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Irregularities of microhyperbolic operators

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Abstract. We consider well-posedness of microhyperbolic Cauchy problems in the category of microlocal ultradistributions. For this purpose, we discuss about the expression of microdifferential operators, and define their irregularities. This enables us to give a general theory about the well-posedness.

1. Introduction.

It is well-known that a microhyperbolic Cauchy problem is always well-posed in the category of microfunctions (c.f. M. Kashiwara and T. Kawai [**3**]). Let us consider its well-posedness in the category of microfunctions which are the singularity spectrums of ultradistributions. There is a fundamental result of K. Kajitani and S. Wakabayashi [**2**] for this problem. However, there are some special but important cases for which their theory does not apply in a satisfactory way. Therefore we want to ameliorate it.

Let $n \geq 2$, let (x, ξ) be the variables of $\sqrt{-1}T^*\boldsymbol{R}^n$, and let $x = (x_1, x') = (x'', x_n) = (x_1, x''', x_n) = (x_1, \cdots, x_n)$. Let $x^* \in \sqrt{-1}T^*\boldsymbol{R}^n$ be the point defined by $x = 0$, $\xi = (0, \cdots, 0, \sqrt{-1})$, and let $x^{*\prime} \in \sqrt{-1}T^*\boldsymbol{R}^{n-1}$ be the point defined by $x' = 0$, $\xi' = (0, \cdots, 0, \sqrt{-1})$. We denote by $\mathscr{B}$, $\mathscr{C}$, $\mathscr{E}$, $\mathscr{O}$ the sheaves of hyperfunctions, microfunctions, microdifferential operators, and holomorphic functions, respectively (c.f. [**9**]). For $1 < s < \infty$ we denote the usual Gevrey functions with compact supports by $\mathscr{D}^{\{s\}}$ and $\mathscr{D}^{(s)}$:

$$\begin{aligned}\mathscr{D}^{\{s\}}(\omega) = \{f(x) \in C^\infty(\omega);\ &\operatorname{supp} f \text{ is compact and there exists}\\ &C > 0 \text{ such that } |\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}\alpha!^s\},\\ \mathscr{D}^{(s)}(\omega) = \{f(x) \in C^\infty(\omega);\ &\operatorname{supp} f \text{ is compact and for } \forall\varepsilon > 0 \text{ there exists}\\ &C_\varepsilon > 0 \text{ such that } |\partial_x^\alpha f(x)| \leq C_\varepsilon\varepsilon^{|\alpha|}\alpha!^s\}\end{aligned}$$

for an open subset ω of $\boldsymbol{R}^n$. Let $\mathscr{D}^{\{s\}\prime}{}_0 = \operatorname{inj}_{0\in\omega}\lim \mathscr{D}^{\{s\}\prime}(\omega)$, $\mathscr{D}^{(s)\prime}{}_0 = \operatorname{inj}_{0\in\omega}\lim \mathscr{D}^{(s)\prime}(\omega)$ be the set of germs of ultradistributions at the origin (c.f. H. Komatsu [**4**]). For the sake of convenience, we denote by $\mathscr{D}^{\{1\}\prime}$ the sheaf of hyperfunctions. We denote by $\mathscr{D}^{\{\infty\}\prime}$ (and also by $\mathscr{D}^{(\infty)\prime}$) the sheaf of distributions.

Let $\operatorname{sp} : \mathscr{B}_{\boldsymbol{R}^n,0} \longrightarrow \mathscr{C}_{\boldsymbol{R}^n,x^*}$ be the canonical map, and let

$$\mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*} = \operatorname{sp}(\mathscr{D}^{\{s\}\prime}{}_0) \quad (1 \leq s \leq \infty), \qquad \mathscr{C}^{(s)}_{\boldsymbol{R}^n,x^*} = \operatorname{sp}(\mathscr{D}^{(s)\prime}{}_0) \quad (1 < s \leq \infty),$$

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which we call microlocal ultradistributions. If $s = 1$, then $\mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*}$ is the usual set of germs of microfunctions.

Let $P(x, D) \in \mathscr{E}_{x^*}$ be written in the form

$$\begin{cases} P(x,D) = D_1^m + \displaystyle\sum_{0 \le j \le m-1} P_j(x, D')D_1^j, \\ \operatorname{ord} P_j \le m - j \qquad \text{for } 0 \le j \le m-1. \end{cases} \tag{1}$$

Here we define $D = \partial/\partial x$. We assume that

$$\begin{cases} \text{for } 1 \le j \le m \text{ there exists } \Lambda_j(x,\xi) = \xi_1 - \lambda_j(x,\xi') \in \mathscr{O}_{\mathbf{C}^{2n},x^*} \\ \text{which is homogeneous in } \xi \text{ of degree 1, vanishing at } x^*, \text{ and} \\ \text{we have } \sigma_m(P) = \displaystyle\prod_{1 \le j \le m} \Lambda_j(x,\xi), \end{cases} \tag{2}$$

where $\sigma_m(P)$ denotes the principal symbol of P. We finally assume that P is microhyperbolic, i.e.,

$$(x, \xi') \in \boldsymbol{R}^n \times \sqrt{-1}\boldsymbol{R}^{n-1} \quad \Longrightarrow \quad \lambda_j(x, \xi') \in \sqrt{-1}\boldsymbol{R} \qquad (1 \le j \le m). \tag{3}$$

We do not assume any further conditions for these characteristic roots.

Let us consider the following Cauchy problem:

$$P(x, D)u(x) = f(x), \quad D_1^{j-1}u(0, x') = v_j(x') \qquad (1 \le j \le m). \tag{4}$$

Remark. In (4) we assume $f(x) \in \mathscr{C}_{\boldsymbol{R}^n,x^*}$ and $v_1(x'), \cdots, v_m(x') \in \mathscr{C}_{\boldsymbol{R}^{n-1},x^{*\prime}}$, and that the support of f is contained in a small neighborhood of x^*. The problem (4) should be formulated more naturally for $u, f \in \rho_!(\mathscr{C}_{\boldsymbol{R}^n}|_L)$ where $L = \sqrt{-1}T^*\boldsymbol{R}^n \bigcap \{x_1 = 0\}$ and

$$\rho : \ L \ni (0, x', \xi') \ \longmapsto \ (x', \xi') \in \sqrt{-1}T^*\boldsymbol{R}^{n-1}.$$

In fact, the traces $D_1^{j-1}u(0, x')$ as microfunctions depend on the spectrum of u along the fiber of ρ. Therefore these traces are defined by a sheaf morphism $\rho_!(\mathscr{C}_{\boldsymbol{R}^n}|_L) \longrightarrow \mathscr{C}_{\boldsymbol{R}^{n-1}}$. Though, u is uniquely determined outside of x^* by the ellipticity of P. Hence, considering the flabbiness of $\mathscr{C}$, one can reduce the solvability of the Cauchy problem to the case that the support of f is contained in a sufficiently small neighborhood of x^* (We decompose $f = f_1 + f_2$ in $\rho_!(\mathscr{C}_{\boldsymbol{R}^n}|_L)$, where $f_2 = 0$ in a neighborhood of x^*. Then, consider the new problem for f_1 as the right term, and $v_j(x') - (D_1^{j-1}P^{-1}f_2)(0, x')$ as the initial values). It is well-known that for any $f(x), v_1(x'), \cdots, v_m(x')$ there exists a unique solution $u(x) \in \mathscr{C}_{\boldsymbol{R}^n,x^*}$ of (4) in this sense.

We say that P is $\{s\}$ well-posed if for any $f(x) \in \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*}$ with small support and $v_1(x'), \cdots, v_m(x') \in \mathscr{C}^{\{s\}}_{\boldsymbol{R}^{n-1},x^{*\prime}}$ there exists $u(x) \in \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*}$ which satisfies (4) in the above sense. Similarly we define (s) well-posedness. K. Kajitani and S. Wakabayashi [**2**] proved the following result:

THEOREM 1. *If* $1 \le s < m/(m-1)$, *then* P *is* $\{s\}$ *well-posed. If* $1 < s \le m/(m-1)$, *then* P *is* (s) *well-posed* (*if* $m = 1$, *we define* $m/(m-1) = \infty$).

To see that we cannot generally improve the ultradistribution order any more, let us consider the following:

EXAMPLE 1. Let $P = D_1^m - D_n^{m-1}$ and let us consider

$$P(x, D)u(x) = 0, \quad D_1^{j-1}u(0, x') = \delta_{j1}v(x') \qquad (1 \le j \le m).$$

It is easy to see that the microfunction solution is given by $u(x) = Q(x, D)v(x')$, where

$$Q(x, D) = \frac{1}{m} \sum_{0 \le j \le m-1} \exp\left(\frac{2\pi\sqrt{-1}j}{m} x_1 D_n^{(m-1)/m}\right).$$

If we restrict ourselves to microlocal ultradistributions, $Q : \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n, x^*} \longrightarrow \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n, x^*}$ is well-defined if, and only if, $1 \le s < m/(m-1)$, and Theorem 1 is the best possible result in this sense.

However, this criterion is not satisfactory for the following cases:

EXAMPLE 2 (regular involutive operators). Let $n \ge 3$ and let $P = D_1(D_1 + D_2) + \alpha D_2$, $\alpha \in \boldsymbol{C}$. Theorem 1 means that if $1 \le s < 2$ (resp. $1 < s \le 2$), then P is $\{s\}$ well-posed (resp. (s) well-posed). However Y. Okada [**8**] proved that it is $\{\infty\}$ well-posed.

EXAMPLE 3 (non-involutive operators). Let $P = D_1(D_1 + x_1^q D_n) + \alpha x_1^{q-1} D_n$. Theorem 1 means the same result as in Example 2 for this case. But it is well-known that P is $\{s\}$ well-posed (resp. (s) well-posed) for any s (Among many papers, we refer to N. Hanges [**1**]).

EXAMPLE 4 (operators with constant multiplicities). Assume that $\lambda_1 = \cdots = \lambda_m = 0$ in (1). H. Komatsu [**5**] defined the irregularity ι for this case by

$$\iota = \max(1,\ \max\{(m-j)/(m-j-\operatorname{ord} P_j);\ 0 \le j \le m-1\})$$

In this case it is known that P is $\{s\}$ well-posed (resp. (s) well-posed) if $1 \le s < \iota/(\iota-1)$ (resp. $1 < s \le \iota/(\iota-1)$). We have $\iota \le m$, and this is a stronger result than Theorem 1. Since the theory which we are going to develop is strongly influenced by [**5**], we briefly sketch the idea of Komatsu:

(i) A hyperbolic partial differential operator P with constant multiplicity can be written in a special form, which he called De Paris decomposition.
(ii) Rewriting P in such a form, we can define its irregularity ι similarly as above.
(iii) P is $\{s\}$ well-posed if $1 \le s < \iota/(\iota-1)$.

As we shall see in the next section, we can extend this theory to the general case.

Our aim is to give a criterion which improves Theorem 1, and is satisfactorily applicable to these examples too. For this purpose we shall define the irregularity of P in the next section, but before such a discussion we first give the main result.

Theorem 2. *If P satisfies* (1)–(3)*, then we can define* $\operatorname{Irr} P$*, which is a rational number satisfying* $1 \le \operatorname{Irr} P \le m$*. Furthermore, if* $1 \le s < \operatorname{Irr} P/(\operatorname{Irr} P - 1)$*, then P is* $\{s\}$ *well-posed, and if* $1 < s \le \operatorname{Irr} P/(\operatorname{Irr} P - 1)$*, then P is* (s) *well-posed.*

Remark. Since $1 \le \operatorname{Irr} P \le m$, Theorem 2 is always better than (or equivalent to) Theorem 1. In the above examples, it will turn out that $\operatorname{Irr} P = m$ in Example 1, $\operatorname{Irr} P = 1$ in Examples 2, 3, $\operatorname{Irr} P = \iota$ (= the above number) in Example 4. This coincides with the well-known results.

2. Lascar decomposition.

We first want to express P in a special form. If $0 \le q \le m$ we define S_{mq} to be the set of all q-tuples $\mu = (\mu_1, \mu_2, \cdots, \mu_q)$ such that $\mu_1, \mu_2, \cdots, \mu_q \in \{1, 2, \cdots, m\}$ are mutually distinctive. Here we distinguish different arrangements of the same set of numbers. Although S_{m0} does not make sense, we assume that it consists of only one element, which we denote by $\varnothing$. We define $S = \bigcup_{0 \le q \le m} S_{mq}$, and $S' = \bigcup_{0 \le q \le m-1} S_{mq}$. If $\mu \in S_{mq}$, then we define $|\mu| = q$, and $\Lambda^\mu(x, D) = \Lambda_{\mu_q}(x, D) \cdots \Lambda_{\mu_1}(x, D)$. Here $\Lambda_j(x, D)$ denotes the microdifferential operator whose complete symbol is $\Lambda_j(x, \xi)$. We also define $\Lambda^\varnothing = 1$. We define $\bar{\mathscr{E}}_{x^*}(j) = \{P \in \mathscr{E}_{x^*};\ [P, x_1] = 0,\ \operatorname{ord} P \le j\}$. By a Lascar decomposition we mean an expression of the following form:

$$\begin{cases} P(x, D) = \Lambda_m(x, D) \cdots \Lambda_1(x, D) + \displaystyle\sum_{\mu \in S'} \big(x_1^{-m+|\mu|} a_\mu(x, D') + b_\mu(x, D')\big) \Lambda^\mu(x, D), \\ a_\mu(x, D') \in \bar{\mathscr{E}}_{x^*}(0), \quad b_\mu(x, D') \in \bar{\mathscr{E}}_{x^*}(m - |\mu| - 1). \end{cases} \tag{5}$$

Here we consider a negative power of x_1 formally. The reason for using a negative power will be explained below. It may happen that μ and ν are different, but Λ^μ and Λ^ν are the same operator. However we distinguish these two expressions. Then it is easy to see that if $m \ge 2$, an arbitrary operator has an infinitely many Lascar decompositions. If $m = 1$, there uniquely exists a Lascar decomposition.

Example 2^{bis}. Let us consider

$$P = D_1(D_1 + D_2) + \alpha D_2 \tag{6}$$

again. Here $\Lambda_1 = D_1 + D_2$, $\Lambda_2 = D_1$, and by a Lascar decomposition we mean an expression of the following form:

$$P = \Lambda_2 \Lambda_1 + (x_1^{-1} a_1 + b_1) \Lambda_1 + (x_1^{-1} a_2 + b_2) \Lambda_2 + (x_1^{-2} a_\varnothing + b_\varnothing),$$

where $\operatorname{ord} a_\mu \le 0$, $\operatorname{ord} b_\mu \le 1 - |\mu|$. Note that (6) is a Lascar decomposition as it stands.

In fact we may take $b_\varnothing = \alpha D_2$, and all the other coefficient operators to be 0. We also have another expression:

$$P = \Lambda_2\Lambda_1 + \alpha\Lambda_1 - \alpha\Lambda_2. \tag{7}$$

This means $b_1 = -b_2 = \alpha$, and all the other coefficient operators are 0. We have still other expressions, but they are not important. Later we shall judge which expression is the best one.

EXAMPLE 3^{bis}. Let $P = D_1(D_1 + x_1^q D_n) + \alpha x_1^{q-1} D_n$, as before. Here $\Lambda_1 = D_1 + x_1^q D_n$, $\Lambda_2 = D_1$. Again this is a Lascar decomposition as it stands. We also have another expression, using a negative power: $P = \Lambda_2\Lambda_1 + \alpha x_1^{-1}\Lambda_1 - \alpha x_1^{-1}\Lambda_2$.

