J. Math. Soc. Japan
Vol. 57, No. 1, 2005

A characterization of symmetric cones through pseudoinverse maps

By Chifune KAI and Takaaki NOMURA

(Received Aug. 8, 2003)
(Revised Dec. 15, 2003)

Abstract. In this paper we characterize symmetric cones among homogeneous convex cones by the condition that the corresponding tube domains are mapped onto the dual tube domains under pseudoinverse maps with parameter. The condition also restricts the parameter to specific ones.

1. Introduction.

We begin this paper with a simple fact. Let Z be a complex $r \times r$ symmetric matrix. Then $\mathrm{Re}\, Z$ is positive definite if and only if Z^{-1} exists and $\mathrm{Re}\, Z^{-1}$ is positive definite. Denoting by $\mathrm{Sym}(r, \boldsymbol{R})^{++}$ the cone of real $r \times r$ positive definite symmetric matrices, we rephrase the above fact as

$$\mathrm{Re}\, Z \in \mathrm{Sym}(r, \boldsymbol{R})^{++} \iff Z^{-1} \text{ exists and } \mathrm{Re}\, Z^{-1} \in \mathrm{Sym}(r, \boldsymbol{R})^{++}.$$

In this way, it is easy to generalize the fact to the case of symmetric cones. Let Ω be a symmetric cone in a real Euclidean vector space V. We recall that V has a Euclidean Jordan algebra structure [**5**], and thus the complexification $W := V_{\boldsymbol{C}}$ is a complex Jordan algebra. Let $z \in W$. Then

$$z \in \Omega + iV \iff \text{Jordan algebra inverse } z^{-1} \text{ exists and } z^{-1} \in \Omega + iV. \tag{1.1}$$

The purpose of the present paper is to show that this equivalence characterizes symmetric cones in a certain sense among homogeneous convex cones.

Symmetric cones form a specific class. Analysis on them and on symmetric tube domains is developed in a fairly explicit manner as described in [**5**]. Thus it is significant to characterize symmetric cones among homogeneous convex cones. Vinberg's characterization [**16**] concerning equal dimensionality of certain eigenspaces is of particular importance. Differential geometric characterizations are given in [**12**] and [**13**]. Another characterization making use of the connection algebra of a homogeneous convex cone is given by [**3**] and [**14**]. Ours is more analytic and motivated by Corollary 2.9 of Rothaus' paper [**11**], where it is investigated if the analytically continued Vinberg's $*$-map preserves the tube domain (see also [**8**, Remark 2.12]).

Let Ω be a homogeneous regular open convex cone in a real vector space V. Associated with Ω and a point $E \in \Omega$, the ambient vector space V has a structure of non-associative algebra with unit element E. This algebra is called a *clan* after Vinberg [**15**]. The multiplication in this algebra V will be denoted as $x \triangle y$, and the left multiplication operator by x as L_x. Then, one

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2000 *Mathematics Subject Classification.* Primary 32M15, Secondary 22E30, 43A85.

Key Words and Phrases. homogeneous cone, symmetric cone, clan, pseudoinverse map, tube domain.

This work was partially supported by the Grant-in-Aid for Scientific Research (B(2), No. 14340044), Japan Society for the Promotion of Science.

knows by [**15**] that $\langle x|y\rangle := \mathrm{Tr}\, L_{x\triangle y}$ defines a positive definite inner product on V, called the trace inner product. Let φ be the characteristic function of the cone Ω:

$$\varphi(x) := \int_{\Omega^*} e^{-\langle x|y\rangle}\, dy \qquad (x \in \Omega),$$

where Ω^* is the dual cone of Ω taken in V relative to the trace inner product:

$$\Omega^* := \{y \in V;\, \langle x|y\rangle > 0 \ \text{ for all } x \in \overline{\Omega} \setminus \{0\}\}.$$

Vinberg's $*$-map $\Omega \to \Omega^*$ is by definition $x^* := -\mathrm{grad}\log\varphi(x)$. One knows that the map $x \mapsto x^*$ extends to a birational map $I : W \to W$, where $W := V_{\boldsymbol{C}}$, and that it is holomorphic on the tube domain $\Omega + iV$. A weaker version of our theorem is the following.

THEOREM 1.1. *Suppose that Ω is irreducible. Then $\Omega = \Omega^*$ if and only if one has $I(\Omega + iV) = \Omega^* + iV$.*

For irreducible symmetric cones, Proposition III.4.3 in [**5**] tells us that x^* is a positive number multiple of the Jordan algebra inverse x^{-1} (see Lemma 5.2 of the present paper for a more precise relation between the $*$-map and the Jordan algebra inverse). Therefore Theorem 1.1 shows that the equivalence in (1.1) can be a characterization of symmetric cones.

Our actual theorem still generalizes Theorem 1.1 by using pseudoinverse maps. We note that Vinberg's $*$-map is a pseudoinverse map with a specific parameter (see subsection 5.2 of this paper with $p = 1$).

Let f be a linear form on the clan V. Then f is said to be *admissible* if the bilinear form $\langle x|y\rangle_f := \langle x\triangle y, f\rangle$ defines a positive definite inner product on V. In Proposition 2.1 of this paper we prove that to every admissible linear form f there corresponds a parameter $\boldsymbol{s} = (s_1, \dots, s_r)$ with $s_1 > 0, \dots, s_r > 0$ so that $f = E^*_{\boldsymbol{s}}$, where r is the rank of V (see (2.6) for $E^*_{\boldsymbol{s}}$). In this case we say that the parameter $\boldsymbol{s}$ is *positive*, and we write $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$ instead of $\langle\cdot|\cdot\rangle_{E^*_{\boldsymbol{s}}}$ for simplicity. By Vinberg [**15**] there exists a split solvable subgroup H in the linear automorphism group $G(\Omega)$ of Ω such that H acts on Ω simply transitively. Let $\mathfrak{h}$ be the Lie algebra of H. Define functions $\Delta_{\boldsymbol{s}}$ on Ω by

$$\Delta_{\boldsymbol{s}}((\exp T)E) := e^{\langle TE,\, E^*_{\boldsymbol{s}}\rangle} \qquad (T \in \mathfrak{h}).$$

Let the parameter $\boldsymbol{s}$ be positive. The pseudoinverse $I_{\boldsymbol{s}}(x)$ of $x \in \Omega$ is defined to be

$$\langle I_{\boldsymbol{s}}(x)|y\rangle_{\boldsymbol{s}} = -\frac{d}{dt}\log\Delta_{-\boldsymbol{s}}(x+ty)\Big|_{t=0} \qquad (y \in V).$$

Let $W := V_{\boldsymbol{C}}$. We extend $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$ to W by complex bilinearity, and denote it by the same symbol. The pseudoinverse map $I_{\boldsymbol{s}} : x \mapsto I_{\boldsymbol{s}}(x)$ extends to a birational map $W \to W$ and has the following properties:

(1) $I_{\boldsymbol{s}}(E) = E$,
(2) $I_{\boldsymbol{s}}(hE) = {}^{\boldsymbol{s}}h^{-1} I_{\boldsymbol{s}}(E)$ for all $h \in H_{\boldsymbol{C}}$, where $H_{\boldsymbol{C}}$ is the complexification of H and ${}^{\boldsymbol{s}}h$ stands for the adjoint operator of h relative to $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$.

If Ω is a symmetric cone and $\boldsymbol{s}$ is a positive number multiple of $\boldsymbol{d}$ (see (2.5) of this paper for the definition of $\boldsymbol{d}$), then $I_{\boldsymbol{s}}$ coincides with a positive number multiple of the Jordan algebra inverse map associated with Ω.

Let $\Omega^{\boldsymbol{s}}$ be the dual cone of Ω realized in V by means of $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$. Our result is as follows:

THEOREM 1.2. *Suppose that Ω is irreducible, and let $\boldsymbol{s} \in \boldsymbol{R}^r$ be positive. Then, the following are equivalent*:

(A) $I_{\boldsymbol{s}}(\Omega + iV) = \Omega^{\boldsymbol{s}} + iV$.
(B) *$\boldsymbol{s}$ is a positive number multiple of $\boldsymbol{d}$, and Ω is a symmetric cone.*
(C) *$\boldsymbol{s}$ is a positive number multiple of $\boldsymbol{d}$, and $\Omega = \Omega^{\boldsymbol{s}}$.*

We now describe the organization of this paper. In Section 2, we summarize basic facts about the clan associated with a homogeneous convex cone. Section 3 is the introduction of the pseudoinverse maps. In Section 4, we present some formulas and norm computations needed in Section 7. The results of this section are valid without any restrictions on clans. In Subsection 5.1, we recall some basic facts about symmetric cones, and they are used in Subsection 5.2 and Section 6. Proof of (C) $\Rightarrow$ (A) in the main theorem is given in Subsection 5.2. In Section 6, we prove the equivalence of (B) and (C), which is valid for homogeneous convex cones which are not necessarily irreducible. Proof of (A) $\Rightarrow$ (B) is accomplished in Section 7 through quite a bit of computations divided into several steps. Our way of the computations is inspired by the one taken in Section 5 of [**10**].

The second author is grateful to Richard Penney, Ewa Damek and Jacques Faraut for conversations about the contents of this paper.

