J. Math. Soc. Japan
Vol. 57, No. 1, 2005

# The Igusa local zeta function of the simple prehomogeneous vector space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$

By Satoshi WAKATSUKI

(Received Jul. 31, 2003)
(Revised Nov. 6, 2003)

**Abstract.** We determine an explicit form of the Igusa local zeta function of the simple prehomogeneous vector space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. As for the Igusa local zeta functions and the $p$-adic $\Gamma$-factors of regular reducible simple prehomogeneous vector spaces with universally transitive open orbits, a table for explicit forms is completed by our result together with the result in Hosokawa [**2**].

## 1. Introduction.

The purpose of this paper is to determine an explicit form of the Igusa local zeta function of the simple prehomogeneous vector space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ $(n \geq 1)$. By this result, we get an explicit form of the $p$-adic $\Gamma$-factor of this space. Therefore by our result together with the result in Hosokawa [**2**], we observe that the $p$-adic $\Gamma$-factor is expressed by the Tate local factor and the $b$-function for every regular reducible simple prehomogeneous vector space with a universally transitive open orbit.

J. Igusa classified all regular irreducible prehomogeneous vector spaces with universally transitive open orbits in [**6**]. T. Kimura, S. Kasai, H. Hosokawa classified all reducible simple or 2-simple prehomogeneous vector spaces with universally transitive open orbits in [**13**]. For all regular irreducible reduced prehomogeneous vector spaces with universally transitive open orbits, J. Igusa gave explicitly their $p$-adic $\Gamma$-factors by calculating their Igusa local zeta functions in [**5**]. Furthermore J. Igusa expressed their $p$-adic $\Gamma$-factors by the Tate local factor and their $b$-functions. In [**2**], H. Hosokawa showed an analogy of Igusa's result in case of regular reducible simple prehomogeneous vector spaces with universally transitive open orbits. He gave explicitly their $p$-adic $\Gamma$-factors by calculating their Igusa local zeta functions except for the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. He also expressed their $p$-adic $\Gamma$-factors by the Tate local factor and their $b$-functions. Furthermore he gave a conjecture [**2**, p. 586 (C-1)] on an explicit form of the Igusa local zeta function of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$, and expected that its $p$-adic $\Gamma$-factor is also expressed by the Tate local factor and its $b$-function. He showed that this conjecture is true when $n = 1, 2$. In this paper, we give a proof of his conjecture for all $n$, and show that its $p$-adic $\Gamma$-factor is also expressed by the Tate local factor and its $b$-function.

The Igusa local zeta function of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ was not determined by the established methods of [**2**], [**5**] and [**8**] (see, [**2**]). Their established methods are to decompose the domain of integration over the residue field. It is difficult to calculate

2000 *Mathematics Subject Classification.* Primary 11S40; Secondary 11S90, 20G25.
*Key Words and Phrases.* Igusa local zeta functions, prehomogeneous vector spaces, universally transitive open orbits.

this local zeta function by their decompositions. Therefore we use Cartan decomposition of alternating forms in order to decompose the domain of integration over the integer ring. We get this idea from the proofs of [**1**, Theorem 10] and [**3**, Theorems 5.1, 5.2]. Furthermore we reduce their integrations to some results of spherical functions of alternating forms of [**1**] by some calculations. This is the point of our calculation. Finally we sum up their values of integrations by a formula of the Hall-Littlewood polynomial. Therefore we get our result.

We shall mention an application of the explicit forms of these Igusa local zeta functions. In [**12**], T. Kimura calculated explicitly the Fourier transform of the complex power over $\boldsymbol{R}$ for the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^*)$ by using the explicit form of the Igusa local zeta function, which was given explicitly in [**2**]. This method is based on the idea of Iwasawa-Tate theory. By [**7**] and [**14**] we see that we can apply this method to all regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits.

The plan of this paper is as follows. In Section 2, we give our main result on an explicit form of the Igusa local zeta function of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. In Section 3, we prove our main result. In Section 4, we give an explicit form of the $p$-adic $\Gamma$-factor of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ by our main result. We show that its $p$-adic $\Gamma$-factor is expressed by the Tate local factor and the $b$-function. In Appendix A, for the sake of convenience, we give a table for explicit forms of the Igusa local zeta functions, the $p$-adic $\Gamma$-factors and the $b$-functions of all regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits.

## 2. Main result.

We shall define the Igusa local zeta functions. Let $K$ be a $p$-adic field i.e. a finite extension of $\boldsymbol{Q}_p$. We denote by $\mathscr{O}_K$ the ring of integers in $K$. We fix a prime element $\pi$ in $\mathscr{O}_K$, and then $\pi\mathscr{O}_K$ is the ideal of nonunits of $\mathscr{O}_K$. The cardinality of the residue field $\mathscr{O}_K/\pi\mathscr{O}_K$ is denoted by $q$. We denote by $|\ |_K$ the absolute value of $K$ normalized by $|\pi|_K = q^{-1}$. Let $dv$ be the Haar measure on $K^n$ normalized by $\int_{\mathscr{O}_K^n} dv = 1$, and $S(K^n)$ the Schwartz-Bruhat space of $K^n$. If $f_1, f_2, \ldots, f_l$ are $K$-valued non-constant polynomial functions on $K^n$, then we put

$$Z(s;\Phi) = \int_{K^n} \prod_{i=1}^{l} |f_i(v)|_K^{s_i} \Phi(v) dv \quad (s = (s_1, \ldots, s_l) \in \boldsymbol{C}^l,\ \mathrm{Re}(s_i) > 0)$$

where $\Phi \in S(K^n)$. The local zeta function $Z(s;\Phi)$ is a rational function of $q^{-s_1}, \ldots, q^{-s_l}$ (see, e.g. [**8**][**16**]). If $\Phi_0$ is the characteristic function of $\mathscr{O}_K^n$, then we put $Z(s) = Z(s;\Phi_0)$. This local zeta function $Z(s)$ is called the Igusa local zeta function.