In (5), P is decomposed into three parts. Firstly, $\Lambda_m \cdots \Lambda_1$ denotes the principal part. The lower order terms are formally written in a form like an element of some $\mathscr{E}_{x^*}$-module generated by Λ^μ, $\mu \in S'$. For the sake of convenience, let us call Λ^μ the generator part, and $x_1^{-m+|\mu|} a_\mu + b_\mu$ the coefficient part. Roughly speaking we have

$$\begin{aligned} P(x, D) &= \text{principal part} + \text{lower order part} \\ &= \text{principal part} + (\text{coefficient part} \times \text{generator part}). \end{aligned}$$

If we calculate the amount of the lower order part (= coefficient part×generator part), we can prove Theorem 1. However we should be able to determine the ultradistribution order of the solution by the amount of the coefficient part alone (which is smaller than the whole lower order part). Of course less amount gives a better result, and such an idea leads us to Theorem 2. However, the coefficient part depends on Lascar decompositions, and we must next compare infinitely many decompositions.

For each Lascar decomposition (5) we define

$$\kappa = \max(1,\ \max\{(m - |\,\mu\,|)/(m - |\,\mu\,| - \operatorname{ord} b_\mu);\ \mu \in S'\}). \tag{8}$$

Clearly we have $1 \le \kappa \le m$. Let us consider the meaning of (8). In (5) we assumed that $\operatorname{ord} b_\mu \le m - |\,\mu\,| - 1$. Increasing this number by one, we consider that the order of b_μ may be at most $m - |\,\mu\,|$, and there remains a capacity of $m - |\,\mu\,| - \operatorname{ord} b_\mu$. Therefore the above fractional number is the reciprocal of the vacancy rate, which is equivalent to the occupancy rate. Anyway, it represents the congestion of the coefficient part. This number depends on the decomposition, and if κ is small, we may say that the corresponding decomposition is concisely written. We define $\operatorname{irr} P$ as the minimum value of κ among all the Lascar decompositions. Although there are infinitely many decompositions, the minimum value is well-defined. In fact from (8) we have $\kappa \in \{p/q;\ 1 \le q \le p \le m\}$, and there are only finitely many possible values. Let us consider the previous examples again.

EXAMPLE 2^{tris}. In (6) we have $m = 2$, and $\operatorname{ord} b_\varnothing = 1$, $|\varnothing| = 0$. Therefore we have $\kappa = \max(1, (2-0)/(2-0-1)) = 2$ for this decomposition. On the other hand, in (7) we have $\operatorname{ord} b_1 = \operatorname{ord} b_2 = 0$, $|1| = |2| = 1$. Therefore we have $\kappa = \max(1, (2-1)/(2-1-0)) = 1$ for this decomposition. This means that (7) is a better expression than (6), and we obtain $\operatorname{irr} P = 1$. We can similarly prove $\operatorname{irr} P = m, 1, \iota$ for Examples 1, 3, 4 respectively.

We next consider permutations in the principal part. Let $\tau \in S_{mm}$, and let us consider the following expression:

$$\begin{cases} P(x,D) = \Lambda^\tau(x,D) + \displaystyle\sum_{\mu \in S'} \left(x_1^{-m+|\mu|} a'_\mu(x,D') + b'_\mu(x,D') \right) \Lambda^\mu(x,D), \\ a'_\mu(x,D') \in \bar{\mathscr{E}}_{x^*}(0), \quad b'_\mu(x,D') \in \bar{\mathscr{E}}_{x^*}(m - |\mu| - 1). \end{cases} \tag{9}$$

We call (9) a Lascar decomposition subordinate to τ. For each expression we define $\kappa' = \max(1, \ \max\{(m - |\mu|)/(m - |\mu| - \operatorname{ord} b'_\mu); \ \mu \in S'\})$, and $\operatorname{irr}_\tau P = \min\{\kappa'; \text{ Lascar decompositions subordinate to } \tau\}$. Finally we define the irregularity $\operatorname{Irr} P$ of P by

$$\operatorname{Irr} P = \max\{\operatorname{irr}_\tau P; \ \tau \in S_{mm}\}. \tag{10}$$

In all the above examples we have $\operatorname{irr} P = \operatorname{irr}_\tau P = \operatorname{Irr} P$ (See Lemma 1 below).

Remark. (i) Although we have infinitely many Lascar decompositions, to construct the fundamental solution we can choose the best decomposition, and neglect all the other expressions. This means that we may use the minimum value of κ. Therefore we define $\operatorname{irr} P = \min\{\kappa; \text{ Lascar decompositions}\}$. To the contrary, we must take the maximum value in (10). This is because we need to consider Lascar decompositions subordinate to $\forall \tau \in S_{mm}$, as will be explained in section 4.

(ii) R. Lascar considered an expression of the form (5) in [**6**]. In his paper he assumed that the characteristic variety of P is regularly involutive, and he assumed that $a_\mu = 0$, $\operatorname{ord} b_\mu \le 0$. Under these assumptions he proved that the wave front set of the distribution solution of $Pu = 0$ propagates along the integral manifold defined by the characteristic variety. His result does not have a direct relation with ours.

The definition of $\operatorname{Irr} P$ consists of three steps. Firstly one must calculate κ for each Lascar decomposition, secondly calculate $\operatorname{irr} P$, and finally $\operatorname{Irr} P$. In some special cases one can skip the third step, and the definition becomes considerably simple. At first we give the following result:

Lemma 1. *Assume that*

$$\{\Lambda_i(x,\xi), \ \Lambda_j(x,\xi)\} \in x_1^{-1} \Lambda_i(x,\xi) \mathscr{O}_{x^*} + x_1^{-1} \Lambda_j(x,\xi) \mathscr{O}_{x^*} \tag{11}$$

for each i and j. Then we have $\operatorname{irr}_\sigma P = \operatorname{irr}_\tau P = \operatorname{Irr} P$ for each $\sigma, \tau \in S_{mm}$.

Proof. Let $\sigma, \tau \in S_{mm}$. From (11) we have $[\Lambda_i(x,D), \Lambda_j(x,D)] \in x_1^{-1} \bar{\mathscr{E}}_{x^*}(0) \Lambda_i(x,D) + x_1^{-1} \bar{\mathscr{E}}_{x^*}(0) \Lambda_j(x,D) + x_1^{-1} \bar{\mathscr{E}}_{x^*}(0)$. It follows that $\Lambda^\tau(x,D) - \Lambda^\sigma(x,D) = \sum_{\mu \in S'} x_1^{-m+|\mu|} a''_\mu(x,D') \Lambda^\mu(x,D)$ for some $a'' \in \bar{\mathscr{E}}_{x^*}(0)$. If $\operatorname{irr}_\tau P = \kappa_\tau$, then we have (9) where $b'_\mu \in \bar{\mathscr{E}}_{x^*}((\kappa_\tau - 1)(m - |\mu|)/\kappa_\tau)$ (If $j \in \boldsymbol{R}$, then we define $\bar{\mathscr{E}}_{x^*}(j) = \bar{\mathscr{E}}_{x^*}([j])$). It follows that $P = \Lambda^\sigma + \sum_{\mu \in S'} (x_1^{-m+|\mu|}(a'_\mu + a''_\mu) + b'_\mu) \Lambda^\mu$. Since we do not have changed b'_μ, we have $\operatorname{irr}_\sigma P \le \operatorname{irr}_\tau P$. Similarly we have $\operatorname{irr}_\sigma P \ge \operatorname{irr}_\tau P$, and we obtain the statement. □

Regularly involutive operators and non-involutive operators satisfy (11). In such cases we only need to calculate $\operatorname{irr} P$ instead of $\operatorname{Irr} P$. We emphasize again that $\operatorname{irr} P$ is easier to calculate than $\operatorname{Irr} P$. The second case is the following result:

LEMMA 2. *If $\sigma, \tau \in S_{mm}$, then we have*

$$\operatorname{irr}_\sigma P \le \max(2, \operatorname{irr}_\tau P), \qquad \operatorname{Irr} P \le \max(2, \operatorname{irr}_\tau P).$$

PROOF. Let $\sigma, \tau, \kappa_\tau$ be as above. We always have $[\Lambda_i(x,D), \Lambda_j(x,D)] \in \bar{\mathscr{E}}_{x^*}(1)$. It follows that $\Lambda^\tau(x,D) - \Lambda^\sigma(x,D) = \sum_{|\mu| \le m-2} b''_\mu(x,D')\Lambda^\mu(x,D)$ for some $b''_\mu \in \bar{\mathscr{E}}_{x^*}(1)$. Similarly as in Lemma 1, from (9) we obtain $P = \Lambda^\sigma + \sum_{\mu \in S'}(x_1^{-m+|\mu|}a'_\mu + (b'_\mu + b''_\mu))\Lambda^\mu$. We have $\operatorname{ord}(b'_\mu + b''_\mu) \le \max((\kappa_\tau - 1)/\kappa_\tau, 1/2) \cdot (m - |\mu|)$. This means that $\operatorname{irr}_\sigma P \le \max(\kappa_\tau, 2)$. The latter statement follows from this. □

This result is very interesting. We are often interested in microlocal ultradistributions of some special order s_0. Theorem 2 means that P is $\{s_0\}$ well-posed if

$$\operatorname{Irr} P < s_0/(s_0 - 1). \tag{12}$$

Assume that $1 \le s_0 < 2$. According to Lemma 2, (12) is equivalent to $\operatorname{irr} P < s_0/(s_0 - 1)$, which means that we can use $\operatorname{irr} P$ instead of $\operatorname{Irr} P$, and otherwise we must calculate $\operatorname{Irr} P$. Therefore the criterion is more complicated if $2 \le s_0 \le \infty$. The author thinks that it coincides with historical experience: The well-posedness is an easy problem in hyperfunction theory (where $s = 1$), and is a difficult problem in distribution theory (where $s = \infty$). Even in the case $2 \le s_0 \le \infty$, the situation is not so bad if either we can use Lemma 1 or m is not large. In distribution theory it is usual to assume such an assumption. Otherwise we need to calculate $\operatorname{irr}_\sigma P$ for many elements σ of S_{mm}. Then the criterion may be complicated.

At the end of this section we consider the case of $m = 2$ as an example. In this case we have $\operatorname{Irr} P \in \{1, 2\}$, and

$$\begin{aligned}
\operatorname{Irr} P = 1 &\iff \operatorname{irr}_{(1,2)} P = \operatorname{irr}_{(2,1)} P = 1 \\
&\iff \begin{cases} P \in \Lambda_2\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_2 + x_1^{-2}\bar{\mathscr{E}}_{x^*}(0), \\ P \in \Lambda_1\Lambda_2 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_2 + x_1^{-2}\bar{\mathscr{E}}_{x^*}(0) \end{cases} \\
&\iff \begin{cases} P \in \Lambda_2\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_2 + x_1^{-2}\bar{\mathscr{E}}_{x^*}(0), \\ [\Lambda_1,\ \Lambda_2] \in x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_2 + x_1^{-2}\bar{\mathscr{E}}_{x^*}(0). \end{cases}
\end{aligned}$$

This is equivalent to

$$P \in \Lambda_2\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_1 + x_1^{-1}\bar{\mathscr{E}}_{x^*}(0)\Lambda_2 + x_1^{-2}\bar{\mathscr{E}}_{x^*}(0), \tag{13}$$

$$\Lambda_1 \text{ and } \Lambda_2 \text{ satisfy (11).} \tag{14}$$

If (13) and (14) are true, then $\operatorname{Irr} P = 1$ and P is $\{s\}$ well-posed for any s. Otherwise $\operatorname{Irr} P = 2$ and P is $\{s\}$ well-posed for $1 \le s < 2$. In other words, according to our result we must assume (13) and (14) for the case $2 \le s \le \infty$. (13) means that the lower order terms must vanish according to some rule, and is not surprising. However as far as our theory applies, we must also assume condition (14) for the principal symbol.

3. Operator theory.

To prove Theorem 2, we need to use a theory of integral operators and symbol functions. They are similar to that of [**3**], but we develop a theory applicable for microlocal ultradistributions. Let $C > 0$ be a large number, $j \in \boldsymbol{Z}_+ = \{0, 1, 2, \cdots\}$, and let

$$\begin{aligned} A_j(C) = \big\{&(x, y_1, \xi') \in \boldsymbol{C}^n \times \boldsymbol{C} \times \boldsymbol{C}^{n-1};\ C|x_1| < 1,\ C|x'| < 1,\ C|x_1 - y_1| < 1, \\ &C|\xi'''| < \operatorname{Im}\xi_n,\ C|\operatorname{Re}\xi_n| < \operatorname{Im}\xi_n,\ C(j+1) < \operatorname{Im}\xi_n\big\}. \end{aligned}$$

Let $0 \le \kappa_1 < 1$. We denote by $\mathscr{R}_{\kappa_1}(C)$ set of formal series $\sum_{j\in\boldsymbol{Z}_+} f_j(x, y_1, \xi')$ such that

(i) f_j is holomorphic on $A_j(C)$,
(ii) for $\exists C' > 0$ and $\exists R \in (0,1)$ we have
$|f_j| \le C'R^j(\operatorname{Im}\xi_n)^{C'} \exp(C|x_1 - y_1|\operatorname{Im}\xi_n + C(\operatorname{Im}\xi_n)^{\kappa_1})$ on $A_j(C)$

for each j. We define $\mathscr{S}_{\kappa_1}(C) \subset \mathscr{R}_{\kappa_1}(C)$ and $\mathscr{N}_{\kappa_1}(C) \subset \mathscr{S}_{\kappa_1}(C)$ by

$$\begin{aligned} \mathscr{S}_{\kappa_1}(C) = \Big\{ &\sum_{j\in\boldsymbol{Z}_+} f_j(x, y_1, \xi') \in \mathscr{R}_{\kappa_1}(C);\ \text{for } \exists C' > 0 \text{ and } \exists R \in (0,1) \text{ we have} \\ &|f_0 + \cdots + f_j| \le C'(\operatorname{Im}\xi_n)^{C'} \exp\big(C(\operatorname{Im}\xi_n)^{\kappa_1}\big) \\ &\times \big\{\exp(C\psi(x, y_1, \xi')) + R^j \exp(C|x_1 - y_1|\operatorname{Im}\xi_n)\big\} \text{ on } A_j(C)\Big\}, \\ \mathscr{N}_{\kappa_1}(C) = \Big\{ &\sum_{j\in\boldsymbol{Z}_+} f_j(x, y_1, \xi') \in \mathscr{R}_{\kappa_1}(C);\ \text{for } \exists C' > 0 \text{ and } \exists R \in (0,1) \text{ we have} \\ &|f_1 + \cdots + f_j| \le C'R^j(\operatorname{Im}\xi_n)^{C'} \exp\big(C|x_1 - y_1|\operatorname{Im}\xi_n + C(\operatorname{Im}\xi_n)^{\kappa_1}\big) \\ &\text{on } A_j(C)\Big\}, \end{aligned}$$

where

$$\begin{aligned} \psi(x, y_1, \xi') = {}&|x_1 - y_1|(|\operatorname{Im}(x, x_1 - y_1)|\operatorname{Im}\xi_n + |\operatorname{Re}\xi'|) \\ &+ |\operatorname{Im}(x_1 - y_1)|(|(x', x_1 - y_1)|\operatorname{Im}\xi_n + |\xi'''|). \end{aligned}$$

We use the following notations for asymptotic expansions $a = \sum_{j\in\boldsymbol{Z}_+} a_j(x, y_1, \xi)$, $b = \sum_{j\in\boldsymbol{Z}_+} b_j(x, y_1, \xi)$. We write $\sum a_j = \sum b_j$ if $a_j = b_j$, for any j. This does not merely mean that their summations coincide. We define $\overline{a} = 0 + a_0 + a_1 + \cdots$, therefore $\overline{a}_j =$

a_{j-1} for $j \geq 1$. We define $\partial_{x_1} a$ by $\partial_{x_1} a = \sum \partial_{x_1} a_j$, and $\overline{\partial}_{x_1}$ by $\overline{\partial}_{x_1} a = \overline{\partial_{x_1} a}$. We define $c_j = \sum_{k+l+|\alpha|=j} [\partial_\xi^\alpha a_k \partial_x^\alpha b_l]_{\xi_1=0}/\alpha!$, and $c'_j = \sum_{k+l+|\alpha'|=j} [\partial_{\xi'}^{\alpha'} a_k \partial_{x'}^{\alpha'} b_l]_{\xi_1=0}/\alpha'!$. We denote $\sum c_j$ (resp. $\sum c'_j$) by $a \circ b$ (resp. $a \mathbin{\bar{\circ}} b$).