2. Preliminaries.

2.1. Clan associated with a homogeneous convex cone.

We begin with the introduction of clan and its normal decomposition. Let V be a finite dimensional vector space over $\boldsymbol{R}$. A regular open convex cone $\Omega \subset V$ is said to be *homogeneous* if the linear Lie group

$$G(\Omega) := \{g \in GL(V); g(\Omega) = \Omega\}$$

of the automorphism group of Ω acts transitively on it. Here by regularity, we mean that Ω does not contain any straight line (not necessarily passing through the origin). In this paper, we assume that Ω is irreducible. By [**15**, Theorem 1] there exists a connected split solvable subgroup H of $G(\Omega)$ acting simply transitively on Ω. Let $\mathfrak{h}$ be the Lie algebra of H. Take any point $E \in \Omega$. Since the orbit map $H \ni h \mapsto hE$ is a diffeomorphism, differentiation at the unit element of H gives a linear isomorphism $\mathfrak{h} \ni T \mapsto TE \in V$. Let us denote by $L : x \mapsto L_x$ its inverse map, so that $L_x E = x$ for all $x \in V$. We define a multiplication $\triangle$ by $x \triangle y := L_x y$ $(x, y \in V)$. Setting $[x \triangle y] := x \triangle y - y \triangle x$, we get by the definition of L

$$[L_x, L_y]E = L_x(L_y E) - L_y(L_x E) = L_x y - L_y x = x \triangle y - y \triangle x,$$

so that,

$$[L_x, L_y] = L_{[x \triangle y]}. \tag{2.1}$$

By [**15**, Chapter II, §1] it holds that

$$\operatorname{Tr} L_{x \triangle x} > 0 \qquad \text{for any non-zero } x. \tag{2.2}$$

Moreover since $\mathfrak{h}$ is split solvable, every linear operator L_x $(x \in V)$ has only real eigenvalues. The space V with $\triangle$ defined in this way is called *the clan associated with Ω*. Since H is maximal among the connected split solvable subgroups of $G(\Omega)$, the Lie algebra $\mathfrak{h}$ contains the identity operator. This together with $L_E E = E$ ensures us that L_E is the identity operator, so that E is a unit element of V. We refer to E as *the base point used in the construction of the clan V associated with Ω*. Conversely, we can construct a homogeneous convex cone from a clan with unit element, and there is a one-to-one correspondence between the set of isomorphic classes of homogeneous convex cones and the set of isomorphic classes of clans with unit element.

Let V be a clan with unit element E. Then, V has a direct sum decomposition called a *normal decomposition*: there exists a positive integer r and idempotents E_i $(i = 1, \dots, r)$ such that

$$V = \sum_{i=1}^{r} \boldsymbol{R}E_i \oplus \sum_{1 \le j < k \le r} V_{kj}, \qquad E = E_1 + \cdots + E_r, \tag{2.3}$$

where we put

$$V_{kj} := \left\{ x \in V; c \triangle x = \frac{1}{2}(\lambda_k + \lambda_j)x, x \triangle c = \lambda_j x \text{ for } c = \sum \lambda_i E_i \ (\forall \lambda_i \in \boldsymbol{R}) \right\}.$$

The integer r is called *the rank of V*. Setting $V_{kk} := \boldsymbol{R}E_k$ for $k = 1, \dots, r$, we have the following multiplication table:

$$\begin{gathered} V_{lk} \triangle V_{kj} \subset V_{lj}, \\ \text{if } k \neq i, j, \text{then } V_{lk} \triangle V_{ij} = 0, \\ V_{lk} \triangle V_{mk} \subset V_{lm} \text{ or } V_{ml} \text{ according to } l \ge m \text{ or } m \ge l. \end{gathered} \tag{2.4}$$

2.2. Inner products defined by positive parameters.

Let V be a clan with unit element E. We keep to the notation of the previous subsection. Let us define linear forms E_i^* $(i = 1, \dots, r)$ on V by

$$\left\langle \sum_{j=1}^{r} x_j E_j + \sum_{j<k} X_{kj}, E_i^* \right\rangle = x_i \qquad (x_j \in \boldsymbol{R},\ X_{kj} \in V_{kj}).$$

We put

$$n_{kj} := \dim V_{kj} \quad (j < k), \qquad d_i := 1 + \frac{1}{2}\sum_{k>i} n_{ki} + \frac{1}{2}\sum_{j<i} n_{ij}. \tag{2.5}$$

For $\boldsymbol{s} = (s_1, \dots, s_r) \in \boldsymbol{R}^r$, we set

$$E_{\boldsymbol{s}}^* := \sum s_i E_i^*, \tag{2.6}$$

and define a bilinear form $\langle \cdot | \cdot \rangle_{\boldsymbol{s}}$ by

$$\langle x | y \rangle_{\boldsymbol{s}} := \langle x \triangle y, E_{\boldsymbol{s}}^* \rangle \qquad (x, y \in V). \tag{2.7}$$

Let $\boldsymbol{d} := (d_1, \dots, d_r)$. Then, taking a basis of V compatible with the normal decomposition (2.3), we know by (2.4) that

$$\operatorname{Tr} L_x = \langle x, E^*_{\boldsymbol{d}} \rangle. \tag{2.8}$$

By (2.2), the bilinear form

$$\langle x|y \rangle_{\boldsymbol{d}} = \langle x \triangle y, E^*_{\boldsymbol{d}} \rangle = \operatorname{Tr} L_{x\triangle y}$$

is a positive definite inner product on V, which we shall call *the trace inner product associated with the clan V*. Then by (2.4) and (2.7), we see easily that if $x, y \in V_{kj}$, then $x \triangle y = d_k^{-1} \langle x|y \rangle_{\boldsymbol{d}} E_k$. Let us assume that $\boldsymbol{s} = (s_1, \dots, s_r) \in \boldsymbol{R}^r$ is *positive*, that is, $s_i > 0$ for all $i = 1, \dots, r$. We obtain by (2.4)

$$\begin{aligned} \langle x|x \rangle_{\boldsymbol{s}} &= \sum_{i=1}^{r} \left\langle x_i^2 E_i + \sum_{\alpha<i} x_{i\alpha} \triangle x_{i\alpha}, s_i E_i^* \right\rangle \\ &= \sum_{i=1}^{r} \left\langle (p_i x_i)^2 E_i + \sum_{\alpha<i} (p_i x_{i\alpha}) \triangle (p_i x_{i\alpha}), d_i E_i^* \right\rangle = \langle x'|x' \rangle_{\boldsymbol{d}}, \end{aligned}$$

where we put $p_i := s_i^{1/2} d_i^{-1/2}$, $x' := \sum_{i=1}^{r} p_i x_i E_i + \sum_{i>1} p_i \sum_{\alpha<i} x_{i\alpha}$. Therefore $\langle \cdot|\cdot \rangle_{\boldsymbol{s}}$ also defines a positive definite inner product on V. This inner product is generic in the following sense:

PROPOSITION 2.1. *Let f be a linear form on V. If the bilinear form $\langle x|y \rangle_f := \langle x \triangle y, f \rangle$ defines a positive definite inner product on V, then there exists a positive parameter $\boldsymbol{s} = (s_1, \dots, s_r) \in \boldsymbol{R}^r$ such that $f = E^*_{\boldsymbol{s}}$.*

PROOF. Take any $x_{kj} \in V_{kj}$ $(j < k)$. Since $\langle \cdot|\cdot \rangle_f$ is a symmetric bilinear form by hypothesis, it holds that $\langle [E_j \triangle x_{kj}], f \rangle = 0$. By the definition of V_{kj}, we have

$$[E_j \triangle x_{kj}] = \frac{1}{2} x_{kj} - x_{kj} = -\frac{1}{2} x_{kj},$$

so that $\langle x_{kj}, f \rangle = 0$. Hence there exists a parameter $\boldsymbol{s} \in \boldsymbol{R}^r$ such that $f = E^*_{\boldsymbol{s}}$. The positive definiteness of $\langle \cdot|\cdot \rangle_f$ gives $s_i > 0$ for all $i = 1, \dots, r$. □

We note that owing to (2.4), the subspaces appearing in the normal decomposition (2.3) are orthogonal with each other relative to $\langle \cdot|\cdot \rangle_{\boldsymbol{s}}$ for any positive $\boldsymbol{s}$.

3. Pseudoinverse maps.

We shall introduce pseudoinverse maps and present their properties briefly. We assume that $\boldsymbol{s} = (s_1, \dots, s_r) \in \boldsymbol{R}^r$ is positive from now on.

We put $H_i := L_{E_i}$ and $\mathfrak{a} := \sum_{i=1}^{r} \boldsymbol{R} H_i$. Then $\mathfrak{a}$ is a commutative Lie subalgebra of $\mathfrak{h}$. For $u = (u_1, \dots, u_r) \in \boldsymbol{R}^r$, we define a one-dimensional representation $\chi_{\boldsymbol{u}}$ of $A := \exp \mathfrak{a}$ by

$$\chi_{\boldsymbol{u}}\Big(\exp\Big(\sum t_i H_i\Big)\Big) := \exp\Big(\sum u_i t_i\Big).$$

Let $\mathfrak{h}_{kj} := \{L_x; x \in V_{kj}\}$ and $\mathfrak{n} := \sum_{j<k} \mathfrak{h}_{kj}$. Then $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{h}$, and $\mathfrak{a}$ acts on $\mathfrak{n}$ semisimply. Put $N := \exp \mathfrak{n}$. Then $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}$, and $H = A \ltimes N$. We extend $\chi_{\boldsymbol{u}}$ to a

one-dimensional representation of H by defining $\chi_{\boldsymbol{u}} = 1$ on N. Recall that H acts on Ω simply transitively and define functions $\Delta_{\boldsymbol{u}}$ ($\boldsymbol{u} \in \boldsymbol{R}^r$) on Ω by

$$\Delta_{\boldsymbol{u}}(hE) = \chi_{\boldsymbol{u}}(h) \qquad (h \in H).$$

If $T = \sum_i t_i H_i + T'$ with $t_i \in \boldsymbol{R}$ and $T' \in \mathfrak{n}$, then it holds from (2.4) that

$$\Delta_{\boldsymbol{u}}((\exp T)E) = \exp\Bigl(\sum u_i t_i\Bigr) = \exp\langle TE, E^*_{\boldsymbol{u}}\rangle.$$

In Introduction we used this relation for the definition of $\Delta_{\boldsymbol{u}}$ for the sake of brevity. Evidently it holds that

$$\Delta_{\boldsymbol{u}}(hx) = \chi_{\boldsymbol{u}}(h)\Delta_{\boldsymbol{u}}(x) \qquad (h \in H,\ x \in \Omega). \tag{3.1}$$

Let D_v be the directional derivative in the direction $v \in V$: for smooth functions f on V,

$$D_v f(x) = \frac{d}{dt} f(x+tv)\Big|_{t=0}.$$

For $x \in \Omega$ we define $I_{\boldsymbol{s}}(x) \in V$ by

$$\langle I_{\boldsymbol{s}}(x)|y\rangle_{\boldsymbol{s}} = -D_y \log \Delta_{-\boldsymbol{s}}(x) \qquad (y \in V).$$