We shall show our main result. We deal with the simple prehomogeneous vector space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. For a commutative ring $R$, we denote by $M(m,n;R)$ the totality of $m \times n$ matrices over $R$, and by $\mathrm{Alt}(n;R)$ the totality of $n \times n$ alternating matrices over $R$ ($m,n \in \boldsymbol{N}$). For any $a \in M(m,n;R)$, ${}^t a$ is the transpose of $a$. We denote by $\mathrm{Pf}(a)$ the Pfaffian of $a \in \mathrm{Alt}(2n;R)$. The group $G = GL(1)^4 \times SL(2n+1)$ acts on $V = \mathrm{Alt}(2n+1) \oplus M(2n+1,1) \oplus M(2n+1,1) \oplus M(2n+1,1)$ by $(x,y,z,w) \mapsto (\alpha g x {}^t g, \beta g y, \gamma g z, \delta g w)$ for $(x,y,z,w) \in V$ and $(\alpha,\beta,\gamma,\delta,g) \in G$. The basic relative invariants $f_1,\ f_2,\ f_3,\ f_4$ of this space are given by

$$f_i(x,y,z,w) = \mathrm{Pf}(\Phi_i(x,y,z,w)) \quad \text{for} \quad (x,y,z,w) \in V,$$

where

$$\Phi_1(x,y,z,w) = \begin{pmatrix} x & y \\ -{}^t y & 0 \end{pmatrix}, \quad \Phi_2(x,y,z,w) = \begin{pmatrix} x & z \\ -{}^t z & 0 \end{pmatrix},$$

$$\Phi_3(x,y,z,w) = \begin{pmatrix} x & w \\ -{}^t w & 0 \end{pmatrix}, \quad \Phi_4(x,y,z,w) = \begin{pmatrix} x & y & z & w \\ -{}^t y & 0 & 0 & 0 \\ -{}^t z & 0 & 0 & 0 \\ -{}^t w & 0 & 0 & 0 \end{pmatrix},$$

$\Phi_1, \Phi_2, \Phi_3 \in \mathrm{Alt}(2n+2)$ and $\Phi_4 \in \mathrm{Alt}(2n+4)$ in [**11**]. We denote by $dx$ the Haar measure on $\mathrm{Alt}(2n+1;K)$ normalized by $\int_{\mathrm{Alt}(2n+1;\mathscr{O}_K)} dx = 1$, and by $dy$, $dz$, $dw$ the Haar measure on $M(2n+1,1;K)$ normalized by $\int_{M(2n+1,1;\mathscr{O}_K)} dy = \int_{M(2n+1,1;\mathscr{O}_K)} dz = \int_{M(2n+1,1;\mathscr{O}_K)} dw = 1$. We define the Igusa local zeta function of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ by

$$Z(s) = \int_{V(\mathscr{O}_K)} \prod_{i=1}^{4} |f_i(x,y,z,w)|_K^{s_i} dxdydzdw.$$

The following theorem is our main result.

THEOREM 2.1. *The Igusa local zeta function $Z(s)$ of the space $(GL(1)^4 \times SL(2n+1),$ $\Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ $(n \geq 1)$ is given by*

$$Z(s) = \prod_{i=1}^{4} \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2n}}{1-q^{-s_4-2n}} \times \prod_{j=1}^{n} \frac{1-q^{-2j-1}}{1-q^{-s_1-s_2-s_3-s_4-2j-1}}.$$

This explicit form is exactly as Hosokawa's conjecture [**2**]. So we prove his conjecture.

## 3. Proof of main result.

In this section, we shall prove our main result. Put

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},$$
$$|\lambda| = \sum_{i=1}^{n} \lambda_i, \quad n(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i.$$

For $\lambda \in \Lambda_n^+$, we put

$$\pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \in \mathrm{Alt}(2n; \mathscr{O}_K),$$

and

$$(\pi^\lambda)' = \begin{pmatrix} \pi^\lambda & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Alt}(2n+1; \mathscr{O}_K).$$

If we put

$$X' = \mathrm{Alt}(2n+1; \mathscr{O}_K) \setminus \{x \in \mathrm{Alt}(2n+1; \mathscr{O}_K); \mathrm{rank}(x) < 2n\},$$

then we have

$$X' = \bigcup_{\lambda \in \Lambda_n^+} GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)' \quad \text{(disjoint union)}.$$

We put

$$y = {}^t(y_1,y_2,\ldots,y_{2n+1}),\ \ z = {}^t(z_1,z_2,\ldots,z_{2n+1}),\ \ w = {}^t(w_1,w_2,\ldots,w_{2n+1}).$$

Then we have

$$\begin{aligned}
Z(s) &= \int_{V(\mathscr{O}_K)} \prod_{i=1}^4 |f_i(x,y,z,w)|_K^{s_i} dxdydzdw \\
&= \sum_{\lambda\in\Lambda_n^+} \int_{(GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)')\oplus M(2n+1,3;\mathscr{O}_K)} \prod_{i=1}^4 |f_i(x,y,z,w)|_K^{s_i} dxdydzdw \\
&= \sum_{\lambda\in\Lambda_n^+} \int_{GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)'} dx\cdot \int_{M(2n+1,3;\mathscr{O}_K)} \prod_{i=1}^4 |f_i((\pi^\lambda)',y,z,w)|_K^{s_i} dydzdw \\
&= \sum_{\lambda\in\Lambda_n^+} \int_{GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)'} dx\cdot \int_{M(2n+1,3;\mathscr{O}_K)} \left|\pi^{|\lambda|}y_{2n+1}\right|_K^{s_1}\cdot\left|\pi^{|\lambda|}z_{2n+1}\right|_K^{s_2} \\
&\qquad \times \left|\pi^{|\lambda|}w_{2n+1}\right|_K^{s_3}\cdot\left|\mathrm{Pf}\begin{pmatrix} (\pi^\lambda)' & y & z & w \\ -{}^t y & 0 & 0 & 0 \\ -{}^t z & 0 & 0 & 0 \\ -{}^t w & 0 & 0 & 0 \end{pmatrix}\right|_K^{s_4} dydzdw \\
&= \sum_{\lambda\in\Lambda_n^+} \int_{GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)'} dx\cdot q^{-(s_1+s_2+s_3)|\lambda|} I(s;\lambda;0,0,0),
\end{aligned}$$

where

$$\begin{aligned}
I(s;\lambda;a_1,a_2,a_3) &= \int_{M(2n+1,3;\mathscr{O}_K)} |y_{2n+1}|_K^{s_1}\cdot|z_{2n+1}|_K^{s_2}\cdot|w_{2n+1}|_K^{s_3} \\
&\quad \times \left|\pi^{a_1}y_{2n+1}\,{}^t z'\pi^\tau w' + \pi^{a_2}z_{2n+1}\,{}^t w'\pi^\tau y' + \pi^{a_3}w_{2n+1}\,{}^t y'\pi^\tau z'\right|_K^{s_4} dydzdw,
\end{aligned}$$

for $a_1,a_2,a_3 \in \mathbf{Z}$ and

$$\tau = (|\lambda|-\lambda_n, |\lambda|-\lambda_{n-1},\ldots,|\lambda|-\lambda_1) \in \Lambda_n^+,$$

$$y' = {}^t(y_{2n},y_{2n-1},\ldots,y_1),\ z' = {}^t(z_{2n},z_{2n-1},\ldots,z_1),\ w' = {}^t(w_{2n},w_{2n-1},\ldots,w_1).$$