REMARK. Let $a = \sum_{j \in \boldsymbol{Z}_+} a_j$, $b = \sum_{j \in \boldsymbol{Z}_+} b_j \in \mathscr{R}_{\kappa_1}(C)$. Since they do not contain ξ_1, we have $a \circ b = a \mathbin{\bar{\circ}} b$, which belongs to $\mathscr{R}_{\kappa_1}(C_1)$, $C_1 \gg C$. We have $a - \overline{a} \in \mathscr{N}_{\kappa_1}(C)$. If $a \in \mathscr{N}_{\kappa_1}(C)$ or $b \in \mathscr{N}_{\kappa_1}(C)$, then we have $a \circ b \in \mathscr{N}_{\kappa_1}(C_1)$.

Let $\sum_{j \in \boldsymbol{Z}_+} f_j \in \mathscr{S}_{\kappa_1}(C)$ and let $C \ll C_1 \ll C_2$. We define

$$\mathscr{F}(f)(x,y) = \sum_{j \in \boldsymbol{Z}_+} \int_{\varDelta_j(C_1)} e^{(x'-y')\cdot\xi'} f_j(x, y_1, \xi') d\xi',$$

where $\varDelta_j(C_1) = \{\xi' \in \sqrt{-1}\boldsymbol{R}^{n-1};\ C_1|\operatorname{Im}\xi'''| < \operatorname{Im}\xi_n,\ C_1(j+1) < \operatorname{Im}\xi_n\}$. Then we have the following result:

LEMMA 3. (i) *$\mathscr{F}(f)(x,y)$ is holomorphic on $\varOmega(C_2) = \{(x,y) \in \boldsymbol{C}^{2n};\ C_2|(x,y)| < 1,\ \operatorname{Im}(x_n - y_n) > C_2|\operatorname{Im}(x, y'')|\}$. Therefore it becomes a defining function of a hyperfunction $g(x,y)$, and in fact we have $g \in \mathscr{D}^{(1/\kappa_1)}{}'_0$.*

(ii) *Let $\omega(C_2) = \{(x,y;\xi,\eta)\infty \in \sqrt{-1}S^*\boldsymbol{R}^{2n};\ C_2|(x,y)| < 1,\ C_2|\operatorname{Im}(\xi'', \xi+\eta)| < \operatorname{Im}\xi_n\}$. Then* S.S.$g \cap \omega(C_2)$ *is contained in*

$$\begin{aligned}\omega_1(C_2) = \big\{(x,y;\xi,\eta)\infty \in \omega(r);\ &|x'-y'| \leq C_2|x_1 - y_1|,\\ &|\operatorname{Im}(\xi+\eta)| \leq C_2|x_1-y_1|\operatorname{Im}\xi_n,\ |\operatorname{Im}\eta_1| \leq C_2(|(x,y_1)|\operatorname{Im}\xi_n + |\operatorname{Im}\xi'''|)\big\}.\end{aligned}$$

(iii) *If $\sum f_j \in \mathscr{N}_{\kappa_1}(C)$, then g is real analytic at the origin.*

PROOF. (i) Let $\varOmega_\varepsilon(C_2) = \{(x,y) \in \varOmega(C_2);\ \operatorname{Im}(x_n - y_n) > C_2|\operatorname{Im}(x,y'')| + \varepsilon\}$ for $0 < \varepsilon < C_2^{-1}$. If $(x,y) \in \varOmega_\varepsilon(C_2)$, there exists $C' > 0$ and $R \in (0,1)$ such that

$$\begin{aligned}|\mathscr{F}(f)(x,y)| &\leq \sum_{j \in \boldsymbol{Z}_+} \left| \int_{\varDelta_j(C_1)\setminus\varDelta_{j+1}(C_1)} e^{(x'-y')\cdot\xi'}(f_0 + \cdots + f_j)(x, y_1, \xi') d\xi' \right| \\ &\leq \sum_{j \in \boldsymbol{Z}_+} \int_{\varDelta_j(C_1)\setminus\varDelta_{j+1}(C_1)} \exp\big(-\operatorname{Im}(x'-y')\cdot\operatorname{Im}\xi' + C(\operatorname{Im}\xi_n)^{\kappa_1}\big) C'(\operatorname{Im}\xi_n)^{C'} \\ &\qquad \times \big\{\exp\{C|x_1 - y_1|\cdot|\operatorname{Im}(x, x_1 - y_1)|\operatorname{Im}\xi_n \\ &\qquad\qquad + C|\operatorname{Im}(x_1 - y_1)|(|(x', x_1 - y_1)|\operatorname{Im}\xi_n + |\xi'''|)\} \\ &\qquad\quad + R^j \exp(C|x_1 - y_1|\operatorname{Im}\xi_n)\big\}|d\xi'| \\ &\leq \sum_{j \in \boldsymbol{Z}_+} \int_{\varDelta_j(C_1)\setminus\varDelta_{j+1}(C_1)} C'(\operatorname{Im}\xi_n)^{C'} \exp\big(C(\operatorname{Im}\xi_n)^{\kappa_1}\big) \\ &\qquad \times \big\{\exp(-\varepsilon\operatorname{Im}\xi_n) + R^j \exp\big(4C_2^{-1}\operatorname{Im}\xi_n\big)\big\}|d\xi'|.\end{aligned}$$

We have $\operatorname{Im}\xi_n \leq C_1(j+2)$ on $\varDelta_j(C_1) \setminus \varDelta_{j+1}(C_1)$. Since C_2 is very large, we have

$C'R^j \exp(4C_2^{-1}\operatorname{Im}\xi_n) \le 2C'\exp(-\varepsilon\operatorname{Im}\xi_n)$ there. It follows that

$$\begin{aligned}|\mathscr{F}(f)(x,y)| &\le 3C' \sum_{j\in \boldsymbol{Z}_+} \int_{\Delta_j(C_1)\setminus\Delta_{j+1}(C_1)} \exp\big(-\varepsilon\operatorname{Im}\xi_n + C(\operatorname{Im}\xi_n)^{\kappa_1}\big)(\operatorname{Im}\xi_n)^{C'}|d\xi'| \\ &\le 3C' \sum_{j\in \boldsymbol{Z}_+} \int_{C_1(j+1)}^{C_1(j+2)} \exp\big(-\varepsilon t + Ct^{\kappa_1}\big) t^{C'+n-2}dt\end{aligned}$$

on $\Omega_\varepsilon(C_2)$ (We have denoted $t = \operatorname{Im}\xi_n$). From

$$\exp(Ct^{\kappa_1}) \le \exp\left\{C\left(\frac{\varepsilon}{3\kappa_1 C}\right)^{\kappa_1/(\kappa_1-1)} + \frac{\varepsilon}{3}t\right\}$$

and

$$t^{C'+n-2} \le \left(\frac{3}{\varepsilon}\right)^{[C']+n-1}([C']+n-1)!\exp\left\{\frac{\varepsilon}{3}t\right\},$$

we obtain

$$\begin{aligned}|\mathscr{F}(f)(x,y)| &\le 3C'\exp\left\{C\left(\frac{\varepsilon}{3\kappa_1 C}\right)^{\kappa_1/(\kappa_1-1)}\right\}\left(\frac{3}{\varepsilon}\right)^{[C']+n-1}([C']+n-1)! \\ &\quad\times \sum_{j\in \boldsymbol{Z}_+}\int_{C_1(j+1)}^{C_1(j+2)} \exp\left\{-\frac{\varepsilon}{3}t\right\}dt \\ &\le 3C'\exp\left\{C\left(\frac{\varepsilon}{3\kappa_1 C}\right)^{\kappa_1/(\kappa_1-1)}\right\}\left(\frac{3}{\varepsilon}\right)^{[C']+n}([C']+n-1)!\end{aligned}$$

on $\Omega_\varepsilon(C_2)$. This means $g \in \mathscr{D}^{(1/\kappa_1)\prime}{}_0$ (c.f. [**4**]).

(ii) Let

$$\mathscr{F}'(f)(x,y,\zeta') = \sum_{j\in \boldsymbol{Z}_+}\int_{C_1(j+1)}^{C_1(j+2)} e^{\sqrt{-1}r(x'-y')\cdot\zeta'}(f_0+\cdots+f_j)\big(x,y_1,\sqrt{-1}r\zeta'\big)r^{n-2}dr.$$

Let $\zeta'^0 = (0,\cdots,0,1) \in \boldsymbol{R}^{n-1}$. We can similarly prove that it is holomorphic on

$$\begin{aligned}&\big\{(x,y,\zeta') \in \boldsymbol{C}^n\times\boldsymbol{C}^n\times\boldsymbol{C}^{n-1};\ C_2|(x,y,\zeta'-\zeta'^0)| < 1, \\ &\quad \operatorname{Im}((x'-y')\cdot\zeta') > C_2|(x_1-y_1)|\cdot|\operatorname{Im}(x,y_1,\zeta')| + C_2|(x,y_1,\zeta''')|\cdot|\operatorname{Im}(x_1-y_1)|\big\},\end{aligned}$$

and is real analytic on

$$\big\{(x,y,\zeta') \in \boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ C_2|(x,y,\zeta'-\zeta'^0)| < 1,\ |(x'-y')\cdot\zeta'| > C_2|x_1-y_1|\big\}.$$

Therefore $\mathscr{F}'(f)$ defines a hyperfunction $g'(x,y,\zeta')$ on $\{(x,y,\zeta') \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^{n-1};\ C_2|(x,y,\zeta'-\zeta'^0)| < 1\}$, and we have

$$\begin{aligned}\mathrm{S.S.}g' \subset \big\{&(x,y,\zeta';\xi,\eta,z')\infty \in \sqrt{-1}S^*(\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^{n-1});\\ &|\zeta'\cdot(x'-y')| \le C_2|x_1-y_1|,\ \eta''' = \zeta_n^{-1}\eta_n\zeta''',\\ &|\mathrm{Im}(\xi+\eta)| \le \sqrt{2}C_2|x_1-y_1|\zeta_n^{-1}(-\mathrm{Im}\,\eta_n),\\ &|\mathrm{Im}(z'+\zeta_n^{-1}\eta_n(x'-y'))| \le C_2|x_1-y_1|\zeta_n^{-1}(-\mathrm{Im}\,\eta_n),\\ &|\mathrm{Im}\,\eta_1| \le C_2(|x_1-y_1|+|(x,y_1,\zeta''')|)\zeta_n^{-1}(-\mathrm{Im}\,\eta_n)\big\},\end{aligned}$$

where (ξ,η,z') is the dual variables of (x,y,ζ'). We can restrict g' to $\{|\zeta'| = 1\}$, and denoting by $\mu(\zeta')$ the canonical volume element on the unit sphere $\sqrt{-1}^n \int \mathrm{sp}(g'(x,y,\zeta')|_{|\zeta'|=1})d\mu(\zeta') \in \mathscr{C}(\omega(C_2))$ is well-defined, whose support is contained in $\omega_1(C_2)$. This coincides with the singularity spectrum of the above hyperfunction $g(x,y)$ on $\omega(C_2)$, and we obtain (ii). We can prove (iii) similarly to (i). $\square$

We finally define $h(x,y) \in \mathscr{C}_{\boldsymbol{R}^{2n},(x^*,-x^*)}$ by $h(x,y) = \mathrm{sp}(Y(x_1-y_1)g(x,y))$, where Y is Heaviside function. Let

$$\begin{aligned}\omega_2(C) = \big\{&(x,y;\xi,\eta)\infty \in \omega(C);\ x_1 \ge y_1,\ |x'-y'| \le C(x_1-y_1),\\ &|\mathrm{Im}(\xi+\eta)| \le C(x_1-y_1)\,\mathrm{Im}\,\xi_n\big\}.\end{aligned}$$

We have $h \in \mathscr{H}_{x^*} \cap \mathscr{C}^{(1/\kappa_1)}_{\boldsymbol{R}^{2n},(x^*,-x^*)}$, where $\mathscr{H}_{x^*} = \mathrm{inj}_{C>0}\lim \varGamma_{\omega_2(C)}(\omega(C);\ \mathscr{C}_{\boldsymbol{R}^{2n}})$. We denote $h(x,y)$ also by $\mathscr{F}''(f)$.

$\mathscr{H}_{x^*}$ was originally defined by [**3**], and has the following properties. If $h_1(x,y)$, $h_2(x,y) \in \mathscr{H}_{x^*}$, then we can define $h_3(x,z) = \int h_1(x,y)h_2(y,z)dy \in \mathscr{H}_{x^*}$. In this way $\mathscr{H}_{x^*}$ becomes a ring with the unit element $\mathrm{sp}\,\delta(x-y)$. Let $\mathscr{C}^+_{x^*} = \{u(x);\ u$ is a microfunction defined on a neighborhood of x^*, whose support is contained in $\{x_1 \ge 0\}\}$. If $h(x,y) \in \mathscr{H}_{x^*},\ u(x) \in \mathscr{C}^+_{x^*}$, then we can define $\int h(x,y)u(y)dy \in \mathscr{C}^+_{x^*}$. In this way $\mathscr{C}^+_{x^*}$ becomes a left $\mathscr{H}_{x^*}$-module.

Let $\sum_{j\in\boldsymbol{Z}_+} f_j(x,y_1,\xi') \in \mathscr{S}_{\kappa_1}(C)$, $Q(x,D') = \sum_{|\alpha'|\le i} Q_{\alpha'}(x)D'^{\alpha'} \in \bar{\mathscr{E}}_{x^*}(i)$, and let $Q(x,\xi') = \sum Q_j(x,\xi')$ be its complete symbol, where Q_j denotes $\sum_{|\alpha'|=i-j} Q_{\alpha'}(x)\xi'^{\alpha'}$. We have $\sum \widetilde{f}_j \underset{\mathrm{def}}{=} Q \circ \sum f_j \in \mathscr{S}_{\kappa_1}(\exists C_1)$, and we have $Q(x,D')\mathscr{F}''(f) = \mathscr{F}''(\widetilde{f})(x,y)$.

If $K_P(x,y) \in \mathscr{H}_{x^*}$ is the kernel function of our microhyperbolic microdifferential operator $P(x,D)$, then K_P has the both-side inverse in $\mathscr{H}_{x^*}$. For these facts, see [**3**].

As for the above $\mathscr{F}''(f)$, we have $\mathscr{F}''(f) \in \mathscr{H}_{x^*} \cap \mathscr{C}^{(1/\kappa_1)}_{\boldsymbol{R}^{2n},(x^*,-x^*)}$, and it is easy to see that if $u(x) \in \mathscr{C}^+_{x^*} \cap \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*}$, $1 \le s < 1/\kappa_1$ (resp. $\mathscr{C}^+_{x^*} \cap \mathscr{C}^{(s)}_{\boldsymbol{R}^n,x^*}$, $1 \le s \le 1/\kappa_1$), then we have $\int h(x,y)u(y)dy \in \mathscr{C}^+_{x^*} \cap \mathscr{C}^{\{s\}}_{\boldsymbol{R}^n,x^*}$ (resp. $\mathscr{C}^+_{x^*} \cap \mathscr{C}^{(s)}_{\boldsymbol{R}^n,x^*}$). We shall prove that K_P^{-1} belongs to $\mathscr{H}_{x^*} \cap \mathscr{C}^{(1/\kappa_1)}_{\boldsymbol{R}^{2n},(x^*,-x^*)}$ with $\kappa_1 = (\mathrm{Irr}\,P-1)/\mathrm{Irr}\,P$. This means that the fundamental solution of P whose support is contained in the forward half space is a microlocal ultradistribution of order $\mathrm{Irr}\,P/(\mathrm{Irr}\,P-1)$, and Theorem 2 is its direct consequence. Therefore it suffices to show that the symbol function of K_P^{-1} belongs to

$\mathscr{S}_{\kappa_1}(C)$.

