_j{}' a_j D_j \right)^2 + \sum_{j,k}{}' a_{jk} D_j D_k \right\} + \sum_j b_j D_j + c$$

where all coefficients are holomorphic, $\sum_j'$ denotes the sum for $2 \le j \le n$ and $\sum_{j,k}'$ the sum for $2 \le j,\ k \le n$. If we set $\delta = \sqrt{-\sum_{j,k}' a_{jk}\zeta_j\zeta_k}$ by taking the branch appropriately, then δ is holomorphic in a neighborhood of $(0, \mu')$ and one may suppose $\lambda^1 = -\sum' a_j\zeta_j + \delta$ and $\lambda^2 = -\sum' a_j\zeta_j - \delta$.

Denoting by $\Phi_j(z)$ $(2 \le j \le n-1)$ the solution to the initial value problem

$$\left(D_1 + \sum_j{}' a_j D_j \right)\Phi = 0, \quad \Phi(0, z') = z_j$$

and by $\Phi_n(z)$ the solution to

$$D_1\Phi + \lambda^1(z, D'\Phi) = 0, \quad \Phi(0, z') = z_n,$$

change the variables by

$$w_1 = z_1, \quad w_j = \Phi_j(z) \quad (j \ge 2).$$

Then, denoting w_j by z_j again, one can write P in the form

$$P = a\left\{ (D_1 + a_n D_n)^2 + \sum_{j,k}{}' a_{jk} D_j D_k \right\} + \sum_j b_j D_j + c.$$

Since $\Phi^1 = \Phi_n = z_n$ is a phase function, we have

$$a \cdot (a_n^2 + a_{nn}) = 0.$$

Now, from the conditions (N2) and (N5) it follows that

$$\left(\frac{\partial P_2}{\partial \zeta_1}\right)_{\zeta_1 = \lambda^1} = 2a\delta = 0 \quad \text{on } \{z_1 = 0\},$$

$$\{X^1, aX^2\}_{\zeta_1 = \lambda^1} = 2\frac{\partial(a\delta)}{\partial z_1} \quad \text{on } \{z_1 = 0\}.$$

Since a, δ are holomorphic, we see by (N3) that there are two cases:

Case 1. $a = O(z_1)$, $D_1 a(0) \ne 0$, $\delta(0, \mu') \ne 0$

Case 2. $\delta = O(z_1)$, $a(0) \ne 0$, $\dfrac{\partial\delta}{\partial z_1}(0, \mu') \ne 0$

In the case 1, by the condition (N5), $\partial(a\delta)/\partial z_1$ is free from ζ'' on $z_1 = 0$ and so is δ. Therefore $a_{jk} = O(z_1)$ for all $j, k \geq 2$ except $j = k = n$. By (N5), we also see $b_j = O(z_1)$ for $2 \leq j \leq n-1$. Thus P can be written in the form

$$P = z_1 \left\{ a_{11}D_1^2 + a_{1n}D_1D_n + z_1 \sum_{j,k}{}' a_{jk}D_jD_k \right\} + b_1D_1 + z_1 \sum_{j=2}^{n-1} b_jD_j + b_nD_n + c$$

where $a_{11}(0) \neq 0$, $a_{1n}(0) \neq 0$, $a_{nn}(z) = 0$, and the indicial equation (N4) is

$$a_{1n}\nu + b_n = 0 \quad \text{on } z_1 = 0.$$

It is easy to see that $P \in \mathscr{L}^{2,1,1}$ and all the conditions of Theorem 1.3 are fulfilled.

In the case 2, $\delta^2 = O(z_1^2)$. Therefore $a_{jk} = O(z_1^2)$ for all $j, k \geq 2$ and $(\partial\delta/\partial z_1) \cdot (0, \mu') = D_1^2 a_{nn}(0) \neq 0$. Besides it follows from (N5) that $a_{jk} = O(z_1^3)$ for $j, k \geq 2$ except $j = k = n$ and $b_j = O(z_1)$ for $2 \leq j \leq n-1$. Thus P can be written in the form

$$P = a_{11}D_1^2 + z_1a_{1n}D_1D_n + z_1^3 \sum_{j,k}{}' a_{jk}D_jD_k + b_1D_1 + z_1 \sum_{j=2}^{n-1} b_jD_j + b_nD_n + c$$

where $a_{11}(0) \neq 0$, $a_{1n}(0) \neq 0$, $a_{nn}(z) = 0$, and the indicial equation (N4) is

$$a_{1n}\nu + b_n = 0 \quad \text{on } z_1 = 0.$$

Therefore $P \in \mathscr{L}^{2,1,1/2}$ and all the conditions of Theorem 1.3 are fulfilled. Thus we have completed the proof in both cases.

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