We denote by $dy'$, $dz'$, $dw'$ the Haar measure on $M(2n,1;K)$ normalized by $\int_{M(2n,1;\mathscr{O}_K)} dy' = \int_{M(2n,1;\mathscr{O}_K)} dz' = \int_{M(2n,1;\mathscr{O}_K)} dw' = 1$. If we split the domain of integration for $I(s;\lambda;0,0,0)$ as $y_{2n+1} \bmod \pi$, then we have

$$\begin{aligned}
&I(s;\lambda;0,0,0) \\
&= \frac{(1-q^{-1})^3}{(1-q^{-s_2-1})(1-q^{-s_3-1})} \int_{M(2n,2;\mathscr{O}_K)} \left|{}^t z'\pi^\tau w'\right|_K^{s_4} dz'dw' + q^{-1-s_1} I(s;\lambda;1,0,0).
\end{aligned}$$

If we repeat this calculation, we have

$$
\begin{aligned}
&I(s;\lambda;1,0,0)\\
&\quad=\frac{(1-q^{-1})^3}{(1-q^{-s_3-1})(1-q^{-s_1-1})}\int_{M(2n,2;\mathscr{O}_K)}\left|{}^t w'\pi^\tau y'\right|_K^{s_4}dw'dy'+q^{-1-s_2}I(s;\lambda;1,1,0),
\end{aligned}
$$

$$
\begin{aligned}
&I(s;\lambda;1,1,0)\\
&\quad=\frac{(1-q^{-1})^3}{(1-q^{-s_1-1})(1-q^{-s_2-1})}\int_{M(2n,2;\mathscr{O}_K)}\left|{}^t y'\pi^\tau z'\right|_K^{s_4}dy'dz'+q^{-1-s_3-s_4}I(s;\lambda;0,0,0).
\end{aligned}
$$

If we put together the above results, we get

$$
\begin{aligned}
Z(s)=&\prod_{i=1}^{3}\frac{1-q^{-1}}{1-q^{-s_i-1}}\times\frac{1-q^{-s_1-s_2-s_3-3}}{1-q^{-s_1-s_2-s_3-s_4-3}}\\
&\times\sum_{\lambda\in\Lambda_n^+}\int_{GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)'}dx\cdot q^{-(s_1+s_2+s_3)|\lambda|}\cdot\int_{M(2n,2;\mathscr{O}_K)}\left|\mathrm{Pf}({}^tT\pi^\tau T)\right|_K^{s_4}dT,
\end{aligned}
$$

where we put $T\in M(2n,2;\mathscr{O}_K)$, and denote by $dT$ the Haar measure on $M(2n,2;K)$ normalized by $\int_{M(2n,2;\mathscr{O}_K)}dT=1$. We need some lemmas to calculate the above integrations. For a positive integer $n$, we put

$$
w_n(t)=\prod_{j=1}^{n}(1-t^j),
$$

$(w_0(t)=1)$. For a non-negative integer $i$ and $\lambda\in\Lambda_n^+$, the number $m_i(\lambda)$ of $\lambda_j$'s which are equal to $i$ is called the multiplicity of $i$ in $\lambda$. For $\lambda\in\Lambda_n^+$, we put

$$
w_\lambda^{(n)}(t)=\prod_{i=0}^{+\infty}w_{m_i(\lambda)}(t).
$$

By [**1**, Corollary of Lemma 2.7] or [**22**, Section 5], we have the following lemma.

LEMMA 3.1. *For $\lambda\in\Lambda_n^+$,*

$$
\int_{GL(2n+1;\mathscr{O}_K)\cdot(\pi^\lambda)'}dx=q^{-4n(\lambda)-3|\lambda|}\cdot(1-q^{-1})^{-1}\cdot w_{2n+1}(q^{-1})\cdot w_\lambda^{(n)}(q^{-2})^{-1}.
$$

For any positive integer $n$, we denote by $\mathscr{S}_n$ the symmetric group in $n$ latters. The Hall-Littlewood polynomial $P_\lambda(x;t)$ is defined by

$$
\begin{aligned}
P_\lambda(x;t)&=P_\lambda(x_1,x_2,\ldots,x_n;t)\\
&=\frac{(1-t)^n}{w_\lambda^{(n)}(t)}\cdot\sum_{\sigma\in\mathscr{S}_n}x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n}\prod_{1\le i<j\le n}\frac{x_{\sigma(i)}-tx_{\sigma(j)}}{x_{\sigma(i)}-x_{\sigma(j)}}
\end{aligned}
$$

for each $\lambda\in\Lambda_n^+$. For $\lambda\in\Lambda_n^+$, $P_\lambda(x;t)$ is a polynomial in $x_1,\ldots,x_n$ and $t$, and the set $\{P_\lambda(x;t);\,\lambda\in\Lambda_n^+\}$ forms a $\boldsymbol{Z}[t]$-basis of the ring $\boldsymbol{Z}[t][x_1,\ldots,x_n]^{\mathscr{S}_n}$ of symmetric polynomials in $x_1,\ldots,x_n$ with coefficients in $\boldsymbol{Z}[t]$ (cf. [**15**]). We get the following lemma by [**1**, Section 3].

