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# On Mordell-Weil lattices of bielliptic fibrations on rational surfaces

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**Abstract.** We study Mordell-Weil lattices for bielliptic fibrations on rational surfaces. We prove theorems on the structure and give an explicit construction of the fibration with maximal Mordell-Weil rank and moreover determine the structures of such lattices.

## 1. Introduction.

Let $X$ be a smooth rational surface defined over $\boldsymbol{C}$ and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal fibration of curves of genus $g \geq 1$ with a section, and let $K$ be the rational function field of $\boldsymbol{P}^1$. We denote by $J_F$ the Jacobian variety of a general fiber $F$ of $\varphi$. The Mordell-Weil group $J_F(K)$ of $K$-rational points on $J_F$ is finitely generated. Its rank $r$ is called the *Mordell-Weil rank*. In [**7**], [**8**] and [**9**], Shioda introduced and developed the theory of Mordell-Weil lattices for $J_F(K)$ (in a more general context). In his theory of Mordell-Weil lattices of the elliptic fibrations, Mordell-Weil lattices with maximal rank which are isometric to $E_8$ play a very important role as a frame lattice.

It is shown in [**6**] that $r \leq 4g+4$ for fibrations of genus $g \geq 2$, and that the fibration with maximal rank $r = 4g+4$ is of hyperelliptic type. In the case of non-hyperelliptic fibrations of genus $g \geq 3$, which are studied in [**5**], $r \leq 3g+6$. The fibration with maximal rank $r = 3g+6$ is either of plane quintic or of trigonal type (so Clifford index 1). Moreover the structure of the corresponding Mordell-Weil lattices are completely determined in these papers.

In this paper we deal with the case of bielliptic fibrations of genus $g \geq 6$ (i.e., when $F$ has a two-to-one map onto an elliptic curve, so Clifford index 2). We first prove the following theorem:

THEOREM 1.1 (cf. Theorem 3.4). *Let $X$ be a smooth rational surface and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal bielliptic fibration of genus $g \geq 6$ with a section. Then*

$$r = \operatorname{rank} J_F(K) \leq 2g+10.$$

*Moreover, the equality $r = 2g+10$ holds if and only if ${K_X}^2 = -2g-2$ and all fibers of $\varphi$ are irreducible.*

We put

$$n = n(g) = \begin{cases} 1 & \text{if } g \text{ is even,} \\ 0 & \text{if } g \text{ is odd,} \end{cases}$$

and let $\Sigma_n$ be the Hirzebruch surface of degree $n$. Let $B_{8,n}$ be a smooth hyperelliptic curve of genus $g-4$ on $\Sigma_n$ and $X_{16}$ the surface obtained as the finite double cover branched along $B_{8,n}$.

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For each $g \geq 6$ we take a general sub-pencil in the pull-back of the anti-canonical system on $\Sigma_n$. Then the blowing up the base points of the pencil whose general members are smooth bielliptic curves of genus $g$ gives us a fibration $\varphi : X \longrightarrow \boldsymbol{P}^1$. We can show that such fibrations have the maximal Mordell-Weil rank $r = 2g + 10$ and 16 disjoint $(-1)$-sections. We have another example. Let $X_{18}$ be a blow-up of $\boldsymbol{P}^2$ at seven points in general position and $K_{X_{18}}$ a canonical divisor. Then we have a bielliptic fibration $\varphi : X \longrightarrow \boldsymbol{P}^1$ of genus 7 whose Mordell-Weil rank is maximal, i.e., $r = 24$ by blowing up the base points of a general sub-pencil in the complete linear system $|-3K_{X_{18}}|$. We say that the fibration $\varphi$ obtained in this way a *fibration of type* $(16; g; n)$ *and* $(18; 7)$, respectively (cf. Proposition 4.9).

THEOREM 1.2 (cf. Theorems 4.3 and 5.1). *Let $X$ be a smooth rational surface and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal bielliptic fibration of genus $g \geq 6$ with a section. Assume that the Mordell-Weil rank is maximal, i.e., $r = 2g + 10$. Then $\varphi$ is a fibration of type $(16; g; n)$ or $(18; 7)$.*

Our final result on the structure of Mordell-Weil lattices with maximal rank $r = 2g + 10$ is stated in the following theorem.

THEOREM 1.3 (cf. Propositions 6.2 and 6.5). *The Mordell-Weil lattices of fibrations of type $(16; g; n)$ and type $(18; 7)$ are unique up to isometry. More precisely, in the case of type $(16; g; n)$ the lattice is the positive-definite odd unimodular lattice ${L_{16,g,n}}^-$ of rank $2g + 10$ whose Dynkin diagram is given by Figure 6.3 and Figure 6.4 in Proposition 6.5. In the case of type $(18; 7)$ the lattice is isometric to the lattice "$\zeta$" in Niemeyer's classification of positive-definite even unimodular lattices of rank 24 (cf.* [**2**, Chapter XVI, §1]).

Let us explain the ideas for the proofs. Theorem 1.1 is a consequence of a slope inequality for bielliptic fibrations (cf. [**1**]). By a refinement of the slope inequality under our situation, the equality $r = 2g + 10$ implies that we have a finite double cover from $X$ to a smooth rational minimal elliptic surface. Theorem 1.2 follows from the analysis of the finite double cover. More precisely, we determine the plane curve model of the branch divisor up to birational maps, which are composite of Cremona transformations (cf. Theorem 4.3), and give an explicit construction of the fibration of each type, all of whose fibers are irreducible, as the bidouble cover of the Hirzebruch surface (cf. Theorem 5.1). To prove Theorem 1.3 we respectively find natural birational morphisms from $X_{16}$ and $X_{18}$ to a Hirzebruch surface and $\boldsymbol{P}^2$ so that we have an explicit description of the Néron-Severi group $\mathrm{NS}(X)$. Then we can determine the structures of Mordell-Weil lattices with maximal rank by calculating intersection pairing of divisors on $X$.

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## 2. Mordell-Weil lattices.

Let $X$ be a smooth rational surface defined over $\boldsymbol{C}$ and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal fibration of curves of genus $g \geq 1$ with a section. We review basic notation and results on Mordell-Weil lattices according to Shioda ([**7**], [**8**] and [**9**]) in the situation we are interested in.

Let $F$ be a general fiber of $\varphi$ and $K = \boldsymbol{C}(\boldsymbol{P}^1)$ the rational function field. We denote by $J_F$ the Jacobian variety of $F$. The Mordell-Weil group of $\varphi$ is the group of $K$-rational points

$J_F(K)$. Then it is a finitely generated abelian group since $X$ is a rational surface. The rank $r$ of this group is called the *Mordell-Weil rank*. There is a natural correspondence between the set of $K$-rational points $F(K)$ and the set of sections of $\varphi$. For $P \in F(K)$ we denote by $(P)$ the section corresponding to $P$ which is regarded as a curve in $X$. In particular, $(O)$ corresponding to the origin of $J_F(K)$ is called the *zero section*. Shioda's main idea in [**7**], [**8**] and [**9**] is to view $J_F(K)$ as a Euclidean lattice with respect to a natural pairing induced by the intersection form on $\mathrm{H}^2(X)$.

We denote by $T$ the subgroup of $\mathrm{NS}(X)$ generated by $(O)$ and all irreducible components of fibers of $\varphi$. With respect to the intersection pairing the sublattice $T$ is called the *trivial sublattice*. The following fundamental result due to Shioda plays an important role in the whole theory.

THEOREM 2.1 (cf. [**7**], [**8**] and [**9**]). *There is a natural isomorphism of groups*

$$J_F(K) \simeq \mathrm{NS}(X)/T. \tag{2.1}$$

As a corollary to Theorem 2.1, we have the following formula:

$$r = \rho - 2 - \sum_{P \in \boldsymbol{P}^1} (v_P - 1), \tag{2.2}$$

where $\rho = \operatorname{rank} \mathrm{NS}(X)$ is the Picard number and $v_P$ denotes the number of irreducible components of the fiber over $P \in \boldsymbol{P}^1$. In particular, if all fibers of $\varphi$ are irreducible, then we have

$$r = \rho - 2.$$

Let $L = T^{\perp} \subset \mathrm{NS}(X)$ be the orthogonal complement of $T$ in $\mathrm{NS}(X)$. The lattice $L$ is called the *essential sublattice*. We define the lattice dual to $L$ by the formula

$$L^* = \{x \in L \otimes \boldsymbol{Q} | x.y \in \boldsymbol{Z} \text{ for all } y \in L\},$$

where $x.y$ denotes the intersection pairing on $\mathrm{NS}(X)$.