$I_{\boldsymbol{s}} : \Omega \to V$ is called the *pseudoinverse map*. Vinberg's $*$-map corresponds to $\boldsymbol{s} = \boldsymbol{d}$. It should be noted that, unlike [**9**], we make the image of the pseudoinverse map within the space V through the inner product (2.7). This slight modification fits to our purpose. Various properties of the original $I_{\boldsymbol{s}}$ proved in [**9**] are easily translated to our modified $I_{\boldsymbol{s}}$. Here we refer the reader to [**2**, p. 536] for the translation of normal j-algebra language to our language of clan. We denote by $\Omega^{\boldsymbol{s}}$ the dual cone of Ω realized in V by means of the inner product (2.7):

$$\Omega^{\boldsymbol{s}} = \left\{x \in V; \langle x|y\rangle_{\boldsymbol{s}} > 0 \text{ for } \forall y \in \overline{\Omega} \setminus \{0\}\right\}.$$

Then, by [**9**, Proposition 3.12], $I_{\boldsymbol{s}}$ gives a diffeomorphism of Ω onto $\Omega^{\boldsymbol{s}}$. The group H acts also on V by the coadjoint action: $x \mapsto {}^{\boldsymbol{s}}h^{-1}x$ ($h \in H, x \in V$), where ${}^{\boldsymbol{s}}T$ stands for the adjoint operator of an operator T with respect to $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$. It is easy to show by using (3.1) that $I_{\boldsymbol{s}}$ is H-equivariant:

$$I_{\boldsymbol{s}}(hx) = {}^{\boldsymbol{s}}h^{-1} I_{\boldsymbol{s}}(x) \qquad (h \in H, x \in \Omega). \tag{3.2}$$

In particular, $I_{\boldsymbol{s}}(\lambda x) = \lambda^{-1} I_{\boldsymbol{s}}(x)$ for all $\lambda > 0$. We have $I_{\boldsymbol{s}}(E) = E$ by [**9**, Lemma 3.10, (ii)], and H acts also on $\Omega^{\boldsymbol{s}}$ simply transitively.

Put $W := V_{\boldsymbol{C}}$. We extend the multiplication $\triangle$ of the clan V to W by complex bilinearity. We also extend $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$ to W by complex bilinearity. We denote the extended multiplication and the bilinear form by the same symbols. For $w \in W$ we denote by R_w the right multiplication by w: $R_w x = x \triangle w$. Then, $R_E = I$. Therefore, $w \mapsto \det R_w$ is a non-zero polynomial function on W. Hence the subset $\mathscr{O} := \{w \in W; \det R_w \neq 0\}$ is a non-empty Zariski-open set. The symbol ${}^{\boldsymbol{s}}T$ for a complex linear operator T on W has an obvious meaning.

LEMMA 3.1 ([**9**, Lemma 3.17]). *The pseudoinverse map $I_{\boldsymbol{s}}$ can be continued analytically to a rational map $W \to W$, and one has $I_{\boldsymbol{s}}(w) = {}^{\boldsymbol{s}}R_w^{-1}E$ for $w \in \mathscr{O}$.*

Recall that H acts on $\Omega^{\boldsymbol{s}}$ simply transitively by the coadjoint action and set for $\boldsymbol{u} \in \boldsymbol{R}^r$

$$\Delta_{\boldsymbol{u}}^*({}^{\boldsymbol{s}}h^{-1}E) := \chi_{\boldsymbol{u}}(h) \qquad (h \in H).$$

$\Delta_{\boldsymbol{u}}^*$ is a function on $\Omega^{\boldsymbol{s}}$ such that $\Delta_{\boldsymbol{u}}^*({}^{\boldsymbol{s}}h^{-1}\xi) = \chi_{\boldsymbol{u}}(h)\Delta_{\boldsymbol{u}}^*(\xi)$ for $h \in H$ and $\xi \in \Omega^{\boldsymbol{s}}$. For $x \in \Omega^{\boldsymbol{s}}$ we define $I_{\boldsymbol{s}}^*(x)$ by

$$\langle I_{\boldsymbol{s}}^*(x)|y\rangle_{\boldsymbol{s}} = -D_y \log \Delta_{\boldsymbol{s}}^*(x) \qquad (y \in V).$$

Then, by [**9**, Proposition 3.15], $I_{\boldsymbol{s}}^*$ gives a diffeomorphism of $\Omega^{\boldsymbol{s}}$ onto Ω. Moreover, $I_{\boldsymbol{s}}^*$ is H-equivariant, that is, $I_{\boldsymbol{s}}^*({}^{\boldsymbol{s}}h^{-1}x) = hI_{\boldsymbol{s}}^*(x)$ for every $h \in H$ and $x \in \Omega^{\boldsymbol{s}}$. We have $I_{\boldsymbol{s}}^*(E) = E$ by [**9**, Lemma 3.13]. $I_{\boldsymbol{s}}^*$ is also continued analytically to a rational map $W \to W$. We know by [**9**, Proposition 3.16] that $I_{\boldsymbol{s}}$ and $I_{\boldsymbol{s}}^*$ are inverse to each other. Thus, $I_{\boldsymbol{s}}$ is a birational map $W \to W$ with $I_{\boldsymbol{s}}^{-1} = I_{\boldsymbol{s}}^*$. By [**9**, Theorem 3.19], $I_{\boldsymbol{s}}$ is holomorphic on $\Omega + iV$, and $I_{\boldsymbol{s}}^*$ on $\Omega^{\boldsymbol{s}} + iV$. Moreover, $I_{\boldsymbol{s}}(\Omega + iV)$ is contained in the holomorphic domain of $I_{\boldsymbol{s}}^*$, and $I_{\boldsymbol{s}}^*(\Omega^{\boldsymbol{s}} + iV)$ in the holomorphic domain of $I_{\boldsymbol{s}}$.

Before closing this section, we would like to mention possible singularities of $I_{\boldsymbol{s}}$. We see from the proof of [**8**, Lemma 2.7] that

$$\det R_{hE} = \det \mathrm{Ad}_W(h) \det \mathrm{Ad}_{\mathfrak{h}_{\boldsymbol{C}}}(h^{-1}) \qquad (h \in H_{\boldsymbol{C}}),$$

so that $w \mapsto \det R_w$ is a holomorphic polynomial function on W relatively invariant under the action of H. Let $\Delta_1, \dots, \Delta_r$ be the basic relative invariants associated with Ω introduced in [**6**, p. 161]. We consider them as holomorphic polynomial functions on W in a natural way. By [**6**, Theorem 2.2], there exist non-negative integers $a_1, \dots, a_r$ and $\alpha \in \boldsymbol{R}$ such that

$$\det R_w = \alpha \Delta_1(w)^{a_1} \cdots \Delta_r(w)^{a_r}.$$

This together with Lemma 3.1 gives

PROPOSITION 3.2. *Let* $\mathscr{N}_i := \{w \in W; \Delta_i(w) = 0\}$ $(i = 1, \dots, r)$. *Then* $I_{\boldsymbol{s}}$ *is holomorphic on* $W \setminus \cup_{i=1}^r \mathscr{N}_i$.

4. Formulas and norm computations.

Put $W_{kj} := (V_{kj})_{\boldsymbol{C}}$ $(j \le k)$. Then the properties similar to (2.4) hold:

$$\begin{gathered} W_{lk} \triangle W_{kj} \subset W_{lj}, \\ \text{if } k \ne i, j, \text{ then } W_{lk} \triangle W_{ij} = 0, \qquad (4.1) \\ W_{lk} \triangle W_{mk} \subset W_{lm} \text{ or } W_{ml} \text{ according to } l \ge m \text{ or } m \ge l. \end{gathered}$$

Note that if $v_{kj}, w_{kj} \in W_{kj}$, then we have

$$v_{kj} \triangle w_{kj} = s_k^{-1} \langle v_{kj}|w_{kj}\rangle_{\boldsymbol{s}} E_k. \qquad (4.2)$$

H-equivariance of $I_{\boldsymbol{s}}$ and $I_{\boldsymbol{s}}^*$ gives

$$I_{\boldsymbol{s}}(hE) = {}^{\boldsymbol{s}}h^{-1}E, \qquad I_{\boldsymbol{s}}^*({}^{\boldsymbol{s}}h^{-1}E) = hE \qquad (h \in H).$$

Moreover these equalities hold for every $h \in H_{\boldsymbol{C}}$ by analytic continuation. Throughout this section we always assume that the integers j,k,l satisfy $1 \le j < k < l \le r$ and write $\langle\cdot|\cdot\rangle$ instead of $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$ for simplicity. We set $\nu[w] := \langle w|w\rangle$ $(w \in W)$ to simplify the description.

Let $w_{lk} \in W_{lk}$, $w_{lj} \in W_{lj}$ and $w_{kj} \in W_{kj}$ in this section.

4.1. Formulas.

LEMMA 4.1. *For every $x = \sum x_i E_i + \sum_{\alpha>\beta} x_{\alpha\beta}$ $(x_i \in \boldsymbol{C},\ x_{\alpha\beta} \in W_{\alpha\beta})$, one has*

$$\begin{aligned}\exp\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)x = x &+ x_j w_{lj} + \sum_{\alpha>j} w_{lj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{lj} \triangle x_{j\beta} \\ &+ x_j w_{kj} + \sum_{\alpha>j} w_{kj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{kj} \triangle x_{j\beta} \\ &+ 2^{-1} x_j \bigl(s_k^{-1}\nu[w_{kj}]E_k + s_l^{-1}\nu[w_{lj}]E_l + (w_{lj}\triangle w_{kj} + w_{kj}\triangle w_{lj})\bigr).\end{aligned}$$

PROOF. We get by (4.1)

$$\begin{aligned}\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)x = x_j w_{lj} &+ \sum_{\alpha>j} w_{lj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{lj} \triangle x_{j\beta} \\ &+ x_j w_{kj} + \sum_{\alpha>j} w_{kj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{kj} \triangle x_{j\beta}.\end{aligned} \tag{4.3}$$

Since $w_{lj} \triangle x_{\alpha j} \in W_{l\alpha}$ or $W_{\alpha l}$, and since $w_{lj} \triangle x_{j\beta} \in W_{l\beta}$, we obtain

$$\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)\Biggl(\sum_{\alpha>j} w_{lj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{lj} \triangle x_{j\beta}\Biggr) = 0.$$

Similarly

$$\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)\Biggl(\sum_{\alpha>j} w_{kj} \triangle x_{\alpha j} + \sum_{\beta<j} w_{kj} \triangle x_{j\beta}\Biggr) = 0.$$

This together with (4.2) and (4.3) yields

$$\begin{aligned}\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)^2 x &= x_j (w_{lj} + w_{kj}) \triangle (w_{lj} + w_{kj}) \\ &= x_j \bigl(s_k^{-1}\nu[w_{kj}]E_k + s_l^{-1}\nu[w_{lj}]E_l + w_{lj}\triangle w_{kj} + w_{kj}\triangle w_{lj}\bigr).\end{aligned}$$

The last term belongs to $\boldsymbol{C}E_k \oplus \boldsymbol{C}E_l \oplus W_{lk}$ by virtue of (4.1), so that we have by (4.1) again

$$\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)^3 x = 0.$$

From these observations we arrive at the lemma easily. □

In what follows, given $w_{lj} \in W_{lj}$ and $w_{kj} \in W_{kj}$, we set

$$S_{lk} := \frac{1}{2}(w_{lj}\triangle w_{kj} + w_{kj}\triangle w_{lj}). \tag{4.4}$$

We have $S_{lk} \in W_{lk}$ by (4.1).

