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On holomorphic mappings of complex manifolds with ball model

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Abstract. We consider holomorphic mappings of complex manifolds with ball model into complex manifolds which are quotients of bounded domains and estimate the dimension of the moduli space of holomorphic mappings in terms of the essential boundary dimension of target manifolds. For this purpose, we generalize a classical uniqueness theorem of Fatou-Riesz for bounded holomorphic functions on the unit disk to one for bounded holomorphic mappings on a bounded C^2 domain. This generalization enables us to establish rigidity and finiteness theorems for holomorphic mappings. We also discuss the rigidity for holomorphic mappings into quotients of some symmetric bounded domains. In the final section, we construct examples related to our results.

1. Introduction.

We consider the rigidity of holomorphic mappings of a complex manifold $M = \boldsymbol{B}^m/\Gamma$, a quotient manifold of the unit ball $\boldsymbol{B}^m$ in $\boldsymbol{C}^m$, into a complex manifold (possibly orbifold) $N = \tilde{N}/G$ which is a quotient of a bounded domain $\tilde{N}$ in $\boldsymbol{C}^n$ by a discrete subgroup G of $\mathrm{Aut}(\tilde{N})$. In this paper, we say that the rigidity of holomorphic mappings holds if two holomorphic mappings on a complex manifold are the same map when they are homotopic to each other.

There are a lot of rigidity theorems for holomorphic mappings which are useful for the study of complex analysis. Under certain conditions, the rigidity of holomorphic mappings yields the finiteness of holomorphic mappings (cf. [**9**], [**10**], [**19**], [**24**] etc.). In [**10**], we have shown a rigidity theorem for holomorphic mappings of Riemann surfaces of finite type to moduli spaces of Riemann surfaces and succeeded in proving Parshin-Arakelov theorem which asserts finiteness of the number of locally non-trivial holomorphic families of Riemann surfaces. Sunada [**24**] estimates the dimension of the space of non-constant holomorphic mappings of M to N in terms of the boundary dimension of $\tilde{N}$ when M is a compact Kähler manifold and N is a compact quotient of a symmetric bounded domain $\tilde{N}$. Noguchi [**18**] investigates holomorphic mappings defined on a Zariski open subset of a compact Kähler manifold to an arithmetic quotient of a symmetric bounded domain. A comprehensive survey of them is given in [**19**]. However, in this paper, we assume that M is a complex hyperbolic manifold of divergence type (see the definition in the next section) but we do not assume that M is compact nor embedded into a compact manifold.

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To study the rigidity we generalize a classical uniqueness theorem of Fatou-Riesz for bounded holomorphic functions (§3 Theorem 3.1). Using this generalization, we estimate the dimension of the moduli space of holomorphic mappings in terms of the (essential) boundary dimension of the target manifold. This result enables us to show rigidity and finiteness theorems for holomorphic mappings from M to N.

Generally speaking, the rigidity is too strong to hold for any case. For example, consider a complex manifold M and put $N = M \times \Delta$, where Δ is the unit disk on $\boldsymbol{C}$. Then, for any $\lambda \in \Delta$ a holomorphic mapping $f_\lambda : M \to N$ defined by $f_\lambda(p) = (p, \lambda)$ is homotopic to f_0, but $f_\lambda \neq f_0$ when $\lambda \neq 0$.

This simple example suggests us that the complex analytic structure of N influences the structure of the space of all non-constant holomorphic mappings of M to N which is denoted by $\mathrm{Hol}(M, N)$. If the image $f(M)$ of a mapping $f \in \mathrm{Hol}(M, N)$ contains a non-empty open set in N, then the mapping is called a *dominant* map. The set of dominant mappings in $\mathrm{Hol}(M, N)$ is denoted by $\mathrm{Hol}_{\mathrm{dom}}(M, N)$.

When M is compact, the space $\mathrm{Hol}(M, N)$ has a natural complex structure so that point evaluation maps $\chi_p(\cdot)$ $(p \in M)$ on $\mathrm{Hol}(M, N)$ defined by $\chi_p(f) = f(p)$ for $f \in \mathrm{Hol}(M, N)$ are holomorphic ([**6**], [**13**]). As for the structure of $\mathrm{Hol}(M, N)$, we shall show the following (see §2 for terminologies):

Theorem 1.1. *Let $M = \boldsymbol{B}^m/\Gamma$ be an m-dimensional complex hyperbolic manifold of divergence type and $N = \tilde{N}/G$ an n-dimensional complex manifold* (*possibly orbifold*), *where $\tilde{N} \subset \boldsymbol{C}^n$ is a bounded domain and G is a discrete subgroup of the set of biholomorphic automorphisms of $\tilde{N}$. Let $\ell(\tilde{N})$ denote the essential boundary dimension of $\tilde{N}$. Then*, *the dimension of holomorphic deformation of any f_0 in* $\mathrm{Hol}(M, N)$ *is not greater than $\ell(\tilde{N})$. More precisely*, *for any holomorphic mapping $f : \Delta^k \times M \to N$ with $f(0, x) = f_0(x)$ $(x \in M)$*,

$$\max_{x \in M}(\mathrm{rank}_{\Delta^k} f(\cdot, x)) \leq \ell(\tilde{N}),$$

where Δ is the unit disk in $\boldsymbol{C}$. In particular, *if M is compact*, *then we have*

$$\dim \mathrm{Hol}(M, N) \leq \ell(\tilde{N}).$$

The proof of Theorem 1.1 gives us a sufficient condition for the rigidity of holomorphic mappings of M in terms of the action of G on $\tilde{N}$.

Theorem 1.2. *Let M and N be the same ones as in Theorem* 1.1. *Furthermore*, *we assume that the following condition* (A)

(A): *For any compact subset K of $\tilde{N}$ and for any infinite sequence $\{g_k\}$ of distinct elements of G*, *we have*

$$\lim_{k \to \infty} \mathrm{diam}(g_k(K)) = 0,$$

where $\mathrm{diam}(E)$ *is the Euclidean diameter of a set E in $\boldsymbol{C}^m$.*

Then, *any two non-constant holomorphic mappings h_1, h_2 of M to N which belong to the same homotopy class are the same holomorphic mapping.*

Remark 1.1. From Lemma 2.2 in §2, we see that if $N = \tilde{N}/G$ admits a non-constant holomorphic map from a complex hyperbolic manifold M of divergence type, then G is an infinite group (Corollary 2.1). Thus, the assumption in the condition (A) is not empty. In Corollary 2.1, we also show that M admits no non-constant positive pluriharmonic function.

The proof of the theorems yields the following two corollaries.

Corollary 1.1. *Let M and N be the same ones as in Theorem* 1.1. *If a holomorphic mapping $f \in \mathrm{Hol}_{\mathrm{dom}}(M,N)$ is homotopic to some $g \in \mathrm{Hol}(M,N)$, then $f = g$.*

Corollary 1.2. *Let M and $N = \tilde{N}/G$ be the same ones as in Theorem* 1.1. *Suppose that $\ell(\tilde{N}) = 0$. If $f, g \in \mathrm{Hol}(M,N)$ are homotopic to each other, then $f = g$.*

From Corollaries 1.1 and 1.2, we obtain a finiteness theorem for holomorphic mappings of complex hyperbolic manifolds of divergence type.

Theorem 1.3. *Let $M = \boldsymbol{B}^m/\Gamma$ be a complex hyperbolic manifold of divergence type and $N = \tilde{N}/G$ an n-dimensional $(n \geq 1)$ complex manifold which is of geometrically finite. Suppose that Γ is of finitely generated and that $\tilde{N}$ is complete with respect to the Kobayashi distance. Then, $\mathrm{Hol}_{\mathrm{dom}}(M,N)$ consists of at most finitely many elements. Furthermore, if $\ell(\tilde{N}) = 0$, then $\mathrm{Hol}(M,N)$ is also a finite set.*

Remark 1.2. S. Kobayashi and T. Ochiai [**16**] show the finiteness of surjective holomorphic mappings of a compact Kähler manifold onto a compact complex space of general type.

When $m = 1$, the complex hyperbolic manifold M in Theorem 1.3 is a topologically finite Riemann surface of divergent type. Therefore, M is a compact Riemann surface with at most finitely many punctures. For $m > 1$, Bowditch ([**4**]) shows the following.

Proposition 1.1. *If a hyperbolic manifold $M = \boldsymbol{B}^m/\Gamma$ is of geometrically finite, then every $\zeta \in \partial\boldsymbol{B}^m$ is a point of approximation of Γ. Thus, Γ is of divergence type.*

Therefore, from Theorem 1.3 we have the following theorem which is a higher dimensional generalization of de Franchis' theorem (cf. [**8**]).

Theorem 1.4. *Let M, N be geometrically finite complex hyperbolic manifolds. Then, $\mathrm{Hol}(M,N)$ consists of at most finitely many elements.*

We shall show Theorem 1.1 in §4 as well as Theorem 1.2 after giving a generalization of Fatou-Riesz theorem in §3. The proof of Theorem 1.3 will be given in §5. In §6, we discuss a rigidity theorem for holomorphic mappings from complex hyperbolic manifolds to some symmetric bounded domains. Examples concerning to these results are constructed in §7.

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2. Definitions and preliminary results.

2.1. Complex hyperbolic geometry.

At first, we shall briefly describe complex hyperbolic spaces. For more detail, see [**5**] or [**7**].