$\mathscr{R}_{\kappa_1}(C)$ defines a formal operator, which was called a "pseudodifferential operator of finite velocity" in [**3**]. We shall at first construct a formal parametrix belonging to $\mathscr{R}_{\kappa_1}(C)$, and afterwards show that it in fact belongs to $\mathscr{S}_{\kappa_1}(C)$.

4. Matrix representation.

As in [**3**], we first construct a formal parametrix belonging to $\mathscr{R}_{\kappa_1}(C)$. We define $L(x,D) \in \mathscr{E}_{x^*}^{m\times m}$ ($= m\times m$ matrix of $\mathscr{E}_{x^*}$) by $L(x,D) = D_1 I_m + L'(x,D')$, where

$$L'(x,D') = \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & -1 \\ P_0(x,D') & & \cdots & & P_{m-1}(x,D') \end{pmatrix}.$$

We denote by $L_{(p,q)}$ the (p,q) component of L'. It is at most of order $p-q+1$, and we define $L'_{j,(p,q)} = \sigma_{p-q+1-j}(L_{(p,q)})$ for $j \in \boldsymbol{Z}_+$. Consequently the complete symbol $\sigma(L')$ has an asymptotic expansion $\sigma(L') \sim \sum_{j\in\boldsymbol{Z}_+} L'_j = \sum_{j\in\boldsymbol{Z}_+}(L'_{j,(p,q)})$. We want to solve $\partial_{x_1}U(x,y_1,\xi') + L'(x,\xi') \mathbin{\bar{\circ}} U(x,y_1,\xi') = O$, $U(x,y_1,\xi')|_{x_1=y_1} = I_m$. In other words, we have $U = \sum_j U_j$, and

$$\partial_{x_1}U_j + \sum_{k+l+|\alpha'|=j} \frac{1}{\alpha'!}\partial_{\xi'}^{\alpha'}L'_k\partial_{x'}^{\alpha'}U_l = O, \qquad U_j(x,y_1,\xi')|_{x_1=y_1} = \delta_{j0}I_m. \tag{15}$$

Let $C \gg 1$. According to [**3**] there uniquely exists a solution $U_j \in \mathscr{O}(A_0(C))^{m\times m}$ of (15), and we have

$$|U_j|\Big(= m\times\max_{p,q}|U_{j,(p,q)}|\Big) \le C^{j+1}j!\,(\operatorname{Im}\xi_n)^{-j+m}\exp(C|x_1-y_1|\operatorname{Im}\xi_n). \tag{16}$$

Let $\operatorname{Irr}P = \kappa_0$. It is convenient to use $\kappa_1 = (\kappa_0-1)/\kappa_0$ instead of κ_0. Note that $1 \le \kappa_0 \le m$ and $0 \le \kappa_1 \le (m-1)/m$. Since $C^{j+1}j!\,(\operatorname{Im}\xi_n)^{-j} \le C^{-j+1}$ on $A_j(C^2)$, we have $\sum U_j \in \mathscr{R}_{\kappa_1}(C_2)^{m\times m}$. This part is very easy and is the same as Proposition 2.2 of [**3**]. The problem is to show that $\sum U_j \in \mathscr{S}_{\kappa_1}(C_2)^{m\times m}$. Assume that this is true. Then as in section 3, we can define $E(x,y) = \mathscr{F}''(U) \in (\mathscr{H}_{x^*}\cap\mathscr{C}^{(1/\kappa_1)}_{\boldsymbol{R}^{2n},(x^*,-x^*)})^{m\times m}$, which satisfies $L(x,D)E(x,y) = \operatorname{sp}\delta(x-y)I_m$. Therefore we have $(L(x,D)E(x,y))_{(p,m)} = \delta_{pm}\operatorname{sp}\delta(x-y)$. It follows that

$$\begin{aligned} &D_1E(x,y)_{(p,m)} = E(x,y)_{(p+1,m)} \qquad \text{for } 1\le p\le m-1,\\ &P(x,D)E(x,y)_{(1,m)} = \operatorname{sp}\delta(x-y). \end{aligned}$$

It follows that $E(x,y)_{(1,m)} \in \mathscr{H}_{x^*}\cap\mathscr{C}^{(1/\kappa_1)}_{\boldsymbol{R}^{2n},(x^*,-x^*)}$ is the inverse of K_P, and it suffices to show $\sum U_j \in \mathscr{S}_{\kappa_1}(C_2)^{m\times m}$. For this purpose we need another matrix expression.

For each $\tau \in S_{mm}$ we have

$$\begin{cases} P(x,D)=\Lambda^{\tau}(x,D)+\displaystyle\sum_{\mu\in S'}\big(x_1^{-m+|\mu|}a_{\mu}^{\tau}(x,D')+b_{\mu}^{\tau}(x,D')\big)\Lambda^{\mu}(x,D),\\ a_{\mu}^{\tau}(x,D')\in\bar{\mathscr{E}}_{x^*}(0),\\ b_{\mu}^{\tau}(x,D')\in\bar{\mathscr{E}}_{x^*}(\kappa_1(m-|\mu|))\ \big(\subset\bar{\mathscr{E}}_{x^*}(m-|\mu|-1)\big).\end{cases}\tag{17}$$

We define $c_{\mu}^{\tau}(x,D')=x_1^{-m+|\mu|}a_{\mu}^{\tau}(x,D')+b_{\mu}^{\tau}(x,D')$ and $\sigma_j(c_{\mu}^{\tau})=x_1^{-m+|\mu|}\sigma_j(a_{\mu}^{\tau})+\sigma_j(b_{\mu}^{\tau})$. We have $\sigma(c_{\mu}^{\tau})=\sum_{j\le[\kappa_1(m-|\mu|)]}\sigma_j(c_{\mu}^{\tau})$. Our aim in the rest of this section is to delete the "generator part" from (17). For this purpose we let $m'=m!\times m$ and rewrite (17) using an $m'\times m'$ matrix.

We enumerate the elements of S_{mm} and let $S_{mm}=\{\tau^1,\cdots,\tau^{m!}\}$. If $1\le j\le m!$, we have $\tau^j=(\tau_1^j,\cdots,\tau_m^j)\in S_{mm}$. Let us define microfunctions $u_1(x),\cdots,u_{m'}(x)$ in the following way. We denote by $\overline{p}\in\{0,1,\cdots,m-1\}$ the remainder of an integer p divided by m. Let $u(x)$ be a solution of (4). Then for any $p\in\{1,2,\cdots,m'\}$ we define

$$u_p(x)=\Lambda_{\tau_l^k}\cdots\Lambda_{\tau_1^k}u(x)$$

where $l=\overline{p-1}$ and $k=1+(p-l-1)/m\in\{1,\cdots,m!\}$. Therefore we have $k\in\{1,2,\cdots,m!\}$ and $p=(k-1)m+l+1$. We define an m'-dimensional column vector $\vec{u}(x)$ by $\vec{u}(x)={}^t(u_1(x),\cdots,u_{m'}(x))$. If $0\le\overline{p-1}(=l)\le m-2$, then we have $\Lambda_{\tau_{l+1}^k}u_p=u_{p+1}$. If $\overline{p-1}(=l)=m-1$, then from (17) we obtain

$$\Lambda_{\tau_{l+1}^k}u_p=\Lambda_{\tau_m^k}\cdots\Lambda_{\tau_1^k}u(x)=\Lambda^{\tau^k}u=-\sum_{\mu\in S'}c_{\mu}^{\tau^k}(x,D')\Lambda^{\mu}(x,D)u+f(x).\tag{18}$$

In (18) we can delete the "generator part" as follows. Note that $\vec{u}$ consists of $\Lambda^{\mu}u$ ($\mu\in S'$), and for any $\mu\in S'$ there exists at least one component u_q such that $u_q=\Lambda^{\mu}u$. For each $\mu\in S'$, we select such a number q, and we can define a map $h:S'\ni\mu\longrightarrow q\in\{1,\cdots,m'\}$. We have the following result:

LEMMA 4. *h is an injection, and we have $\overline{h(\mu)-1}=|\mu|$.*

PROOF. Since $h(\mu)=q$ means $u_q(x)=\Lambda^{\mu}u(x)$, the injectivity is clear. If $q=(k-1)m+l+1$, $1\le k\le m!$, $0\le l\le m-1$, then we have $u_q=\Lambda^{\mu}u=\Lambda_{\tau_l^k}\cdots\Lambda_{\tau_1^k}u$, which means $\mu=(\tau_1^k,\cdots,\tau_l^k)$, and $|\mu|=l=\overline{q-1}$. □

Now we can rewrite (18) as $\Lambda_{\tau_{l+1}^k}u_p=-\sum_{\mu\in S'}c_{\mu}^{\tau^k}(x,D')u_{h(\mu)}(x)+f(x)$. We have the following system for $p=(k-1)m+l+1\in\{1,\cdots,m'\}$:

$$\begin{aligned}\Lambda_{\tau_{l+1}^k}u_p&=u_{p+1}, && p\notin m\boldsymbol{Z},\\ \Lambda_{\tau_{l+1}^k}u_p&=-\sum_{\mu\in S'}c_{\mu}^{\tau^k}(x,D')u_{h(\mu)}(x)+f(x)\\ &=-\sum_{1\le q\le m'}\ \sum_{\mu\in h^{-1}(\{q\})}c_{\mu}^{\tau^k}(x,D')u_q(x)+f(x), && p\in m\boldsymbol{Z}.\end{aligned}$$

Let us rewrite this by use of an $m' \times m'$ matrix $M(x, D) = M'(x, D) + M''(x, D')$. $M'(x, D)$ is a diagonal matrix, and if $p = (k-1)m + l + 1,\ 1 \le k \le m!,\ 0 \le l \le m-1$, then its (p, p) component $M'_{(p,p)}$ is given by $M'_{(p,p)} = \Lambda_{\tau^k_{l+1}}$, which we also denote by $M'_{(p)}$. If $p \notin m\boldsymbol{Z}$, then $M''_{(p,q)}$ is given by $M''_{(p,q)} = -\delta_{p+1,q}$, and if $p \in m\boldsymbol{Z}$, then $M''_{(p,q)} = -\sum_{\mu \in h^{-1}(\{q\})} c^{\tau^{p/m}}_{\mu}(x, D')$. Since the rows of M'' repeat similar forms periodically with period m, we write the p-th rows for $(k-1)m + 1 \le p \le km$:

$$
\begin{array}{cl}
\left(\begin{array}{ccccccccc}
 & & & & \text{———————} & & & & \\
 & 0 & -1 & 0 & \cdots & 0 & 0 & & \\
0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \\
 & & & \cdots & & & & & \\
 & 0 & 0 & 0 & \cdots & 0 & -1 & & \\
* \quad \cdots & & & & -\displaystyle\sum_{\mu \in h^{-1}(\{q\})} c^{\tau^k}_{\mu} & & & \cdots & * \\
 & & & & \text{———————} & & & &
\end{array}\right) & \begin{array}{l} \\ (\quad (k-1)m+1 \\ \\ \\ (\quad km \\ \\ \end{array} \\
\begin{array}{ccc} \overset{\frown}{(k-1)m+1} \qquad\qquad & \overset{\frown}{q} \qquad\qquad & \overset{\frown}{km} \end{array} &
\end{array}
$$

M has the following properties. Firstly, the principal part M' is a diagonal matrix. Secondly the lower order part M'' consists of the "coefficient part" alone, and we have the following result:

Lemma 5. $M''_{(p,q)} \in x_1^{-\overline{p-1}+\overline{q-1}-1} \bar{\mathscr{E}}_{x^*}(0) + \bar{\mathscr{E}}_{x^*}\big(\kappa_1(\overline{p-1} - \overline{q-1} + 1)\big)$.

Proof. Let $p \notin m\boldsymbol{Z}$. We have $M''_{(p,q)} = -\delta_{p+1,q} \in \bar{\mathscr{E}}_{x^*}(\kappa_1(\overline{p-1} - \overline{q-1} + 1))$. Let $p \in m\boldsymbol{Z}$. From (17) and Lemma 4 we have

$$
\begin{aligned}
M''_{(p,q)} &= -\sum_{\mu \in h^{-1}(\{q\})} c_{\sigma^{p/m},\mu}(x, D') \in x_1^{-m+\overline{q-1}} \bar{\mathscr{E}}_{x^*}(0) + \bar{\mathscr{E}}_{x^*}\big(\kappa_1(m - \overline{q-1})\big) \\
&= x_1^{-\overline{p-1}+\overline{q-1}-1} \bar{\mathscr{E}}_{x^*}(0) + \bar{\mathscr{E}}_{x^*}\big(\kappa_1(\overline{p-1} - \overline{q-1} + 1)\big). \qquad \square
\end{aligned}
$$

We define

$$
\begin{cases}
M'_{j,(p,q)}(x, \xi) = \delta_{pq}\delta_{j0} M'_{(p,p)}, \\
M''_{j,(p,q)}(x, \xi') = \sigma_{[\kappa_1(\overline{p-1} - \overline{q-1} + 1)] - j}\big(M''_{(p,q)}\big),
\end{cases} \tag{19}
$$

and $M_j = M'_j + M''_j$ for $j \in \boldsymbol{Z}_+$. Therefore we have $M = \sum_{j \in \boldsymbol{Z}_+} M_j \in x_1^{-m} \mathscr{R}_{\kappa_1}(C)^{m' \times m'}$, $M' = \sum_{j \in \boldsymbol{Z}_+} M'_j$, $M'' = \sum_{j \in \boldsymbol{Z}_+} M''_j$. Note that $M''_{j,(p,q)} = 0$ if $\overline{p-1} + 2 \le \overline{q-1}$.

We have constructed a matrix $U = \sum U_j$ satisfying (15). Let us define an $m' \times m'$ matrix $V = \sum V_j \in \mathscr{R}_{\kappa_1}(C)^{m' \times m'}$ by

$$
\begin{cases}
V_{(p,q)} = U_{(1,q)}, & p \in m\boldsymbol{Z} + 1, \\
V_{(p,q)} = M'_{(p-1)} \circ V_{(p-1,q)}, & p \notin m\boldsymbol{Z} + 1.
\end{cases} \tag{20}
$$

Then we have the following result:

LEMMA 6. $\partial_{x_1} V + M \mathbin{\bar{\circ}} V \in x_1^{-m} \mathscr{N}_{\kappa_1}(C)^{m' \times m'}$.