LEMMA 3.2.

$$\int_{M(2n,2;\mathcal{O}_K)} \left|\mathrm{Pf}({}^tT\pi^\tau T)\right|_K^{s_4} dT = \frac{1-q^{-1}}{1-q^{-s_4-1}} \times \frac{1-q^{-2n}}{1-q^{-s_4-2n}} \times \frac{w_\lambda^{(n)}(q^{-2})}{w_n(q^{-2})} \times q^{-(n-1)|\lambda|+2n(\lambda)} P_\tau(q^{z_1}, q^{z_2}, \dots, q^{z_n}; q^{-2}),$$

*where we put*

$$(z_1+n, z_2+n, z_3+n, \dots, z_n+n) = (-s_4+1, 3, 5, \dots, 2n-1).$$

We can get the following lemma by the homogeneity of the Hall-Littlewood polynomials.

LEMMA 3.3.

$$P_\tau(x_1, x_2, \dots, x_n; t) = (x_1 x_2 \cdots x_n)^{|\lambda|} P_\lambda(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}; t).$$

By Lemmas 3.1, 3.2, 3.3, and

$$\sum_{\lambda \in \Lambda_n^+} t^{n(\lambda)} P_\lambda(x_1, x_2, \dots, x_n; t) = \prod_{i=1}^n (1-x_i)^{-1}$$

(cf. [**15**, Chapter 3, Section 4, Example 1]), we have

$$\begin{aligned} Z(s) &= \prod_{i=1}^4 \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2n}}{1-q^{-s_4-2n}} \times \frac{1-q^{-s_1-s_2-s_3-3}}{1-q^{-s_1-s_2-s_3-s_4-3}} \times \frac{w_{2n+1}(q^{-1})}{(1-q^{-1})w_n(q^{-2})} \\ &\quad \times \sum_{\lambda \in \Lambda_n^+} q^{-(n+2)|\lambda|-2n(\lambda)-(s_1+s_2+s_3)|\lambda|} P_\tau(q^{z_1}, q^{z_2}, \dots, q^{z_n}; q^{-2}) \\ &= \prod_{i=1}^4 \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2n}}{1-q^{-s_4-2n}} \times \frac{1-q^{-s_1-s_2-s_3-3}}{1-q^{-s_1-s_2-s_3-s_4-3}} \times \frac{w_{2n+1}(q^{-1})}{(1-q^{-1})w_n(q^{-2})} \\ &\quad \times \sum_{\lambda \in \Lambda_n^+} q^{-2n(\lambda)} P_\lambda(q^{-s_1-s_2-s_3-3}, q^{-s_1-s_2-s_3-s_4-5}, \dots, q^{-s_1-s_2-s_3-s_4-2n-1}; q^{-2}) \\ &= \prod_{i=1}^4 \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2n}}{1-q^{-s_4-2n}} \times \prod_{j=1}^n \frac{1-q^{-2j-1}}{1-q^{-s_1-s_2-s_3-s_4-2j-1}}. \end{aligned}$$

Therefore we obtain our main result.

## 4. *p*-adic *Γ*-factor.

In this section, we give an explicit form of the $p$-adic $\Gamma$-factor of the space $(GL(1)^4 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ by Theorem 2.1. We show that its $p$-adic $\Gamma$-factor is expressed by the Tate local factor and the $b$-function. As for all regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits, we can treat the Igusa zeta functions, the $p$-adic $\Gamma$-factors and the $b$-functions in a uniform way by the results of Igusa [**5**], Hosokawa [**2**] and Theorem 2.1. So we shall discuss these spaces uniformly.

Let $G$ be a connected linear algebraic group defined over $\boldsymbol{Q}$, $V$ a finite dimensional vector spaces with $\boldsymbol{Q}$-structure, and $\rho : G \to GL(V)$ a rational representation of $G$ on $V$ defined over $\boldsymbol{Q}$. Throughout this section, we assume that the triplet $(G, \rho, V)$ is a regular irreducible reduced or regular simple prehomogeneous vector space with a universally transitive open orbit. Let $f_1,\dots,f_l$ be the basic relative invariants of $(G,\rho,V)$, and $\chi_i$ the rational character of $G$ corresponding to $f_i$, i.e. $f_i(\rho(g)v)=\chi_i(g)f_i(v)$ for all $g\in G$ and all $v\in V$. We denote by $dv$ the Haar measure on $V(K)$ normalized by $\int_{V(\mathscr{O}_K)}dv=1$, and $S(V(K))$ the Schwartz-Bruhat space of $V(K)$. We define the $p$-adic local zeta function $Z(s;\Phi)$ of $(G,\rho,V)$ by

$$Z(s;\Phi)=\int_{V(K)}\prod_{i=1}^{l}|f_i(v)|_K^{s_i}\Phi(v)dv \quad (s=(s_1,\dots,s_l)\in\boldsymbol{C}^l,\ \mathrm{Re}(s_i)>0)$$

where $\Phi\in S(V(K))$. We put $Z(s)=Z(s;\Phi_0)$ for the characteristic function $\Phi_0$ of $V(\mathscr{O}_K)$. This local zeta function $Z(s)$ is called the Igusa local zeta function of $(G,\rho,V)$. From the results of [**5**], [**2**] and Theorem 2.1, the Igusa local zeta function of $(G,\rho,V)$ is given by

$$Z(s)=\prod_{j=1}^{N}\frac{1-q^{-\alpha_j}}{1-q^{-\eta_j(s)}} \quad (\eta_j(0)=\alpha_j), \tag{1}$$

where

$$\eta_j(s)=\sum_{i=1}^{l}\eta_{ij}s_i+\alpha_j, \quad (\eta_{ij}=0 \text{ or } 1,\ \alpha_j\in\boldsymbol{Z}_{>0}).$$

Let $V^*$ be the dual space of $V$, and $\rho^*$ the contragredient representation of $\rho$. It is known that $(G,\rho^*,V^*)$ is a prehomogeneous vector space, and there exist the basic relative invariants $f_1^*,f_2^*,\dots,f_l^*$ of $(G,\rho^*,V^*)$ such that the character $\chi_i^{-1}$ corresponds to $f_i^*$. Let $dv^*$ be the Haar measure on $V^*(K)$ normalized by $\int_{V^*(\mathscr{O}_K)}dv^*=1$, and $S(V^*(K))$ the Schwartz-Bruhat space of $V^*(K)$. We define the $p$-adic local zeta function $Z^*(s;\Phi^*)$ of $(G,\rho^*,V^*)$ by

$$Z^*(s;\Phi)=\int_{V^*(K)}\prod_{i=1}^{l}|f_i^*(v^*)|_K^{s_i}\Phi^*(v^*)dv^* \quad (s=(s_1,\dots,s_l)\in\boldsymbol{C}^l,\ \mathrm{Re}(s_i)>0)$$

where $\Phi^*\in S(V^*(K))$.