Using (2.1), we define a symmetric bilinear form $\langle , \rangle$ on $J_F(K)$, which induces the structure of a positive-definite lattice on $J_F(K)/J_F(K)_{\mathrm{tor}}$. The lattice $(J_F(K)/J_F(K)_{\mathrm{tor}}, \langle , \rangle)$ is called the *Mordell-Weil lattice* of the fibration $\varphi : X \longrightarrow \boldsymbol{P}^1$. The *narrow Mordell-Weil lattice* $J_F(K)^0$ is a sublattice of the Mordell-Weil lattice $J_F(K)$ such that $J_F(K)^0 \simeq L/T \subset \mathrm{NS}(X)/T$ under the isomorphism (2.1).

THEOREM 2.2 (cf. [**7**], [**8**] and [**9**]). *There is the following commutative diagram in which the natural morphisms are isometries*:

$$\begin{array}{ccc} J_F(K)/J_F(K)_{\mathrm{tor}} & \simeq & (L^-)^* \\ \cup & & \cup \\ J_F(K)^0 & \simeq & L^-, \end{array}$$

*where the opposite lattice $L^-$ is obtained from $L$ by putting the minus sign on the intersection pairing on $L$.*

*In particular, if all fibers of $\varphi$ are irreducible, then*

$$J_F(K) \simeq J_F(K)^0 \simeq L^-$$

*is a unimodular lattice of rank $r = \rho - 2$.*

## 3. Bounds of Mordell-Weil rank.

In this section, we will give an upper bound of the Mordell-Weil rank for bielliptic fibrations of genus $g \geq 6$ on rational surfaces. The important result we need in this section is a slope inequality for bielliptic fibrations due to Barja [**1**].

Let $F$ be a smooth curve of genus $g$. The curve $F$ is called bielliptic if $F$ admits a two-to-one map onto a smooth elliptic curve. Let $\varphi : X \longrightarrow C$ be a fibration of genus $g$. We say that $\varphi$ is bielliptic if the general fiber $F$ of $\varphi$ is a bielliptic curve. The following result clarifies the structure of such fibrations.

PROPOSITION 3.1 (cf. [**1**, Proposition 1.1]). *Let $\varphi : X \longrightarrow C$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Then $X$ is a rational double cover of an elliptic surface over $C$.*

Let $\phi : S \longrightarrow C$ be the relatively minimal elliptic surface as in Proposition 3.1. We let $\sigma : \widetilde{X} \longrightarrow X$ be a minimal succession of blowing-ups which eliminates the indeterminacy of the rational double cover $X \dashrightarrow S$. Let $\widetilde{\varpi} : \widetilde{X} \longrightarrow S$ denote the resulting morphism of degree 2. Let $\varpi_0 \circ u : \widetilde{X} \longrightarrow X_0 \longrightarrow S$ be the Stein factorization of $\widetilde{\varpi}$, where $u$ is birational, $\varpi_0$ is finite and $X_0$ normal. Now consider the diagram as in Figure 3.1, where $\varpi_k : X_k \longrightarrow S_k$ is the canonical resolution of singularities of $\varpi_0 : X_0 \longrightarrow S_0$ and $\overline{\sigma} : X_k \longrightarrow X$ is the birational morphism to the relative minimal model $X$. Denote $\chi_\varphi = \deg \varphi_* \mathscr{O}_X(K_{X/C})$ where $K_{X/C}$ is a relative canonical divisor of $\varphi$.

$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "\sigma"'] \arrow[drrr, "u"] & & & \\
& \overline{X} = X_k \arrow[r, "\sigma_k"'] \arrow[dl, "\overline{\sigma}"'] \arrow[dd, "\varpi_k"] & \cdots \arrow[r, "\sigma_1"'] & X_0 \arrow[dd, "\varpi_0"] \\
X \arrow[dd, "\varphi"'] & & \cdots & \\
& \overline{S} = S_k \arrow[r] & \cdots \arrow[r] & S_0 = S \arrow[dlll, "\phi"] \\
C & & &
\end{tikzcd}$$

Figure 3.1.

THEOREM 3.2 (cf. [**1**, Theorem 2.1]). *Let $\varphi : X \longrightarrow C$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let $S$ be the relative minimal model of the elliptic fibration as above. Then*

$$K_{X/C}{}^2 - 4\chi_\varphi \geq 2(g-5)\chi(\mathscr{O}_S). \tag{3.1}$$

*$K_{X/C}{}^2 - 4\chi_\varphi = 2(g-5)\chi(\mathscr{O}_S)$ if and only if $X$ is the minimal desingularization of the double cover $X_0 \longrightarrow S$ whose branch divisor has at most* negligible *singularities* (*i.e., all the multiplicities of singular points of the branch divisor of $\varpi_i$ in Figure* 3.1 *are* 2 *or* 3).

If $X$ is a rational surface, then $S$ in Figure 3.1 is also rational. So we have the following:

LEMMA 3.3. *Let $X$ be a smooth rational surface and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal bielliptic fibration of genus $g \geq 6$. Then*

$$K_X{}^2 \geq -2g-2. \tag{3.2}$$

*If $K_X{}^2 = -2g-2$ and all fibers of $\varphi$ are irreducible, then there are the smooth rational minimal elliptic surface $\phi : S \longrightarrow \boldsymbol{P}^1$ and a finite double cover $\varpi : X \longrightarrow S$ such that $\varphi = \phi \circ \varpi$* (cf. Figure 3.2).

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\varpi"'] & & \boldsymbol{P}^1 \\
& S \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Figure 3.2.

PROOF. Since $X$ is a smooth rational surface, we have $\chi(\mathscr{O}_S) = 1$. By Leray spectral sequence,

$$\chi(\mathscr{O}_X) = \chi(\varphi_* \mathscr{O}_X) - \chi(R^1\varphi_* \mathscr{O}_X) = \chi(\mathscr{O}_{\boldsymbol{P}^1}) - \chi(R^1\varphi_* \mathscr{O}_X).$$

It follows from Grothendieck duality and Riemann-Roch theorem that $\chi(R^1\varphi_* \mathscr{O}_X) = g\chi(\mathscr{O}_{\boldsymbol{P}^1}) - \chi_\varphi$. Therefore we have $\chi_\varphi = g$. It follows that $K_{X/\boldsymbol{P}^1}{}^2 = K_X{}^2 - 8(g-1)(0-1) = K_X{}^2 + 8g - 8$. So (3.2) follows from Theorem 3.2.

Assume that $K_X{}^2 = -2g-2$. Then the equality sign holds in (3.1), hence $X$ satisfies the equivalent condition in Theorem 3.2. We use the notation as in Figure 3.1. We have $X = X_k$. If the branch divisor of $\varpi_0$ were singular, then we would have a $(-2)$-curve on $X$ which is a component of a fiber. Hence if all the fibers are irreducible, then $k = 0$. $\square$

THEOREM 3.4. *Let $X$ be a smooth rational surface, $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal bielliptic fibration of genus $g \geq 6$ with a section, and let $r$ be the Mordell-Weil rank of $\varphi$. Then*

$$r \leq 2g + 10.$$

*Moreover, $r = 2g+10$ if and only if $K_X{}^2 = -2g-2$ and all fibers of $\varphi$ are irreducible.*

PROOF. Since $X$ is a rational surface, the Picard number $\rho(X)$ is equal to the second Betti number $b_2(X)$. Moreover, $b_1(X) = 2q(X) = 0$ and $\chi(\mathscr{O}_X) = 1$. So from Noether's formula, we have $\rho(X) = 10 - K_X{}^2$. Hence (2.2) and (3.2) imply that $r \leq 2g+10$. The equality $r = 2g+10$ holds if and only if $K_X{}^2 = -2g-2$ and all fibers of $\varphi$ are irreducible. $\square$

## 4. Fibrations with maximal Mordell-Weil rank.

Let $X$ be a smooth rational surface and $\varphi : X \longrightarrow \boldsymbol{P}^1$ a relatively minimal bielliptic fibration of genus $g \geq 6$ with a section. In this section, we assume that the Mordell-Weil rank $r$ of $\varphi$ is maximal, i.e., $r = 2g+10$, and analyze the structure of $\varphi$. We use the following notation in the rest of the paper: Put

$$n = n(g) = \begin{cases} 1 & \text{if } g \text{ is even,} \\ 0 & \text{if } g \text{ is odd,} \end{cases}$$

and let $pr : \Sigma_n = \boldsymbol{P}(\mathscr{O}_{\boldsymbol{P}^1} \oplus \mathscr{O}_{\boldsymbol{P}^1}(n)) \longrightarrow \boldsymbol{P}^1$ be the Hirzebruch surface of degree $n$, $C_n$ the minimal section, and $f_n$ a fiber of $pr$.