PROPOSITION 4.2. *Let $t_j, t_k, t_l \in \boldsymbol{R}$. Then one has*

$$\begin{aligned}
&\exp\big(L_{w_{lj}}+L_{w_{kj}}\big)\exp\big(L_{w_{lk}}\big)\exp(t_jH_j+t_kH_k+t_lH_l)E \\
&\quad=\sum_{m\neq j,k,l}E_m+e^{t_j}E_j+\big(e^{t_k}+(2s_k)^{-1}e^{t_j}\nu[w_{kj}]\big)E_k \\
&\qquad+\big(e^{t_l}+(2s_l)^{-1}e^{t_k}\nu[w_{lk}]+(2s_l)^{-1}e^{t_j}\nu[w_{lj}]\big)E_l+e^{t_j}w_{lj}+e^{t_j}w_{kj}+(e^{t_j}S_{lk}+e^{t_k}w_{lk}).
\end{aligned}$$

PROOF. We see easily that

$$\exp(t_jH_j+t_kH_k+t_lH_l)E=\sum_{m\neq j,k,l}E_m+e^{t_j}E_j+e^{t_k}E_k+e^{t_l}E_l.$$

For $m=1,\dots,r$, we have by Lemma 4.1

$$\exp\left(L_{w_{lk}}\right)E_m=E_m+\delta_{mk}\left((2s_l)^{-1}\nu[w_{lk}]E_l+w_{lk}\right).$$

Hence it holds that

$$\begin{aligned}
&\exp\big(L_{w_{lk}}\big)\exp(t_jH_j+t_kH_k+t_lH_l)E \\
&\quad=\sum_{m\neq j,k,l}E_m+e^{t_j}E_j+e^{t_k}E_k+\left(e^{t_l}+(2s_l)^{-1}e^{t_k}\nu[w_{lk}]\right)E_l+e^{t_k}w_{lk}.
\end{aligned}$$

Now by Lemma 4.1 we have for $m=1,\dots,r$

$$\begin{aligned}
&\exp\big(L_{w_{lj}}+L_{w_{kj}}\big)E_m \\
&\quad=E_m+\delta_{mj}(w_{lj}+w_{kj})+2^{-1}\delta_{mj}\left(s_k^{-1}\nu[w_{kj}]E_k+s_l^{-1}\nu[w_{lj}]E_l+(w_{lj}\triangle w_{kj}+w_{kj}\triangle w_{lj})\right).
\end{aligned}$$

Moreover it holds that

$$\exp\big(L_{w_{lj}}+L_{w_{kj}}\big)w_{lk}=w_{lk}.$$

The proposition follows from these formulas. $\square$

LEMMA 4.3. *One has*

$$\begin{aligned}
&{}^{s}\big(\exp\big(L_{w_{lj}}+L_{w_{kj}}\big)\big)^{-1}E_m \\
&\quad=E_m+\delta_{mk}\big((2s_j)^{-1}\nu[w_{kj}]E_j-w_{kj}\big)+\delta_{ml}\big((2s_j)^{-1}\nu[w_{lj}]E_j-w_{lj}\big).
\end{aligned}$$

PROOF. Take $x=\sum x_iE_i+\sum_{\alpha>\beta}x_{\alpha\beta}$ $(x_i\in\boldsymbol{C},x_{\alpha\beta}\in W_{\alpha\beta})$. Since the spaces $W_{\alpha\beta}$ are orthogonal to each other relative to $\langle\cdot|\cdot\rangle$, Lemma 4.1 yields

$$\begin{aligned}
&\big\langle x\big|{}^{s}(\exp(L_{w_{lj}}+L_{w_{kj}}))^{-1}E_m\big\rangle=\big\langle\exp\big(-L_{w_{lj}}-L_{w_{kj}}\big)x\big|E_m\big\rangle \\
&\quad=\big\langle x-w_{lj}\triangle x_{lj}-w_{kj}\triangle x_{kj}+2^{-1}x_j(s_k^{-1}\nu[w_{kj}]E_k+s_l^{-1}\nu[w_{lj}]E_l)\big|E_m\big\rangle \\
&\quad=\langle x|E_m\rangle+\delta_{mk}\big\langle(2s_k)^{-1}x_j\nu[w_{kj}]E_k-w_{kj}\triangle x_{kj}\big|E_k\big\rangle+\delta_{ml}\big\langle(2s_l)^{-1}x_j\nu[w_{lj}]E_l-w_{lj}\triangle x_{lj}\big|E_l\big\rangle.
\end{aligned}$$

Here $w_{kj}\triangle x_{kj}=s_k^{-1}\langle w_{kj}|x_{kj}\rangle E_k$ by (4.2), so that $\|E_k\|^2=s_k$ implies

$$\left\langle (2s_k)^{-1} x_j \nu[w_{kj}] E_k - w_{kj} \triangle x_{kj} \middle| E_k \right\rangle = 2^{-1} x_j \nu[w_{kj}] - \langle w_{kj} | x_{kj} \rangle = \left\langle x \middle| (2s_j)^{-1} \nu[w_{kj}] E_j - w_{kj} \right\rangle.$$

A similar computation gives

$$\left\langle (2s_l)^{-1} x_j \nu[w_{lj}] E_l - w_{lj} \triangle x_{lj} \middle| E_l \right\rangle = \left\langle x \middle| (2s_j)^{-1} \nu[w_{lj}] E_j - w_{lj} \right\rangle,$$

which completes the proof. □

LEMMA 4.4. *One has* ${}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} \in W_{kj}$ *and* ${}^{\boldsymbol{s}}L_{w_{kj}} w_{lk} \in W_{lj}$.

PROOF. We put $W' := \sum \boldsymbol{C} E_i \oplus \sum_{(\alpha,\beta)\neq(k,j)} W_{\alpha\beta}$, so that W' is complement to W_{kj} in W. For any $x = \sum x_i E_i + \sum_{(\alpha,\beta)\neq(k,j)} x_{\alpha\beta} \in W'$, it follows from (4.1) that

$$\begin{aligned} \left\langle {}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} | x \right\rangle &= \left\langle w_{lk} | L_{w_{lj}} x \right\rangle \\ &= \left\langle w_{lk} \middle| w_{lj} \triangle \left(x_j E_j + \sum_{\alpha > j, \alpha \neq k} x_{\alpha j} + \sum_{\beta < j} x_{j\beta} \right) \right\rangle = 0. \end{aligned}$$

Hence we have ${}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} \in W_{kj}$. The proof for ${}^{\boldsymbol{s}}L_{w_{kj}} w_{lk} \in W_{lj}$ is similar and omitted. □

LEMMA 4.5. *One has*

$$\begin{aligned} &{}^{\boldsymbol{s}}\left(\exp\left(L_{w_{lj}} + L_{w_{kj}}\right)\right)^{-1} w_{lk} \\ &\quad = w_{lk} + (2s_j)^{-1} \langle w_{lj} \triangle w_{kj} + w_{kj} \triangle w_{lj} | w_{lk} \rangle E_j - {}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} - {}^{\boldsymbol{s}}L_{w_{kj}} w_{lk}. \end{aligned}$$

PROOF. Take $x = \sum x_i E_i + \sum_{\alpha > \beta} x_{\alpha\beta}$ $(x_i \in \boldsymbol{C}, x_{\alpha\beta} \in W_{\alpha\beta})$. Discussing as in the proof of Lemma 4.3, we get

$$\begin{aligned} &\left\langle x | {}^{\boldsymbol{s}}\left(\exp\left(L_{w_{lj}} + L_{w_{kj}}\right)\right)^{-1} w_{lk} \right\rangle \\ &\quad = \langle x | w_{lk} \rangle + (2s_j)^{-1} \langle x | E_j \rangle \langle w_{lj} \triangle w_{kj} + w_{kj} \triangle w_{lj} | w_{lk} \rangle - \left\langle x_{kj} | {}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} \right\rangle - \left\langle x_{lj} | {}^{\boldsymbol{s}}L_{w_{kj}} w_{lk} \right\rangle. \end{aligned}$$

Lemma 4.4 shows that the last two terms are equal to $-\left\langle x | {}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} + {}^{\boldsymbol{s}}L_{w_{kj}} w_{lk} \right\rangle$. Hence we obtain the lemma. □

PROPOSITION 4.6. Let S_{lk} be as in (4.4). Then we have

$$\begin{aligned} &{}^{\boldsymbol{s}}\left(\exp\left(L_{w_{lj}} + L_{w_{kj}}\right) \exp\left(L_{w_{lk}}\right) \exp(t_j H_j + t_k H_k + t_l H_l)\right)^{-1} E \\ &\quad = \sum_{m \neq j,k,l} E_m + \Big(e^{-t_j} + (2s_j)^{-1} \left(e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]\right) \nu[w_{kj}] \\ &\qquad\qquad + (2s_j)^{-1} e^{-t_l} \nu[w_{lj}] - s_j^{-1} e^{-t_l} \langle S_{lk} | w_{lk} \rangle \Big) E_j \\ &\qquad + \left(e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]\right) E_k + e^{-t_l} E_l \\ &\qquad + \left(e^{-t_l}\, {}^{\boldsymbol{s}}L_{w_{lj}} w_{lk} - (e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]) w_{kj}\right) \\ &\qquad + e^{-t_l} \left({}^{\boldsymbol{s}}L_{w_{kj}} w_{lk} - w_{lj}\right) - e^{-t_l} w_{lk}. \end{aligned}$$