Let $\psi$ be an additive character of $K$ such that $\psi$ is non-trivial on $\pi^{-1}\mathscr{O}_K$ and trivial on $\mathscr{O}_K$. We define the Fourier transform $\widehat{\Phi}^*$ of $\Phi^*\in S(V^*(K))$ by

$$\widehat{\Phi}^*(v)=\int_{V^*(K)}\Phi^*(v^*)\psi(v^*(v))dv^*.$$

By the regularity of $(G,\rho,V)$, there exists an element $\kappa=(\kappa_1,\kappa_2,\dots,\kappa_l)\in(1/2)\cdot\boldsymbol{Z}^l$ satisfying $\det(\rho(g))^2=\chi_1(g)^{2\kappa_1}\cdots\chi_l(g)^{2\kappa_l}$ (cf. [**19**], [**20**], [**17**]). By [**12**, Theorem 3.3], [**7**] and [**14**], we have the functional equation

$$Z(s-\kappa;\widehat{\Phi}^*)=\gamma(s)Z^*(-s;\Phi^*),$$

where $s-\kappa=(s_1-\kappa_1,\dots,s_l-\kappa_l)$ and $\gamma(s)$ is independent of $\Phi^*$. We call $\gamma(s)$ the $p$-adic $\Gamma$-factor of $(G,\rho,V)$. For $p$-adic local functional equations of prehomogeneous vector spaces which do not have universally transitive open orbits, we refer to [**4**] and [**18**]. Since the Fourier transform $\widehat{\Phi}_0$ of $\Phi_0$ is equal to $\Phi_0$, we have $\gamma(s)=Z(s-\kappa)/Z(-s)$. By the explicit forms of the Igusa local zeta functions, the $p$-adic $\Gamma$-factor $\gamma(s)$ of $(G,\rho,V)$ is given by

$$\gamma(s)=\prod_{j=1}^{N}\gamma^T(\eta_j(s-\kappa)), \quad \left(\gamma^T(s)=\frac{1-q^{-(1-s)}}{1-q^{-s}}\right). \tag{2}$$

This factor $\gamma^T(s)$ is called the Tate local factor.

We put $f^m := \prod_{i=1}^l f_i^{m_i}$, $f^{*m} := \prod_{i=1}^l f_i^{*m_i}$ for $m=(m_1,m_2,\dots,m_l)\in \boldsymbol{Z}^l$. Fix a $\boldsymbol{Q}$-basis of $V$, and identify $V(\boldsymbol{Q})$ with $\boldsymbol{Q}^n$ $(\dim V = n)$. We also identify $V^*(\boldsymbol{Q})$ with $\boldsymbol{Q}^n$ by the $\boldsymbol{Q}$-basis of $V^*$ dual to the fixed basis of $V$. Then we put $v=(v_1,\dots,v_n)$, $\mathrm{grad}_v = (\partial/\partial v_1,\dots,\partial/\partial v_n)$. By [**19**, Proposition 12], for any l-tuple $m=(m_1,m_2,\dots,m_l)\in(\boldsymbol{Z}_{\geq 0})^l$, there exists a polynomial $b_m(s)$ such that

$$f^{*m}(\mathrm{grad}_v)f^{s+m}(v) = b_m(s)f^s(v),$$

where $s+m=(s_1+m_1,\dots,s_l+m_l)$. We call $b_m(s)$ the $b$-function of $(G,\rho,V)$. In [**11**], T. Kimura calculate explicitly the $\Gamma$-factor over $\boldsymbol{R}$ for the space $(GL(1)^4\times SL(2n+1),\Lambda_2\oplus\Lambda_1\oplus\Lambda_1^*\oplus\Lambda_1^*)$ by using the explicit form of the Igusa local zeta function, which was given explicitly in [**2**]. By [**7**] and [**14**] we can apply this method to all regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits. By [**17**, p. 459 (5–8)], we can get $b$-functions from the $\Gamma$-factors over $\boldsymbol{R}$. Therefore the $b$-function of $(G,\rho,V)$ is given by

$$b_m(s) = \prod_{j=1}^{N}\frac{\Gamma(\eta_j(s+m))}{\Gamma(\eta_j(s))} \tag{3}$$

uniquely up to constant, where $\Gamma(s)$ is the gamma function. As for regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits, the $b$-functions were given explicitly by other methods in [**10**], [**9**] and [**21**] except for the space $(GL(1)^4\times SL(2n+1),\Lambda_2\oplus\Lambda_1\oplus\Lambda_1^*\oplus\Lambda_1^*)$.

From the above argument, we see that the $p$-adic $\Gamma$-factor of $(G,\rho,V)$ is expressed by the Tate local factor and the set

$$\left\{\eta_j(s)\,;1\leq j\leq N\right\},$$

which is determined by the $b$-function of $(G,\rho,V)$ (cf. Equations (2) and (3)).

We shall consider the space $(GL(1)^4\times SL(2n+1),\Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1)$. We have $\kappa = (1,1,1,2n)$ by easy calculation. Hence we have the following result by Theorem 2.1, Equations (1) and (2).

THEOREM 4.1. *The p-adic $\Gamma$-factor $\gamma(s)$ of the space $(GL(1)^4\times SL(2n+1),\Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1)$ $(n\geq 1)$ is given by*

$$\gamma(s) = \prod_{i=1}^{4}\gamma^T(s_i)\times\gamma^T(s_4-2n+1)\times\prod_{j=1}^{n}\gamma^T(s_1+s_2+s_3+s_4-2j).$$

By Equation (3) we see that the $b$-function of the space $(GL(1)^4\times SL(2n+1),\Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1)$ is given by

$$\begin{aligned} b_m(s) = \prod_{i=1}^{4}\frac{\Gamma(s_i+1+m_i)}{\Gamma(s_i+1)}\times\frac{\Gamma(s_4+2n+m_4)}{\Gamma(s_4+2n)} \\ \times\prod_{j=1}^{n}\frac{\Gamma(s_1+s_2+s_3+s_4+2j+1+m_1+m_2+m_3+m_4)}{\Gamma(s_1+s_2+s_3+s_4+2j+1)}. \end{aligned}$$

This $b$-function was known already and given explicitly in [**21**], but our method is different as we explained above. We note that the above $p$-adic $\Gamma$-factor is expressed by the Tate local factor and the set $\{s_i+1, s_4+2n, s_1+s_2+s_3+s_4+2j+1\,;\, i=1,2,3,4,\ j=1,2,\ldots,n\}$, which is determined by the $b$-function.