Theorem 3.4 says that ${K_X}^2 = -2g-2$ and all fibers of $\varphi$ are irreducible. Moreover there are a rational elliptic surface $\phi : S \longrightarrow \boldsymbol{P}^1$ and a finite double cover $\varpi : X \longrightarrow S$ such that $\varphi = \phi \circ \varpi$ by Lemma 3.3.

LEMMA 4.1. *Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a bielliptic fibration of genus $g \geq 6$ whose Mordell-Weil rank is maximal, i.e., $r = 2g+10$. Let $\phi : S \longrightarrow \boldsymbol{P}^1$ denote the smooth rational minimal elliptic surface and $\varpi : X \longrightarrow S$ the finite double cover in Lemma* 3.3. *Then $\phi : S \longrightarrow \boldsymbol{P}^1$ has a section and satisfies the following*:

(a) *$S$ is obtained by blowing up nine points of $\boldsymbol{P}^2$.*
(b) *The elliptic fibration $\phi$ is the anti-canonical map of $S$ and it has no reducible fibers.*
(c) *A section of $\phi$ is a $(-1)$-curve on $S$, and vice versa.*

*Furthermore, $C^2 \geq -1$ for any smooth rational curves $C$ on $S$. Consequently, the nine points of $\boldsymbol{P}^2$ as in* (a) *are in general position, that is, no three of them are collinear and no six lie on a conic.*

PROOF. The direct image as a divisor of a section of $\varphi$ by $\varpi$ is a section of $\phi$ from the projection formula. If $\phi$ has a reducible fiber, then so does $\varphi$. So, $\phi$ cannot have reducible fibers. Then from [**4**] we have (a) and (b). Let $C$ be a smooth rational curve on $S$. Then we have $C^2 = -2 - C.K_S$ by the genus formula. Since $\phi$ has no reducible fiber, $C$ must be horizontal with respect to $\phi$. So $C$ intersects a fiber of $\phi$ which is the anti-canonical map by (b). It follows that $C.(-K_S) \geq 1$ and we have $C^2 \geq -1$. Note that we have $C^2 = -1$ if and only if $C.(-K_S) = 1$. This gives (c). The rest may be clear. $\square$

Let $B$ be the branch divisor of the finite double cover $\varpi : X \longrightarrow S$. Then $B$ is smooth and divisible by 2 in $\mathrm{Pic}(S)$. In our situation $\mathrm{Pic}(S)$ is torsion free since $S$ is a smooth rational surface. So there is a unique element $\delta \in \mathrm{Pic}(S)$ with $B \sim 2\delta$. From Lemma 4.1, we already know that $S$ can be obtained as a nine-points blow-up of $\boldsymbol{P}^2$. Hence we can transform the pair $(S, B)$ to $(\boldsymbol{P}^2, C)$. However since there are many choices of disjoint nine $(-1)$-curves on $S$ to obtain $\boldsymbol{P}^2$, we have various plane curve models of $B$. We want to choose among them the canonical one. For this purpose, we prove the following lemma needed later.

LEMMA 4.2. *Let $(S, B)$ be the pair as above. Then there exists a blow-down $\varepsilon : (S, B) \longrightarrow (\boldsymbol{P}^2, C)$ of disjoint nine $(-1)$-curves $e_1, \ldots, e_9$ such that*

$$\deg C \geq m_9 + m_8 + m_7, \tag{4.1}$$

$$m_9 \geq m_8 \geq m_7 \geq \cdots \geq m_1 \geq 0, \tag{4.2}$$

*where the $m_i$ $(i = 1, \ldots, 9)$ denotes the multiplicity of $C$ at $P_i = \varepsilon(e_i)$.*

PROOF. Let $e_1, \ldots, e_9$ be disjoint nine $(-1)$-curves on $S$, $\mu : S \longrightarrow \boldsymbol{P}^2$ the blow-down which contracts $e_i$'s and put $P_i = \mu(e_i) \in \boldsymbol{P}^2$. In particular, we have $P_i \neq P_j$ if $i \neq j$. Let $d$ be the degree of $\mu_* B$ and $m_i$ $(i = 1, \ldots, 9)$ the multiplicity of $\mu_* B$ at $P_i$. We can assume that (4.2) holds.

Being branch divisor of a finite double cover, $B$ does not contain a $(-1)$-curve. It follows that $\mu_* B$ does not contain the line $l_{i,j}$ through two points $P_i$, $P_j$ for any $i \neq j$. We get the plane curve of degree $2d - m_9 - m_8 - m_7$ after the Cremona transformation of $\mu_* B$ at $P_9$, $P_8$ and $P_7$. Then the composite of $\mu$ and the Cremona transformation gives us a new blow-down $\mu' : S \longrightarrow \boldsymbol{P}^2$ replacing the role of $e_9$, $e_8$ and $e_7$ by the strict transform of $l_{9,8}$, $l_{8,7}$ and $l_{7,9}$ by $\mu$.

Hence after a finite number of succession of such transformations, we get a blow-down $\varepsilon$ satisfying (4.1). □

A $m$-fold point $P$ of a plane curve $C$ is called a *simple singular point*, if the strict transform of $C$ by the blowing up at $P$ is smooth over $P$.

THEOREM 4.3. *Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a relatively minimal bielliptic fibration of genus $g \geq 6$ with a section whose Mordell-Weil rank is maximal, i.e., $r = 2g + 10$. Let $\phi : S \longrightarrow \boldsymbol{P}^1$ denote the smooth rational minimal elliptic surface, $\varpi : X \longrightarrow S$ the finite double cover obtained by Lemma* 3.3, *and $B$ the branch divisor of $\varpi$. Then there is a blow-down $\varepsilon : (S, B) \longrightarrow (S_9, B_9)$ such that $S_9 \simeq \boldsymbol{P}^2$ and $B_9$ is one of the following*:

Type $(16; g; 1)$: *a curve of degree $g - 2$ with a simple singular point of multiplicity $g - 4$ ($g$ is even).*

Type $(16; g; 0)$: *a curve of degree $g - 1$ with a simple singular point of multiplicity $g - 3$ and a node or cusp ($g$ is odd).*

Type $(18; 7)$: *a smooth quartic curve ($g = 7$).*

*In particular, $B$ is a smooth irreducible curve of genus $g - 4$.*

PROOF. Let $\varepsilon : (S, B) \longrightarrow (S_9, B_9)$ be a blow-down as in Lemma 4.2. Let $P_i \in \boldsymbol{P}^2$ $(i = 1, \ldots, 9)$ denote the contracted point by $\varepsilon$, $e_i$ the $(-1)$-curve on $S$ corresponding to $P_i$, $d$ the degree of $B_9$ and $m_i$ the multiplicity of $B_9$ at $P_i$. We have

$$
\begin{aligned}
&\operatorname{Pic}(S) \simeq \boldsymbol{Z}\varepsilon^* \mathscr{O}_{\boldsymbol{P}^2}(1) \oplus \bigoplus_{i=1}^{9} (\boldsymbol{Z} e_i), \\
&\varepsilon^* \mathscr{O}_{\boldsymbol{P}^2}(1).e_i = e_j.e_k = 0 \ (1 \leq i, j, k \leq 9,\ j \neq k)
\end{aligned} \tag{4.3}
$$

from Lemma 4.1. Let $\delta$ be a divisor with $2\delta \sim B$. Then $B.\varepsilon^* \mathscr{O}_{\boldsymbol{P}^2}(1) = 2\delta.\varepsilon^* \mathscr{O}_{\boldsymbol{P}^2}(1)$ and $B.e_i = 2\delta.e_i$. Since $B \in |\varepsilon^* B_9 - \sum_{i=1}^{9} m_i e_i|$, we see that $d$ and $m_i$'s are all even. We put $b = d/2$, $n_i = m_i/2$ $(1 \leq i \leq 9)$. It follows from Lemma 4.2 that

$$
n_9 \geq n_8 \geq n_7 \geq \cdots \geq n_1 \geq 0, \tag{4.4}
$$

$$
b \geq n_9 + n_8 + n_7. \tag{4.5}
$$

Restricting the finite double cover $\varpi : X \longrightarrow S$ to a general fiber of $\varphi : X \longrightarrow \boldsymbol{P}^1$, we have the finite double cover of an elliptic curve. By Hurwitz's formula,

$$
B.K_S = 2 - 2g. \tag{4.6}
$$

So we have

$$
-3b + \sum_{i=1}^{9} n_i = 1 - g \tag{4.7}
$$

from $K_S \sim \varepsilon^* \mathcal{O}_{\boldsymbol{P}^2}(-3) + \sum_{i=1}^{9} e_i$. Since $\varpi : X \longrightarrow S$ is the finite double cover of $S$ branched along $B$, we have $-2g-2 = {K_X}^2 = \varpi^*(K_S+\delta)^2 = 2(K_S+\delta)^2 = 2{K_S}^2 + 2B.K_S + 2\delta^2$. The equality (4.6) and ${K_S}^2 = 0$ imply $\delta^2 = g-3$, that is,

$$b^2 - \sum_{i=1}^{9} {n_i}^2 = g-3. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$b(3-b) + \sum_{i=1}^{9} n_i(n_i-1) = 2. \tag{4.9}$$