PROOF. If $p \notin m\boldsymbol{Z}$, then from (20) we have

$$\begin{aligned}(M \circ V)_{(p,q)} &= M_{(p,p)} \circ V_{(p,q)} + M_{(p,p+1)} \circ V_{(p+1,q)} \\ &= M'_{(p)} \circ V_{(p,q)} - V_{(p+1,q)} = 0,\end{aligned}$$

and

$$\begin{aligned}\partial_{x_1} V_{(p,q)} + (M \mathbin{\bar{\circ}} V)_{(p,q)} &= \partial_{x_1} V_{(p,q)} - \overline{\partial_{x_1}} V_{(p,q)} + (M \circ V)_{(p,q)} \\ &= \partial_{x_1} V_{(p,q)} - \overline{\partial_{x_1}} V_{(p,q)} \in \mathscr{N}_{\kappa_1}(C).\end{aligned}$$

Let us consider the case $p = mp' \in m\boldsymbol{Z}$. If $1 \le p'' \le m$, (20) means $V_{((p'-1)m+p'',q)} = M'_{((p'-1)m+p''-1)} \circ \cdots \circ M'_{((p'-1)m+1)} \circ U_{(1,q)}$. If $h(\mu) = (p'-1)m + p'' - 1$, then we have $|\mu| = p'' - 1$ and $V_{((p'-1)m+p'',q)} = (\Lambda^{\circ\mu}) \circ U_{(1,q)}$. Here $\Lambda^{\circ\mu}$ denotes $\Lambda_{\mu_{p''-1}} \circ \cdots \circ \Lambda_{\mu_1}$. Therefore we have

$$\begin{aligned}(M \circ V)_{(p,q)} &= M'_{(p)} \circ V_{(p,q)} + \sum_{1 \le r \le m'} M''_{(p,r)} \circ V_{(r,q)} \\ &= \left(\Lambda^{\circ\tau^{p'}}\right) \circ U_{(1,q)} + \sum_{\mu \in S'} c_\mu \circ (\Lambda^{\circ\mu}) \circ U_{(1,q)}.\end{aligned}$$

We regard $P(x,\xi)$ as a formal series $P = \sum_{j \ge 0} \sigma_{m-j}(P)$. We have

$$P(x,D) = \Lambda^{\tau^{p'}}(x,D) + \sum_{\mu \in S'} c_\mu(x,D') \Lambda^\mu(x,D)$$

and

$$P(x,\xi) \equiv \Lambda^{\circ\tau^{p'}} + \sum_{\mu \in S'} c_\mu \circ (\Lambda^{\circ\mu}) \qquad \text{modulo } x_1^{-m} \mathscr{N}_{\kappa_1}(C).$$

It follows that

$$\begin{aligned}(M \circ V)_{(p,q)} &= \left(\Lambda^{\circ\tau^{p'}}\right) \circ U_{(1,q)} + \sum_{\mu \in S'} c_\mu \circ (\Lambda^{\circ\mu}) \circ U_{(1,q)} \\ &\equiv P \circ U_{(1,q)} \qquad \text{modulo } x_1^{-m} \mathscr{N}_{\kappa_1}(C).\end{aligned}$$

On the other hand, we have $L \circ U = \overline{\partial}_{x_1} U + L' \circ U = \overline{\partial}_{x_1} U - \partial_{x_1} U \in \mathscr{N}_{\kappa_1}(C)^{m \times m}$, and

$$P \circ U_{(1,q)} \equiv (L \circ U)_{(m,q)} \equiv 0 \qquad \text{modulo } \mathscr{N}_{\kappa_1}(C).$$

It follows that $(M \circ V)_{(p,q)} \in x_1^{-m} \mathscr{N}_{\kappa_1}(C)$, and $\partial_{x_1} V_{(p,q)} + (M \mathbin{\overline{\circ}} V)_{(p,q)} = \partial_{x_1} V_{(p,q)} - \overline{\partial}_{x_1} V_{(p,q)} + (M \circ V)_{(p,q)} \in x_1^{-m} \mathscr{N}_{\kappa_1}(C)$. $\square$

It will turn out that the negative powers of x_1 does not have any influence on the ultradistribution order. Neglecting them, the diagonal elements of M'' are at most of order κ_1 (Since we are not considering any operators of fractional orders, they are in fact at most of order 0). The orders of the off-diagonal components vary according to their positions, but we may say that the matrix order of M'' is equal to κ_1. Using these facts, we shall show that $V \in \mathscr{S}_{\kappa_1}(C)^{m' \times m'}$, which implies $U \in \mathscr{S}_{\kappa_1}(C)^{m \times m}$.

5. Construction of the real parametrix.

To see that $V \in \mathscr{S}_{\kappa_1}(C)^{m' \times m'}$, we first consider phase functions. Let $r \in \boldsymbol{N} = \{1, 2, 3, \cdots\}$. We call $I = (i_1, \cdots, i_r) \in \{1, \cdots, m'\}^r$ a multi index of length r, and we define $|I| = r$. We denote $M'_{(p,p)}(x, \xi')$ also by $\xi_1 - m_p(x, \xi')$. Therefore we have $m_p = \lambda_j$ for some j. We define the phase function $\varphi_I(x, t, \xi')$ where $t = (t_1, \cdots, t_r)$ and $I = (i_1, \cdots, i_r)$, by induction on r. If $r = 1$, then $\varphi_I(x, t_1, \xi')$ is the solution of

$$\partial_{x_1} \varphi_I(x, t_1, \xi') - m_{i_1}\big(x, \xi' + \nabla_{x'} \varphi_I(x, t_1, \xi')\big) = 0, \quad \varphi_I(x, t_1, \xi')|_{x_1 = t_1} = 0.$$

Assume that $r \geq 2$ and that φ_I for $|I| \leq r - 1$ have already been defined. Let $|I| = r$. We define $I'' = (i_1, \cdots, i_{r-1})$ and $t'' = (t_1, \cdots, t_{r-1})$. We define φ_I as the solution of

$$\begin{cases} \partial_{x_1} \varphi_I(x, t, \xi') - m_{i_r}\big(x, \xi' + \nabla_{x'} \varphi_I(x, t, \xi')\big) = 0, \\ \varphi_I(x, t, \xi')|_{x_1 = t_r} = \varphi_{I''}(x, t'', \xi')|_{x_1 = t_r}. \end{cases} \tag{21}$$

Here t_1 corresponds to y_1 in the previous notation, and $t_2, \cdots, t_r$ are parameters which in fact move between y_1 and x_1.

Let $C > 0$ and let

$$A_j^r(C) = \Big\{ (x, t, \xi') \in \boldsymbol{C}^n \times \boldsymbol{C}^r \times \boldsymbol{C}^{n-1};\ (x, t_1, \xi') \in A_j(C), \\ C \sum_{1 \leq r' \leq r-1} |t_{r'} - t_{r'+1}| + C|t_r - x_1| \leq 1 \Big\}$$

for $r \in \boldsymbol{N},\ j \in \boldsymbol{Z}_+$. Then we have the following result:

Lemma 7. *If C is large enough, then φ_I is holomorphic on $A_0^r(C)$ for $r = |I|$, and we have $|\varphi_I| \leq C \sum_{1 \leq r' \leq r-1} |t_{r'} - t_{r'+1}| + C|t_r - x_1|$ there. Here we can choose the same C for any I.*

Proof. Let $(x, t, \xi') \in \boldsymbol{C}^n \times \boldsymbol{C}^r \times \boldsymbol{C}^{n-1}$, and let $\gamma(x_1, t)$ be the union of line segments connecting $t_1, \cdots, t_r, x_1$ in this order. γ contains r line segments, and we denote by $\gamma_{r'}$ that one from $t_{r'}$ to $t_{r'+1}$ (t_{r+1} denotes x_1). Let $\varphi'_I(s, x', \xi') = \varphi_{(i_1, \cdots, i_{r'})}(s, x', t_1, \cdots, t_{r'}, \xi')$, and $m'_I(s, x', \xi') = m_{i_{r'}}(s, x', \xi')$ if $s \in \gamma_{r'}$. Then we have

$$\partial_s \varphi'_I(s,x',\xi') - m'_I\big(s,x',\xi' + \nabla_{x'}\varphi'_I(s,x',\xi')\big) = 0, \quad \varphi'_I(s,x',\xi')|_{s=t_1} = 0$$

if $s \in \gamma \setminus \{t_1, \cdots, t_r\}$. For any I and γ, $m'_I(s,x',\xi')$ are Lipschitz continuous with respect to ξ', and we can take the same Lipschitz constant for them all, and we obtain the uniform domain $A_0^r(C)$ and the uniform estimate for $|\varphi_I|$ (For example, we can apply [**7**] to the present context). $\square$

We next remark the following result:

Lemma 8. *Let $C > 0$ be large. If $|I| = r$, then we have*

$$\partial_{x'}^{\alpha'}\big(\exp(\varphi_I)\big) = \sum_{0 \le j \le |\alpha'|} e_{I,\alpha',j}(x,t,\xi')\exp(\varphi_I),$$

$$\left|\partial_{x'}^{\beta'} e_{I,\alpha',j}(x,t,\xi)\right| \le C^{4|\alpha'|+2|\beta'|}\Bigg(\sum_{1 \le r' \le r} |t_{r'} - t_{r'+1}| \operatorname{Im}\xi_n\Bigg)^{|\alpha'|-j}(j + |\beta'|)!$$

where $e_{I,\alpha',j}(x,t,\xi')$ is a function for $(x,t,\xi') \in A_j^r(C)$. Here we have denoted $t_{r+1} = x_1$. Furthermore, we have $e_{I,\alpha',0} = \prod_{2 \le k \le n}(\partial_{x_k}\varphi_I)^{\alpha_k}$.

Proof. If $|\alpha'| = 0$, then the statements are trivial. Let $p \ge 1$, and assume that the statements are true for $|\alpha'| = p-1$. Let us consider the case $|\alpha'| = p$. We assume that $\alpha'^1 + \alpha'^2 = \alpha'$, $|\alpha'^1| = 1$, $|\alpha'^2| = p-1$. Then by the assumption of induction we have

$$\begin{aligned}\partial_{x'}^{\alpha'}\big(\exp(\varphi_I)\big) &= \partial_{x'}^{\alpha'^1}\Bigg(\sum_{0 \le j \le |\alpha'|-1} \exp(\varphi_I) e_{I,\alpha'^2,j}\Bigg)\\ &= \sum_{0 \le j \le |\alpha'|-1}\Big\{\partial_{x'}^{\alpha'^1} e_{I,\alpha'^2,j} + e_{I,\alpha'^2,j}\partial_{x'}^{\alpha'^1}\varphi_I\Big\}\exp(\varphi_I).\end{aligned}$$

Therefore we define $e_{I,\alpha',j} = \partial_{x'}^{\alpha'^1} e_{I,\alpha'^2,j-1} + e_{I,\alpha'^2,j}\partial_{x'}^{\alpha'^1}\varphi_I$, where $e_{I,\alpha'^2,-1} = e_{I,\alpha'^2,p} = 0$. We can easily prove the estimate for the derivatives of these functions. $\square$

Let

$$\begin{aligned}A'^r_j(C) = \big\{(x,t,\xi') \in A_j^r(C);\ |t_{r'}| > (\operatorname{Im}\xi_n)^{-1} \qquad &\text{for } 1 \le r' \le r+1,\\ |\arg t_{r'} - \arg t_1| < \pi \qquad &\text{for } 2 \le r' \le r+1\big\}.\end{aligned}$$

We denote $A'^1_j(C)$ also by $A'_j(C)$. Replacing $A_j(C)$ by $A'_j(C)$ in the definition of $\mathscr{R}_{\kappa_1}(C)$, $\mathscr{S}_{\kappa_1}(C)$, $\mathscr{N}_{\kappa_1}(C)$ and considering single-valued holomorphic functions on $A'_j(C)$, one can define new classes of formal series in the same way, which we denote by $\mathscr{R}'_{\kappa_1}(C)$, $\mathscr{S}'_{\kappa_1}(C)$, $\mathscr{N}'_{\kappa_1}(C)$ respectively. Restricting ourselves to $A'_j(C) \subset A_j(C)$, we can prove the following result:

Proposition 1. *There exists $W = \sum_{j \in \boldsymbol{Z}_+} W_j(x,t_1,\xi') \in \mathscr{S}'_{\kappa_1}(C)^{m' \times m'}$ such that $V - W \in \mathscr{N}'_{\kappa_1}(C)^{m' \times m'}$. Therefore we have $V \in \mathscr{S}'_{\kappa_1}(C)^{m' \times m'}$.*

In the rest of this section we define W and prove $W \in \mathscr{S}'_{\kappa_1}(C)^{m' \times m'}$. In the next section we shall prove $V - W \in \mathscr{N}'_{\kappa_1}(C)^{m' \times m'}$.

We use amplitude functions $\widetilde{W}_{j,I}(x', t, \xi') \in \mathscr{O}(A'^r_j(C))^{m' \times m'}$ for $|I| = r \le j+1$ which we shall define below, and define $W_j(x, t_1, \xi')$ in the following way:

$$W_j(x,t_1,\xi') = \sum_{|I|=1} \exp\big(\varphi_I(x,t_1,\xi')\big)\widetilde{W}_{j,I}(x',t_1,\xi') + \sum_{2\le|I|=r\le j+1} \int_{t_1}^{t_{r+1}} \cdots \int_{t_1}^{t_3} \exp\big(\varphi_I(x,t,\xi')\big)\widetilde{W}_{j,I}(x',t,\xi')dt_2\cdots dt_r.$$

Here $t = (t_1, \cdots t_r)$ and $t_{r+1} = x_1$, as before. Of course we want to let $M \circ W \sim O$. Let us discuss precisely. We have $M = M' + M''$, and from Lemma 5 we have $M''_{(p,q)} = M^1_{(p,q)} + M^2_{(p,q)}$, where $M^1_{(p,q)} \in x_1^{-\overline{p-1}+\overline{q-1}-1}\bar{\mathscr{E}}_{x^*}(0)$ and $M^2_{(p,q)} \in \bar{\mathscr{E}}_{x^*}(\kappa_1(\overline{p-1} - \overline{q-1} + 1))$. We define

$$M^1_{j,(p,q)} = \sigma_{-j}\big(M^1_{(p,q)}\big)\Big(= x_1^{-\overline{p-1}+\overline{q-1}-1}\sigma_{-j}\big(x_1^{\overline{p-1}-\overline{q-1}+1}M^1_{(p,q)}\big)\Big),$$
$$M^2_{j,(p,q)} = \sigma_{[\kappa_1(\overline{p-1}-\overline{q-1}+1)]-j}\big(M^2_{(p,q)}\big)$$

for $j \in \boldsymbol{Z}_+$. From (19) we have $M^1_{j,(p,q)} = M^2_{j,(p,q)} = 0$ if $\overline{p-1} + 2 \le \overline{q-1}$.

Let us define the amplitude function $\widetilde{W}_{j,I}(x', t, \xi')$. We define the (p,q) component of $\widetilde{W}_{j,I}$ by

$$\widetilde{W}_{j,I,(p,q)}(x',t_1,\xi') = \begin{cases} V_{j,(p,q)}(x,t_1,\xi')|_{x_1=t_1}, & i_1 = p, \\ 0 & i_1 \ne p \end{cases} \tag{22}$$

if $|I| = 1$, and

$$\widetilde{W}_{j,I,(p,q)}(x',t_1,\xi') = \begin{cases} F_{j,I,(p,q)}(x,t'',\xi')|_{x_1=t_r}, & i_r = p, \\ 0 & i_r \ne p \end{cases} \tag{23}$$

if $|I| \ge 2$. Here we have written $t'' = (t_1, \cdots, t_{r-1})$ as before, and $F_{j,I}(x,t'',\xi') = \sum_{0\le h\le 2} F^h_{j,I}(x,t'',\xi')$ is defined by

$$F^0_{j,I}(x,t'',\xi') = \sum_{(24)} \frac{1}{\beta'!\,\gamma'!}\partial_{\xi'}^{\beta'+\gamma'} m_{i_{r-1}}(x,\xi') e_{I''\beta'k}(x,t'',\xi')\partial_{x'}^{\gamma'}\widetilde{W}_{j'',I''}(x',t'',\xi')$$

where the summation is taken for

$$k + j'' + |\gamma'| = j,\ k \le |\beta'|,\ k + |\gamma'| \ne 0, \tag{24}$$

and

$$F^h_{j,I}(x,t'',\xi') = -\sum_{(25)} \frac{1}{\beta'!\,\gamma'!}\partial_{\xi'}^{\beta'+\gamma'} M^h_{j'}(x,\xi') e_{I''\beta'k}(x,t'',\xi')\partial_{x'}^{\gamma'}\widetilde{W}_{j'',I''}(x',t'',\xi'),$$

where

$$k + j' + j'' + |\gamma'| + 1 = j, \; k \le |\beta'| \tag{25}$$

for $h = 1, 2$. In (24) and (25) we have $j'' \le j-1$, and

$$(x, t, \xi') \in A'^{r}_{j}(C) \Longrightarrow (x, t'', \xi'), (t_r, x', t'', \xi') \in A'^{r-1}_{j}(C). \tag{26}$$

If $\widetilde{W}_{j''I''}$ is already defined on $A'^{r-1}_{j''}(C) \subset A'^{r-1}_{j}(C)$, then we can in this way define $\widetilde{W}_{jI}$ on $A'^{r}_{j}(C)$ by induction on j. Furthermore, we have $\widetilde{W}_{jI} = O$ if $|I| \ge j+2$. In fact, assume that $j_0 \ge 1$ and this is true for $0 \le j \le j_0 - 1$. Let $j = j_0$. If (24) or (25) is true, then we have $\widetilde{W}_{j''I''} = O$ for $|I''| (= |I| - 1) \ge j'' + 2$. This means $F_{jI} = O$ and $\widetilde{W}_{jI} = O$ for $|I| \ge j+2$. In this way we can define $\widetilde{W}_{jI} \in \mathscr{O}(A'^{r}_{j}(C))^{m' \times m'}$ for $|I| = r \le j+1$.