PROOF. First we see easily that

$$^{s}\!\big(\exp(t_jH_j+t_kH_k+t_lH_l)\big)^{-1}E=\sum_{m\neq j,k,l}E_m+e^{-t_j}E_j+e^{-t_k}E_k+e^{-t_l}E_l.$$

On the other hand, Lemma 4.3 says that

$$^{s}\!\big(\exp\big(L_{w_{lk}}\big)\big)^{-1}E_m=E_m+\delta_{ml}\big((2s_k)^{-1}\nu[w_{lk}]E_k-w_{lk}\big).$$

Hence we have

$$\begin{aligned}&^{s}\!\big(\exp\big(L_{w_{lk}}\big)\exp(t_jH_j+t_kH_k+t_lH_l)\big)^{-1}E\\&=\sum_{m\neq j,k,l}E_m+e^{-t_j}E_j+\big(e^{-t_k}+(2s_k)^{-1}e^{-t_l}\nu[w_{lk}]\big)E_k+e^{-t_l}E_l-e^{-t_l}w_{lk}.\end{aligned}$$

Therefore Lemmas 4.3 and 4.5 give

$$\begin{aligned}&^{s}\!\big(\exp\big(L_{w_{lj}}+L_{w_{kj}}\big)\exp\big(L_{w_{lk}}\big)\exp(t_jH_j+t_kH_k+t_lH_l)\big)^{-1}E\\&=\sum_{m\neq j,k,l}E_m+e^{-t_j}E_j+e^{-t_l}\left(E_l+(2s_j)^{-1}\nu[w_{lj}]E_j-w_{lj}\right)\\&\quad+\left(e^{-t_k}+(2s_k)^{-1}e^{-t_l}\nu[w_{lk}]\right)\left(E_k+(2s_j)^{-1}\nu[w_{kj}]E_j-w_{kj}\right)\\&\quad-e^{-t_l}\left(w_{lk}+s_j^{-1}\langle S_{lk}|w_{lk}\rangle E_j-{}^{s}L_{w_{lj}}w_{lk}-{}^{s}L_{w_{kj}}w_{lk}\right).\end{aligned}$$

The proposition follows from this easily. □

4.2. Norm computations.

LEMMA 4.7. *$\|v_{lk}\triangle v_{kj}\|^2=(2s_k)^{-1}\|v_{lk}\|^2\|v_{kj}\|^2$ for every $v_{lk}\in V_{lk}$ and $v_{kj}\in V_{kj}$.*

PROOF. Put $z:=v_{lk}\triangle v_{kj}\in V_{lj}$. Then (2.1) and (2.4) give $[L_z,L_{v_{lk}}]=L_{[z\triangle v_{lk}]}=0$, so that

$$\begin{aligned}z\triangle z&=L_zL_{v_{lk}}v_{kj}=L_{v_{lk}}L_zv_{kj}=L_{v_{lk}}\big(L_{v_{kj}\triangle v_{lk}}+\big[L_{v_{lk}},L_{v_{kj}}\big]\big)v_{kj}\\&=L^2_{v_{lk}}(v_{kj}\triangle v_{kj})-L_{v_{lk}}L_{v_{kj}}z,\end{aligned}$$

because $v_{kj}\triangle v_{lk}=0$. Moreover, by (2.1) and (4.2) the last term is equal to

$$\begin{aligned}s_k^{-1}\|v_{kj}\|^2L^2_{v_{lk}}E_k-\big(L_{v_{kj}}L_{v_{lk}}+L_{v_{lk}\triangle v_{kj}}\big)z&=s_k^{-1}\|v_{kj}\|^2L_{v_{lk}}v_{lk}-z\triangle z\\&=s_k^{-1}s_l^{-1}\|v_{kj}\|^2\|v_{lk}\|^2E_l-z\triangle z.\end{aligned}$$

Hence we get $z\triangle z=(2s_ks_l)^{-1}\|v_{kj}\|^2\|v_{lk}\|^2E_l$. Since $z\triangle z=s_l^{-1}\|z\|^2E_l$ by (4.2), we obtain the lemma. □

LEMMA 4.8. (1) *If $n_{kj}\neq 0$, then one has $n_{lj}\geq n_{lk}$.*
(2) *If $n_{lk}\neq 0$, then one has $n_{lj}\geq n_{kj}$.*

PROOF. Let us assume $n_{kj} \neq 0$. Take any non-zero $v_{kj} \in V_{kj}$, and consider the linear map $V_{lk} \ni v_{lk} \mapsto v_{lk} \triangle v_{kj} \in V_{lj}$. We see that this map is injective by virtue of Lemma 4.7. Hence we get $n_{lj} \geq n_{lk}$. The proof for (2) is similar. □

Given $v_{lj} \in V_{lj}$, $v_{kj} \in V_{kj}$, we set

$$U_{lk} := \frac{1}{2}(v_{lj} \triangle v_{kj} + v_{kj} \triangle v_{lj}). \tag{4.5}$$

By (2.4) we know that $U_{lk} \in V_{lk}$.

LEMMA 4.9. $\|U_{lk}\|^2 \leq (2s_k)^{-1} \|v_{lj}\|^2 \|v_{kj}\|^2$.

PROOF. Since $v_{lj} \triangle v_{kj} \in V_{lk}$ by (2.4), we get by (2.1) and (2.4)

$$[L_{(v_{lj} \triangle v_{kj})}, L_{v_{lj}}] = L_{[(v_{lj} \triangle v_{kj}) \triangle v_{lj}]} = 0.$$

Hence it follows from (2.7) that

$$\|v_{lj} \triangle v_{kj}\|^2 = \left\langle L_{(v_{lj} \triangle v_{kj})}(v_{lj} \triangle v_{kj}), E_{\mathbf{s}}^* \right\rangle = \left\langle L_{(v_{lj} \triangle v_{kj})} L_{v_{lj}} v_{kj}, E_{\mathbf{s}}^* \right\rangle = \left\langle L_{v_{lj}} L_{(v_{lj} \triangle v_{kj})} v_{kj}, E_{\mathbf{s}}^* \right\rangle.$$

Since $L_{v_{lj}} L_{(v_{lj} \triangle v_{kj})} v_{kj} = v_{lj} \triangle ((v_{lj} \triangle v_{kj}) \triangle v_{kj})$, we have by (2.7)

$$\begin{aligned} \left\langle L_{v_{lj}} L_{v_{lj} \triangle v_{kj}} v_{kj}, E_{\mathbf{s}}^* \right\rangle &= \langle v_{lj} | (v_{lj} \triangle v_{kj}) \triangle v_{kj} \rangle \leq \|v_{lj}\| \|(v_{lj} \triangle v_{kj}) \triangle v_{kj}\| \\ &= (2s_k)^{-1/2} \|v_{lj}\| \|v_{lj} \triangle v_{kj}\| \|v_{kj}\|, \end{aligned}$$

where the last equality follows from Lemma 4.7. Thus we get

$$\|v_{lj} \triangle v_{kj}\| \leq (2s_k)^{-1/2} \|v_{lj}\| \|v_{kj}\|. \tag{4.6}$$

On the other hand, since $v_{kj} \triangle v_{lj} \in V_{lk}$, it follows from (2.4) that

$$[L_{v_{kj}}, L_{v_{lj}}](v_{kj} \triangle v_{lj}) = 0,$$

so that we have by (2.7) and (2.1)

$$\|v_{kj} \triangle v_{lj}\|^2 = \left\langle L_{(v_{kj} \triangle v_{lj})}(v_{kj} \triangle v_{lj}), E_{\mathbf{s}}^* \right\rangle = \left\langle L_{(v_{lj} \triangle v_{kj})}(v_{kj} \triangle v_{lj}), E_{\mathbf{s}}^* \right\rangle. \tag{4.7}$$

By (2.7), the last term is equal to

$$\langle v_{lj} \triangle v_{kj} | v_{kj} \triangle v_{lj} \rangle = \left\langle L_{(v_{kj} \triangle v_{lj})}(v_{lj} \triangle v_{kj}), E_{\mathbf{s}}^* \right\rangle.$$

Discussing as in (4.7), we see that this is equal to $\|v_{lj} \triangle v_{kj}\|^2$, so that we obtain $\|v_{kj} \triangle v_{lj}\| = \|v_{lj} \triangle v_{kj}\|$. Then we see that $\|U_{lk}\| \leq \|v_{lj} \triangle v_{kj}\|$. Now (4.6) completes the proof. □

5. Proof of (C) ⇒ (A) in the main theorem.

We are now able to begin the proof of our main theorem (Theorem 1.2). We first need to quote two lemmas for the proof of (C) ⇒ (A).

5.1. Some facts about symmetric cones.

Let V_1 be a real Euclidean vector space with an inner product $\langle\cdot|\cdot\rangle$ and $\Omega_1 \subset V_1$ a self-dual cone with respect to this inner product. The characteristic function φ_1 of Ω_1 is defined by

$$\varphi_1(x) := \int_{\Omega_1} e^{-\langle x|y\rangle}\, dy \qquad (x \in \Omega_1). \tag{5.1}$$

Let us define Vinberg's $*$-map $\Omega_1 \to V_1$ by

$$\langle x^*|y\rangle = -D_y \log \varphi_1(x) \qquad (x \in \Omega_1, y \in V_1).$$

It is known that the $*$-map has a unique fixed point e_1 ([**5**, Proposition I.3.5]). Since Ω_1 is a symmetric cone, V_1 has a Jordan algebra structure with unit element e_1. In this case, we have the following lemma ([**5**, Chapter 3, Exercise 5]):

LEMMA 5.1. *Let $L'(v)$ be the multiplication by v in the Jordan algebra V_1. Then*

$$\operatorname{Tr} L'(uv) = D_u D_v \log \varphi_1(e_1) = \langle u|v\rangle.$$

Therefore, $\langle\cdot|\cdot\rangle$ coincides with $\langle\cdot|\cdot\rangle_{\mathrm{Tr}} : (u,v) \mapsto \operatorname{Tr} L'(uv)$. We note here that even if we replace the inner product $\langle\cdot|\cdot\rangle$ by its positive number multiple in Definition (5.1) of φ_1, $D_y \log \varphi_1(x)$ is the same.