Moreover, (4.7) and $g \geq 6$ give

$$3b - \sum_{i=1}^{9} n_i \geq 5. \tag{4.10}$$

Now, Lemma 4.1 says that the singularities of $B_9$ are simple. So we have the classification as in Theorem 4.3 by the following:

CLAIM 4.4. *The solutions of the simultaneous inequalities given by* (4.4), (4.5), (4.9) *and* (4.10) *are*

$$(b, n_9, n_8, n_7, \ldots, n_1) = (2, 0, \ldots, 0),\ (2, 1, 0, \ldots, 0),$$
$$(k, k-1, 0, \ldots, 0),\ (k, k-1, 1, 0, \ldots, 0),\ (k \geq 3, k \in \mathbf{Z}).$$

PROOF OF CLAIM 4.4. If $b = 1$, the simultaneous inequalities has no solution. If $b = 2$, we have $n_i = 0$ $(i = 1, \ldots, 9)$ or $n_9 = 1$, $n_i = 0$ $(i = 1, \ldots, 8)$. Suppose $b \geq 3$. Then it follows from (4.5) that

$$\begin{aligned} b(b-3) &\geq (n_9+n_8+n_7)(n_9+n_8+n_7-3) \\ &\geq n_9(n_9-1) + n_9(n_8-1) + n_9(n_7-1) \\ &\quad + n_8(n_9-1) + n_8(n_8-1) + n_8(n_7-1) \\ &\quad + n_7(n_9-1) + n_7(n_8-1) + n_7(n_7-1). \end{aligned} \tag{4.11}$$

Assume $n_7 \geq 1$. We have $n_9 - 1 \geq n_8 - 1 \geq n_7 - 1 \geq 0$ from (4.4), and, hence

$$b(b-3) \geq \sum_{i=1}^{9} n_i(n_i-1),$$

contradicting to (4.9).

Assume $n_i = 0$ $(i = 1, \ldots, 7)$. Then (4.11) becomes

$$b(b-3) \geq n_9(n_9-1) + n_8(n_8-1) + n_9(n_8-2) + n_8(n_9-2).$$

If $n_8 \geq 2$, then we have similarly $n_9 - 2 \geq n_8 - 2 \geq 0$ and $0 \geq b(3-b) + n_9(n_9-1) + n_8(n_8-1)$, which leads us to a contradiction as before.

If $n_8 = 0$ or 1, then it follows from (4.9) that $(n_9 - b + 1)(n_9 + b - 2) = 0$. Thus we have $n_9 = b - 1$, and Claim 4.4 is proved. $\square$

It remains to show the irreducibility of $B$. The case $(18;7)$ is clear, since a smooth quartic curve on $\boldsymbol{P}^2$ is irreducible. In the other cases, we argue as follows.

If $B_9$ is a curve of degree $g-2$ with a simple singular point of multiplicity $g-4$ at $P_9$, let $\sigma_1 : (S_{8,1}, B_{8,1}) \longrightarrow (S_9, B_9)$, $S_{8,1} \simeq \Sigma_1$, be the blow-up of $S_9$ at $P_9$. Then $B_{8,1}$ is a smooth curve which is linearly equivalent to $2C_1 + (g-2) f_1$.

If $B_9$ is a curve of degree $g-1$ with a simple singular point of multiplicity $g-3$ at $P_9$ and a node or cusp at $P_8$, let $\sigma_2 : (S_7, B_7) \longrightarrow (S_9, B_9)$ be the blow-up at $P_9$ and $P_8$, and $l_{9,8}$ the strict transform by $\sigma_2$ of the line through $P_9$ and $P_8$. Then $l_{9,8}$ is a $(-1)$-curve which is disjoint from $B_7$, and let $\varsigma : (S_7, B_7) \longrightarrow (S_{8,0}, B_{8,0})$, $S_{8,0} \simeq \Sigma_0$, be the blow-down contracting $l_{9,8}$. Then $B_{8,0}$ is a smooth curve which is linearly equivalent $2C_0 + (g-3) f_0$.

CLAIM 4.5. *Let $B_{8,n}$ be a smooth curve on $\Sigma_n$ whish is linearly equivalent to $2C_n + (g-3+n) f_n$ and $g \geq 6$. Then $B_{8,n}$ is irreducible.*

PROOF. Let $G$ be an irreducible curve on $\Sigma_n$ which is not $C_n$. Assume that $G$ is linearly equivalent to $\alpha C_n + \beta f_n$. From $G.C_n \geq 0$ and $G.f_n \geq 0$, we have $\alpha \geq 0$ and $\beta \geq n\alpha \geq 0$. If $B_{8,n}$ contains a fiber $f$ as an irreducible component, then $B_{8,n} - f$ does not contain $f$ since $B_{8,n}$ is a reduced curve. From $(B_{8,n} - f).f = 2$, $B_{8,n}$ cannot be smooth. Hence $B_{8,n}$ consists of horizontal components. Since $B_{8,n}.f_n = 2$, we conclude that $B_{8,n}$ has at most two components. We assume that $B_{8,n}$ has two components and write $B_{8,n} = G_1 + G_2$, where $G_1 \sim C_n + \gamma f_n$ and $G_2 \sim C_n + (g-3+n-\gamma) f_n$. Then $G_1.G_2 = g-3 \geq 3$, which is absurd because $B_{8,n}$ is smooth. $\square$

This completes the proof of Theorem 4.3. $\square$

If $B_9$ is of type $(16;g;1)$, then $\varepsilon_1 = {\sigma_1}^{-1} \circ \varepsilon : (S,B) \longrightarrow (S_{8,1}, B_{8,1})$, $S_{8,1} \simeq \Sigma_1$, is the blow-down contracting $e_i$ $(i = 1, \ldots, 8)$. If $B_9$ is of type $(16;g;0)$, then $\varepsilon_0 = \varsigma \circ {\sigma_2}^{-1} \circ \varepsilon : (S,B) \longrightarrow (S_{8,0}, B_{8,0})$, $S_{8,0} \simeq \Sigma_0$, is the blow-down contracting $l_{9,8}$ and $e_i$ $(i = 1, \ldots, 7)$.

LEMMA 4.6. *Let $\varepsilon_n$ be as above. Any $(-1)$-curve which is contracted by $\varepsilon_n$ is disjoint from $B$. In particular, the model $(S_{8,n}, B_{8,n})$ is unique.*

PROOF. Let $e$ be a $(-1)$-curve that is contracted by $\varepsilon_n$. Then $B.e$ is even. Therefore $B.e \neq 0$ implies that $B_{8,n}$ has a singular point, which is absurd.

Let $e_1, \ldots, e_8$ be the contracted $(-1)$-curves by $\varepsilon_n$. Assume that there exists another model ${\varepsilon_n}' : (S,B) \longrightarrow ({S_{8,n}}', {B_{8,n}}')$. We may assume that $e_1$ is not contracted by ${\varepsilon_n}'$. Then ${\varepsilon_n}'(e_1)$ is an irreducible curve and we have ${\varepsilon_n}'(e_1).{B_{8,n}}' > 0$ because ${B_{8,n}}'$ is ample. Now the center of the blow-up ${\varepsilon_n}'$ is disjoint from ${B_{8,n}}'$. This implies that $e_1.B \neq 0$, which is a contradiction. $\square$

DEFINITION 4.7. The birational morphism $\varepsilon_n : (S,B) \longrightarrow (S_{8,n}, B_{8,n})$ as above is called the canonical non-singular minimal model of type $(16;g;n)$.

We next consider the minimal model of $(X,F)$. Since $\varphi$ is a relatively minimal fibration, we have only to consider the section of $\varphi$ whose self-intersection is minus one. We call such a section a $(-1)$-*section of* $\varphi$.

LEMMA 4.8. *Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a bielliptic fibration of genus $g \geq 6$ with maximal Mordell-Weil rank, $\phi : S \longrightarrow \boldsymbol{P}^1$ the smooth rational elliptic surface, $\varpi : X \longrightarrow S$ the finite double cover as in Lemma* 4.1, *and B the branch divisor of $\varpi$. Then the direct image of a $(-1)$-section of $\varphi$ by $\varpi$ is a $(-1)$-curve on S which is disjoint from B. Conversely the pull-back of a $(-1)$-curve on S which is disjoint from B consists of two disjoint $(-1)$-sections of $\varphi$.*

PROOF. Let $\delta$ be a divisor with $2\delta \sim B$. Since $\phi$ is the anti-canonical map and $K_X \sim \varpi^*(K_S + \delta)$, we have

$$F \sim -\varpi^* K_S \sim -K_X + \varpi^* \delta. \tag{4.12}$$

Let $\mathscr{E}$ be a $(-1)$-section of $\varphi$. Since $\mathscr{E}.F = 1$ and $\mathscr{E}.K_X = -1$, we have $\mathscr{E}.\varpi^*\delta$ =0 by (4.12). Thus the projection formula implies that $\varpi_*\mathscr{E}$ is a $(-1)$-curve on $S$ which is disjoint from $B$.