Let $V = V^{1,m}$ $(m \geq 1)$ denote the vector space $\boldsymbol{C}^{m+1}$ with the Hermitian form

$$\Phi(z, w) = -z^0\overline{w^0} + \sum_{j=1}^{m} z^j\overline{w^j}$$

for $z = (z^0, \dots, z^m)$ and $w = (w^0, \dots, w^m)$ in V. A linear isomorphism g of V satisfying

$$\Phi(g(z), g(w)) = \Phi(z, w) \quad (z, w \in V)$$

is called a *unitary transformation*. The set of unitary transformations is denoted by $U(1, m : \boldsymbol{C})$.

Let $P(V)$ be the complex projective space obtained from V and the projective map $P : V - \{0\} \to P(V)$. For $V_- = \{z \in V \mid \Phi(z, z) < 0\}$, we define $H^n(\boldsymbol{C}) = P(V_-)$. It is just a higher dimensional complex analog of Klein's model of the 2-dimensional real hyperbolic space.

Since $g(V_-) = V_-$, and $g(cz) = cg(z)$ for $g \in U(1, m : \boldsymbol{C})$, $U(1, m : \boldsymbol{C})$ acts on $H^m(\boldsymbol{C})$. In fact, $U(1, m : \boldsymbol{C})$ acts transitively on $H^m(\boldsymbol{C})$.

For $z = (z^0, z^1, \dots, z^m) \in V_-$, we have $z^0 \neq 0$. Thus, $H^m(\boldsymbol{C})$ is identified with the unit ball

$$\boldsymbol{B}^m = \left\{ \zeta = (\zeta^1, \dots, \zeta^m) \in \boldsymbol{C}^m \mid \|\zeta\| = \sqrt{\sum_{j=1}^{m} \zeta^j\overline{\zeta^j}} < 1 \right\}$$

via

$$z \mapsto \zeta = (z^1/z^0, \dots, z^m/z^0).$$

Hence, a unitary transformation is regarded as a biholomorphic self-mapping of $\boldsymbol{B}^m$. Actually, the action is realized by an element of $PU(m, 1)$, which is an isometry for the Bergman metric on $\boldsymbol{B}^m$.

Definition 2.1. A complex manifold M is called *a complex manifold with ball model* or *a complex hyperbolic manifold* if it is represented as $\boldsymbol{B}^m/\Gamma$, where Γ is a discrete torsion-free subgroup of $PU(m, 1)$.

A typical example of a complex hyperbolic manifold is a hyperbolic Riemann surface represented by a Fuchsian group. For Fuchsian groups and Riemann surfaces, there is a notion "divergence type" as follows.

Definition 2.2. A Fuchsian group Γ acting on the unit disk Δ is said to be *divergence type* if

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|) = +\infty \quad (z \in \Delta).$$

It is easily seen that the definition does not depend on the choice of z in Δ.

Definition 2.3. A hyperbolic Riemann surface R is called *divergence type* if it is represented by a Fuchsian group of divergence type.

Any compact Riemann surface of genus $g \geq 1$ is of divergence type and an open Riemann surface with "small boundary" can be of divergence type. That is, the following holds (cf. [**17**], [**26**]).

LEMMA 2.1. *Let $R = \Delta/\Gamma$ be a hyperbolic Riemann surface. Then the following conditions are equivalent.*

(1) *R is a Riemann surface of divergence type.*
(2) *R has no Green's functions.*
(3) *Almost every point on $\partial\Delta$ is a point of approximation, that is, for almost every x on $\partial\Delta$ there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of Γ such that $\{\gamma_n(z)\}_{n=1}^{\infty}$ converges to x conically for all $z \in \Delta$.*

As for complex hyperbolic manifolds, we have a similar notion.

DEFINITION 2.4. Let Γ be a subgroup of $PU(m,1)$ acting on $\boldsymbol{B}^m$. It is called a group of *divergence type* if

$$\sum_{\gamma \in \Gamma} (1 - \|\gamma(z)\|)^m = +\infty$$

for one (and all) $z \in \boldsymbol{B}^m$. A hyperbolic manifold $M = \boldsymbol{B}^m/\Gamma$ is called of divergence type if Γ is of divergence type.

Recently, S. Kamiya [**12**] shows a characterization of divergence subgroups of $PU(m,1)$ which is similar to that of Lemma 2.1. To state the result, we need the notion of "points of approximation" for Γ.

For $\alpha > 1$ and $\zeta = (\zeta^1, \dots, \zeta^m) \in \partial\boldsymbol{B}^m$, we define $D_\alpha(\zeta)$ as the set of $z \in \boldsymbol{B}^m$ satisfying

$$\left| 1 - \sum_{j=1}^{n} z^j \bar{\zeta}^j \right| < \frac{\alpha}{2}(1 - \|z\|^2).$$

DEFINITION 2.5. Let Γ be a discrete subgroup of $PU(m,1)$. A point $\zeta \in \partial\boldsymbol{B}^m$ is called a *point of approximation* if there exist a sequence $\{\gamma_k\}_{k=1}^{\infty}$ of Γ and $\alpha > 1$ such that $\{\gamma_k\}_{k=1}^{\infty}$ converges to ζ from the inside of $D_\alpha(\zeta)$ for some and any z in $\boldsymbol{B}^m$.

The following is shown in [**12**].

PROPOSITION 2.1. *Let Γ be a discrete subgroup of $PU(m,1)$. Then Γ is a group of divergence type if and only if the set of points of approximation has full Lebesgue measure on $\partial\boldsymbol{B}^m$.*

2.2. Bounded holomorphic functions on $\boldsymbol{B}^m$.

Here, we note Fatou-Riesz type theorems for bounded holomorphic functions on $\boldsymbol{B}^m$.

DEFINITION 2.6. A function f on $\boldsymbol{B}^m$ is said to have a K-limit at $\zeta \in \partial\boldsymbol{B}^m$ if the limit

$$f^*(\zeta) = \lim_{j\to\infty} f(z_j)$$

exists for every $\alpha > 1$ and for every sequence $\{z_j\}$ in $D_\alpha(\zeta)$ which converges to ζ.

As for K-limits of holomorphic functions on $\boldsymbol{B}^m$, the following results are known (cf. [**21**], Theorems 5.5.9 and 5.6.4).

Proposition 2.2. *Let f be a bounded holomorphic function (or H^p-function, more generally) on $\boldsymbol{B}^m$. Then it has K-limits $f^*(\zeta)$ at almost all points $\zeta \in \partial\boldsymbol{B}^m$.*

Proposition 2.3. *Let f be a bounded holomorphic function (or H^p-function, more generally) on $\boldsymbol{B}^m$. If there exists a measurable set $E \subset \partial\boldsymbol{B}^m$ with positive Lebesgue measure such that $f^* \equiv 0$ on E, then $f \equiv 0$.*

Let $f : M \to N$ be a holomorphic mapping of a complex hyperbolic manifold $M = \boldsymbol{B}^m/\Gamma$ of divergence type to a complex manifold $N = \tilde{N}/G$, where $\tilde{N}$ is a bounded domain of $\boldsymbol{C}^n$. Then a lift F of f is a bounded holomorphic mapping of $\boldsymbol{B}^m$. From Proposition 2.2, F^* has a K-limit $F^*(\zeta)$ in $\tilde{N} \cup \partial\tilde{N}$ at almost all point ζ in $\partial\boldsymbol{B}^m$. Here, we show that the mapping F is almost proper.

Lemma 2.2. *For almost all points ζ in $\partial\boldsymbol{B}^m$, $F^*(\zeta) \in \partial\tilde{N}$ if $f : M \to N$ is a non-constant holomorphic mapping.*

Proof. Since M is a complex hyperbolic manifold of divergence type, almost all points in $\partial\boldsymbol{B}^m$ are points of approximation for Γ. Therefore, the mapping F has K-limits at almost all points $\zeta \in \partial\boldsymbol{B}^m$ which are points of approximation for Γ. Let $E \subset \partial\boldsymbol{B}^m$ denote the set of such points ζ.

For any $\zeta \in E$, there exists a sequence $\{\gamma_k\}_{k=1}^\infty$ of Γ such that $\{\gamma_k(z)\}_{k=1}^\infty$ converges to ζ from the inside of $D_\alpha(\zeta)$ for any $z \in \boldsymbol{B}^m$ and for some $\alpha > 1$. Since $F : \boldsymbol{B}^m \to \tilde{N}$ is a lift of a holomorphic mapping f of $M = \boldsymbol{B}^m/\Gamma$ to $N = \tilde{N}/G$, there exists a homomorphism θ of Γ to G such that

$$F(\gamma(z)) = \theta(\gamma)(F(z)) \tag{1}$$

holds for any $\gamma \in \Gamma$ and for any $z \in \boldsymbol{B}^m$. Thus, we have

$$F^*(\zeta) = \lim_{k\to\infty} F(\gamma_k(z)) = \lim_{k\to\infty} \theta(\gamma_k)(F(z)), \tag{2}$$

for any $\zeta \in E$ and for any $z \in \boldsymbol{B}^m$. On the other hand, F is non-constant, $F(z) \neq F(z')$ for some $z, z' \in \boldsymbol{B}^m$. Hence, we have

$$\begin{aligned} 0 < d_{\tilde{N}}(F(z), F(z')) &= d_{\tilde{N}}(\theta(\gamma_k)(F(z)), \theta(\gamma_k)(F(z'))) \\ &= d_{\tilde{N}}(F(\gamma_k(z)), F(\gamma_k(z'))), \end{aligned} \tag{3}$$

where $d_{\tilde{N}}(\cdot,\cdot)$ is the Kobayashi distance on $\tilde{N}$. Thus, if $F^*(\zeta) \in \tilde{N}$, then we have a contradiction $0 < d_{\tilde{N}}(F^*(\zeta), F^*(\zeta)) = 0$ by letting $k \to \infty$ in (3). The proof of Lemma 2.2 is completed. $\square$

From the above argument, we have the following.