Suppose now that Ω_1 is irreducible. Then V_1 is simple. We know by Proposition III.4.2 of [**5**] that $\operatorname{Tr} L'(x) = (n_1/r_1)\operatorname{tr}(x)$, where $\operatorname{tr}(x)$ is the trace of x in the Jordan algebra V_1, and r_1 and n_1 are the rank and the dimension of V_1 respectively.

LEMMA 5.2. $x^* = x^{-1}$ *for every invertible* $x \in V_1$.

PROOF. Denoting by x^{tr} the $*$-map used in [**5**, Proposition III.4.3], we have $x^{\mathrm{tr}} = (n_1/r_1)x^{-1}$. On the other hand, the discussion done just before the present lemma gives $(n_1/r_1)x^* = x^{\mathrm{tr}}$. Now the lemma follows. $\square$

5.2. Proof of (C) $\Rightarrow$ (A).

Now we assume that (C) in Theorem 1.2 holds. Proceeding as in Subsection 5.1 with V, Ω and $\langle\cdot|\cdot\rangle_{\boldsymbol{s}}$, we see that V has a Jordan algebra structure and we have a $*$-map $\Omega \to V$. We shall show that $I_{\boldsymbol{s}}$ is a positive number multiple of the $*$-map in this situation. By assumption, we have $\boldsymbol{s} = p\boldsymbol{d}$ $(p > 0)$, so that $\Delta_{-\boldsymbol{s}}(x) = \Delta_{-\boldsymbol{d}}(x)^p$ for every $x \in \Omega$. On the other hand it is easy to see that $\operatorname{Det} h = \chi_{\boldsymbol{d}}(h)$ $(h \in H)$. Let φ be the characteristic function of Ω. Since $\varphi(hE) = (\operatorname{Det} h)^{-1}\varphi(E)$ ([**5**, Proposition I.3.1]), it holds that $\varphi(x) = \Delta_{-\boldsymbol{d}}(x)\varphi(E)$ $(x \in \Omega)$. Thus, for every $x \in \Omega$ and $y \in V$ one has

$$\langle I_{\boldsymbol{s}}(x)|y\rangle_{\boldsymbol{s}} = -D_y \log \Delta_{-\boldsymbol{s}}(x) = -pD_y \log \varphi(x) = \langle px^*|y\rangle_{\boldsymbol{s}}.$$

Hence we get $I_{\boldsymbol{s}}(x) = px^*$. From Lemma 5.2 it follows that $I_{\boldsymbol{s}}(x) = px^{-1}$. Since the inverse map $w \mapsto w^{-1}$ in the complexified Jordan algebra $W = V_{\boldsymbol{C}}$ is an involutive holomorphic automorphism of $\Omega + iV$ by [**5**, Theorem X.1.1], we obtain $I_{\boldsymbol{s}}(z) = pz^{-1}$ for all $z \in \Omega + iV$, and (A) of Theorem 1.2 follows.

6. Equivalence of (B) and (C).

The implication (C) $\Rightarrow$ (B) is trivial. In [**15**, Chapter III, §6] the dual cone of an irreducible symmetric cone Ω is realized in V by means of the trace inner product of the corresponding clan and we see easily from [**16**, Chapter II, §2] that it coincides with Ω.

In this section, we give a proof of equivalence of (B) and (C) that is valid for homogeneous convex cones which are not necessarily irreducible. Let us assume that Ω is self-dual with respect to an inner product $\langle\cdot|\cdot\rangle_0$ of V.

Let φ_0 be the characteristic function of Ω, and E_0 the unique fixed point of the $*$-map. Discussing as in Subsection 5.1, V has a Jordan algebra structure with unit element E_0. One has by Lemma 5.1

$$D_x D_y \log \varphi_0(E_0) = \langle x|y\rangle_0. \tag{6.1}$$

In §2 we took E as the base point in the construction of the clan V. We shall denote this clan by (V,E). Now, taking E_0 as the base point, we obtain a new clan (V,E_0). It follows from [**15**, Chapter II, §1] that there exists an algebra isomorphism $\Phi : (V,E) \to (V,E_0)$ such that $\Phi(\Omega) = \Omega$. Let $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$ be the trace inner product of the clan (V,E_0). We have by [**15**, Chapter II, §1]

$$D_x D_y \log \varphi_0(E_0) = \langle x|y\rangle_{\mathrm{tr}}. \tag{6.2}$$

Hence we get from (6.1) and (6.2) that $\langle\cdot|\cdot\rangle_0$ coincides with $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$. Therefore Ω is self-dual with respect to $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$, too.

LEMMA 6.1. *An algebra isomorphism between two clans is a unitary map when both clans are equipped with their respective trace inner products.*

PROOF. Let V, V' be two clans, and $\Psi : V \to V'$ an algebra isomorphism. We denote the multiplications of V, V' by $\triangle$, $\triangle'$, the left-multiplication operators by L, L', and the trace inner products by $\langle\cdot|\cdot\rangle_1$, $\langle\cdot|\cdot\rangle_2$ respectively. Since Ψ is an algebra isomorphism, we see easily that $L'_{\Psi(x)} = \Psi L_x \Psi^{-1}$. Therefore, $\mathrm{Tr}\, L_x = \mathrm{Tr}\, L'_{\Psi(x)}$, so that we get

$$\langle \Psi(x)|\Psi(y)\rangle_2 = \mathrm{Tr}\, L'_{(\Psi(x)\triangle'\Psi(y))} = \mathrm{Tr}\, L'_{\Psi(x\triangle y)} = \mathrm{Tr}\, L_{x\triangle y} = \langle x|y\rangle_1.$$

Hence the proof is complete. □

Let $\Omega^{\boldsymbol{d}}$, Ω^{tr} be the dual cones of Ω realized in V by means of the trace inner products of (V,E), (V,E_0) respectively. Since Ω is self-dual with respect to $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$, we get

$$\begin{aligned}\Omega &= \Phi^{-1}(\Omega) = \left\{\Phi^{-1}(x); \langle x|y\rangle_{\mathrm{tr}} > 0 \text{ for } \forall y \in \overline{\Omega}\setminus\{0\}\right\}\\ &= \left\{\Phi^{-1}(x); \langle \Phi^{-1}(x)|\Phi^{-1}(y)\rangle_{\boldsymbol{d}} > 0 \text{ for } \forall \Phi^{-1}(y) \in \overline{\Omega}\setminus\{0\}\right\}\\ &= \Omega^{\boldsymbol{d}}.\end{aligned}$$

Therefore Ω is also self-dual with respect to the trace inner product of (V,E). This completes the proof of (B) $\Rightarrow$ (C).

7. Proof of (A) ⇒ (B).

We assume that (A) of Theorem 1.2 holds. In particular, we have

$$\mathrm{Re}\, I_{\boldsymbol{s}}(E+iV) \subset \Omega^{\boldsymbol{s}}, \qquad \mathrm{Re}\, I_{\boldsymbol{s}}^{*}(E+iV) \subset \Omega. \tag{7.1}$$

Since $\sum e^{t_j} E_j \in \Omega$ and $\sum e^{t_j} E_j \in \Omega^{\boldsymbol{s}}$ for all $t_j \in \boldsymbol{R}$, it follows that

$$E_m \in \overline{\Omega} \cap \overline{\Omega^{\boldsymbol{s}}} \qquad (m = 1, \dots, r). \tag{7.2}$$

We assume that the integers j, k, l satisfy $1 \le j < k < l \le r$ throughout this section.

7.1. First step.

LEMMA 7.1. *If $n_{kj} \neq 0$, then one has $s_j \geq s_k$.*

PROOF. Take any $v_{kj} \in V_{kj}$. In Proposition 4.2, we put

$$t_j = t_l = 0, \quad t_k = \log\left(1 + (2s_k)^{-1} \|v_{kj}\|^2\right),$$

$$w_{lj} = w_{lk} = 0, \quad w_{kj} = i v_{kj},$$

and $\eta := \exp L_{iv_{kj}} \exp(t_k H_k)$. Then the formula becomes $\eta E = E + i v_{kj}$. By Proposition 4.6 we obtain

$${}^{\boldsymbol{s}}\eta^{-1} E = \sum_{m \neq j,k,l} E_m + \left(1 - (2s_j)^{-1} e^{-t_k} \|v_{kj}\|^2\right) E_j + e^{-t_k} E_k + E_l - i e^{-t_k} v_{kj}.$$

Since $I_{\boldsymbol{s}}(E + i v_{kj}) = I_{\boldsymbol{s}}(\eta E) = {}^{\boldsymbol{s}}\eta^{-1} I_{\boldsymbol{s}}(E) = {}^{\boldsymbol{s}}\eta^{-1} E$, we get

$$\mathrm{Re}\, I_{\boldsymbol{s}}(E + i v_{kj}) = \sum_{m \neq j,k,l} E_m + \left(1 - (2s_j)^{-1} e^{-t_k} \|v_{kj}\|^2\right) E_j + e^{-t_k} E_k + E_l.$$

Since we have (7.1), the coefficients of E_m are all positive by (7.2). Hence we obtain $1 - (2s_j)^{-1} e^{-t_k} \|v_{kj}\|^2 > 0$, that is,

$$2s_j > \left(1 + (2s_k)^{-1} \|v_{kj}\|^2\right)^{-1} \|v_{kj}\|^2.$$

Limiting procedure $\|v_{kj}\| \to \infty$ yields $s_j \geq s_k$. □

LEMMA 7.2. *If $n_{kj} \neq 0$, then one has $s_k \geq s_j$.*

PROOF. Take any $v_{kj} \in V_{kj}$. In Proposition 4.6 we put

$$t_j = -\log\left(1 + (2s_j)^{-1} \|v_{kj}\|^2\right), \quad t_k = t_l = 0,$$

$$w_{lj} = w_{lk} = 0, \quad w_{kj} = -i v_{kj},$$

and $\eta^* := \exp L_{(-iv_{kj})} \exp(t_j H_j)$. Then the formula becomes ${}^{\boldsymbol{s}}(\eta^*)^{-1} E = E + i v_{kj}$. By Proposition 4.2 we have

$$\eta^* E = \sum_{m \neq j,k,l} E_m + e^{t_j} E_j + \left(1 - (2s_k)^{-1} e^{t_j} \|v_{kj}\|^2\right) E_k + E_l - i e^{t_j} v_{kj}.$$