Conversely, let $e$ be a $(-1)$-curve on $S$ which is disjoint from $B$. Since $\varpi$ is unramified over $e$, we can write $\varpi^* e = \mathscr{E}_1 + \mathscr{E}_2$ with disjoint non-singular rational curves $\mathscr{E}_1$ and $\mathscr{E}_2$. Then $e.K_S = -1$ implies $F.(\mathscr{E}_1 + \mathscr{E}_2) = F.\varpi^* e = 2$ by (4.12). It follows $F.\mathscr{E}_i = 1$. □

The fibrations of each type are characterized by the following proposition:

PROPOSITION 4.9. *For a bielliptic fibration with maximal Mordell-Weil rank, there exists uniquely the* maximal set of disjoint $(-1)$-sections*. It consists of sixteen disjoint $(-1)$-sections in the case of type $(16; g; n)$, and eighteen in the case of type $(18; 7)$. In particular, the maximal set of disjoint $(-1)$-sections induces the canonical non-singular minimal model.*

PROOF. Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a bielliptic fibration of type $(16; g; n)$. We use the same notation as in Lemma 4.6. There exist eight disjoint $(-1)$-curves on $S$ each of which does not meet $B$ by Definition 4.7. Pulling back these eight $(-1)$-curves, we have sixteen disjoint $(-1)$-sections of $\varphi$ by Lemma 4.8.

Assume that there exist more than sixteen disjoint $(-1)$-sections of $\varphi$. From Lemma 4.8 these $(-1)$-sections of $\varphi$ give at least nine $(-1)$-curves on $S$ disjoint from $B$. It contradicts to Lemma 4.6.

Similarly, in the case of type $(18; 7)$, there exist exactly eighteen disjoint $(-1)$-sections of $\varphi$. □

COROLLARY 4.10. *Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a fibration of type $(16; g; n)$ or $(18; 7)$, and let $\phi : S \longrightarrow \boldsymbol{P}^1$ and $\varpi : X \longrightarrow S$ be as in Lemma* 4.1, *and B the branch divisor of $\varpi$. Let $\varepsilon_n : (S, B) \longrightarrow (S_{8,n}, B_{8,n})$ and $\varepsilon : (S, B) \longrightarrow (S_9, B_9)$ denote the canonical non-singular minimal model for $\varphi$ of type $(16; g; n)$ and $(18; 7)$ as in Definition* 4.7 *and Theorem* 4.3, *and let $\varepsilon_n : X \longrightarrow X_{16}$ and $\varepsilon : X \longrightarrow X_{18}$ be the blow-down which contracts disjoint $(-1)$-sections in the maximal set as in Proposition* 4.9, *respectively. Then there are the natural diagrams in Figure* 4.1, *where $\pi : X_{16} \longrightarrow S_{8,n}$ and $\pi_n : X_{18} \longrightarrow S_9$ are the finite double cover branched along $B_{8,n}$ and $B_9$ respectively.*

PROOF. Let $\varphi$ be a bielliptic fibration of type $(16; g; n)$. Let $\{\mathscr{E}_1, \ldots, \mathscr{E}_{16}\}$ denote the maximal set of $(-1)$-sections of $\varphi$, and $\{e_1, \ldots, e_8\}$ the set of the direct image curves by $\varpi$. Since the blow-down $\varepsilon_n : (S, B) \longrightarrow (S_{8,n}, B_{8,n})$ contracts $e_1, \ldots, e_8$, the morphism $\varepsilon_n \circ \varpi$ contracts $\mathscr{E}_1, \ldots, \mathscr{E}_{16}$ to eight points $\varepsilon_n(e_1), \ldots, \varepsilon_n(e_8)$. So $\varepsilon_n \circ \varpi$ factors through the blow-down $\varepsilon_n : X \longrightarrow$

Type $(16;g;n)$ Type $(18;7)$

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\varpi"] \arrow[dd, "\varepsilon_n"'] & & \boldsymbol{P}^1 \\
& S \arrow[ur, "\phi"'] \arrow[dd, "\varepsilon_n"] & \\
X_{16} \arrow[dr, "\pi_n"'] & & \\
& S_{8,n} &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\varpi"] \arrow[dd, "\varepsilon"'] & & \boldsymbol{P}^1 \\
& S \arrow[ur, "\phi"'] \arrow[dd, "\varepsilon"] & \\
X_{18} \arrow[dr, "\pi"'] & & \\
& S_9 &
\end{tikzcd}$$

Figure 4.1.

$X_{16}$ of $\mathscr{E}_1, \dots, \mathscr{E}_{16}$. Let $\pi_n$ be the morphism such that $\pi_n \circ \varepsilon_n = \varepsilon_n \circ \varpi$. Because $\varpi(\mathscr{E}_1), \dots, \varpi(\mathscr{E}_{16})$ are disjoint from $B$, $\pi_n : X_{16} \longrightarrow S_{8,n}$ is the finite double cover branched along $B_{8,n}$.

We can argue similarly in the case of type $(18;7)$. □

## 5. An explicit construction of a bielliptic fibration with maximal rank.

In this section, we give an explicit construction of smooth rational surfaces with the bielliptic fibration of genus $g \geq 6$ whose Mordell-Weil rank is maximal, i.e., $r = 2g+10$.

Put $S_9 = \boldsymbol{P}^2$ and let $B_9$ be an irreducible plane curve of type as in Theorem 4.3. Since the singularities of $B_9$ are at most two simple singular points, we have a blow-up $\sigma : (S_3, B_3) \longrightarrow (S_9, B_9)$ at six points in general position so that $B_3$ is a smooth curve. Let $\varsigma : (S_2, B_2) \longrightarrow (S_3, B_3)$ be a blow-up of $S_3$ at a general point $P_3$. Let $\psi : (S_2, B_2) \longrightarrow (Z, D)$ denote the anti-canonical map of $S_2$, which is the finite double cover of $Z \simeq \boldsymbol{P}^2$. Take a sufficiently general pencil $\mathscr{L}$ of lines on $Z$. Then we have the diagram as in Figure 5.1. Here $\varepsilon_2 : (S, B) \longrightarrow (S_2, B_2)$ denote the blow-up at the base points of $\psi^*\mathscr{L}$, $\Phi_{\mathscr{L}}$ and $\Phi_{\psi^*\mathscr{L}}$ the rational maps corresponding to $\mathscr{L}$ and $\psi^*\mathscr{L}$, respectively, and $\phi : S \longrightarrow \boldsymbol{P}^1$ the anti-canonical map of $S$, which is an elliptic

$$\begin{tikzcd}
X \arrow[drrr, "\varphi"] \arrow[dr, "\varpi"'] & & & \\
& (S,B) \arrow[rr, "\phi"] \arrow[d, "\varepsilon_2"'] & & \boldsymbol{P}^1 \\
& (S_2,B_2) \arrow[urr, dashed, "\Phi_{\psi^*\mathscr{L}}"'] \arrow[d, "\varsigma"'] \arrow[ddr, "\psi"] & & \\
\boldsymbol{P}^3 & (S_3,B_3) \arrow[l, hook'] \arrow[d, "\sigma"'] \arrow[dr, dashed, "\upsilon"'] & & \\
& (S_9,B_9) & (Z,D) \arrow[uuur, dashed, "\Phi_{\mathscr{L}}"'] &
\end{tikzcd}$$

Figure 5.1.

fibration. Let $\varpi : X \longrightarrow S$ be the finite double cover branched along $B$. Then we get a fibration $\varphi = \phi \circ \varpi : X \longrightarrow \boldsymbol{P}^1$.

THEOREM 5.1. *The fibration $\varphi : X \longrightarrow \boldsymbol{P}^1$ obtained by $\mathscr{L}$ as above is a bielliptic fibration of genus $g$ whose Mordell-Weil rank is maximal, i.e., ${K_X}^2 = -2g-2$ and all fibers are irreducible.*

*In particular, the fibration of each type, i.e., type $(16;g;n)$ or $(18;7)$, exists.*

We can check the numerically conditions easily.