Corollary 2.1. *Let $M = \boldsymbol{B}^m/\Gamma$ be a complex hyperbolic manifold of divergence type. If a complex manifold $N = \tilde{N}/G$ admits a non-constant holomorphic mapping from M, then the group G is infinite. Moreover, there are no non-constant positive pluriharmonic functions on M. In other words, every complex hyperbolic manifold of divergence type belongs to O_{HP}.*

Proof. If G is a finite group, then so is $\theta(\Gamma)$, where θ is a homomorphism defined by (1). Therefore, $\{\theta(\gamma)(F(z))\}_{\gamma \in \Gamma} = \{F(\gamma(z))\}_{\gamma \in \Gamma}$ is a finite subset of $\tilde{N}$. Hence, $F^*(\zeta)$ is in $\tilde{N}$ for all $\zeta \in \partial \boldsymbol{B}^m$. It contradicts Lemma 2.2.

Let u be a positive pluriharmonic function on M. Then, we may take a lift U of u, which is a pluriharmonic function on $\boldsymbol{B}^m$. It follows from a theorem of Forelli (cf. [**21**], Theorem 4.4.4) that there exists a pluriharmonic function V on $\boldsymbol{B}^m$ such that $F(z) = U(z) + \sqrt{-1}V(z)$ is a holomorphic function on $\boldsymbol{B}^m$. Put $G(z) = \exp(-F(z))$, then G is a bounded holomorphic function on $\boldsymbol{B}^m$. Proposition 2.3 guarantees that $G(z)$ has the K-limit $G^*(\zeta)$ at almost all $\zeta \in \partial \boldsymbol{B}^m$ and so does $U(z) = -\log|G(z)|$.

Now, assume that $U(z)$ is not a constant function. Then, there exist z_1, z_2 in $\boldsymbol{B}^m$ such that $U(z_1) \neq U(z_2)$. Since U is a lift of u, we have

$$U(\gamma(z)) = U(z),$$

for every $\gamma \in \Gamma$. We may assume that $\zeta \in \partial \boldsymbol{B}^m$ is a point of approximation of Γ. Hence, there exists a sequence $\{\gamma_k\}_{k=1}^{\infty}$ such that

$$U(z_j) = \lim_{k \to \infty} U(\gamma_k(z_j)) = U^*(\zeta) \quad (j = 1, 2).$$

Thus, we have a contradiction. □

Remark 2.1. Kamiya ([**12**]) shows that if M is a complex hyperbolic manifold of divergence type, then it has no non-constant bounded M-harmonic function. Since the set of M-harmonic functions is a subclass of the set of pluriharmonic functions (cf. [**21**]), our result is an extension of his one.

2.3. Essential boundary dimensions.

The boundary dimension of a bounded domain D in $\boldsymbol{C}^n$ is the maximal of the dimensions of analytic spaces in ∂D. Here, we introduce another notion, the essential boundary dimension, to study boundary behavior of holomorphic mappings.

Definition 2.7. Let E be a subset of $\boldsymbol{C}^n$. The set E is called a *pluripolar set* if there exists a plurisuperharmonic function s in $\boldsymbol{C}^n$ such that $s(p) = +\infty$ for every $p \in E$. The set E is a *complete pluripolar set* if there exists a plurisuperharmonic function s in $\boldsymbol{C}^n$ such that $E = \{p \in \boldsymbol{C}^n \mid s(p) = +\infty\}$.

Remark 2.2. Usually, the definition of pluripolarity is local, that is, a subset E of $\boldsymbol{C}^n$ is pluripolar if for each $z \in E$ there exist a neighbourhood U of z and a plurisuperharmonic function s in U such that $E \cap U = \{p \in U \mid s(p) = +\infty\}$. This definition seems to be different from Definition 2.7 using a global plurisuperharmonic function. In fact, both definitions are the same from a theorem of B. Josefson which shows that local pluripolarity means global one (cf. [**14**], Theorem 4.7.4).

We define the essential boundary dimension by the following way (cf. [**25**]).

Definition 2.8. Let D be a bounded domain in $\boldsymbol{C}^n$. Consider a family $\{R_j\}_{j=1}^{\infty}$ of countable complete pluripolar sets with $R_j \cap \partial D = \emptyset$ $(j = 1, 2, \ldots)$. We denote by $\ell(D; \{R_j\}_{j=1}^{\infty})$ the maximal dimension of analytic spaces contained in $\partial D - \bigcup_{j=1}^{\infty} R_j$. We define the *essential boundary dimension of* D, which is denoted by $\ell(D)$, by

$$\ell(D) = \inf \ell(D; \{R_j\}_{j=1}^{\infty}),$$

where the infimum is taken over all families $\{R_j\}_{j=1}^{\infty}$ of countable complete pluripolar sets as above.

It is not hard to see that $\ell(\boldsymbol{B}^n) = 0$ and $\ell(\Delta^n) = n - 1$. Let T_g $(g > 1)$ denote the Teichmüller space of compact Riemann surfaces of genus g. It is well known that T_g is regarded as a bounded domain in $\boldsymbol{C}^{3g-3}$ by Bers' embedding (cf. [**11**]). In [**22**], we show that the essential boundary dimension $\ell(T_g)$ of T_g is zero (see Example 7.4). Similarly, we may show that $\ell(T_g \times \boldsymbol{B}^n) = 0$ and $\ell(T_g \times \Delta^n) = n - 1$.

2.4. Geometrically finite manifolds.

Here, we define geometrically finite manifolds which appear in Theorem 1.3.

For every connected subset S of $N = \tilde{N}/G$, we say that a set $\tilde{S}$ in $\tilde{N}$ is a *lift* of S if it is a connected component of $\pi^{-1}(S)$, where π is the canonical projection of $\tilde{N}$ onto N.

Definition 2.9. Let $N = \tilde{N}/G$ be a complex manifold (possibly orbifold) which is a quotient space of a bounded domain $\tilde{N}$ in $\boldsymbol{C}^n$ by a discrete subgroup G of $\mathrm{Aut}(\tilde{N})$. An end V of N is called a *parabolic end* of N if there exist a lift $\tilde{V}$ of V in $\tilde{N}$ and at most countably many pluripolar sets $\{R_j\}_{j=1}^{\infty}$ in $\boldsymbol{C}^n - \tilde{N}$ such that $\bar{\tilde{V}} \cap \partial\tilde{N} \subset \bigcup_{j=1}^{\infty} R_j$.

The manifold N is said to be *of geometrically finite* if it has only finitely many ends and all of them are parabolic ends.

3. Fatou-Riesz theorem for holomorphic mappings.

Let D be a bounded domain in $\boldsymbol{C}^m$. We assume that D is a C^2-domain, that is, there exists a real valued C^2 function λ in a neighbourhood of $\bar{D}$ such that $D = \{z \mid \lambda(z) < 0\}$, $\partial D = \{z \mid \lambda(z) = 0\}$ and

$$\frac{\partial \lambda}{\partial v_z} > 0, \quad (z \in \partial D)$$

where v_z is the outward unit normal vector at z.

For each $\alpha > 0$ and for each $\zeta \in \partial D$, we define an approach region $\mathscr{A}_\alpha(\zeta)$ by

$$\mathscr{A}_\alpha(\zeta) = \{z \in D \mid |(z - \zeta, v_\zeta)| < (1 + \alpha)\delta_\zeta(z), \|z - \zeta\|^2 < \alpha\delta_\zeta(z)\}, \tag{4}$$

where $(\cdot, \cdot)$ means the standard inner product in $\boldsymbol{C}^m$ and $\delta_\zeta(z)$ is the minimum of the distances from z to ∂D and from z to the tangent plane at ζ. We shall say that a mapping F has an *admissible limit* $F^*(\zeta)$ at $\zeta \in \partial D$ if the limit

$$F^*(\zeta) = \lim_{j \to \infty} F(z_j)$$

exists for every sequence $\{z_j\}$ converging to ζ in $\mathscr{A}_\alpha(\zeta)$ and for any $\alpha > 0$. Note that when D is the unit ball $\boldsymbol{B}^m$ the admissible limit $F^*(\zeta)$ is nothing but the K-limit at $\zeta \in \partial \boldsymbol{B}^m$. The following generalization of Proposition 2.2 holds (cf. Stein [**23**]).

PROPOSITION 3.1. *Let f be a bounded holomorphic function in a bounded C^2 domain D in $\boldsymbol{C}^m$. Then f has admissible limits at almost all points on ∂D.*

In this section, we shall extend Proposition 2.3 for holomorphic mappings defined on a bounded C^2 domain D in $\boldsymbol{C}^m$. The proposition says that if f is non-constant bounded holomorphic function on the unit ball $\boldsymbol{B}^m$, then $E_f^* = (f^*)^{-1}(0) \subset \partial \boldsymbol{B}^m$ is of measure zero. In other words, the preimage of a small set via f^* is also small in $\partial \boldsymbol{B}^m$. Therefore, to extend this result to one for bounded holomorphic mappings of D to $\boldsymbol{C}^n$, we need to obtain a notion of small sets in $\boldsymbol{C}^n$. We use complete pluripolar sets.