Since $I_{\mathbf{s}}^*(E+iv_{kj}) = I_{\mathbf{s}}^*({}^{\mathbf{s}}(\eta^*)^{-1}E) = \eta^* I_{\mathbf{s}}^*(E) = \eta^* E$, it holds that

$$\operatorname{Re} I_{\mathbf{s}}^*(E+iv_{kj}) = \sum_{m\neq j,k,l} E_m + e^{t_j}E_j + \left(1-(2s_k)^{-1}e^{t_j}\|v_{kj}\|^2\right)E_k + E_l. \tag{7.3}$$

The assumption (7.1) together with (7.2) shows that the coefficients of E_m in (7.3) are positive for all m. Hence we get $1-(2s_k)^{-1}e^{t_j}\|v_{kj}\|^2 > 0$, that is,

$$2s_k > \left(1+(2s_j)^{-1}\|v_{kj}\|^2\right)^{-1}\|v_{kj}\|^2.$$

Taking the limit as $\|v_{kj}\| \to \infty$, we arrive at $s_k \geq s_j$. □

Lemmas 7.1 and 7.2 give

PROPOSITION 7.3. *If $n_{kj} \neq 0$, then one has $s_k = s_j$.*

Now, Asano's theorem [**1**, Theorem 4] tells us that Ω is irreducible if and only if for each pair (j,k) with $1 \leq j < k \leq r$, there exists a series $j_0,\dots,j_m$ of distinct positive integers such that $j_0 = k$, $j_m = j$ and $n_{j_{\lambda-1}j_\lambda} \neq 0$ for any $\lambda = 1,\dots,m$, where if $j_{\lambda-1} < j_\lambda$, then one puts $n_{j_{\lambda-1}j_\lambda} := n_{j_\lambda j_{\lambda-1}}$. Therefore we arrive at

PROPOSITION 7.4. *The numbers s_m for $m = 1,\dots,r$ are independent of m.*

7.2. Second step.

We next show that if $n_{lk} \neq 0$, then $n_{lj} = n_{kj}$. Before starting, we present three lemmas which hold in general.

LEMMA 7.5. *Let $v_{kj} \in V_{kj}$. Then the following two statements are equivalent*:

(i) $\sum a_m E_m + v_{kj} \in \Omega$,
(ii) $a_m > 0$ $(m = 1,\dots,r)$ *and* $a_j a_k - (2s_k)^{-1}\|v_{kj}\|^2 > 0$.

PROOF. We assume that (i) holds. It follows from (7.2) that $a_m > 0$ for $m = 1,\dots,r$. Put $w_{kj} := -a_j^{-1}v_{kj} \in V$ and $z := \left(\exp L_{w_{kj}}\right)\left(\sum_m a_m E_m + v_{kj}\right)$. Lemma 4.1 and (4.2) give

$$\begin{aligned} z &= \sum_m a_m E_m + v_{kj} + a_j w_{kj} + w_{kj} \triangle v_{kj} + \frac{1}{2}(a_j s_k)^{-1}\|v_{kj}\|^2 E_k \\ &= \sum_{m\neq j,k} a_m E_m + a_j E_j + \left(a_k - (2a_j s_k)^{-1}\|v_{kj}\|^2\right)E_k. \end{aligned} \tag{7.4}$$

Now the assumption implies $z \in \Omega$, so that (7.2) gives $a_j a_k - (2s_k)^{-1}\|v_{kj}\|^2 > 0$.

Conversely we assume that (ii) holds. Then (7.4) tells us that $z \in \Omega$, so that

$$\sum a_m E_m + v_{kj} = \left(\exp L_{(-w_{kj})}\right)z \in \Omega,$$

whence the proof is complete. □

Discussing as in the proof of Lemma 7.5, we get

LEMMA 7.6. *Let $v_{kj} \in V_{kj}$. Then the following two statements are equivalent*:

(i) $\sum_m a_m E_m + v_{kj} \in \Omega^{\boldsymbol{s}}$,
(ii) $a_m > 0$ $(m = 1, \ldots, r)$ *and* $a_j a_k - (2s_j)^{-1} \|v_{kj}\|^2 > 0$.

LEMMA 7.7. *Let $v_{lk} \in V_{lk}$ and $v_{lj} \in V_{lj}$. Then one has ${}^{\boldsymbol{s}}L_{v_{lk}} v_{lj} = {}^{\boldsymbol{s}}L_{v_{lj}} v_{lk}$.*

PROOF. Note that since v_{lk}, v_{lj} remain in V, both ${}^{\boldsymbol{s}}L_{v_{lk}} v_{lj}$ and ${}^{\boldsymbol{s}}L_{v_{lj}} v_{lk}$ are in V_{kj} by Lemma 4.4. Take any $x \in V_{kj}$. We obtain by (2.7) and (2.1)

$$\begin{aligned}\left\langle {}^{\boldsymbol{s}}L_{v_{lk}} v_{lj} | x \right\rangle &= \langle v_{lj} | v_{lk} \triangle x \rangle = \left\langle L_{v_{lj}} L_{v_{lk}} x, E_{\boldsymbol{s}}^* \right\rangle \\ &= \left\langle (L_{v_{lk}} L_{v_{lj}} + L_{[v_{lj} \triangle v_{lk}]}) x, E_{\boldsymbol{s}}^* \right\rangle.\end{aligned}$$

Since $v_{lj} \triangle v_{lk} = v_{lk} \triangle v_{lj} = 0$ by (2.4), the last term is equal to

$$\left\langle L_{v_{lk}} L_{v_{lj}} x, E_{\boldsymbol{s}}^* \right\rangle = \langle v_{lk} | v_{lj} \triangle x \rangle = \left\langle {}^{\boldsymbol{s}}L_{v_{lj}} v_{lk} | x \right\rangle.$$

Therefore we obtain ${}^{\boldsymbol{s}}L_{v_{lk}} v_{lj} = {}^{\boldsymbol{s}}L_{v_{lj}} v_{lk}$. □

Let us return to the proof of our main theorem. In view of Proposition 7.4 we put $s = s_m$, independent of m, from now on.

LEMMA 7.8. *If $n_{lk} \neq 0$, then one has $n_{kj} \geq n_{lj}$.*

PROOF. If $n_{lj} = 0$, then the conclusion of the lemma is trivially true. Thus we assume $n_{lj} \neq 0$ as well as $n_{lk} \neq 0$. Take any $v_{lk} \in V_{lk}$ and $v_{lj} \in V_{lj}$. In Proposition 4.6 we put

$$\begin{aligned} w_{lk} := -i v_{lk}, \quad w_{lj} := -i v_{lj}, \quad w_{kj} := -{}^{\boldsymbol{s}}L_{v_{lj}} v_{lk}, \\ t_j := -\log\left(1 + (2s)^{-1} \|w_{kj}\|^2 + (2s)^{-1} \|v_{lj}\|^2\right), \\ t_k := -\log\left(1 + (2s)^{-1} \|v_{lk}\|^2\right), \quad t_l = 0. \end{aligned} \tag{7.5}$$

It should be noted here that $w_{kj} \in V_{kj}$ just as in the proof of Lemma 7.7. Let us see what the right-hand side of the formula in Proposition 4.6 looks like. By definition we get

$$\langle w_{lj} \triangle w_{kj} | w_{lk} \rangle = -\langle v_{lj} \triangle w_{kj} | v_{lk} \rangle = \|w_{kj}\|^2. \tag{7.6}$$

Since $L_{v_{lk}} v_{lj} = 0$ and $L_{(w_{kj} \Delta v_{lk})} = 0$ by (2.4), we have $v_{lk} \triangle (w_{kj} \triangle v_{lj}) = (v_{lk} \triangle w_{kj}) \triangle v_{lj}$ by (2.1). This gives

$$\begin{aligned}\langle w_{kj} \triangle w_{lj} | w_{lk} \rangle &= -\langle w_{kj} \triangle v_{lj} | v_{lk} \rangle = -\left\langle v_{lk} \triangle (w_{kj} \triangle v_{lj}), E_{\boldsymbol{s}}^* \right\rangle \\ &= -\left\langle (v_{lk} \triangle w_{kj}) \triangle v_{lj}, E_{\boldsymbol{s}}^* \right\rangle = -\langle v_{lk} \triangle w_{kj} | v_{lj} \rangle \\ &= -\left\langle w_{kj} | {}^{\boldsymbol{s}}L_{v_{lk}} v_{lj} \right\rangle.\end{aligned}$$

Lemma 7.7 shows that the last term equals $-\left\langle w_{kj} | {}^{\boldsymbol{s}}L_{v_{lj}} v_{lk} \right\rangle$, so that we obtain

$$\langle w_{kj} \triangle w_{lj} | w_{lk} \rangle = \|w_{kj}\|^2. \tag{7.7}$$

Let S_{lk} be as in (4.4). Then it follows from (7.6) and (7.7) that

$$\langle S_{lk}|w_{lk}\rangle = \|w_{kj}\|^2.$$

Let us put

$$\eta^* := \exp\bigl(L_{w_{lj}} + L_{w_{kj}}\bigr)\exp\bigl(L_{w_{lk}}\bigr)\exp(t_jH_j + t_kH_k).$$

Then we see without difficulty that the formula in Proposition 4.6 becomes

$$^{\boldsymbol{s}}(\eta^*)^{-1}E = E + i\bigl(v_{lk} - {}^{\boldsymbol{s}}L_{w_{kj}}v_{lk} + v_{lj}\bigr).$$

Now we have

$$I_{\boldsymbol{s}}^*\bigl(E + i\bigl(v_{lk} - {}^{\boldsymbol{s}}L_{w_{kj}}v_{lk} + v_{lj}\bigr)\bigr) = I_{\boldsymbol{s}}^*\bigl({}^{\boldsymbol{s}}(\eta^*)^{-1}E\bigr) = \eta^* I_{\boldsymbol{s}}^*(E) = \eta^* E,$$

and Proposition 4.2 gives

$$\begin{aligned}\eta^*E = \sum_{m\neq j,k,l} & E_m + e^{t_j}E_j + \bigl((2s)^{-1}e^{t_j}\|w_{kj}\|^2 + e^{t_k}\bigr)E_k \\ & + \bigl(1 - (2s)^{-1}e^{t_j}\|v_{lj}\|^2 - (2s)^{-1}e^{t_k}\|v_{lk}\|^2\bigr)E_l + e^{t_j}w_{kj} \\ & - i\bigl(e^{t_j}v_{lj} + e^{t_k}v_{lk} + 2^{-1}e^{t_j}(v_{lj}\triangle w_{kj} + w_{kj}\triangle v_{lj})\bigr).\end{aligned}$$