LEMMA 5.2. *Let $(S,B)$ be a pair as above, and let $\varpi : X \longrightarrow S$ denote the finite double cover of $S$ branched along $B$. Assume that $\phi : S \longrightarrow \boldsymbol{P}^1$ satisfies* (a), (b) and (c) *of Lemma* 4.1. *Then $X$ is a smooth rational surface with ${K_X}^2 = -2g-2$ and $\varphi := \varpi \circ \phi$ is a bielliptic fibration of genus $g$.*

PROOF. There exists $B$ as above by Bertini's theorem. Let $(S,B)$ be a pair obtained by a plane curve model of type $(16;g;n)$ as in Theorem 4.3. Then there is a blow-down $\varepsilon_n : (S,B) \longrightarrow (S_{8,n}, B_{8,n})$ of disjoint eight $(-1)$-curves each of which does not meet $B$ so that $B_{8,n}$ is smooth and

$$B_{8,n} \in |2C_n + (g-3+n) f_n|, \; S_{8,n} \simeq \Sigma_n.$$

Hence we have $B^2 = {B_{8,n}}^2 = 4g-12$ and $B.K_S = B_{8,n}.K_{S_{8,n}} = 2-2g$. From Hurwitz's formula, the general fiber of $\varphi$ is a smooth bielliptic curve of genus $g$. Consider the finite double cover $\varpi : X \longrightarrow S$ branched along $B$. Then ${K_X}^2 = -2g-2$ and $\chi(\mathscr{O}_X) = 1$. Moreover, the projection formula implies $\varpi_* \mathscr{O}(2K_X) \simeq \mathscr{O}(2K_S + B) \oplus \mathscr{O}(2K_S + B/2)$. So we have

$$H^0(X,\, 2K_X) \simeq H^0(S,\, 2K_S + B) \oplus H^0(S,\, 2K_S + B/2).$$

It follows that

$$ {\varepsilon_n}^* f_n.(2K_S + B) = -2, \quad {\varepsilon_n}^* f_n.(2K_S + B/2) = -3.$$

Since ${\varepsilon_n}^* f_n$ is nef, $h^0(S,\, 2K_S + B) = h^0(S,\, 2K_S + B/2) = 0$. So $p_2(X) = 0$ follows. Therefore $X$ is a rational surface by Castelnuovo's rationality criterion. We can prove similarly in the case of type $(18;7)$. □

There exists a $(-1)$-curve $e$ which is disjoint from $B$. Then $\varpi^* e$ is a union of two disjoint $(-1)$-section of $\varphi$ (cf. Lemma 4.8). Moreover, if $\varphi$ has no reducible fiber, then $\varphi$ is relatively minimal. In order to see that $\varphi$ has no reducible fiber, it suffices to show that any fiber of $\phi$ meets $B$ transversely at least at one point.

Let $A$ be the branch divisor of $\psi$, which is a smooth quartic curve on $Z$. Since $\mathscr{L}$ is sufficiently general, we may assume that any base points of $\mathscr{L}$ is not on $A$. Thus $\psi^* \mathscr{L}$ has two distinct base points. By blowing up $\varepsilon_2 : (S,B) \longrightarrow (S_2, B_2)$ at these points, we have a rational elliptic surface $\phi : S \longrightarrow \boldsymbol{P}^1$ satisfying (a) of Lemma 4.1. We now consider the dual curve of the plane curve $A$. Since the dual curve has at most finite number of singular points, the number of bitangent lines and pluritangent lines to $A$ is finite. Hence we can assume that any line of $\mathscr{L}$ meets $A$ transversely at least at one point. Thus we have the following lemma.

LEMMA 5.3. *$\phi : S \longrightarrow \boldsymbol{P}^1$ obtained by $\mathscr{L}$ as above is a smooth rational minimal elliptic surface satisfying* (a), (b) and (c) *of Lemma* 4.1.

CLAIM 5.4. *Assume that $\mathscr{L}$ is sufficiently general. Then any line $l \in \mathscr{L}$ meets $D$ transversely at least at one point $R_l$ which is not on $A$.*

PROOF. The number of singular points of the dual curve of $D$ are at most finite. Similarly as in Lemma 5.3, we can assume that any line $l \in \mathscr{L}$ meets $D$ transversely at least at one point $R_l$.

The intersection of $A$ and $D$ is a finite set. For a point $R'_l$ on $A \cap D$, the number of tangent lines to $D$ is finite, since the upper bound of the number of such tangent lines is given by the degree of the dual curve of $D$. Therefore the number of lines which meets $D$ transversely only on $A \cap D$ is at most finite. Hence we may assume that $R_l \notin A$. □

CLAIM 5.5. *$\psi^* D$ is reducible.*

PROOF. Since $\mathscr{L}$ is sufficiently general, we may assume that the base point of $\mathscr{L}$ is not on $D$. Assume that $\psi^* D$ is irreducible. Then ${\varepsilon_2}^* \psi^* D = B$ by definition. We have $2D^2 = B^2 = 4g - 12$, and hence $D^2 = 2g - 6$. On the other hand, we have $\deg D = \deg \Phi_{\mathscr{L}}|_D = g - 1$ by $2 \deg \Phi_{\mathscr{L}}|_D = \deg \phi|_B = -K_S.B = 2g - 2$. This implies that $D^2 = (g-1)^2$, which is absurd. □

The anti-canonical embedding of $S_3$ is a del Pezzo surface of degree three. Under this identification, $\upsilon = \psi \circ \varsigma^{-1} : S_3 \dashrightarrow \boldsymbol{P}^2$ is the point-projection from $P_3$. Let us prove that the transversality on $Z$ lifts on $S_2$.

CLAIM 5.6. *Assume that $\mathscr{L}$ is sufficiently general. Then any elliptic curve $\psi^* l \in \psi^* \mathscr{L}$ meets $B_2$ transversely at least at one point $P_l$.*

PROOF. By Claims 5.4 and 5.5, $\psi^{-1}(R_l) \cap B_2$ is one point, say $P_l$. Now we may regard points out of $P_3$ on $S_3$ as points out of the exceptional curve of $\varsigma$ on $S_2$ by blowing up $\varsigma : S_2 \longrightarrow S_3$. Under this identification, in particular we have $P_l \neq P_3$ since the image of the exceptional curve of $\varsigma$ is a bitangent line to $A$. For our purpose, it suffices to show that any $\psi^* l \cap S_3$ meets $B_3$ transversely at $P_l$. Since $P_3$ is the center of projection and $R_l \notin A$, we have

$$P_3 \notin \mathscr{T}_{S_3, P_l}, \tag{5.1}$$

where $\mathscr{T}_{S_3,P_l}$ is the tangent space of $S_3$ at $P_l$ in $\boldsymbol{P}^3$. On the other hand, $\upsilon^* l \cap \upsilon^* \mathscr{T}_{D,R_l}$ is the line through $P_3$ and $R_l$. This and $(5.1)$ implies that

$$\upsilon^* l \cap \upsilon^* \mathscr{T}_{D,R_l} \cap \mathscr{T}_{S_3,P_l} = \{P_l\}. \tag{5.2}$$

We now recall

$$\mathscr{T}_{S_3 \cap \psi^* l, P_l} = \mathscr{T}_{S_3,P_l} \cap \upsilon^* l, \quad \mathscr{T}_{B_3,P_l} = \mathscr{T}_{S_3,P_l} \cap \upsilon^* \mathscr{T}_{D,R_l}.$$

In fact (5.2) means that $B_3$ and $\psi^* l \cap S_3$ meet transversely at $P_l$. □

PROOF OF THEOREM 5.1. Consider the construction as in Figure 5.1. It follows that $X$ is a smooth rational surface with ${K_X}^2 = -2g-2$ and $\varphi : X \longrightarrow \boldsymbol{P}^1$ is a bielliptic fibration of genus $g$ with a section by Lemmas 5.2 and 5.3. The base points of $\psi^* \mathscr{L}$ are not on $B_2$ since we take sufficiently general $\mathscr{L}$ for $D$. This implies that transversality of $B_2$ and $\psi^* l$ on $S_2$ lifts to that of $B$ and fibers of $\phi$ on $S$ by $\varepsilon_2$. Therefore all fibers of $\varphi$ are irreducible and $\varphi$ is relatively minimal. Thus the Mordell-Weil rank of $\varphi$ is $2g + 10$ by Theorem 3.4. □

## 6. Mordell-Weil lattices with maximal rank.

In this section, we shall determine the structure of the Mordell-Weil lattices for the fibrations of each type. For this purpose, return to the situation considered in Corollary 4.10.

Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a fibration of type $(18;7)$. Recall that $\pi : X_{18} \longrightarrow S_9$ as in Corollary 4.10 is a finite double cover branched along a smooth quartic curve. Hence $X_{18}$ is obtained by blowing up seven points of $\boldsymbol{P}^2$ in general position and $\pi$ is the anti-canonical map of $X_{18}$. Let $\eta : X_{18} \longrightarrow X_{25}, X_{25} \simeq \boldsymbol{P}^2$, be the blow-up as above, and $E_i$, $1 \le i \le 7$, the $(-1)$-curves contracted by $\eta$. Considering the diagram in Figure 4.1, we have the following lemma.