### Appendix A. Table of Igusa local zeta functions, *p*-adic *Γ*-factors and *b*-functions.

In this appendix, we give a table for the set $\left\{\eta_j(s)\,;1\le j\le N\right\}$ of all regular irreducible reduced or regular simple prehomogeneous vector spaces with universally transitive open orbits. From the set $\left\{\eta_j(s)\,;1\le j\le N\right\}$ of each space $(G,\rho,V)$, the Igusa local zeta function $Z(s)$ of $(G,\rho,V)$ is given by Equation (1) in Section 4, the $p$-adic $\Gamma$-factor $\gamma(s)$ of $(G,\rho,V)$ is given by Equation (2) in Section 4, and the $b$-function $b_m(s)$ of $(G,\rho,V)$ is given by Equation (3) in Section 4. The explicit forms of the Igusa local zeta functions and the $p$-adic $\Gamma$-factors are due to [**5**], [**2**], Theorems 2.1 and 4.2. The explicit forms of the $b$-functions are due to [**10**], [**9**], [**12**] and [**21**].

We shall define some notations in the table. We define the irreducible representation $\Lambda_1$ of $SL(n)$ by $\Lambda_1(g)x=gx$ for $g\in SL(n)$, $x\in M(n,1)$, the irreducible representation $\Lambda_2$ of $SL(n)$ by $\Lambda_2(g)x=gx{}^tg$ for $g\in SL(n)$, $x\in \mathrm{Alt}(n)$, and the irreducible representation $\Lambda_1$ of $SO(2n)$ (resp. $Sp(n)$) by $\Lambda_1(g)x=gx$ for $g\in SO(2n)$ (resp. $Sp(n)$), $x\in M(2n,1)$. For the irreducible representations of $Spin(n)$ and $E(6)$, we refer to [**20**]. We denote by $V(n)$ an $n$-dimensional vector space.

**Regular irreducible reduced case.**

(1) $(H\times GL(n), \rho\otimes\Lambda_1, V(n)\otimes V(n))$ $(n\ge 1)$. $l=1$.
$\chi_1(h,g)=\det(g)$ for $(h,g)\in H\times GL(n)$. $\kappa_1=n$.

$$\{s_1+j\,;\, j=1,2,\ldots,n\}.$$

In this case, $\rho$ is an $n$-dimensional irreducible representation of a connected algebraic group $H$.

(2) $(GL(2n), \Lambda_2, V(n(2n-1)))$ $(n\ge 2)$. $l=1$.
$\chi_1(g)=\det(g)$ for $g\in GL(2n)$. $\kappa_1=2n-1$.

$$\{s_1+2j-1\,;\, j=1,2,\ldots,n\}.$$

(3) $(GL(1)\times SO(2n), \Lambda_1\otimes\Lambda_1, V(1)\otimes V(2n))$ $(n\ge 1)$. $l=1$.
$\chi_1(a,g)=a^2$ for $(a,g)\in GL(1)\times SO(2n)$. $\kappa_1=n$.

$$\{s_1+1, s_1+n\}.$$

In this case, we put $SO(2n)=\left\{g\in SL(2n)\,;{}^tg\begin{pmatrix}0 & 1_n\\ 1_n & 0\end{pmatrix}g=\begin{pmatrix}0 & 1_n\\ 1_n & 0\end{pmatrix}\right\}$.

(4) $(Sp(n)\times GL(2m), \Lambda_1\otimes\Lambda_1, V(2n)\otimes V(2m))$ $(n\ge 2m\ge 1)$. $l=1$.
$\chi_1(h,g)=\det(g)$ for $(h,g)\in Sp(n)\times GL(2m)$. $\kappa_1=2n$.

$$\{s_1+2j-1, s_1+2n-2j+2\,;\, j=1,2,\ldots,m\}.$$

(5) $(GL(1)\times Spin(7), \Lambda_1\otimes(\text{spin rep.}), V(1)\otimes V(8))$. $l=1$.
$\chi_1(a,g)=a^2$ for $(a,g)\in GL(1)\times Spin(7)$. $\kappa_1=4$.

$$\{s_1+1, s_1+4\}.$$

(6) $(GL(1) \times Spin(9), \Lambda_1 \otimes (\text{spin rep.}), V(1) \otimes V(16))$. $l = 1$.
$\chi_1(a,g) = a^2$ for $(a,g) \in GL(1) \times Spin(9)$. $\kappa_1 = 8$.

$$\{s_1+1, s_1+8\}.$$

(7) $(Spin(10) \times GL(2), (\text{half spin rep.}) \otimes \Lambda_1, V(16) \otimes V(2))$. $l = 1$.
$\chi_1(h,g) = \det(g)^2$ for $(h,g) \in Spin(10) \times GL(2)$. $\kappa_1 = 8$.

$$\{s_1+1, s_1+4, s_1+5, s_1+8\}.$$

(8) $(GL(1) \times E(6), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(27))$. $l = 1$.
$\chi_1(a,g) = a^3$ for $(a,g) \in GL(1) \times E(6)$. $\kappa_1 = 9$.

$$\{s_1+1, s_1+5, s_1+9\}.$$

**Regular reducible simple case.**

(1) $(GL(1)^2 \times SL(n), \Lambda_1 \oplus \Lambda_1^*, V(n) \oplus V(n)^*)$ $(n \geq 2)$. $l = 1$.
$\chi_1(a_1,a_2,g) = a_1 a_2$ for $(a_1,a_2,g) \in GL(1)^2 \times SL(n)$. $\kappa_1 = n$.