We next estimate these amplitude functions. We define

$$\begin{aligned} K_r = \{(k_1, \cdots, k_r) \in \boldsymbol{Z}_+^r;\ & \text{for each } r' \text{ satisfying } 1 \le r' \le r \text{ we have} \\ & 0 \le k_{r'} \le m \text{ and } k_1 + \cdots + k_{r'} \le m + r'\} \\ K_{ri} = \{(k_1, \cdots, k_r) \in K_r;\ & k_{r'} = 0 \text{ if } r - i + 1 \le r' \le r\} \\ K_{rij} = \{(k_1, \cdots, k_r) \in K_r;\ & k_1 + \cdots + k_r \le j\} \end{aligned}$$

for $r \in \boldsymbol{N},\ i \in \boldsymbol{Z}_+,\ j \in \boldsymbol{Z}_+$. It is easy to see that these sets are not empty, and therefore we can define $\| t \|_{rij} = \max_{(k_1, \cdots, k_r) \in K_{rij}} |t_1^{-k_1}\ t_2^{-k_2} \cdots t_r^{-k_r}|$ for $t \in (\boldsymbol{C} \setminus \{0\})^r$. Then we have the following result:

LEMMA 9. *Let $r \in \boldsymbol{N},\ i \in \boldsymbol{Z}_+,\ j \in \boldsymbol{Z}_+,\ 0 \le l \le \min(i, m-1)$ and $(t, x_1) \in (\boldsymbol{C} \setminus \{0\})^r \times (\boldsymbol{C} \setminus \{0\})$. Then we have*

(i) $\| t \|_{rij} \ge 1$,
(ii) $\| t \|_{rij} \le \| (t, x_1) \|_{r+1, i+1, j} \le \| (t, x_1) \|_{r+1, i, j}$,
(iii) $|x_1|^{-l-1} \| t \|_{rij} \le \| (t, x_1) \|_{r+1, 0, j+l+1}$.

PROOF. (i) We have $(0, \cdots, 0) \in K_{rij}$ for any i, j, and we have $\| t \|_{rij} \ge |t_1^0 \cdots t_r^0| = 1$

(ii) If $(k_1, \cdots, k_r) \in K_{rij}$, then we have $(k_1, \cdots, k_r, 0) \in K_{r+1, i+1, j} \subset K_{r+1, i, j}$. This means

$$\begin{aligned} \| t \|_{rij} &= \max_{(k_1, \cdots, k_r) \in K_{rij}} \left| t_1^{-k_1} \cdots t_r^{-k_r} x_1^0 \right| \\ &\le \max_{(k_1, \cdots, k_{r+1}) \in K_{r+1, i+1, j}} \left| t_1^{-k_1} \cdots t_r^{-k_r} x_1^{-k_{r+1}} \right| = \| (t, x_1) \|_{r+1, i+1, j} \\ &\le \max_{(k_1, \cdots, k_{r+1}) \in K_{r+1, i, j}} \left| t_1^{-k_1} \cdots t_r^{-k_r} x_1^{-k_{r+1}} \right| = \| (t, x_1) \|_{r+1, i, j}. \end{aligned}$$

We can prove (iii) similarly. □

Now we can prove the following result:

PROPOSITION 2. *Let $C_1 \gg C$. If $1 \le |I| = r \le j+1$, then we have*

$$\left|\partial_{x'}^{\alpha'} \widetilde{W}_{j,I,(p,q)}(x',t,\xi')\right| \le \sum_{\substack{l+l' \le r+\overline{p-1} \\ l,l' \ge 0}} \left(|\alpha'| + r + \overline{p-1} - l - l'\right)!\ C_1^{r+1+\frac{1}{2}|\alpha'|-\frac{j}{5}}$$
$$\times \|\, t\,\|_{r,m-1-\overline{p-1},l} (\operatorname{Im}\xi_n)^{\kappa_1 l' + 2m}$$

on $A'^{r}_{j+|\alpha'|}(C_1)$.

PROOF. If $r = 1$, from (16) and (22) we obtain $|\widetilde{W}_{j,I,(p,q)}| \le C_1^{1+\frac{1}{2}|\alpha'|-\frac{j}{5}} \alpha'!\ (\operatorname{Im}\xi_n)^{2m}$, and the statement is true.

We assume that $r_0 \ge 2$, and that the statement is true if $1 \le r \le r_0 - 1$. Let us consider the case $r = r_0$. If $(x,t,\xi') \in A'^{r}_{j+|\alpha'|}(C_1)$, then we have

$$\left|\partial_{x'}^{\alpha'} F^0_{j,I,(p,q)}(x,t'',\xi')\right| \le \sum_{(27)} \frac{\alpha'!}{\alpha'^1!\ \alpha'^2!\ \alpha'^3!\ \beta'!\ \gamma'!}$$
$$\times \left|\partial_{x'}^{\alpha'^1} \partial_{\xi'}^{\beta'+\gamma'} m_{i_{r-1}}\right| \cdot \left|\partial_{x'}^{\alpha'^2} e_{I''\beta' k}\right| \cdot \left|\partial_{x'}^{\alpha'^3+\gamma'} \widetilde{W}_{j'',I'',(p,q)}\right|,$$

where the summation is taken for

$$\alpha'^1 + \alpha'^2 + \alpha'^3 = \alpha',\ k + j'' + |\gamma'| = j,\ k \le |\beta'|,\ k + |\gamma'| \ne 0. \tag{27}$$

In (27) we have $j'' + |\alpha'^3 + \gamma'| \le j + |\alpha'|$. If $(x,t,\xi') \in A'^{r}_{j+|\alpha'|}(C_1)$, then by (26) we have $(x,t'',\xi') \in A'^{r-1}_{j+|\alpha'|}(C_1) \subset A'^{r}_{j''+|\alpha'^3+\gamma'|}(C_1)$. Therefore we can apply the statement to $\partial_{x'}^{\alpha'^3+\gamma'} \widetilde{W}_{j'',I'',(p,q)}(x',t'',\xi')$ if $(x,t,\xi') \in A'^{r}_{j+|\alpha'|}(C_1)$. Combining this with Lemma 8 we obtain

$$\left|\partial_{x'}^{\alpha'} F^0_{j,I,(p,q)}(x,t'',\xi')\right| \le \sum_{(27)} \sum_{\substack{l+l' \le r-1+\overline{p-1} \\ l,l' \ge 0}} \frac{\alpha'!}{\alpha'^1!\ \alpha'^2!\ \alpha'^3!\ \beta'!\ \gamma'!}$$
$$\times C^{|\alpha'^1+\beta'+\gamma'|+1} \alpha'^1!\ \beta'!\ \gamma'!\ (\operatorname{Im}\xi_n)^{1-|\beta'+\gamma'|}$$
$$\times (2nC^4)^{|\alpha'^2+\beta'|} \left(C_1^{-1} \operatorname{Im}\xi_n\right)^{|\beta'|-k}\ k!\ \alpha'^2!$$
$$\times \left(|\alpha'^3 + \gamma'| + r - 1 + \overline{p-1} - l - l'\right)!\ C_1^{r+\frac{1}{2}|\alpha'^3+\gamma'|-\frac{j''}{5}}$$
$$\times \|\, t''\,\|_{r-1,m-1-\overline{p-1},l} (\operatorname{Im}\xi_n)^{\kappa_1 l' + 2m}.$$

We have

$$k!\,\bigl(|\,\alpha'^3+\gamma'\,|+r-1+\overline{p-1}-l-l'\bigr)!$$
$$\le \bigl(|\,\alpha'^3\,|+r+\overline{p-1}-l-l'\bigr)!(m/C_1\operatorname{Im}\xi_n)^{k+|\gamma'|-1}$$

on $A'^r_{j+|\alpha'|}(C_1)$. Furthermore, from Lemma 9 we have $\|\,t''\,\|_{r-1,m-1-\overline{p-1},l} \le \|\,(t'',x_1)\,\|_{r,m-1-\overline{p-1},l}$. It follows that

$$\Bigl|\partial_{x'}^{\alpha'}F^0_{j,I,(p,q)}(x,t'',\xi')\Bigr| \le C_1^{-\frac{1}{5}} \sum_{\substack{l+l'\le r+\overline{p-1}\\ l,l'\ge 0}} \bigl(|\,\alpha'\,|+r+\overline{p-1}-l-l'\bigr)!\; C_1^{r+1+\frac{1}{2}|\alpha'|-\frac{j}{5}}$$
$$\times \|\,(t'',x_1)\,\|_{r,m-1-\overline{p-1},l}\bigl(\operatorname{Im}\xi_n\bigr)^{\kappa_1 l'+2m}.$$

Let $p\notin m\boldsymbol{Z}$. From (19) we may assume that $M^1_{(p,q)}=0$, and we have $F^1_{(p,q)}=0$. Therefore we only need to consider $F^1_{(p,q)}$ for $p\in m\boldsymbol{Z}$, and for this case we have

$$\Bigl|\partial_{x'}^{\alpha'}F^1_{j,I,(p,q)}(x,t'',\xi')\Bigr| \le \sum_{(28)} \frac{\alpha'!}{\alpha'^1!\;\alpha'^2!\;\alpha'^3!\;\beta'!\;\gamma'!}$$
$$\times \Bigl|\partial_{x'}^{\alpha'^1}\partial_{\xi'}^{\beta'+\gamma'}M^1_{j',(p,p')}\Bigr|\cdot\Bigl|\partial_{x'}^{\alpha'^2}e_{I''\beta'k}\Bigr|\cdot\Bigl|\partial_{x'}^{\alpha'^3+\gamma'}\widetilde{W}_{j'',I'',(p',q)}\Bigr|,$$

where the summation is taken for

$$\begin{cases}\alpha'^1+\alpha'^2+\alpha'^3=\alpha',\ k+j'+j''+|\,\gamma'\,|+1=j,\\ k\le|\,\beta'\,|,\ 1\le p'\le m',\ \overline{p'-1}\le\overline{p-1}+1.\end{cases} \tag{28}$$

It follows that

$$\Bigl|\partial_{x'}^{\alpha'}F^1_{j,I,(p,q)}(x,t'',\xi')\Bigr| \le \sum_{(28)} \sum_{\substack{l+l'\le r-1+\overline{p'-1}\\ l,l'\ge 0}} \frac{\alpha'!}{\alpha'^1!\;\alpha'^2!\;\alpha'^3!\;\beta'!\;\gamma'!}$$
$$\times C^{j'+|\alpha'^1+\beta'+\gamma'|+1}j'!\;\alpha'^1!\;\beta'!\;\gamma'!\;|\,x_1\,|^{-\overline{p-1}+\overline{p'-1}-1}$$
$$\times\bigl(\operatorname{Im}\xi_n\bigr)^{-j'-|\beta'+\gamma'|}(2nC^4)^{|\alpha'^2+\beta'|}\bigl(C_1^{-1}\operatorname{Im}\xi_n\bigr)^{|\beta'|-k}\;k!\;\alpha'^2!$$
$$\times\bigl(|\,\alpha'^3+\gamma'\,|+r-1+\overline{p'-1}-l-l'\bigr)!$$
$$\times C_1^{r+\frac{1}{2}|\alpha'^3+\gamma'|-\frac{j''}{5}}\|\,t''\,\|_{r-1,m-1-\overline{p'-1},l}\bigl(\operatorname{Im}\xi_n\bigr)^{\kappa_1 l'+2m}.$$

We have

$$j'!\;k!\;\bigl(|\,\alpha'^3+\gamma'\,|+r-1+\overline{p'-1}-l-l'\bigr)!$$
$$\le\bigl(|\,\alpha'^3\,|+r-1+\overline{p'-1}-l-l'\bigr)!(m\operatorname{Im}\xi_n/C_1)^{j'+k+|\gamma'|}.$$

Since $p \in m\boldsymbol{Z}$, from Lemma 9 it follows that

$$\begin{aligned}|x_1|^{-\overline{p-1}+\overline{p'-1}-1}\| t'' \|_{r-1,m-1-\overline{p'-1},l} &\le \| (t'', x_1) \|_{r,0,l+\overline{p-1}-\overline{p'-1}+1} \\ &= \| (t'', x_1) \|_{r,m-1-\overline{p-1},l+\overline{p-1}-\overline{p'-1}+1} .\end{aligned}$$

Denoting $l + \overline{p-1} - \overline{p'-1} + 1$ by l'', we obtain

$$\begin{aligned}\left|\partial_{x'}^{\alpha'} F^1_{j,I,(p,q)}(x, t'', \xi')\right| \le C_1^{-\frac{1}{5}} &\sum_{\substack{l''+l' \le r+\overline{p-1} \\ l'', l' \ge 0}} \left(|\alpha'| + r + \overline{p-1} - l'' - l'\right)! \\ &\times C_1^{r+1+\frac{1}{2}|\alpha'|-\frac{j}{5}} \| (t'', x_1) \|_{r,m-1-\overline{p-1},l''} (\operatorname{Im} \xi_n)^{\kappa_1 l' + 2m} .\end{aligned}$$

Similarly we can prove the same result for $|\partial_{x'}^{\alpha'} F^2_{j,I,(p,q)}(x', t'', \xi')|$. Since this part is easier, we leave it to the reader. From (23) we obtain

$$\begin{aligned}\left|\widetilde{W}_{j,I,(p,q)}(x', t, \xi')\right| &\le \sum_{0 \le h \le 2} \left|\partial_{x'}^{\alpha'} F^h_{j,I,(p,q)}(t_r, x', t'', \xi')\right| \\ &\le \sum_{\substack{l+l' \le r+\overline{p-1} \\ l, l' \ge 0}} \left(|\alpha'| + r + \overline{p-1} - l - l'\right)!\ C_1^{r+1+\frac{1}{2}|\alpha'|-\frac{j}{5}} \\ &\quad \times \| t \|_{r,m-1-\overline{p-1},l} (\operatorname{Im} \xi_n)^{\kappa_1 l' + 2m} .\end{aligned}$$

□

We next define

$$W_{jI}(x, t_1, \xi') = \exp(\varphi_I(x, t_1, \xi'))\widetilde{W}_{j,I}(x', t_1, \xi'),$$

for $|I| = 1$, and

$$W_{jI}(x, t_1, \xi') = \int_{t_1}^{t_{r+1}} \cdots \int_{t_1}^{t_3} \exp(\varphi_I(x, t, \xi'))\widetilde{W}_{j,I}(x', t, \xi') dt_2 \cdots dt_r$$

for $|I| = r \ge 2$. Therefore we have $W_j = \sum_{|I| \le j+1} W_{jI}$. To estimate W_{jI}, we must determine the path of integration for the case $|I| \ge 2$. Let $(x, t_1, \xi') \in A'_j(C)$. If $\varepsilon > 0$ is sufficiently small, we have $|t_1|, |x_1| > (\operatorname{Im} \xi_n)^{-1} + \varepsilon$. For such an ε, we define a continuous curve $\Gamma_\varepsilon(t_1, x_1)$ from t_1 to x_1 in the following way. Let $\gamma(a, b)$ be the line segment from a to b. If we have $|s| \ge (\operatorname{Im} \xi_n)^{-1} + \varepsilon$ for any $s \in \gamma(t_1, x_1)$, we define $\Gamma_\varepsilon(t_1, x_1) = \gamma(t_1, x_1)$. Otherwise, there are two points $s_1, s_2 \in \gamma(t_1, x_1)$ such that $|s_1| = |s_2| = (\operatorname{Im} \xi_n)^{-1} + \varepsilon$. We assume t_1, s_1, s_2, x_1 are located on $\gamma(t_1, x_1)$ in this order. We define $\Gamma_\varepsilon(t_1, x_1) = \gamma(t_1, s_1) \cup \gamma'(s_1, s_2) \cup \gamma(s_2, x_1)$, where $\gamma'(s_1, s_2) = \{s \in \boldsymbol{C};\ |s| = (\operatorname{Im} \xi_n)^{-1} + \varepsilon,\ \arg s$ varies from $\arg s_1$ to $\arg s_2\}$. We finally define $\Gamma^r_\varepsilon(t_1, x_1) = \{(t_2, \cdots, t_r) \in \Gamma_\varepsilon(t_1, x_1) \times \cdots \times \Gamma_\varepsilon(t_1, x_1);\ t_1, t_2, \cdots, t_r, x_1$ are located on

$\Gamma_\varepsilon(t_1, x_1)$ in this order$\}$.