LEMMA 6.1. *Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a fibration of type $(18;7)$. In the notation as above,*

$$\mathrm{NS}(X) \simeq \boldsymbol{Z}(\eta \circ \varepsilon)^* \mathscr{O}_{X_{25}}(1) \oplus \bigoplus_{i=1}^{7} (\boldsymbol{Z}E_i) \oplus \bigoplus_{i=1}^{18} (\boldsymbol{Z}\mathscr{E}_i),$$
$$F = 9(\eta \circ \varepsilon)^* \mathscr{O}_{X_{25}}(1) - 3\sum_{i=1}^{7} E_i - \sum_{i=1}^{18} \mathscr{E}_i. \tag{6.1}$$

PROOF. Since $\phi$ and $\pi$ are the anti-canonical maps of $S$ and $X_{18}$, respectively, we have

$$F \sim \varpi^*(-K_S) \sim \varpi^*\left(\varepsilon^*(-K_{S_9}) - \sum_{i=1}^{9} e_i\right) \sim \varepsilon^*\pi^*(-K_{S_9}) - \sum_{i=1}^{18} \mathscr{E}_i,$$
$$\pi^*(-K_{S_9}) \sim 3\pi^*\mathscr{O}_{S_9}(1) \sim -3K_{X_{18}},$$

where $e_i$, $1 \le i \le 9$, are the $(-1)$-curves contracted by $\varepsilon$. Thus the lemma follows. □

Since $F.\mathscr{E}_i = 1$ $(i = 1, \dots, 18)$, the rational curves $\mathscr{E}_i$ become sections of $\varphi$, and we take $\mathscr{E}_{18}$ as the zero section $(O)$. Then by definition, the trivial sublattice $T_{18,7} \subset \mathrm{NS}(X)$ is generated by $\mathscr{E}_{18}$ and $F$. From Theorem 2.2, the Mordell-Weil lattice $(J_F(K), \langle , \rangle)$ is isomorphic to $L_{18,7}{}^{-}$ where $L_{18,7}$ is the orthogonal complement of $T_{18,7}$. The following proposition determines the structure of the lattice $L_{18,7}{}^{-}$.

PROPOSITION 6.2. *For the fibration of type $(18;7)$, the lattice $L_{18,7}{}^{-}$ is a positive-definite even unimodular lattice of rank* 24 *whose Dynkin diagram is given by Figure* 6.1.

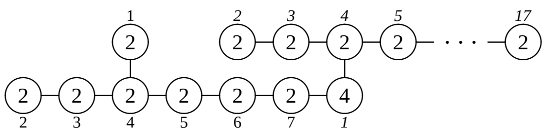

Figure 6.1.

*Here the numbers in the circles denote the self parings of elements, and a line between two circles shows that the paring of two elements is equal to $-1$.*

PROOF. Let the notation be as in Lemma 6.1. In particular, $F$ is given by (6.1) and $(O) = \mathscr{E}_{18}$. Take the following elements whose numbers correspond to Figure 6.1 from $T_{18,7}{}^{\perp}$:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & & & & & & & & \\ 0 & 1 & -1 & 0 & & & & & & & & \\ 0 & 0 & 1 & -1 & & & & & & & \text{\huge 0} & \\ & & & \ddots & \ddots & & & & & & & \\ & & & & 1 & -1 & 0 & 0 & 0 & & & \\ & & & & 0 & 1 & -1 & -1 & -1 & & & \\ & & & & 0 & 0 & 1 & -1 & 0 & & & \\ & & \text{\huge 0} & & & 0 & 0 & 1 & -1 & & & \\ & & & & & \vdots & \vdots & & \ddots & \ddots & & \\ & & & & & 0 & 0 & & & 1 & -1 & 0 \\ 9 & -3 & -3 & -3 & \cdots & -3 & -1 & \cdots & \cdots & -1 & -1 & -1 \\ & & & & & & & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} (\eta\circ\varepsilon)^*\mathscr{O}_{X_{25}}(1) \\ E_1 \\ \vdots \\ E_7 \\ \mathscr{E}_1 \\ \vdots \\ \mathscr{E}_{18} \end{pmatrix}$$

Figure 6.2.

$$\begin{aligned} H_1 &= (\eta\circ\varepsilon)^*\mathscr{O}_{X_{25}}(1) - E_1 - E_2 - E_3, \\ H_k &= E_{k-1} - E_k \ (k = 2, \ldots, 7), \\ \mathscr{H}_1 &= E_7 - \mathscr{E}_1 - \mathscr{E}_2 - \mathscr{E}_3, \\ \mathscr{H}_k &= \mathscr{E}_{k-1} - \mathscr{E}_k \ (k = 2, \ldots, 17). \end{aligned} \tag{6.2}$$

Therefore the matrix representing the base change to ${}^t(H_1, \ldots, H_7, \mathscr{H}_1, \ldots, \mathscr{H}_{17}, F, (O))$ is given by Figure 6.2. On the other hand, we have

$$\begin{aligned} \mathscr{E}_{17} = F + (O) - 9H_1 - 6H_2 - 12H_3 - 18H_4 - 15H_5 - 12H_6 - 9H_7 \\ -6\mathscr{H}_1 - 5\mathscr{H}_2 - 10\mathscr{H}_3 - \sum_{k=4}^{17}(19-k)\mathscr{H}_k, \end{aligned}$$

hence $(6.2)$ is a basis for $L_{18,7}$. □

REMARK 6.3. The sublattice $\langle H_1, \ldots, H_7, F + (O) - \mathscr{E}_1, \mathscr{H}_2, \ldots, \mathscr{H}_{17}\rangle \subset L_{18,7}{}^-$ is the root lattice $E_7 + A_{17}$. This characterizes $L_{18,7}{}^-$ among the positive-definite even unimodular lattices of rank 24 (cf. [**2**] or [**3**]).

Let $\varphi : X \longrightarrow \boldsymbol{P}^1$ be a fibration of type $(16; g; n)$. We consider the finite double cover $\pi_n : X_{16} \longrightarrow S_{8,n}$ branched along $B_{8,n}$ as in Corollary 4.10 similarly.

LEMMA 6.4. *Keep the same assumptions and notation as above.*
*Then there exists a birational morphism $\eta : X_{16} \longrightarrow X_{2g+10}$ such that $X_{2g+10} \simeq \Sigma_d$ for some $d$ and $\pi^* f_n \sim \eta^*\Gamma$, where $\Gamma$ is a fiber of $X_{2g+10}$.*

PROOF. Restricting the projection $pr : S_{8,n} \longrightarrow \boldsymbol{P}^1$ to $B_{8,n}$, we have a double covering $pr|_{B_{8,n}} : B_{8,n} \longrightarrow \boldsymbol{P}^1$. Since the genus of $B_{8,n}$ is $g-4$, there are $2g-6$ distinct branch points of $pr|_{B_{8,n}}$. Consider $pr\circ\pi_n : X_{16} \longrightarrow \boldsymbol{P}^1$. Then this is a conic bundle with $2g-6$ reducible conics over the fibers through the branch points of $pr|_{B_{8,n}}$. Let $\{E_i^+, E_i^-\}_{i=1}^{2g-6}$ be irreducible components of these reducible conics such that $pr\circ\pi(E_i^+) = pr\circ\pi(E_i^-)$. It is easy to see that each curve $E_i^\pm$

is a $(-1)$-curve, hence for each $1 \le i \le 2g-6$, we can contract one of $E_i^{\pm}$'s and obtain a smooth rational ruled surface $\Sigma_d$. □

For simplicity, we also denote the total transforms of a minimal section $\Delta$, a fiber $\Gamma$ by the birational morphism $\eta \circ \varepsilon_n : X \longrightarrow X_{2g+10}$ by the same letters. Then $\mathrm{NS}(X)$ is isomorphic to the free module

$$\mathrm{NS}(X) \simeq \mathbf{Z}\Delta \oplus \mathbf{Z}\Gamma \oplus \bigoplus_{i=1}^{2g-6} (\mathbf{Z}E_i) \oplus \bigoplus_{i=1}^{16} (\mathbf{Z}\mathscr{E}_i).$$

Moreover from Corollary 4.10, Lemma 6.4 and $(4.12)$, in $\mathrm{NS}(X)$, we have the relation:

$$F = 4\Delta + (2d+g-1)\Gamma - 2\sum_{i=1}^{2g-6} E_i - \sum_{i=1}^{16} \mathscr{E}_i. \tag{6.3}$$

Since $F.\mathscr{E}_i = 1$ $(i = 1,\dots,16)$, the rational curves $\mathscr{E}_i$ become sections of $\varphi$, and we take $\mathscr{E}_{16}$ as the zero section $(O)$. Then $\mathscr{E}_{16}$ and F generate the trivial sublattice $T_{18,7} \subset \mathrm{NS}(X)$.