$$\{s_1+1, s_1+n\}.$$

(2) $(GL(1)^n \times SL(n), \Lambda_1 \overbrace{\oplus \cdots \oplus}^{n} \Lambda_1, V(n) \overbrace{\oplus \cdots \oplus}^{n} V(n))$ $(n \geq 2)$. $l = 1$.
$\chi_1(a_1,\ldots,a_n,g) = a_1 \cdots a_n$ for $(a_1,\ldots,a_n,g) \in GL(1)^n \times SL(n)$. $\kappa_1 = n$.

$$\{s_1+j\,;\ j = 1,2,\ldots,n\}.$$

(3) $(GL(1)^2 \times SL(2n+1), \Lambda_2 \oplus \Lambda_1, V(n(2n+1)) \oplus V(2n+1))$ $(n \geq 1)$. $l = 1$.
$\chi_1(a_1,a_2,g) = a_1^n a_2$ for $(a_1,a_2,g) \in GL(1)^2 \times SL(2n+1)$. $\kappa_1 = 2n+1$.

$$\{s_1+2j-1\,;\ j = 1,2,\ldots,n+1\}.$$

(4) $(GL(1)^2 \times Sp(n), \Lambda_1 \oplus \Lambda_1, V(2n) \oplus V(2n))$ $(n \geq 1)$. $l = 1$.
$\chi_1(a_1,a_2,g) = a_1 a_2$ for $(a_1,a_2,g) \in GL(1)^2 \times Sp(n)$. $\kappa_1 = 2n$.

$$\{s_1+1, s_1+2n\}.$$

(5) $(GL(1)^2 \times Spin(10), (\text{half spin rep.}) \oplus (\text{half spin rep.}), V(16) \oplus V(16))$. $l = 1$.
$\chi_1(a_1,a_2,g) = a_1^2 a_2^2$ for $(a_1,a_2,g) \in GL(1)^2 \times Spin(10)$. $\kappa_1 = 8$.

$$\{s_1+1, s_1+4, s_1+5, s_1+8\}.$$

(6) $(GL(1)^3 \times SL(2n), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1, V(n(2n-1)) \oplus V(2n) \oplus V(2n))$ $(n \geq 1)$. $l = 2$.
$\chi_1(a_1,a_2,a_3,g) = a_1^n$, $\chi_2(a_1,a_2,a_3,g) = a_1^{n-1} a_2 a_3$ for $(a_1,a_2,a_3,g) \in GL(1)^3 \times SL(2n)$. $\kappa_1 = 1, \kappa_2 = 2n$.

$$\{s_1+1, s_2+1, s_2+2n, s_1+s_2+2j+1\,;\ j = 1,2,\ldots,n-1\}.$$

(7) $(GL(1)^3 \times SL(2n), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^*, V(n(2n-1)) \oplus V(2n) \oplus V(2n)^*)$ $(n \geq 1)$. $l = 2$.
$\chi_1(a_1,a_2,a_3,g) = a_1^n$, $\chi_2(a_1,a_2,a_3,g) = a_2 a_3^{-1}$ for $(a_1,a_2,a_3,g) \in GL(1)^3 \times SL(2n)$. $\kappa_1 = 2n-1, \kappa_2 = 2n$.

$$\{s_1+2j-1, s_2+1, s_2+2n;\ j=1,2,\ldots,n\}.$$

(8) $(GL(1)^3\times SL(2n), \Lambda_2\oplus\Lambda_1^*\oplus\Lambda_1^*, V(n(2n-1))\oplus V(2n)^*\oplus V(2n)^*)$ $(n\geq 2)$. $l=2$. $\chi_1(a_1,a_2,a_3,g)=a_1^n$, $\chi_2(a_1,a_2,a_3,g)=a_1a_2^{-1}a_3^{-1}$ for $(a_1,a_2,a_3,g)\in GL(1)^3\times SL(2n)$. $\kappa_1=2n-3, \kappa_2=2n$.

$$\{s_1+2j-1, s_2+1, s_2+2n, s_1+s_2+2n-1;\ j=1,2,\ldots,n-1\}.$$

(9) $(GL(1)^2\times Spin(8), (\text{vec. rep.})\oplus(\text{half spin rep.}), V(8)\oplus V(8))$. $l=2$. $\chi_1(a_1,a_2,g)=a_1^2$, $\chi_2(a_1,a_2,g)=a_2^2$ for $(a_1,a_2,g)\in GL(1)^2\times Spin(8)$. $\kappa_1=4, \kappa_2=4$.

$$\{s_1+1, s_1+4, s_2+1, s_2+4\}.$$

(10) $(GL(1)^2\times Spin(10), (\text{vec. rep.})\oplus(\text{half spin rep.}), V(10)\oplus V(16))$. $l=2$. $\chi_1(a_1,a_2,g)=a_1^2$, $\chi_2(a_1,a_2,g)=a_1a_2^2$ for $(a_1,a_2,g)\in GL(1)^2\times Spin(10)$. $\kappa_1=1, \kappa_2=8$.

$$\{s_1+1, s_2+1, s_2+8, s_1+s_2+5\}.$$

(11) $(GL(1)^4\times SL(2n+1), \Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1, V(n(2n+1))\oplus V(2n+1)\oplus V(2n+1)\oplus V(2n+1))$ $(n\geq 1)$. $l=4$. $\chi_1(a,g)=a_1^na_2$, $\chi_2(a,g)=a_1^na_3$, $\chi_3(a,g)=a_1^na_4$, $\chi_4(a,g)=a_1^{n-1}a_2a_3a_4$ for $(a,g)=(a_1,a_2,a_3,a_4,g)\in GL(1)^4\times SL(2n+1)$. $\kappa_1=1, \kappa_2=1, \kappa_3=1, \kappa_4=2n$.

$$\{s_i+1, s_4+2n, s_1+s_2+s_3+s_4+2j+1;\ i=1,2,3,4,\ j=1,2,\ldots,n\}.$$

(12) $(GL(1)^4\times SL(2n+1), \Lambda_2\oplus\Lambda_1\oplus\Lambda_1^*\oplus\Lambda_1^*, V(n(2n+1))\oplus V(2n+1)\oplus V(2n+1)^*\oplus V(2n+1)^*)$ $(n\geq 1)$. $l=4$. $\chi_1(a,g)=a_1^na_2$, $\chi_2(a,g)=a_2a_3^{-1}$, $\chi_3(a,g)=a_2a_4^{-1}$, $\chi_4(a,g)=a_1a_3^{-1}a_4^{-1}$ for $(a,g)=(a_1,a_2,a_3,a_4,g)\in GL(1)^4\times SL(2n+1)$. $\kappa_1=2n-1, \kappa_2=1, \kappa_3=1, \kappa_4=2n$.