THEOREM 3.1. *Let φ be a non-constant bounded holomorphic mapping of a bounded C^2 domain D in $\boldsymbol{C}^m$ to $\boldsymbol{C}^n$ and E a countable union of complete pluripolar sets in $\boldsymbol{C}^n$. Let $E_\varphi^* \subset \partial D$ denote the preimage of E via the admissible limit φ^* of φ. Suppose that $\varphi(D) \cap E^c$ is not empty. Then, E_φ^* is of measure zero.*

PROOF. Let $g(\cdot, z_0)$ be Green's function for D with the pole at z_0. Then for sufficiently small $\varepsilon > 0$, $D_\varepsilon = \{z \in D \mid g(z, z_0) > \varepsilon\}$ defines an approximating region of D, and $g_\varepsilon = g - \varepsilon$ is Green's function for D_ε. It is known that $-\partial g/\partial v_\zeta$ and $-\partial g_\varepsilon/\partial v_\zeta^\varepsilon$ are the Poisson kernels for D and D_ε, respectively, where v_ζ^ε is the outward unit normal vector at $\zeta \in \partial D_\varepsilon$.

For each $\zeta \in \partial D$ and for small $\varepsilon > 0$, there exists $k = k(\varepsilon) < 0$ such that $\zeta_\varepsilon = \zeta + k v_\zeta$ belongs to ∂D_ε and a mapping $\zeta \mapsto \zeta_\varepsilon$ is surjective from ∂D to ∂D_ε.

Now, we assume that E itself is a complete pluripolar set. Let s be a plurisuperharmonic function on $\boldsymbol{C}^n$ defining the complete pluripolar set E. We may assume that there exists an open set U in $\boldsymbol{C}^n$ such that U contains the closure of $\varphi(D)$ and s is positive in U. Since φ is holomorphic, $s \circ \varphi$ is also plurisuperharmonic and $\not\equiv +\infty$. Since $\varphi(D) \cap E^c \neq \emptyset$, there exists a point $a_0 \in D$ such that $s \circ \varphi(a_0) < +\infty$.

From the superharmonicity, we have

$$\int_{\partial D_\varepsilon} s \circ \varphi(\zeta_\varepsilon)\left(-\frac{\partial g(\zeta_\varepsilon, a_0)}{\partial v_{\zeta_\varepsilon}^\varepsilon}\right) d\sigma_\varepsilon(z) \leq s \circ \varphi(a_0) < +\infty, \tag{5}$$

where $d\sigma_\varepsilon$ is the induced measure on ∂D_ε.

Since $\partial g(\zeta_\varepsilon, a_0)/\partial v_{\zeta_\varepsilon}^\varepsilon$ uniformly converges to $\partial g(\zeta, a_0)/\partial v_\zeta$ as $\varepsilon \to 0$, we have

$$\begin{aligned}\liminf_{\varepsilon \to 0} \int_{\partial D_\varepsilon} s \circ \varphi(\zeta_\varepsilon)\left(-\frac{\partial g(\zeta_\varepsilon, a_0)}{\partial v_{\zeta_\varepsilon}^\varepsilon}\right) d\sigma_\varepsilon(\zeta_\varepsilon) \\ \geq \int_{\partial D} \liminf_{\varepsilon \to 0} s \circ \varphi(\zeta_\varepsilon)\left(-\frac{\partial g(\zeta, a_0)}{\partial v_\zeta}\right) d\sigma(\zeta),\end{aligned} \tag{6}$$

where $d\sigma$ is the induced measure on ∂D. If E_φ^* has positive measure, then

$$\int_{\partial D} \liminf_{\varepsilon \to 0} s \circ \varphi(\zeta_\varepsilon)\, d\sigma(\zeta) = +\infty,$$

since $\zeta_\varepsilon \in \mathscr{A}_\alpha(\zeta)$ for any $\varepsilon > 0$. Combining (5) and (6) with this equation, we have a contradiction.

Now, we consider a general case, $E = \bigcup_{k=1}^{\infty} E_k$, where E_k $(k = 1, 2, \dots)$ are complete pluripolar sets in $\boldsymbol{C}^n$. If E_φ^* is of positive measure, then so is $(E_k)_\varphi^*$ for some k because $E_\varphi^* = \bigcup_{k=1}^{\infty} (E_k)_\varphi^*$. From the above argument, we have a contradiction. $\square$

Remark 3.1. The assumption that $\varphi(D) \cap E^c$ is not empty is necessary. For example, if E is the zero set of a holomorphic function φ on $\boldsymbol{C}^n$ with the dimension $\geq m$, then E is complete pluripolar and there exists an embedding ι of D into E such that $\overline{\iota(D)} \subset E$. Obviously $E_\iota^* = \partial D$, but ι is not constant. Thus, the conclusion of Theorem 3.1 does not hold.

4. Proofs of rigidity theorems and their corollaries.

In this section, we shall prove Theorems 1.1, 1.2 and Corollaries 1.1, 1.2.

4.1. Proof of Theorem 1.1.

Suppose that there exists a holomorphic mapping $f(\cdot,\cdot) : \Delta^k \times M \to N$ for some $k \in \boldsymbol{N}$ such that

$$\max_{x \in M}(\operatorname{rank}_{\lambda \in \Delta^k} f(\lambda, x)) > \ell(\tilde{N}). \tag{7}$$

and $f(0,\cdot) : M \to N$ is a non-constant holomorphic mapping.

Let $F(\cdot,\cdot) : \Delta^k \times \boldsymbol{B}^m \to \tilde{N}$ be a holomorphic mapping which is a lift of f. Hence, it satisfies

$$F(\lambda, \gamma(z)) = \theta(\gamma)(F(\lambda, z))$$

for all $\gamma \in \Gamma$, where $\theta : \Gamma \to G$ is a homomorphism induced by f. Note that θ does not depend on λ because of the discreteness of G.

From Proposition 2.2, for each $\lambda \in \Delta^k$ there exists a measurable subset E_λ of $\partial \boldsymbol{B}^m$ with full Lebesgue measure such that $F(\lambda,\cdot)$ has a K-limit $F^*(\lambda,\zeta)$ at every ζ in E_λ. Take a countable dense subset $\{\lambda_j\}_{j=1}^{\infty}$ of Δ^k and set

$$E = \bigcap_{j=1}^{\infty} E_{\lambda_j}.$$

Then, the set E is also a measurable subset of $\partial \boldsymbol{B}^m$ of full Lebesgue measure and $F(\lambda_j,\cdot)$ has a K-limit $F^*(\lambda_j,\zeta)$ at every ζ in E for each $j \geq 1$.

Since $F(\cdot, z)$ $(z \in \boldsymbol{B}^m)$ is a bounded holomorphic mapping, a family $\mathscr{F} = \{F(\cdot, z) \mid z \in \boldsymbol{B}^m\}$ is equicontinuous and it is a normal family. Hence we see that when $z \to \zeta$ in a $D_\alpha(\zeta)$ for any $\zeta \in E$, $F(\cdot, z)$ converges to a function $F^*(\cdot,\zeta)$ uniformly on every compact subset of Δ^k. This implies that $F^*(\cdot,\zeta)$ is a holomorphic mapping for each $\zeta \in E$. Furthermore, from Lemma 2.2 it is a holomorphic mapping of Δ^k to $\partial\tilde{N}$.

From (7) there exists a point $z_0 \in \boldsymbol{B}^m$ such that

$$\operatorname{rank}_{\lambda \in \Delta^k} F(\lambda, z_0) = \ell > \ell(\tilde{N}). \tag{8}$$

For $F = (F^1, \dots, F^n)$ and $\lambda = (\lambda^1, \dots, \lambda^k)$, we set two matrices by

$$A(\lambda, z) = \left(\frac{\partial F^j}{\partial \lambda^i}(\lambda, z) \right)_{1 \le i \le k, 1 \le j \le n}$$

and

$$A^*(\lambda, \zeta) = \left(\left(\frac{\partial F^j}{\partial \lambda^i} \right)^* (\lambda, \zeta) \right)_{1 \le i \le k, 1 \le j \le n}$$

for $\zeta \in E$. Then from (8), we may take a point $\lambda_0 \in \Delta^k$ such that $\operatorname{rank} A(\lambda_0, z_0) = \ell > \ell(\tilde{N})$. Therefore, there exists an $(\ell \times \ell)$ submatrix $a(\lambda_0, z)$ of $A(\lambda_0, z)$ such that

$$\det a(\lambda_0, z_0) \neq 0. \tag{9}$$

Since $F(\lambda, z)$ converges to $F^*(\lambda, \zeta)$ uniformly on every compact subset of Δ^k for each $\zeta \in E$ as $z(\in D_\alpha(\zeta)) \to \zeta$, we see that

$$\frac{\partial F^j}{\partial \lambda^i}(\lambda, z) \to \left(\frac{\partial F_j}{\partial \lambda^i} \right)^* (\lambda, \zeta)$$

uniformly on every compact subset of Δ^k when $z(\in D_\alpha(\zeta))$ converges to ζ $(i = 1, \dots, k; j = 1, \dots, n)$. In particular, the matrix $a(\lambda, z)$ converges to $a^*(\lambda, \zeta)$ of $A^*(\lambda, \zeta)$ consisting of elements with the same indices as $a(\lambda, z)$. In fact, $a^*(\lambda, \zeta)$ is a K-limit of $a(\lambda, z)$ at ζ.