PROPOSITION 6.5. *For a fibration of type* $(16; g; n)$, $g \ge 6$, *the lattice* $L_{16,g,n}{}^{-}$ *is a positive-definite odd unimodular lattice of rank* $2g+10$ *whose Dynkin diagram is given by Figure* 6.3 *in the case $g$ is even and Figure* 6.4 *in the case $g$ is odd. In particular* $L_{16,g,n}{}^{-}$ *is independent on $d$. Here the notation is the same as in Proposition* 6.2.

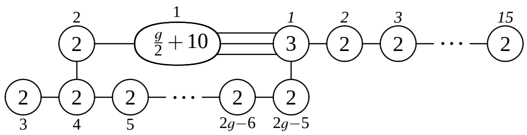


Figure 6.3.

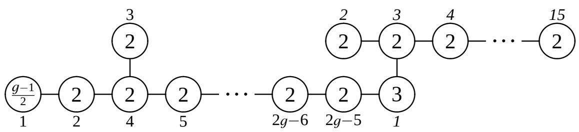


Figure 6.4.

PROOF. Let us keep the notation as above. In particular, $F$ is given by (6.3) and $(O) = \mathscr{E}_{16}$. Take the following elements from $T_{16,g,n}{}^{\perp}$:

$$H_k = E_{k-2} - E_{k-1} \quad (k = 4,5,\dots,2g-5),$$

$$\mathscr{H}_1 = E_{2g-6} - \mathscr{E}_1 + (n-1)\mathscr{E}_2 - nF - n(O),$$

$$\mathscr{H}_k = \mathscr{E}_{k-1} - \mathscr{E}_k \quad (k = 2,3,\dots,15).$$

Moreover take $H_1$, $H_2$, $H_3$ according to the following rule:
(i) The case $\alpha = (2d-g+3-3n)/4 \in \mathbf{Z}$:

$$H_1 = \Delta + \alpha\Gamma - E_1 + 3nF + 3n(O),$$

$$H_2 = E_1 - E_2,$$

$$H_3 = \Gamma - E_1 - E_2.$$

(ii) The case $\beta = (2d - g + 1 - 3n)/4 \in \mathbf{Z}$:

$$H_1 = \Delta + \beta\Gamma + 3nF + 3n(O),$$

$$H_2 = \Gamma - E_1 - E_2,$$

$$H_3 = E_1 - E_2.$$

Here the numbers attached to elements correspond to those in Figure 6.3 or Figure 6.4 according to the parity of $g$. Therefore the matrix representing the base change from ${}^t(\Delta, \Gamma, E_1, \ldots, E_{2g-6}, \mathscr{E}_1, \ldots, \mathscr{E}_{16})$ to ${}^t(H_1 - 3n(O) - 3nF, H_2, \ldots, H_{2g-5}, \mathscr{H}_1 + n(O) + nF, \mathscr{H}_2, \ldots, \mathscr{H}_{15}, F, (O))$ is given by Figure 6.5 in the case (i). In the case (ii), changing the second and third rows, we have the matrix similar to the one in Figure 6.5 which is triangular modulo off the row corresponding to $F$. If $g$ is even, we have

$$\begin{aligned}\mathscr{E}_{15} = & -F - (O) + 4(H_1 - 3(O) - 3F) + (2g+1)H_2 + (2g-1)H_3 \\ & + \sum_{k=4}^{2g-5} (4g + 6 - 2k)H_k + 14(\mathscr{H}_1 + (O) + F) + \sum_{k=2}^{14} (15 - k)\mathscr{H}_k,\end{aligned}$$

and, if $g$ is odd, we have

$$\begin{aligned}\mathscr{E}_{15} = & F + (O) - 4H_1 - (2g-2)H_2 - (2g-4)H_3 \\ & - \sum_{k=4}^{2g-5} (4g - 2k)H_k - 8\mathscr{H}_1 - 7\mathscr{H}_2 - \sum_{k=3}^{15} (17 - k)\mathscr{H}_k.\end{aligned}$$

So $\{H_1, \ldots, H_{2g-5}, \mathscr{H}_1, \ldots, \mathscr{H}_{15}\}$ is a basis for $L_{16,g,n}$.

$$\begin{pmatrix}
1 & \alpha & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & -1 & 0 & 0 & & & & & & & & \\
0 & 1 & -1 & -1 & 0 & 0 & & & & & \text{\Large 0} & & & \\
0 & 0 & 0 & 1 & -1 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & -1 & & & & & & & & \\
 & & & & & \ddots & \ddots & & & & & & & \\
 & & & & & & 1 & -1 & 0 & 0 & & & & \\
 & & & & & & 0 & 1 & -1 & n-1 & 0 & & & \\
 & & \text{\Large 0} & & & & & 0 & 1 & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 1 & -1 & & & \\
 & & & & & & & \vdots & \vdots & & \ddots & \ddots & & \\
 & & & & & & & 0 & 0 & & & 1 & -1 & 0 \\
4 & 2d+g-5 & -2 & -2 & -2 & -2 & \cdots & -2 & -1 & \cdots & \cdots & -1 & -1 & -1 \\
 & & & & & & & & & & & & 0 & 1
\end{pmatrix}$$

Figure 6.5.

Then taking the minus sign on the pairing on $L_{16,g,n}$ into account, we can easily check that the Gram matrix of $L_{16,g,n}{}^{-1}$

$$\left(\begin{array}{c|c} (-H_i.H_j)_{1\le i,j\le 2g-5} & (-H_i.\mathscr{H}_j)_{1\le i\le 2g-5, 1\le j\le 15} \\ \hline (-\mathscr{H}_i.H_j)_{1\le i\le 15, 1\le j\le 2g-5} & (-\mathscr{H}_i.\mathscr{H}_j)_{1\le i,j\le 15} \end{array}\right)$$

is given by Figure 6.6 or Figure 6.7 according as $g$ is even or odd, and all other statements follow from this.

$$\left(\begin{array}{cccccccc|cccccc}
g/2+10 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -3 & 0 & & & & \\
0 & 2 & 0 & -1 & 0 & 0 & & & 0 & & & & & \\
-1 & 0 & 2 & -1 & 0 & 0 & & & \vdots & & & & & \\
0 & -1 & -1 & 2 & -1 & 0 & & & \vdots & & & \mathbf{0} & & \\
0 & 0 & 0 & -1 & 2 & -1 & & & \vdots & & & & & \\
0 & 0 & 0 & 0 & -1 & 2 & \ddots & & \vdots & & & & & \\
\vdots & & & & & \ddots & \ddots & -1 & 0 & & & & & \\
0 & & & & & & -1 & 2 & -1 & 0 & & & & \\
\hline
-3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 3 & -1 & 0 & & & \\
0 & & & & & & & 0 & -1 & 2 & -1 & & & \\
 & & & & & & & & 0 & -1 & 2 & \ddots & & \\
 & & & \mathbf{0} & & & & & & & \ddots & \ddots & -1 & 0 \\
 & & & & & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & & & & 0 & -1 & 2
\end{array}\right)$$

Figure 6.6.

$$\left(\begin{array}{cccccccc|cccccccc}
(g-1)/2 & 0 & -1 & 0 & 0 & 0 & & & & & & & & & & \\
0 & 2 & 0 & -1 & 0 & 0 & & & & & & & & & & \\
-1 & 0 & 2 & -1 & 0 & 0 & & & & & & & & & & \\
0 & -1 & -1 & 2 & -1 & 0 & & & & & & & \mathbf{0} & & & \\
0 & 0 & 0 & -1 & 2 & -1 & & & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 2 & \ddots & & & & & & & & & \\
 & & & & & \ddots & \ddots & -1 & 0 & 0 & & & & & & \\
 & & & & & & -1 & 2 & -1 & 0 & 0 & & & & & \\
\hline
 & & & & & & 0 & -1 & 3 & 0 & -1 & 0 & & & & \\
 & & & & & & 0 & 0 & 0 & 2 & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & -1 & -1 & 2 & -1 & 0 & & & \\
 & & & & & & & & 0 & 0 & -1 & 2 & -1 & & & \\
 & & & & & & & & & 0 & 0 & -1 & 2 & \ddots & & \\
 & & & \mathbf{0} & & & & & & & & & \ddots & \ddots & -1 & 0 \\
 & & & & & & & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & & & & & & 0 & -1 & 2
\end{array}\right)$$

Figure 6.7.

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