By (7.1), the real part of this belongs to Ω. Hence by Lemma 7.5

$$1 - (2s)^{-1}e^{t_j}\|v_{lj}\|^2 - (2s)^{-1}e^{t_k}\|v_{lk}\|^2 > 0.$$

Rewriting this by using (7.5), we arrive at

$$(2s)^{-1}\|v_{lk}\|^2\|v_{lj}\|^2 - 2s < \|w_{kj}\|^2. \tag{7.8}$$

We observe here that (7.8) forces $n_{kj} \neq 0$, because we are assuming $n_{lj} \neq 0$ and $n_{lk} \neq 0$ and note that v_{lk} and v_{lj} are arbitrary. Let $\{e_m\}_{m=1}^{n_{kj}}$ be an orthonormal basis of V_{kj}. Since Lemma 7.7 yields

$$\begin{aligned}\|w_{kj}\|^2 &= \sum_{m=1}^{n_{kj}}\langle w_{kj}|e_m\rangle^2 = \sum_{m=1}^{n_{kj}}\bigl\langle {}^{\boldsymbol{s}}L_{v_{lj}}v_{lk}|e_m\bigr\rangle^2 \\ &= \sum_{m=1}^{n_{kj}}\bigl\langle {}^{\boldsymbol{s}}L_{v_{lk}}v_{lj}|e_m\bigr\rangle^2 = \sum_{m=1}^{n_{kj}}\langle v_{lj}|v_{lk}\triangle e_m\rangle^2,\end{aligned}$$

(7.8) is equivalent to the inequality

$$(2s)^{-1}\|v_{lk}\|^2\|v_{lj}\|^2 - 2s < \sum_{m=1}^{n_{kj}}\langle v_{lj}|v_{lk}\triangle e_m\rangle^2. \tag{7.9}$$

We make v_{lj} run over an orthonormal basis of V_{lj} in (7.9) and sum up the resulting formulas. We get

$$n_{lj}\bigl((2s)^{-1}\|v_{lk}\|^2 - 2s\bigr) < \sum_{m=1}^{n_{kj}}\|v_{lk}\triangle e_m\|^2.$$

Here we have $\|v_{lk}\triangle e_m\|^2 = (2s)^{-1}\|v_{lk}\|^2$ by Lemma 4.7, so that we obtain

$$\|v_{lk}\|^{-2}\left(\|v_{lk}\|^2-(2s)^2\right)n_{lj} < n_{kj}.$$

Taking the limit as $\|v_{lk}\|\to\infty$, we obtain $n_{lj}\le n_{kj}$. □

Lemma 7.8 together with the statement (2) of Lemma 4.8 yields

PROPOSITION 7.9. *If $n_{lk}\neq 0$, then one has $n_{lj}=n_{kj}$.*

7.3. Third step.

We show that if $n_{kj}\neq 0$, then $n_{lk}=n_{lj}$. Let U_{lk} be as in (4.5). Under (7.1) the norm of U_{lk} can be calculated.

LEMMA 7.10. $\|U_{lk}\|^2 = (2s)^{-1}\|v_{lj}\|^2\|v_{kj}\|^2$.

PROOF. In view of Lemma 4.9, it suffices to show

$$\|U_{lk}\|^2 \geq (2s)^{-1}\|v_{lj}\|^2\|v_{kj}\|^2.$$

This inequality is trivial if $n_{lj}=0$ or $n_{kj}=0$. Therefore we assume that $n_{lj}\neq 0$ and $n_{kj}\neq 0$. In Proposition 4.2 we put

$$t_j := 0, \quad t_k := \log\left(1+(2s)^{-1}\|v_{kj}\|^2\right),$$

$$t_l := \log\left(1+(2s)^{-1}\|v_{lj}\|^2-(2s+\|v_{kj}\|^2)^{-1}\|U_{lk}\|^2\right),$$

$$w_{lj} := iv_{lj}, \quad w_{kj} := iv_{kj}, \quad w_{lk} := e^{-t_k}U_{lk},$$

where we note that Lemma 4.9 guarantees that t_l is actually a real number as is easily seen. Put

$$\eta := \exp\left(L_{w_{lj}}+L_{w_{kj}}\right)\exp\left(L_{w_{lk}}\right)\exp(t_kH_k+t_lH_l).$$

Then we see that the formula in Proposition 4.2 is the following:

$$\eta E = E + i(v_{lj}+v_{kj}).$$

We have $I_{\mathbf{s}}\left(E+i(v_{lj}+v_{kj})\right) = I_{\mathbf{s}}(\eta E) = {}^{\mathbf{s}}\eta^{-1}I_{\mathbf{s}}(E) = {}^{\mathbf{s}}\eta^{-1}E$ as before, and Proposition 4.6 gives

$$\begin{aligned}
{}^{\mathbf{s}}\eta^{-1}E = \sum_{m\neq j,k,l} E_m &+ \Big(1-(2s)^{-1}(e^{-t_k}+(2s)^{-1}e^{-t_l}\|w_{lk}\|^2)\|v_{kj}\|^2 \\
&\qquad -(2s)^{-1}e^{-t_l}\|v_{lj}\|^2+s^{-1}e^{-t_l}\langle U_{lk}|w_{lk}\rangle\Big)E_j \\
&+(e^{-t_k}+(2s)^{-1}e^{-t_l}\|w_{lk}\|^2)E_k+e^{-t_l}E_l-e^{-t_l}w_{lk} \\
&+i\Big(-e^{-t_k}v_{kj}+e^{-t_l}\left({}^{\mathbf{s}}L_{v_{lj}}w_{lk}-(2s)^{-1}\|w_{lk}\|^2v_{kj}+{}^{\mathbf{s}}L_{v_{kj}}w_{lk}-v_{lj}\right)\Big).
\end{aligned}$$

Since the real part of this belongs to $\Omega^{\mathbf{s}}$, it follows from Lemma 7.6 that the coefficient of E_j is positive. We put $\alpha := \|U_{lk}\|^2$, $\beta := \|v_{lj}\|^2$ and $\gamma := \|v_{kj}\|^2$ for simplicity. Then we have after some simplification

$$(\text{the coefficient of } E_j) \times e^{t_l} e^{2t_k} = (2s)^{-2}\big((2s+\beta)(2s+\gamma) - 2s\alpha\big) - (2s)^{-3}\beta(2s+\gamma)^2$$
$$+(2s^2)^{-1}(2s+\gamma)\alpha - (2s)^{-2}\alpha\gamma. \tag{7.10}$$

Let $x > 0$ be arbitrary, and replace v_{lj} and v_{kj} with xv_{lj} and xv_{kj} respectively in (7.10), so that α, β and γ are replaced by αx^4, βx^2 and γx^2 respectively. Let us denote by $F(x)$ the right-hand side of (7.10). We see that $F(x)$ is a polynomial of degree 6 and

$$(\text{the coefficient of } x^6 \text{ in } F(x)) = (2s)^{-3}\gamma(-\beta\gamma + 2s\alpha). \tag{7.11}$$

Since $F(x) > 0$ for every $x \geq 0$, it is necessary for the right-hand side of (7.11) to be non-negative. Hence it follows that $2s\alpha \geq \beta\gamma$. This completes the proof. □

PROPOSITION 7.11. *If $n_{kj} \neq 0$, then one has $n_{lk} = n_{lj}$.*

PROOF. If $n_{kj} \neq 0$, then we choose $v_{kj} \neq 0$, so that the linear map $v_{lj} \mapsto U_{lk}$ from V_{lj} to V_{lk} is injective by virtue of Lemma 7.10. Thus $n_{lk} \geq n_{lj}$. The reverse inequality follows from (1) of Lemma 4.8. □

7.4. Last step.

The concluding step is parallel to that of [**10**, Subsection 5.5].

LEMMA 7.12. *If at least two of n_{lk}, n_{lj}, n_{kj} are non-zero, they are all equal.*

PROOF. In view of Propositions 7.9 and 7.11, the proof is completely similar to that of [**10**, Lemma 5.15]. □

PROPOSITION 7.13. *The numbers n_{kj} are independent of j, k.*

PROOF. We first show that $n_{k1} \neq 0$ for any k with $2 \leq k \leq r$. By Asano's theorem, there exists a series of distinct positive integers such that $j_0 = k$, $j_m = 1$, $n_{j_{\lambda-1}j_\lambda} \neq 0$. Since $n_{j_0j_1} \neq 0$ and $n_{j_1j_2} \neq 0$, we get by Lemma 7.12 that $n_{j_0j_1} = n_{j_1j_2} = n_{j_0j_2} \neq 0$. Then, since $n_{j_0j_2} \neq 0$ and $n_{j_2j_3} \neq 0$, we obtain $n_{j_0j_3} = n_{j_0j_2} = n_{j_2j_3} \neq 0$. Continuing this argument, we have $n_{j_0j_m} \neq 0$, that is, $n_{k1} \neq 0$.

Now, we see that n_{k1} are independent of k by Lemma 7.12. Take two integers j, k with $1 < j < k \leq r$. Since $n_{j1}, n_{k1} \neq 0$, Lemma 7.12 gives $n_{j1} = n_{k1} = n_{kj}$, whence the conclusion. □

Now the following proposition due to Vinberg completes the proof of (A) $\Rightarrow$ (B).

PROPOSITION 7.14 ([**16**, Proposition 3]). *The irreducible homogeneous convex cone Ω is self-dual if and only if the numbers n_{kj} are independent of j, k.*

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Chifune KAI
Department of Mathematics, Faculty of Science
Kyoto University
Sakyo-ku 606-8502, Kyoto
Japan
E-mail: kai@math.kyoto-u.ac.jp

Takaaki NOMURA
Department of Mathematics, Faculty of Science
Kyoto University
Sakyo-ku 606-8502, Kyoto
Japan
E-mail: nomura@math.kyoto-u.ac.jp