REMARK. (i) We denote by $\rho_\varepsilon(s)$ the length from t_1 to $s \in \Gamma_\varepsilon(t_1, x_1)$ along $\Gamma_\varepsilon(t_1, x_1)$. It is easy to see $\rho_\varepsilon(s) \leq \pi |s - t_1|/2$.

(ii) If $(x, t_1, \xi') \in A'_j(\pi C/2)$ and $(t_2, \cdots, t_r) \in \Gamma^r_\varepsilon(t_1, x_1)$, then we have $(x, t, \xi') \in A'^r_j(C)$.

Now we have the following result:

LEMMA 10. *Let $r \geq 2,\ i \geq 0,\ 0 \leq j \leq m + r - 1$, and assume $(x, t_1, \xi') \in A'_j(C)$. If $0 < \varepsilon < 1/\operatorname{Im} \xi_n$, then we have*

$$\int_{\Gamma^r_\varepsilon(t_1, x_1)} \| t \|_{rij} |dt'| \leq \frac{(16\pi)^r}{(r-1)!} (\operatorname{Im} \xi_n)^{m+1} (\log(\operatorname{Im} \xi_n))^j |(x_1, t_1)|^{m+r-j}.$$

Here we denote $(t_2, \cdots, t_r)$ by t'.

PROOF. There are two cases: the case $\Gamma_\varepsilon(t_1, x_1) = \gamma(t_1, x_1)$, and the case $\Gamma_\varepsilon(t_1, x_1) \neq \gamma(t_1, x_1)$. Let us consider the second case (The first case is easier, and is essentially contained in the second one).

We first assume $0 \leq j \leq m$. We have $1/\operatorname{Im} \xi_n \leq |s| \leq |(x_1, t_1)|$ for $s \in \gamma_\varepsilon(t_1, x_1)$, and therefore $\| t \|_{rij} \leq (\operatorname{Im} \xi_n)^j \leq (\operatorname{Im} \xi_n)^{m+1} |(x_1, t_1)|^{m-j+1}$. It follows that

$$\begin{aligned} \int_{\Gamma^r_\varepsilon(t_1, x_1)} \| t \|_{rij} |dt'| &\leq (\operatorname{Im} \xi_n)^{m+1} |(x_1, t_1)|^{m-j+1} \int_{\Gamma^r_\varepsilon(t_1, x_1)} |dt'| \\ &\leq \frac{(16\pi)^r (\operatorname{Im} \xi_n)^{m+1} |(x_1, t_1)|^{m-j+r}}{(r-1)!}. \end{aligned}$$

We next assume $m + 1 \leq j \leq m + r - 1,\ j \leq m - p + q + 1$. Let s_1, s_2 be the points determined above. Let $\Gamma^{pqr}_\varepsilon(t_1, x_1)$, $1 \leq p \leq q \leq r$, be the subset of $\Gamma^r_\varepsilon(t_1, x_1)$ defined as follows: (i) $t_2, \cdots, t_p$ are on $\gamma(t_1, s_1)$, (ii) $t_{p+1}, \cdots, t_q$ are on $\gamma'(s_1, s_2)$, (iii) $t_{q+1}, \cdots, t_r$ are on $\gamma(s_2, x_1)$. Let $t^1 = (t_2, \cdots, t_p)$, $t^2 = (t_{p+1}, \cdots, t_q)$, and $t^3 = (t_{q+1}, \cdots, t_r)$. Furthermore we denote by $\Gamma^{pqr1}_\varepsilon(t_1, x_1)$ the image of the projection $\Gamma^{pqr}_\varepsilon(t_1, x_1) \ni (t_2, \cdots, t_r) \mapsto t^1 \in \boldsymbol{C}^{p-1}$. We define $\Gamma^{pqr2}_\varepsilon(t_1, x_1)$ and $\Gamma^{pqr3}_\varepsilon(t_1, x_1)$ similarly. Note that

$$\Gamma^{pqr}_\varepsilon(t_1, x_1) = \prod_{1 \leq l \leq 3} \Gamma^{pqrl}_\varepsilon(t_1, x_1),$$

$$\Gamma^r_\varepsilon(t_1, x_1) = \bigcup_{1 \leq p \leq q \leq r} \Gamma^{pqr}_\varepsilon(t_1, x_1).$$

We have $\| t \|_{rij} \leq ((\operatorname{Im} \xi_n)^{-1} + \varepsilon)^{-j}$ and

$$\int_{\Gamma^{pqr}_\varepsilon(t_1, x_1)} \| t \|_{rij} |dt'| \leq \left((\operatorname{Im} \xi_n)^{-1} + \varepsilon\right)^{-j} \prod_{1 \leq l \leq 3} \int_{\Gamma^{pqrl}_\varepsilon(t_1, x_1)} |dt^l|.$$

Furthermore, we have

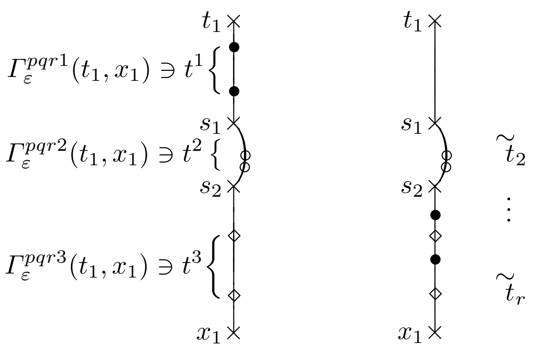
$$\prod_{l=1,3}\int_{\Gamma_\varepsilon^{pqrl}(t_1,x_1)}|\,dt^l\,| \le \frac{|\,x_1-t_1\,|^{p-1}}{(p-1)!}\cdot\frac{|\,x_1-t_1\,|^{r-q}}{(r-q)!},$$

$$\int_{\Gamma_\varepsilon^{pqr2}(t_1,x_1)}|\,dt^2\,| \le \frac{\bigl(\pi\bigl((\operatorname{Im}\xi_n)^{-1}+\varepsilon\bigr)\bigr)^{q-p}}{(q-p)!}.$$

It follows that

$$\begin{aligned}\int_{\Gamma_\varepsilon^{pqr}(t_1,x_1)}\|\,t\,\|_{rij}|\,dt'\,| &\le \frac{(4\pi)^{r-1}}{(r-1)!}(\operatorname{Im}\xi_n)^{j+p-q}|\,(x_1,t_1)\,|^{p-q+r-1}\\ &\le \frac{(4\pi)^{r-1}}{(r-1)!}(\operatorname{Im}\xi_n)^{m+1}|\,(x_1,t_1)\,|^{r+m-j}.\end{aligned}$$

We finally assume $m-p+q+2\le j\le m+r-1$. In addition we assume $|\,x_1\,|\ge|\,t_1\,|$ (The case $|\,x_1\,|\le|\,t_1\,|$ is similar). Let $(u_2,\cdots,u_r)$ be a permutation of $(t_2,\cdots,t_r)\in\Gamma_\varepsilon^{pqr}(t_1,x_1)$ such that $|\,u_2\,|\le\cdots\le|\,u_r\,|$. Since we have $|\,t_1\,|\ge\cdots\ge|\,t_p\,|\ge|\,t_{p+1}\,|=\cdots=|\,t_q\,|=(\operatorname{Im}\xi_n)^{-1}+\varepsilon\le|\,t_{q+1}\,|\le\cdots\le|\,t_r\,|\le|\,x_1\,|$, we may assume $(u_2,\cdots,u_{q-p+1})=(t_{p+1},\cdots,t_q)$, and $t_p,\cdots,t_2$ (resp. $t_{q+1},\cdots,t_r$) appear in $u_{q-p+2},\cdots,u_r$ in this order. Therefore we only have the choice if $u_{r'}$ represents a component of t^1 or of t^3, for $q-p+2\le r'\le r$. Let $T^{pqr}\subset S_{r-1,r-1}$ be the set of permutations which may appear. T^{pqr} consists of at most $2^{p-q+r-1}$ elements, and we have determined a map $h'':\Gamma_\varepsilon^{pqr}(t_1,x_1)\ni t'\mapsto\tau\in T^{pqr}$. Let $\Gamma_{\varepsilon\tau}^{pqr}(t_1,x_1)=h''^{-1}(\tau)$. If $(t_2,\cdots,t_r)\in\Gamma_{\varepsilon\tau}^{pqr}(t_1,x_1)$, then we have $|\,t_{\tau_2}\,|\le\cdots\le|\,t_{\tau_r}\,|$. By a rotation around the origin, we can map $t_2,\cdots,t_{p-1}\in\gamma(t_1,s_1)$ into $\gamma(s_2,x_1)$, and we obtain an injection $\theta:\Gamma_{\varepsilon\tau}^{pqr}(t_1,x_1)\ni(t^1,t^2,t^3)=t'\mapsto\widetilde{t}'=(\widetilde{t}_2,\cdots,\widetilde{t}_r)\in\Gamma_\varepsilon^{1,q-p+1,r}(t_1,x_1)$ (We do not move t^2 and t^3, see the figure below).



$$\theta:\Gamma_{\varepsilon\tau}^{pqr}(t_1,x_1)\ni t'\mapsto\widetilde{t}'\in\Gamma_{\varepsilon\tau}^{1,q-p+1,r}(t_1,x_1)$$

If $t'\in\Gamma_\varepsilon^{1,q-p+1,r}(t_1,x_1)$, then we have $|\,t_2\,|\le\cdots\le|\,t_r\,|$, and

$$\|t\|_{rij} \le \|t\|_{r0j} = 1/\big(|t_2|^{m+1}|t_3|\cdots|t_{j-m}|\big)$$
$$= 1/\Big(\big((\operatorname{Im}\xi_n)^{-1}+\varepsilon\big)^{m+q-p+1}|t_{q-p+2}|\cdots|t_{j-m}|\Big).$$

It follows that

$$\int_{\Gamma^{pqr}_{\varepsilon\tau}(t_1,x_1)} \|t\|_{rij}|dt'| \le \int_{\Gamma^{1,q-p+1,r}_{\varepsilon}(t_1,x_1)} \|t\|_{rij}|dt'|$$
$$= \frac{1}{\big((\operatorname{Im}\xi_n)^{-1}+\varepsilon\big)^{m-p+q+1}} \int_{\Gamma^{1,q-p+1,r}_{\varepsilon}(t_1,x_1)} \frac{|dt'|}{|t_{q-p+2}|\cdots|t_{j-m}|}$$
$$\le \frac{\Big(\pi\big((\operatorname{Im}\xi_n)^{-1}+\varepsilon\big)\Big)^{-p+q}}{(q-p)!\,\big((\operatorname{Im}\xi_n)^{-1}+\varepsilon\big)^{m-p+q+1}} \int_{\Gamma^{1,q-p+1,r,3}_{\varepsilon}(t_1,x_1)} \frac{|d(t_{q-p+2},\cdots,t_r)|}{|t_{q-p+2}|\cdots|t_{j-m}|}$$
$$\le \frac{\pi^{q-p}(\operatorname{Im}\xi_n)^{m+1}}{(q-p)!} \int_{0\le v_{q-p+2}\le\cdots\le v_r\le|x_1-s_2|} \frac{d(v_{q-p+2},\cdots,v_r)}{\sqrt{\big(s_0^2+v_{q-p+2}^2\big)\cdots\big(s_0^2+v_{j-m}^2\big)}}$$
$$\le \frac{\pi^{q-p}(\operatorname{Im}\xi_n)^{m+1}(\log(1/|s_2|))^{j-m+p-q-1}|x_1|^{r+m-j}}{(q-p)!\,(j-m+p-q-1)!\,(r+m-j)!}.$$

Here we have denoted $v_{r'} = |t_{r'}-s_2|$. Since $|s_2| \ge 1/|\xi_n|$, it follows that

$$\int_{\Gamma^{pqr}_{\varepsilon\tau}(t_1,x_1)} \|t\|_{rij}|dt'|$$
$$\le (\operatorname{Im}\xi_n)^{m+1}\frac{(\pi+2)^{r-1}}{(r-1)!}(\log(\operatorname{Im}\xi_n))^{j-m-q+p-1}|(x_1,t_1)|^{r+m-j}.$$

Summing up these inequalities for p, q, τ we obtain the statement. □

PROPOSITION 3. *We have* $\sum_{j\in\boldsymbol{Z}_+} W_j \in \mathscr{S}'_{\kappa_1}(C_1^2)^{m'\times m'}$ (*This is the first part of Proposition* 1).

PROOF. It suffices to prove

$$|W_j| \le C_1^{3-\frac{1}{5}j}(\operatorname{Im}\xi_n)^{C_1^3}\exp\big(C\psi(x,t_1,\xi')+C(\operatorname{Im}\xi_n)^{\kappa_1}\big)$$

on $A'^1_j(C_1^2)$. For this purpose, we first prepare the following result:

LEMMA 11. (i) *If* $(x,t_1,\xi'_1)\in A'^1_j(C_1^2)$, *then we have* $\operatorname{Re}\varphi_I(x,t,\xi') \le C^4\psi(x,t_1,\xi')$ *for* $|I|=1$.

(ii) *If* $(t_2,\cdots,t_r) \in \Gamma^r_\varepsilon(t_1,x_1)$ *in addition, then we have* $\operatorname{Re}\varphi_I(x,t,\xi') \le C^4\psi(x,t_1,\xi')+\varepsilon\operatorname{Im}\xi_n+1$ *for* $|I|=r\ge 2$.