From the definition of $\ell(\tilde{N})$, $\operatorname{rank} A^*(\lambda, \zeta) \le \ell(\tilde{N})$ for any $\zeta \in E$. Indeed, if $\operatorname{rank} A^*(\lambda, \zeta) > \ell(\tilde{N})$, then $\dim F(\Delta^k, \zeta) > \ell(\tilde{N})$ and we have $F^*(\lambda, E) \subset \bigcup_{k=1}^{\infty} R_j$, where $\{R_j\}_{j=1}^{\infty}$ is a family of countable complete pluripolar sets with $\ell(\tilde{N}; \{R_j\}_{j=1}^{\infty}) = \ell(\tilde{N})$. Hence, it follows from Theorem 3.1 that $F^*(\lambda, \cdot)$ is a constant. It is a contradiction.

Thus,

$$\det a^*(\lambda_0, \zeta) = 0 \tag{10}$$

for any $\zeta \in E$. Therefore, it follows from Proposition 2.3 that

$$a(\lambda_0, z) = 0$$

for any $z \in \boldsymbol{B}^m$. It contradicts (9). The proof of Theorem 1.1 is completed.

4.2. Proofs of Theorem 1.2, Corollaries 1.1 and 1.2.

The proofs of Theorem 1.2, Corollaries 1.1 and 1.2 are done simultaneously.

Let f_1, f_2 be non-constant holomorphic mappings of M to N which are homotopic to each other. Then f_1, f_2 induces the same monodromy. That is, there exists a homomorphism $\theta : \Gamma \to G$ such that

$$F_i(\gamma(z)) = \theta(\gamma)(F_i(z)) \quad (i = 1, 2), \tag{11}$$

for all $\gamma \in \Gamma$ and for all $z \in \boldsymbol{B}^m$, where F_i are lifts of f_i $(i = 1, 2)$. By the same argument as in the proof of Theorem 1.1, we may find a measurable subset E of $\partial \boldsymbol{B}^m$ with full Lebesgue measure such that every $\zeta \in E$ is a point of approximation for Γ and admits K-limits $F^*(\zeta)$ and $F_2(\zeta)$ for F_1 and F_2, respectively.

Let $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence for $\zeta \in E$ such that $\gamma_k(z)$ converges to ζ from the inside of $D_\alpha(\zeta)$ for some $\alpha > 1$ as $k \to \infty$. From Lemma 2.2 and (11), we have

$$\theta(\gamma_k)(F_1(z)) = F_1(\gamma_k(z)) \to F_1^*(\zeta) \in \partial\tilde{N} \tag{12}$$

and

$$\theta(\gamma_k)(F_2(z)) = F_2(\gamma_k(z)) \to F_2^*(\zeta) \in \partial\tilde{N}. \tag{13}$$

Thus, $\{\theta(\gamma_k)\}_{k=1}^{\infty}$ is an infinite sequence of distinct elements of G.

Now we suppose that N satisfies the condition (A) in Theorem 1.2. Applying the condition (A) for $g_k = \theta(\gamma_k)$ and $K = \{F_1(z), F_2(z)\}$, we verify that $F_1^*(\zeta) = F_2^*(\zeta)$. Hence, from Proposition 2.3 we conclude that $F_1 = F_2$ and $f_1 = f_2$. The proof of Theorem 1.2 is completed.

Next, we show Corollary 1.2. Take two holomorphic mappings $F_1, F_2 : \boldsymbol{B}^m \to \tilde{N}$ as above. We note that the mapping $\theta(\gamma) : \tilde{N} \to \tilde{N}$ $(\gamma \in \Gamma)$ defined in (11) is a bounded holomorphic mapping. Thus, $\{\theta(\gamma_k)\}_{k=1}^{\infty}$ forms a normal family on $\tilde{N}$ and we may assume that $\{\theta(\gamma_k)\}_{k=1}^{\infty}$ converges to a holomorphic mapping $g_\zeta : \tilde{N} \to \tilde{N} \cup \partial\tilde{N}$ uniformly on every compact subset of $\tilde{N}$ as $k \to \infty$. Since G is discrete, the mapping g_ζ is a holomorphic mapping of $\tilde{N}$ to the boundary ∂N for every $\zeta \in E$. We assert that g_ζ is a constant mapping for almost every $\zeta \in \partial\boldsymbol{B}^m$.

Indeed, if not, we may take a measurable set $E' \subset E$ with positive measure so that g_ζ is not a constant for every $\zeta \in E'$. Then $g_\zeta(\tilde{N}) \subset \bigcup_{j=1}^{\infty} R_j$ from the definition of $\ell(\tilde{N}) = 0$, where R_j $(j = 1, 2, \dots)$ are complete pluripolar sets in Definition 2.8. Since $\{R_j\}_{j=1}^{\infty}$ is countable, we may assume that $g_\zeta(\tilde{N}) \subset R_{j_0}$ for some j_0 and for every $\zeta \in E'$. Therefore, we have that

$$F_i^*(\zeta) = g_\zeta(F_i(z)) \in R_{j_0}, \quad (i = 1, 2)$$

for every $\zeta \in E'$ from (12) and (13). It follows from Theorem 3.1 that both F_1 and F_2 are constant. Thus, we have a contradiction and we conclude that g_ζ are constants for almost all $\zeta \in \partial\boldsymbol{B}^m$.

Using (12) and (13) again, we have

$$F_1^*(\zeta) = g_\zeta(F_1(z)) = g_\zeta(F_2(z)) = F_2^*(\zeta)$$

because g_ζ is a constant. Hence, we verify that $f_1 = f_2$ and we complete the proof of Corollary 1.2.

Finally, we show Corollaries 1.1. Suppose that $f_1 \in \mathrm{Hol}_{\mathrm{dom}}(M, N)$ and $f_2 \in \mathrm{Hol}(M, N)$ are homotopic to each other. Then, there exist an open subset O of $\tilde{N}$ with $O \subset F_1(\boldsymbol{B}^m)$. Noting that (12) and (13) hold for any $z \in \boldsymbol{B}^m$, we see that $g_\zeta(O) = \{F_1^*(\zeta)\}$ and $F_2^*(\zeta) \in g_\zeta(\tilde{N})$ for every $\zeta \in E$, where g_ζ is a holomorphic mapping obtained in the proof of Corollary 1.2. Since O is an open subset of $\tilde{N}$, the mapping g is a constant mapping and we have $F_1^*(\zeta) = F_2^*(\zeta)$. Hence, we obtain that $f_1 = f_2$ and the proof of Corollary 1.1 is completed.

5. Proof of finiteness theorem.

We prove the finiteness theorem (Theorem 1.3) by using the rigidity of holomorphic mappings. Here, we only show that $\mathrm{Hol}_{\mathrm{dom}}(M, N)$ is finite by using Corollary 1.1 since the same argument and Corollary 1.2 give the proof of the theorem for the case $\ell(\tilde{N}) = 0$.

Let $\pi : \boldsymbol{B}^m \to M = \boldsymbol{B}^m/\Gamma$ and $\pi' : \tilde{N} \to N = \tilde{N}/G$ denote the natural projections on M and N, respectively. Take p_0 in M as $p_0 = \pi(0)$. Since N is of geometrically finite, there exists a compact subset K of N such that $N - K$ are contained in the union of finitely many parabolic ends, say $V_1, \dots, V_J$.

First, we shall show that there are only finitely many holomorphic mappings $f \in \mathrm{Hol}_{\mathrm{dom}}(M, N)$ with $f(p_0) \in K$. We take a compact subset $\tilde{K}$ of $\tilde{N}$ and lifts F of f so that $\pi'(\tilde{K}) = K$ and $F(0) \in \tilde{K}$. Let θ_F denote a monodromy homomorphism defined by F, that is, θ_F is a group homomorphism of Γ to G with

$$F \circ \gamma = \theta_F(\gamma) \circ F$$

for all $\gamma \in \Gamma$. Because of Corollary 1.1, it suffices to show that there are only finitely many possible homomorphisms for the monodromies. Since Γ is finitely generated, we may take $\{\delta_1, \delta_2, \dots, \delta_\ell\}$ as a system of generators of Γ. We show that there are only finitely many possible elements in G for $\theta_F(\delta_i)$ $(i = 1, 2, \dots, \ell)$.

Set

$$a = \max_{i=1,2,\dots,\ell} d_{\boldsymbol{B}^m}(0, \delta_i(0)), \tag{14}$$

where $d_{\boldsymbol{B}^m}(\cdot\,,\cdot)$ stands for the Kobayashi distance of $\boldsymbol{B}^m$. For the Kobayashi distance $d_{\tilde{N}}(\cdot\,,\cdot)$ of $\tilde{N}$ we have

$$d_{\boldsymbol{B}^m}(0, \delta_i(0)) \geq d_{\tilde{N}}(F(0), F(\delta_i(0))) = d_{\tilde{N}}(F(0), \theta_F(\delta_i)(F(0)))$$

from the decreasing property of holomorphic mappings with respect to the Kobayashi distances. Therefore,

$$d_{\tilde{N}}(F(0), \theta_F(\delta_i)(F(0))) \leq a.$$

Since $F(0) \in \tilde{K} \subset \tilde{N}$, we verify that $\{\theta_F(\delta_i)(F(0))\}$ is in a subset $\tilde{K}_a = \{q \in \tilde{N} \mid d_{\tilde{N}}(\tilde{K}, q) \leq a\}$. Since $\tilde{N}$ is complete with respect to the Kobayashi distance, $\tilde{K}_a$ is compact in $\tilde{N}$. Noting that G acts properly discontinuously on $\tilde{N}$, we verify that there exists a finite subset G' of G such that $g(F(0)) \in \tilde{K}_a$ implies $g \in G'$. Therefore, $\theta_F(\delta_i)$ is in G' $(i = 1, 2, \dots, \ell)$. This shows that there are only finitely many possible elements in G for $\theta_F(\delta_i)$ $(i = 1, 2, \dots, \ell)$.