$$\{s_1+2j-1, s_i+1, s_4+2n, s_1+s_2+s_3+s_4+2n+1;\ i=2,3,4,\ j=1,2,\ldots,n\}.$$

(13) $(GL(1)^{n+1}\times SL(n), \Lambda_1\overbrace{\oplus\cdots\oplus}^{n+1}\Lambda_1, V(n)\overbrace{\oplus\cdots\oplus}^{n+1}V(n))$ $(n\geq 2)$. $l=n+1$. $\chi_1(a,g)=a_2\cdots a_{n+1}$, $\chi_2(a,g)=a_1a_3\cdots a_{n+1}$, $\ldots$, $\chi_{n+1}(a,g)=a_1\cdots a_n$ for $(a,g)=(a_1,\ldots,a_{n+1},g)\in GL(1)^{n+1}\times SL(n)$. $\kappa_1=\kappa_2=\cdots=\kappa_{n+1}=1$.

$$\{s_i+1, s_1+s_2+\cdots+s_{n+1}+j;\ i=1,2,\ldots,n+1,\ j=2,3,\ldots,n\}.$$

(14) $(GL(1)^{n+1}\times SL(n), \Lambda_1\overbrace{\oplus\cdots\oplus}^{n}\Lambda_1\oplus\Lambda_1^*, V(n)\overbrace{\oplus\cdots\oplus}^{n}V(n)\oplus V(n)^*)$ $(n\geq 2)$. $l=n+1$. $\chi_1(a,g)=a_1a_{n+1}^{-1}$, $\ldots$, $\chi_n(a,g)=a_na_{n+1}^{-1}$, $\chi_{n+1}(a,g)=a_1\cdots a_n$ for $(a,g)=(a_1,\ldots,a_{n+1},g)\in GL(1)^{n+1}\times SL(n)$. $\kappa_1=\kappa_2=\cdots=\kappa_n=1, \kappa_{n+1}=n-1$.

$$\{s_i+1, s_{n+1}+j, s_1+s_2+\cdots+s_{n+1}+n;\ i=1,2,\ldots,n,\ j=1,2,\ldots,n-1\}.$$

## References

[ 1 ] Y. Hironaka and F. Sato, Spherical functions and local densities of alternating forms, Amer. J. Math., **110** (1988), 473–512.

[ 2 ] H. Hosokawa, Some results on Igusa local zeta functions associated with simple prehomogeneous vector spaces, J. Math. Soc. Japan, **49** (1997), 565–587.

[ 3 ] T. Ibukiyama and H. Saito, On zeta functions associated to symmetric matrices, I: An explicit form of zeta functions, Amer. J. Math., **117** (1995), 1097–1155.

[ 4 ] J. Igusa, Some results on *p*-adic complex powers, Amer. J. Math., **106** (1984), 1103–1032.

[ 5 ] J. Igusa, On functional equations of complex powers, Invent. Math., **85** (1986), 1–29.

[ 6 ] J. Igusa, On a certain class of prehomogeneous vector spaces, J. Pure Appl. Algebra, **47** (1987), 265–282.

[ 7 ] J. Igusa, Zeta distributions associated with some invariants, Amer. J. Math., **109** (1987), 1–34.

[ 8 ] J. Igusa, An Introduction to the theory of local zeta functions, Stud. Adv. Math., **14**, Amer. Math. Soc., Providence, RI, 2000.

[ 9 ] S. Kasai, On the *b*-functions of prehomogeneous vector spaces of simple algebraic groups (in Japanese), Sūrikaisekikenkyūsho Kōkyūroku, **629** (1987), 31–81.

[10] T. Kimura, The *b*-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, Nagoya Math. J., **85** (1982), 1–80.

[11] T. Kimura, A classification of prehomogeneous vector spaces of simple algebraic groups with scalar multiplications, J. Algebra, **83** (1983), 72–100.

[12] T. Kimura, Arithmetic calculus of Fourier transforms by Igusa local zeta functions, Trans. Amer. Math. Soc., **346** (1994), 297–306.

[13] T. Kimura, S. Kasai and H. Hosokawa, Universal transitivity of simple and 2-simple prehomogeneous vector spaces, Ann. Inst. Fourier (Grenoble), **38** (1988), 11–41.

[14] T. Kimura and T. Kogiso, On adelic zeta functions of prehomogeneous vector spaces with a finitely many adelic open orbits, In: Zeta functions in geometry, Tokyo, 1990, (eds. N. Kurokawa and T. Sunada), Adv. Stud. Pure Math., **21**, Kinokuniya, 1992, pp. 21–31.

[15] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Math. Monogr., Clarendon Press, Oxford, 1979.

[16] D. Meuser, On the rationality of certain generating functions, Math. Ann., **256** (1981), 303–310.

[17] F. Sato, Zeta functions in several variables associated with prehomogeneous vector spaces I: functional equations, Tohoku Math. J., **34** (1982), 437–483.

[18] F. Sato, On functional equations of zeta distributions, In: Automorphic forms and geometry of arithmetic varieties, Aichi, (eds. K. Hashimoto and Y. Namikawa), Adv. Stud. Pure Math., **15** (1989), 465–508.

[19] M. Sato, Theory of prehomogeneous vector spaces (notes by T. Shintani in Japanese), Sugaku no Ayumi, **15** (1970), 85–157.

[20] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their invariants, Nagoya Math. J., **65** (1977), 1–155.

[21] K. Sugiyama, *b*-Functions of regular simple prehomogeneous vector spaces, Tsukuba J. Math., **26** (2002), 407–451.

[22] S. Wakatsuki, Igusa local zeta functions and integration formulas associated to equivariant maps, to appear in Osaka J. Math.

Satoshi WAKATSUKI
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1
Toyonaka, Osaka, 560-0043
Japan
E-mail: wakatsuki@gaia.math.wani.osaka-u.ac.jp