Since $\{\delta_1, \delta_2, \dots, \delta_\ell\}$ is a system of generators of Γ, we see that there are only finitely many possible homomorphisms for $\{\theta_F\}$. From Corollary 1.1, we verify that there are only finitely many holomorphic mappings f with $f(p_0) \in K$.

Next, we suppose that $f(p_0) \in V_j - K$ $(1 \leq j \leq J)$. We may assume that $j = 1$. We take a lift $\tilde{V}_1 \subset \tilde{N}$ of V_1 so that $\overline{\tilde{V}_1} \cap \overline{\partial \tilde{N}}$ is contained in $\bigcup_{j=1}^{\infty} R_j$, where $\{R_j\}_{j=1}^{\infty}$ is the set of complete pluripolar sets as in Definition 2.9. Considering a larger K if necessary, we may assume that

$$d_N(f(p_0), \partial V_1) > a, \tag{15}$$

where $a > 0$ is a constant defined by (14) and $d_N(\cdot\,,\cdot)$ is the Kobayashi distance on N. From (14), there exist closed curves L_i $(i = 1, 2, \dots, \ell)$ on M corresponding to δ_i such that $L_i \ni p_0$ and the length of L_i with respect to the Kobayashi distance is not greater than a. The length of $f(L_i)$ is also not greater than a from the decreasing property of the Kobayashi distance. Hence, it follows from (15) that $f(L_i)$ is entirely contained in V_1.

We take a lift F of f so that $F(0) \in \tilde{V}_1$. Since the curve L_i corresponds to δ_i, we see that $F(\delta_i(0)) = \theta_F(\delta_i)(F(0)) \in \tilde{V}_1$. Hence, we have $\theta_F(\delta_i)(\tilde{V}_1) = \tilde{V}_1$ $(i = 1, 2, \dots, \ell)$ and we verify that $\theta_F(\Gamma)(\tilde{V}_1) = \tilde{V}_1$ because Γ is generated by $\delta_1, \dots, \delta_\ell$. In particular, $\theta_F(\gamma)(F(0)) \in \tilde{V}_1$ for every $\gamma \in \Gamma$.

Now, we may take a measurable subset E in $\partial \boldsymbol{B}^m$ with full measure as before such that every $\zeta \in E$ is a point of approximation for Γ and F has a K-limit $F^*(\zeta)$ at ζ. There exists a sequence $\{\gamma_k\}_{k=1}^\infty$ in Γ such that $\gamma_k(0)$ converges to ζ in D_α $(\alpha > 1)$ and

$$F(\gamma_k(0)) = \theta_F(\gamma_k)(F(0)) \to F^*(\zeta)$$

as $k \to \infty$. Since $F^*(\zeta) \in \partial \tilde{N}$ (Lemma 2.2) and $\theta_F(\gamma_k)(F(0)) \in \tilde{V}_1$, we see that $F^*(\zeta) \in \overline{\tilde{V}_1} \cap \partial \tilde{N} \subset \bigcup_{j=1}^\infty R_j$ for every $\zeta \in E$. Thus, it follows from Theorem 3.1 that the mapping F must be a constant and we have a contradiction.

6. Classical domains.

In this section, we consider irreducible symmetric bounded domains and discuss a sufficient condition to hold a rigidity theorem for holomorphic mappings of a complex hyperbolic manifold M to a quotient manifold of an irreducible symmetric bounded domain.

Let D be an irreducible symmetric bounded domain. According to a work of E. Cartan, the domain D is biholomorphic to one of the following types if it is not exceptional.

I: $R_I = \{Z \in M_{m,n} \mid I_m - ZZ^* > 0\}$,

II: $R_{II} = \{Z \in M_n \mid Z = {}^tZ, I_n - ZZ^* > 0\}$,

III: $R_{III} = \{Z \in M_n \mid Z = -{}^tZ, I_n - ZZ^* > 0\}$,

IV: $R_{IV} = \{z = (z_1, \dots, z_n) \in \boldsymbol{C}^n \mid |z^tz|^2 + 1 - 2\bar{z}^tz > 0, |z^tz| < 1\}$,

where $M_{m,n}$ is the space of $m \times n$-matrices with complex coefficients, $M_n = M_{n,n}$, and $Z^* = {}^t\bar{Z}$.

Note that via the Cayley transform, R_{II} is biholomorphically equivalent to the Siegel upper half space $\mathscr{H}_n$ of degree n, where

$$\mathscr{H}_n = \{Z \in M_n \mid Z = {}^tZ, \operatorname{Im} Z > 0\}.$$

Definition 6.1. A set $V \subset \boldsymbol{C}^n$ is said to be *holomorphically connected* if for any points $z, z' \in V$ there exist finitely many holomorphic mappings $f_1, \dots, f_k$ from the unit disk Δ to V such that

$$f_1(0) = z', \quad f_k(0) = z' \quad \text{and} \quad f_j(\Delta) \cap f_{j+1}(\Delta) \neq \emptyset.$$

A subset V of a set U in $\boldsymbol{C}^n$ is called *a holomorphic component* of U if it is a maximal set in the family of holomorphically connected subsets of U containing a common point in U.

Now, we assume that $M = \boldsymbol{B}^m/\Gamma$ is a complex hyperbolic manifold of divergence type and $N = D/G$ is a quotient manifold of $D = R_I, R_{II}$, or R_{III}.

Let $f : M \to N$ be a non-constant holomorphic mapping and $F : \boldsymbol{B}^m \to D$ a lift of f. Then, as we noted in Lemma 2.2, the mapping F has K-limits $F^*(\zeta)$ at almost all points ζ of $\partial \boldsymbol{B}^m$ and the images $F^*(\zeta)$ belong to ∂D. Under these circumstances, we may show

Theorem 6.1. *Suppose that there exists a measurable subset E of $\partial \boldsymbol{B}^m$ with positive Lebesgue measure such that $F^*(E) \subset \partial D$ and $F^*(E)$ intersects with at most countably many holomorphic components of ∂D. Then, f is rigid, that is, if a non-constant holomorphic mapping $g : M \to N$ is homotopic to f, then $g = f$.*

Proof. We give a proof only for the case $D = R_I$ because the following argument works also for $D = R_{II}$ and $D = R_{III}$.

Take any $Z_0 \in \partial D$ and fix it. From the definition, the matrix $I_m - Z_0 Z_0^*$ is semi-positive definite but it is not positive definite. Therefore, there exists an $x_0 \in \boldsymbol{C}^m - \{0\}$ such that

$$\|x_0\|^2 - x_0 Z_0 Z_0^{*\,t}\overline{x_0} = 0$$

while

$$\|x\|^2 - x Z_0 Z_0^{*\,t}\bar{x} \geq 0$$

for any $x \in \boldsymbol{C}^m - \{0\}$. We may assume that $\|x_0\| = 1$. Thus, we have

$$1 - x_0 Z_0 Z_0^{*\,t}\overline{x_0} = 0 \tag{16}$$

and

$$1 - x_0 Z Z^{*\,t}\overline{x_0} > 0 \tag{17}$$

for every $Z \in D$.

We define a holomorphic mapping of $M_{m,n}$ to $\boldsymbol{C}^m$ by

$$\Phi(Z) = x_0 Z. \tag{18}$$

From (16), $\Phi(Z_0) \in \partial \boldsymbol{B}^m$ and for any $Z \in D$ we have $\Phi(Z) \in \boldsymbol{B}^m$ from (17).

We assume that there exists a holomorphic mapping $\varphi : \Delta \to \partial D$ such that $\varphi(0) = Z_0$ and put

$$\psi(\lambda) = \Phi \circ \varphi(\lambda) \quad (\lambda \in \Delta).$$

Then, $\psi(\Delta) \subset \boldsymbol{B}^m \cup \partial \boldsymbol{B}^m$ and $\psi(0) \in \partial \boldsymbol{B}^m$. It follows from the maximum principle that ψ is a constant mapping. In other words,

$$\Phi(Z) = \Phi(Z_0)$$

for any $Z \in \varphi(\Delta)$. Repeating this argument, we have

Lemma 6.1. *Let V be a holomorphic component of ∂D containing Z_0. Then $\Phi(Z) = \Phi(Z_0)$ for any $Z \in V$.*

Let h be a holomorphic mapping of M to N homotopic to f. Then, both f and h induce the same monodromy θ. Hence, we have

$$F \circ \gamma = \theta(\gamma) \circ F$$

$$H \circ \gamma = \theta(\gamma) \circ H$$

for some lift H of h.

Since Γ is of divergence type and F, H is bounded holomorphic functions on $\boldsymbol{B}^m$, we may assume that every point of $E \subset \partial \boldsymbol{B}^m$ is a point of approximation for Γ and F, H have K-limits $F^*(\zeta), H^*(\zeta)$ at every $\zeta \in E$. Therefore, for each $\zeta \in E$ there exists $\{\gamma_k\}_{k=1}^{\infty} \subset \Gamma$ such that

$$\lim_{k\to\infty} \theta(\gamma_k)(F(z)) = \lim_{k\to\infty} F(\gamma_k(z)) = F^*(\zeta), \tag{19}$$

$$\lim_{k\to\infty} \theta(\gamma_k)(H(z)) = \lim_{k\to\infty} H(\gamma_k(z)) = H^*(\zeta) \tag{20}$$

hold for every $z \in \boldsymbol{B}^m$.

On the other hand, since D is a bounded domain in $\boldsymbol{C}^{mn}$, $\{\theta(\gamma_k)\}_{k=1}^{\infty}$ is a normal family on D and it converges to a holomorphic mapping $h_\zeta : D \to D \cup \partial D$ uniformly on every compact subset of D. Hence, from (19) and (20) we have

$$h_\zeta(F(z)) = F^*(\zeta), \tag{21}$$

$$h_\zeta(H(z)) = H^*(\zeta). \tag{22}$$

Here, we assume that there exists a subset E' of E with positive Lebesgue measure such that h_ζ is not a constant function if ζ belongs to E'.

Noting that $F^*(\zeta), H^*(\zeta) \in \partial D$ (Lemma 2.2), we verify that $F^*(\zeta)$ and $H^*(\zeta)$ belong to the same holomorphic component on ∂D. From the assumption, only countably many holomorphic components of ∂D intersects with $F^*(E)$. Hence, there exist a holomorphic component V of ∂D and a subset E'' of E' with positive Lebesgue measure such that $F^*(\zeta)$ and $H^*(\zeta)$ belong to V for every $\zeta \in E''$.

We consider a holomorphic mapping Φ of $M_{m,n}$ to $\boldsymbol{C}^m$ for V as (18). Then, from Lemma 6.1, there exists a constant C_V in $\boldsymbol{C}^m$ with $\|C_V\| = 1$ such that

$$\Phi \circ F^*(\zeta) = \Phi \circ H^*(\zeta) = C_V$$

holds for any $\zeta \in E''$. Two values $\Phi \circ F^*(\zeta)$ and $\Phi \circ H^*(\zeta)$ are still K-limits at $\zeta \in E''$ of $\Phi \circ F$ and $\Phi \circ H$, respectively. Therefore, from Proposition 2.3, we conclude that both $\Phi \circ F$ and $\Phi \circ H$ are constant functions on $\boldsymbol{B}^m$ and $\|\Phi \circ F(z)\| = \|\Phi \circ H(z)\| = 1$ for all $z \in \boldsymbol{B}^m$. This is a contradiction because $F(0), H(0) \in D$ and $\|\Phi(F(0))\|, \|\Phi(H(0))\| < 1$ from (17).

Therefore, h_ζ must be a constant function for almost all ζ in E, then we have $F^*(\zeta) = H^*(\zeta)$ for almost all $\zeta \in E$. It follows from Proposition 2.3 that $F = H$ and $f = h$. □

7. Examples.

In this section, we shall exhibit some examples related to our arguments.

EXAMPLE 7.1. *The complex unit ball.*

It is easy to see that $\ell(\boldsymbol{B}^n) = 0$. Thus, all of our results are valid for holomorphic mappings of M to $\boldsymbol{B}^n/\Gamma$ if M is of divergence type. In particular, a generalization of de Franchis' theorem (Theorem 1.4) holds.

We discuss a complex manifold $N = \tilde{N}/G$ which admits a non-constant holomorphic mapping from a complex hyperbolic manifold M of divergence type. From Lemma 2.2 the discrete group G consists of infinitely many elements. Thus, the group of biholomorphic automorphisms $\mathrm{Aut}(\tilde{N})$ of $\tilde{N}$ must be non-compact. From this point of view, the complex unit ball $\boldsymbol{B}^n$ is somewhat general because any strongly pseudo-convex bounded domain bounded by C^2 boundary with non-compact automorphisms is automatically biholomorphic to a complex unit ball (cf. a theorem of Wong-Rosay [**20**]).

There are another natural domains which have non-compact automorphism groups.

Example 7.2. *The complex ellipsoid.*

For $\boldsymbol{m} = (m_1, \dots, m_{n-1}) \in \boldsymbol{N}^{n-1}$ with $m_1 \le m_2 \le \cdots \le m_{n-1}$ which is not $(1, \dots, 1)$, we consider a complex ellipsoid $E_{\boldsymbol{m}}$ in $\boldsymbol{C}^n$,

$$E_{\boldsymbol{m}} = \{z = (z^1, z^2, \dots, z^n) \in \boldsymbol{C}^n \mid |z^1|^2 + |z^2|^{2m_1} + \cdots + |z^n|^{2m_{n-1}} < 1\}.$$

It is known that $E_{\boldsymbol{m}}$ is not biholomorphic to the unit ball $\boldsymbol{B}^n$ and $\mathrm{Aut}(E_{\boldsymbol{m}})$ is non-compact. Obviously, $\ell(E_{\boldsymbol{m}}) = 0$, thus the rigidity holds for the space of non-constant holomorphic mappings of a complex hyperbolic manifold of divergence type to a quotient manifold of $E_{\boldsymbol{m}}$.

Bedford-Pinchuk [**1**] shows that in $\boldsymbol{C}^2$ the converse is true if the domain is bounded by real analytic boundary, that is, they show that if $\mathrm{Aut}(D)$ is non-compact for a bounded domain D in $\boldsymbol{C}^2$ with real analytic boundary, then the domain D is biholomorphic to either the unit ball or a complex ellipsoid.

As we noted in §1, if $\tilde{N}$ has the product structure, then the rigidity of holomorphic mappings of M to $N = \tilde{N}/G$ is hard to hold. However, using Theorem 1.2, we may construct a discrete group G so that the rigidity holds for $\mathrm{Hol}(M, N)$ even if $\tilde{N}$ is a product space.

Example 7.3. *Product space with good action.*

Let $M = \boldsymbol{B}^m/\Gamma$ be a complex hyperbolic manifold of divergence type and $\tilde{N} = \boldsymbol{B}^{m_1} \times \boldsymbol{B}^{m_2} \times \cdots \times \boldsymbol{B}^{m_t}$. Suppose that there exist homomorphisms α_j $(j = 1, 2, \dots, t)$ from an infinite group Γ to $\mathrm{Aut}(\boldsymbol{B}^{m_j})$ such that $\alpha_j(\Gamma)$ is a discrete subgroup of $\mathrm{Aut}(\boldsymbol{B}^{m_j})$ and $\ker \alpha_j$ is a finite subgroup of Γ. We set a homomorphism $\alpha : \Gamma \to \mathrm{Aut}(\tilde{N})$ by

$$\alpha(\gamma) = \tilde{\gamma} = (\alpha_1(\gamma), \alpha_2(\gamma), \dots, \alpha_t(\gamma)).$$

Since $\ker \alpha_j$ $(j = 1, 2, \dots, t)$ are finite subgroups, we verify that if $\{\tilde{\gamma}_k\}_{k=1}^{\infty}$ is an infinite sequence of distinct elements of $\alpha(\Gamma)$, then so is each $\{\alpha_j(\gamma_k)\}_{k=1}^{\infty}$ $(1 \le j \le t)$. It is easily seen that each $\alpha_j(\Gamma)$ has the property (A) in Theorem 1.2. Hence, $\alpha(\Gamma)$ has also the property (A) and the statement of Theorem 1.2 (the rigidity of $\mathrm{Hol}(M, N)$) holds for $N = \tilde{N}/\alpha(\Gamma)$.

Example 7.4. *Teichmüller spaces.*

Let T_g be the Bers embedding of the Teichmüller space of compact Riemann surfaces of genus $g > 1$. It is known that T_g is a bounded domain in $\boldsymbol{C}^{3g-3}$ and the group of biholomorphic automorphisms of T_g is the Teichmüller modular group Mod_g, the mapping class group. As for the fundamental facts of T_g and Mod_g, see [**2**] and [**11**].

The boundary ∂T_g contains the set of so-called regular b-groups which are Kleinian groups representing pinched surfaces (stable curves). We may consider a non-trivial complex analytic deformation space of a regular b-group on the boundary of Teichmüller space unless the group is a terminal regular b-group. More precisely, a boundary point called a cusp may have its non-trivial deformation space on the boundary of the Teichmüller space. Since a set of countably many algebraic equations defines the set of cusps, each cusp belongs to a complete pluripolar set in $\boldsymbol{C}^{3g-3}$. On the other hand, it is known (cf. [**2**]) that any boundary point which is not a cusp is a totally degenerate group. Also, we know (cf. [**22**]) that any totally degenerate group has no non-trivial complex analytic deformation. Thus, we verify that $\ell(T_g) = 0$. We also use the same argument as in §5 (cf. [**3**]) and we may show the following (cf. [**10**], [**25**]).

Proposition 7.1. *Let $M = \boldsymbol{B}^m/\Gamma$ be a complex hyperbolic manifold of divergence type. If two mappings $f, g \in \mathrm{Hol}(M, T_g/\mathrm{Mod}_g)$ are homotopic to each other, then $f = g$. Moreover, the space $\mathrm{Hol}(M, T_g/\mathrm{Mod}_g)$ consists of at most finitely many elements if Γ is of finitely generated.*

Let M_g denote the moduli space of compact Riemann surfaces of genus $g > 0$. For a canonical homology basis $\chi(R)$ of $R \in M_g$, we have a period matrix $Z(\chi(R))$ in $\mathscr{H}_g$, where $\mathscr{H}_g$ is the Siegel upper half space of degree g, that is, it is the space of $g \times g$ symmetric matrices whose imaginary parts are positive definite.

From a theorem of Torelli, a period matrix in $\mathscr{H}_g$ determines a unique point in M_g. Changing homology basis arises the action of $PSp(g, \boldsymbol{Z}) \subset PSp(g, \boldsymbol{R})$ on $\mathscr{H}_g$ (see the definition below for $PSp(g, \boldsymbol{R})$). Thus, we have a natural identification

$$M_g = P_g/PSp(g, \boldsymbol{Z}) = T_g/\mathrm{Mod}_g,$$

where $P_g \subset \mathscr{H}_g$ is the space of period matrices of all $R \in M_g$ and for all $\chi(R)$. Therefore, if one sees Proposition 7.1, it is natural to expect that the rigidity holds for $\mathrm{Hol}(M, \mathscr{H}_g/G)$ and for a discrete subgroup G of $PSp(2, \boldsymbol{R})$. Unfortunately, we can show by the following example that the rigidity does not hold for holomorphic mappings of a complex hyperbolic manifold to a manifold which is a quotient space of $\mathscr{H}_g$.

Example 7.5. *Non-rigid holomorphic mappings to Siegel upper half spaces.*

Let M be a hyperbolic Riemann surface. The Riemann surface M is represented by $M = \boldsymbol{H}/\Gamma$, where Γ is a discrete group of $PSL(2, \boldsymbol{R})$. We consider the Siegel upper half space $\mathscr{H}_2$ of degree 2. It is well known that the space $\mathscr{H}_2$ is biholomorphic to a bounded domain D_2 in $\boldsymbol{C}^2$, where D_2 is the space of 2×2 symmetric matrices with

$$I_2 - ZZ^* > 0.$$

The biholomorphic automorphism group of $\mathscr{H}_2$ is the projective symplectic group $PSp(2, \boldsymbol{R})$ defined by $Sp(2, \boldsymbol{R})/\{\pm\boldsymbol{1}\}$, where $Sp(g, \boldsymbol{R})$ $(g > 0)$ is the space of real $2g \times 2g$ matrices m satisfying

$$ {}^t m j m = j \tag{23} $$

for

$$ j = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}. $$

The space of all matrices m in $PSp(g, \boldsymbol{R})$ with integral entries is denoted by $PSp(g, \boldsymbol{Z})$.

For $m \in PSp(g, \boldsymbol{R})$ with $g \times g$ blocks,

$$ m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, $$

the equation (23) holds if and only if ${}^t AC, {}^t BD$ are symmetric and

$$ {}^t AD - {}^t CD = I_g. $$

For $m \in PSp(g, \boldsymbol{R})$, the action of m for $Z \in \mathscr{H}_g$ is defined by

$$ Z \mapsto m(Z) = (AZ + B)(CZ + D)^{-1}. $$

Taking $\tau \in \boldsymbol{H}$, we define a holomorphic mapping F_τ of $\boldsymbol{H}$ to $\mathscr{H}_2$ by

$$ F_\tau(z) = \begin{pmatrix} \tau & 0 \\ 0 & z \end{pmatrix} \quad (z \in \boldsymbol{H}). $$

For each $\gamma(z) = (az + b)(cz + d)^{-1} \in \Gamma$, we set

$$ \theta(\gamma) = \begin{pmatrix} A(\gamma) & B(\gamma) \\ C(\gamma) & D(\gamma) \end{pmatrix}, $$

where

$$ A(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad B(\gamma) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, $$

$$ C(\gamma) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \quad D(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}. $$

Then, θ is an isomorphism of Γ into $PSp(2, \boldsymbol{R})$, and the image $\theta(\Gamma)$ is a discrete subgroup of $PSp(2, \boldsymbol{R})$. We also see that

$$ F_\tau(\gamma(z)) = \theta(\gamma)(F_\tau(z)). \tag{24} $$

Therefore, the mapping F_τ is regarded as a lift of a holomorphic mapping f_τ of $M = \boldsymbol{H}/\Gamma$ to $N = \mathscr{H}_2/\theta(\Gamma)$.

Since θ does not depend on $\tau \in \boldsymbol{H}$, $f_{\tau'}$ is a holomorphic mapping of M to N for another $\tau'(\neq \tau) \in \boldsymbol{H}$. Obviously, $f_{\tau'}$ is homotopic to f_τ and the rigidity does not hold.

Finally, we exhibit an example of a holomorphic mapping from a hyperbolic Riemann surface to a quotient space of the Siegel upper half space which satisfied the condition of Theorem 6.1. Hence the mapping is rigid.

Example 7.6. Let $M = \boldsymbol{H}/\Gamma$ be a Riemann surface as above. We consider a holomorphic mapping $\hat{F}$ of $\boldsymbol{H}$ to $\mathscr{H}_2$ by

$$\hat{F}(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \quad (z \in \boldsymbol{H}).$$

For each $\gamma(z) = (az+b)(cz+d)^{-1} \in \Gamma$, we set

$$\hat{\theta}(\gamma) = \begin{pmatrix} \hat{A}(\gamma) & \hat{B}(\gamma) \\ \hat{C}(\gamma) & \hat{D}(\gamma) \end{pmatrix},$$

where

$$\hat{A}(\gamma) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \hat{B}(\gamma) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},$$

$$\hat{C}(\gamma) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \hat{D}(\gamma) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

Then, $\hat{\theta}$ is an isomorphism of Γ into $PSp(2, \boldsymbol{R})$, $\hat{\theta}(\Gamma)$ is discrete and

$$\hat{\theta}(\gamma)(\hat{F}(z)) = \hat{F}(\gamma(z))$$

holds for any $z \in \boldsymbol{H}$. Hence $\hat{F}$ is regarded as a lift of a holomorphic mapping $\hat{f}$ of M to $\hat{N} = \mathscr{H}_2/\hat{\theta}(\Gamma)$ as in Example 7.5. However, it is easily seen that any point in $\hat{F}^*(\partial \boldsymbol{H})$ is not contained in any holomorphic component in $\partial \mathscr{H}_2$. Hence it follows from Theorem 6.1 that $\hat{f} : M \to \hat{N}$ is rigid.

References

[1] E. Bedford and S. Pinchuk, Domains in $\boldsymbol{C}^2$ with Noncompact Automorphism Groups, Indiana Univ. Math. J., **47** (1998), 199–222.

[2] L. Bers, On boundaries of Teichmüller spaces and on kleinian groups I, Ann. of Math., **91** (1970), 570–600.

[3] L. Bers, A remark on Mumford's compactness theorem, Israel J. Math., **12** (1972), 400–407.

[4] B. H. Bowditch, Geometrical finiteness with variable negative curvature, Duke Math. J., **77** (1995), 229–274.

[5] S. S. Chen and L. Greenberg, Hyperbolic Spaces, In: Contributions to Analysis, (ed. L. V. Ahlfors, I. Kra, B. Maskit and L. Nirenberg), Academic Press, 1974, 49–88.

[6] A. Douady, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier (Grenoble), **16** (1964), 1–95.

[7] W. G. Goldman, Complex Hyperbolic Geometry, Oxford Math. Monogr., Clarendon Press Oxford, 1999.

[8] Y. Imayoshi, Generalizations of de Franchis theorem, Duke Math. J., **50** (1983), 393–408.

[9] Y. Imayoshi, Holomorphic maps of projective algebraic manifolds into compact C-hyperbolic manifolds, J. Math. Soc. Japan, **46** (1994), 289–307.

[10] Y. Imayoshi and H. Shiga, A finiteness theorem for holomorphic families of Riemann surfaces, In: Holomorphic Functions and Moduli II, (ed. S. S. Chern, I. Kaplansky et al.), Math. Sci. Res. Inst. Publ., Springer, 1988, 207–219.

[11] Y. Imayoshi and M. Taniguchi, Introduction to Teichmüller Spaces, Springer, Tokyo, 1992.

[12] S. Kamiya, Notes on discrete subgroups of divergence type of $U(1, n; \boldsymbol{C})$, Math. Japon., **38** (1993), 583–587.

[13] W. Kaup, Holomorphic mappings of complex spaces, Sympos. Math., **2** (1968), 333–340.

[14] M. Klimek, Pluripotential Theory, (ed. P. H. Cohn and H. G. Dales), London Math. Soc. Monogr. (N.S.), **6**, Clarendon Press, Oxford-New York-Tokyo, 1991.
[15] S. Kobayashi, Hyperbolic Complex Spaces, Grundlehren Math. Wiss., **318**, Springer, Berlin-Heidelberg-New York, 1998.
[16] S. Kobayashi and T. Ochiai, Meromorphic mappings into compact complex spaces of general type, Invent. Math., **31** (1975), 7–16.
[17] P. J. Nicholls, The ergodic theory of discrete groups, London Math. Soc. Lecture Note Ser., **143**, Cambridge Univ. Press, 1989.
[18] J. Noguchi, Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and locally symmetric spaces, Invent. Math., **93** (1988), 15–34.
[19] J. Noguchi, Hyperbolic manifolds and Diophantine geometry, Sugaku Expositions, **4** (1991), 63–81.
[20] J. P. Rosay, Sur une caratérisation de la boule parmi les domaines de C^n par son groupe d'automorphismes, Ann. Inst. Fourier (Grenoble), **29** (1979), 91–97.
[21] W. Rudin, Function Theory in the Unit Ball of C^n, (ed. M. Artin, S. S. Chern et al.), Grundlehren Math. Wiss., **241**, Springer, New York-Heidelberg-Berlin, 1980.
[22] H. Shiga, Characterization of quasi-disks and Teichmüller spaces, Tohoku Math. J. (2), **37** (1985), 541–552.
[23] E. M. Stein, Boundary behavior of holomorphic functions of several complex variables, Mathematical Notes, No. 11, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1972.
[24] T. Sunada, Holomorphic mappings into a compact quotient of symmetric bounded domain, Nagoya Math. J., **64** (1976), 159–175.
[25] H. Tanigawa, Rigidity and boundary behavior of holomorphic mappings, J. Math. Soc. Japan, **44** (1992), 131–143.
[26] M. Tsuji, Potential theory in modern function theory, Maruzen, 1959.

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