PROOF. We have $|\varphi_I| \le C\sum_{1\le r'\le r}|t_{r'+1}-t_{r'}|\operatorname{Im}\xi_n$, where t_{r+1} denotes x_1, as before. Since φ_I satisfies (21) and $m_{i_r}(x^*)=0$, we have

$$|\varphi_I(x,t,\xi')| \le C^2 \sum_{1\le r'\le r} |t_{r'+1}-t_{r'}| \Big(\Big(\sum_{1\le r'\le r} |t_{r'+1}-t_{r'}| + |x|\Big) \operatorname{Im}\xi_n + |\xi'''|\Big).$$

However, we have $\operatorname{Re}\varphi_I(x,t,\xi') = 0$ if $(x,t,\xi') \in \boldsymbol{R}^n \times \boldsymbol{R}^r \times \sqrt{-1}\boldsymbol{R}^{n-1}$. This means

$$\begin{aligned}\operatorname{Re}\varphi_I \le C^3 \sum_{1\le r'\le r} |t_{r'+1}-t_{r'}| \Big(\Big(\sum_{1\le r'\le r} |\operatorname{Im}(t_{r'+1}-t_{r'})| + |\operatorname{Im} x|\Big) \operatorname{Im}\xi_n + |\operatorname{Re}\xi'|\Big)\\ + C^3 \sum_{1\le r'\le r} |\operatorname{Im}(t_{r'+1}-t_{r'})| \Big(\Big(\sum_{1\le r'\le r} |t_{r'+1}-t_{r'}| + |x'|\Big) \operatorname{Im}\xi_n + |\xi'''|\Big)\end{aligned}$$

for $(x,t,\xi') \in \boldsymbol{C}^n \times \boldsymbol{C}^r \times \boldsymbol{C}^{n-1}$, and we obtain (i).

Let $(x,t_1,\xi_1') \in {A'}_j^1(C_1^2)$ and $(t_2,\cdots,t_r) \in \Gamma_\varepsilon^r(t_1,x_1)$. We have

$$\sum_{1\le r'\le r} |t_{r'+1}-t_{r'}| \le \pi|x_1-t_1| + 2(\operatorname{Im}\xi_n)^{-1} + 2\varepsilon$$

and

$$\sum_{1\le r'\le r} |\operatorname{Im}(t_{r'+1}-t_{r'})| \le \pi|\operatorname{Im}(x_1-t_1)| + 2(\operatorname{Im}\xi_n)^{-1} + 2\varepsilon.$$

It follows that $\operatorname{Re}\varphi_I(x,t,\xi') \le C^4\psi(x,t_1,\xi') + \varepsilon \operatorname{Im}\xi_n + 1$, and we obtain (ii). $\square$

Continued proof of Proposition 3. Let $(x,t_1,\xi') \in {A'}_j^1(C_1^2)$. Let $0<\varepsilon\ll 1$. From Proposition 2, Lemma 10, and Lemma 11 we have

$$\begin{aligned}|W_j(x,t_1,\xi')| \le &\sum_{|I|=1} \exp(\operatorname{Re}\varphi_I(x,t,\xi'))|\widetilde{W}_{jI}(x,t,\xi')|\\ &+ \sum_{2\le|I|=r\le j+1} \int_{t_1}^{t_{r+1}} \cdots \int_{t_1}^{t_3} \exp\big(\operatorname{Re}\varphi_I(x,t,\xi')\big)|\widetilde{W}_{jI}(x,t,\xi')||dt_2\cdots dt_r|\\ \le m' &\sum_{\substack{1\le|I|=r\le j+1\\ l+l'\le r+m-1\\ l,l'\ge 0}} C_1^{r+1-\frac{j}{5}} \frac{(r+m-1)!}{l!\,l'!}\cdot\frac{(16\pi)^{r-1}}{(r-1)!}(\log(\operatorname{Im}\xi_n))^l\\ &\times |(x_1,t_1)|^{m+r-l}(\operatorname{Im}\xi_n)^{\kappa_1 l'+3m+1}\exp\big(C_1\psi(x,t_1,\xi')+\varepsilon\operatorname{Im}\xi_n+1\big).\end{aligned}$$

Here we can let $\varepsilon \longrightarrow +0$, and it follows that

$$|W_j(x,t_1,\xi')| \le C_1^{3-\frac{1}{5}j}(\operatorname{Im}\xi_n)^{C_1^3}\exp\big(C_1\psi(x,t_1,\xi')+C_1(\operatorname{Im}\xi_n)^{\kappa_1}\big). \qquad \square$$

6. Asymptotic equivalence.

To prove the latter part of Proposition 1, we discuss about asymptotic expansions. We first note the following result:

LEMMA 12. *If $C \gg 1$, we have $\partial_{x_1} V + M \mathbin{\bar{\circ}} V$, $\partial_{x_1} W + M \mathbin{\bar{\circ}} W \in \mathscr{N}'_{\kappa_1}(C)^{m' \times m'}$.*

PROOF. We have $\partial_{x_1} W + M \circ W = \sum_j G_j$, where

$$G_j = \partial_{x_1} W_j + \sum_{j'+j''+|\alpha'|=j} \frac{1}{\alpha'!} \partial_{\xi'}^{\alpha'} M_{j'} \partial_{x'}^{\alpha'} W_{j''}.$$

Let us denote $M' = M^0$. It follows that

$$\begin{aligned} G_{j,(p,q)}(x, t_1, \xi') = {} & \sum_{|I|=r=1} \sum_{(29)} \exp\big(\varphi_I(x, t_1, \xi')\big) G_{ij'j''kI\beta'\gamma'pp'q}(x, t_1, \xi') \\ & + \sum_{|I|=r\geq 2} \sum_{(29)} \int_{t_1}^{t_{r+1}} \cdots \int_{t_1}^{t_3} \exp\big(\varphi_I(x, t, \xi')\big) \\ & \times G_{ij'j''kI\beta'\gamma'pp'q}(x, t, \xi') dt_2 \cdots dt_r, \end{aligned}$$

where

$$G_{ij'j''kI\beta'\gamma'pp'q} = \frac{1}{\beta'!\,\gamma'!}\ \partial_{\xi'}^{\beta'+\gamma'} M^i_{j',(p,p')}\ e_{I\beta'k}\ \partial_{x'}^{\gamma'} \widetilde{W}_{j'',I,(p',q)},$$

and the summation is taken for

$$\begin{cases} 0 \le i \le 2,\ j' + j'' + |\beta' + \gamma'| = j,\ 0 \le k \le |\beta'|, \\ 1 \le p' \le m',\ \overline{p'-1} \le \overline{p-1} + 1,\ r \le j'' + 1 \end{cases} \tag{29}$$

We need to show $\sum G_j, \sum G'_j \in \mathscr{R}'_{\kappa_1}(C)$, where $G'_j = G_0 + \cdots + G_j$. We have defined W in such a way that it satisfies

$$\sum_{(31)} G_{ij'j''kI\beta'\gamma'pp'q}(x, t, \xi') = 0 \tag{30}$$

for

$$i = 0,\ j' = 0,\ k + j'' + |\gamma'| = j,\ k \le |\beta'|,\ p = p',\ r \le j'' + 1. \tag{31}$$

(30) is also true if we replace (31) by

$$\begin{cases} 1 \le i \le 2,\ k + j' + j'' + |\gamma'| + 1 = j,\ k \le |\beta'|, \\ 1 \le p' \le m',\ \overline{p'-1} \le \overline{p-1} + 1,\ r \le j'' + 1. \end{cases} \tag{31a}$$

Therefore $G'_j = G_0 + \cdots + G_j$ is given by

$$
\begin{aligned}
G'_{j,(p,q)}(x,t_1,\xi') = & \sum_{|I|=r=1} \exp\big(\varphi_I(x,t_1,\xi')\big)\Big\{ -\sum_{(32)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t_1,\xi') \\
& -\sum_{(33)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t_1,\xi') + \sum_{(34)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t_1,\xi')\Big\} \\
& + \sum_{|I|=r\geq 2} \int_{t_1}^{t_{r+1}} \cdots \int_{t_1}^{t_3} \exp\big(\varphi_I(x,t,\xi')\big)\Big\{ -\sum_{(32)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t,\xi') \\
& -\sum_{(33)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t,\xi') + \sum_{(34)} G_{ij'j''kI\beta'\gamma'pp'q}(x,t,\xi')\Big\} dt_2 \cdots dt_r,
\end{aligned}
$$

where

$$
\begin{cases} i = j' = 0,\ j'' + |\beta'+\gamma'| \geq j+1,\ k + j'' + |\gamma'| \leq j, \\ 0 \leq k \leq |\beta'|,\ p = p',\ r \leq j''+1, \end{cases} \tag{32}
$$

$$
\begin{cases} 1 \leq i \leq 2,\ j'+j''+|\beta'+\gamma'| \geq j+1,\ k+j'+j''+|\gamma'|+1 \leq j, \\ 0 \leq k \leq |\beta'|,\ \overline{p'-1} \leq \overline{p-1}+1,\ r \leq j''+1, \end{cases} \tag{33}
$$

$$
\begin{cases} 1 \leq i \leq 2,\ j'+j''+|\beta'+\gamma'| \leq j,\ k+j'+j''+|\gamma'| = j, \\ 0 \leq k \leq |\beta'|,\ \overline{p'-1} \leq \overline{p-1}+1,\ r \leq j''+1, \end{cases} \tag{34}
$$

respectively. From Proposition 2 we obtain

$$
\begin{aligned}
& |G_{ij'j''kI\beta'\gamma'pp'q}(x,t,\xi')| \\
& \quad \leq \sum_{\substack{l+l' \leq r+m-1 \\ l,l' \geq 0}} C^{r+2-\frac{1}{5}(j'+j''+k+|\beta'+\gamma'|)} \frac{(r+m)!}{l!\ l'!} \| t \|_{r0l} (\operatorname{Im}\xi_n)^{\kappa_1 l' + 3m}
\end{aligned}
$$

on $A'^{r}_{j'+j''+k+|\gamma'|}(C)$, $C \gg 1$. From Lemma 10 and Lemma 11 we obtain

$$
|G_j(x,t_1,\xi')|,\ |G'_j(x,t_1,\xi')| \leq C'C^{-\frac{j}{5}} (\operatorname{Im}\xi_n)^{C'} \exp\big(C\psi(x,t_1,\xi') + C(\operatorname{Im}\xi_n)^{\kappa_1}\big)
$$

on $A'_j(C')$ for $C' \gg C$, just in the same way as the last part of the proof of Proposition 3. This means $\partial_{x_1} W + M \mathbin{\bar{\circ}} W \in \mathscr{N}'_{\kappa_1}(C)$. We have already proved $\partial_{x_1} V + M \mathbin{\bar{\circ}} V \in \mathscr{N}'_{\kappa_1}(C)$ in Lemma 6. □

We next prove the following result:

LEMMA 13. *If* $X = \sum_j X_j(x,y_1,\xi')$, $Y = \sum_j Y_j(x',y_1,\xi') \in \mathscr{R}'_{\kappa_1}(C)^{m'\times m'}$, *then there uniquely exists* $Z = \sum_j Z_j(x,y_1,\xi') \in \mathscr{R}'_{\kappa_1}(C_1)^{m'\times m'}$ *for* $C_1 \gg C$, *such that*

$$
\partial_{x_1} Z + M \mathbin{\bar{\circ}} Z = X,\ Z|_{x_1=y_1} = Y. \tag{35}
$$

PROOF. We need to solve

$$\partial_{x_1} Z_j + \sum_{j'+j''+|\alpha'|=j} \frac{1}{\alpha'!} \partial_{\xi'}^{\alpha'} M_{j'} \partial_{x'}^{\alpha'} Z_j'' = X_j, \ Z_j|_{x_1=y_1} = Y_j \tag{36}$$

on $A'_j(C_1)$ for $j \in \boldsymbol{Z}_+$. We solve this by successive approximation. We consider

$$\partial_{x_1} Z_{jk} + \sum_{j'+j''+|\alpha'|=j} \frac{1}{\alpha'!} \partial_{\xi'}^{\alpha'} M_{j'} \partial_{x'}^{\alpha'} Z_{j'',k-1} = \delta_{k0} X_j, \ Z_{jk}|_{x_1=y_1} = \delta_{k0} Y_j$$

for $j, k \in \boldsymbol{Z}_+$. Here we have denoted $Z_{j,-1} = O$. Let us prove that $Z_j = \sum_k Z_{j,k}$ converges for each j, and $Z = \sum_j Z_j \in \mathscr{R}'_{\kappa_1}(C_1)^{m' \times m'}$. We have

$$\begin{aligned} Z_{jk}(x, y_1, \xi') = \int_{y_1}^{x_1} \Bigl\{ - &\sum_{j'+j''+|\alpha'|=j} \frac{1}{\alpha'!} \partial_{\xi'}^{\alpha'} M_{j'}(s, x', y_1, \xi') \partial_{x'}^{\alpha'} Z_{j'',k-1}(s, x', y_1, \xi') \\ &+ \delta_{k0} X_j(s, x', y_1, \xi) \Bigr\} ds + \delta_{k0} Y_j(x', y_1, \xi'). \end{aligned}$$

Let us prove

$$\begin{aligned} |\, \partial_{x'}^{\alpha'} Z_{j,k,(p,q)} \,| \le \frac{\alpha'!}{k!} C^{4k+3|\alpha'|} C_1 R^j (\rho_\varepsilon(x_1) \operatorname{Im} \xi_n)^k (\operatorname{Im} \xi_n)^{\overline{p-1}+C_1} \\ \times \exp\bigl(C_1 |\, x_1 - y_1 \,| \operatorname{Im} \xi_n + C (\operatorname{Im} \xi_n)^{\kappa_1}\bigr) \end{aligned} \tag{37}$$

for $\exists R \in (0,1)$ on $A'_{j+|\alpha'|}(C_1)$. Here $0 < \varepsilon \ll 1$ and $\rho_\varepsilon(x_1)$ denotes the distance from y_1 to x_1 along $\Gamma_\varepsilon(y_1, x_1)$, defined in section 5. If $k = 0$, then (37) is trivial. Assume $k_0 \ge 1$ and (37) is true if $0 \le k \le k_0 - 1$. Let us consider the case $k = k_0$. We have

$$\begin{aligned} &|\, \partial_{x'}^{\alpha'} Z_{j,k,(p,q)} \,| \\ &\le \sum_{(38) \left(\substack{j'+j''+|\beta'|=j \\ \alpha'^1+\alpha'^2=\alpha' \\ \overline{p'-1} \le \overline{p-1}+1} \right)} \int_{\Gamma_\varepsilon(y_1,x_1)} \frac{\alpha'!}{\alpha'^1!\ \alpha'^2!\ \beta'!} \\ &\quad \times \Bigl| \partial_{x'}^{\alpha'^1} \partial_{\xi'}^{\beta'}{}_{j',(p,p')} M_{j',(p,p')}(s, x', y_1, \xi') \Bigr| \Bigl| \partial_{x'}^{\alpha'^2+\beta'} Z_{j'',k-1,(p',q)}(s, x', y_1, \xi') \Bigr| d\rho_\varepsilon(s) \\ &\le \sum_{(38)} \int_{\Gamma_\varepsilon(y_1,x_1)} \frac{\alpha'!}{\alpha'^1!\ \alpha'^2!\ \beta'!} C^{|\alpha'^1+\beta'|+j'+1} \alpha'^1!\ \beta'!\ j'! (\operatorname{Im} \xi_n)^{\overline{p-1}-\overline{p'-1}+1-j'-|\beta'|} \\ &\quad \times \frac{(\alpha'^2+\beta')!}{(k-1)!} C^{4k-4+3|\alpha'^2+\beta'|} C_1 R^{j''} (\rho_\varepsilon(x_1) \operatorname{Im} \xi_n)^{k-1} (\operatorname{Im} \xi_n)^{\overline{p'-1}+C_1} \\ &\quad \times \exp\bigl(C_1 |\, s - y_1 \,| \operatorname{Im} \xi_n + C (\operatorname{Im} \xi_n)^{\kappa_1}\bigr) d\rho_\varepsilon(s). \end{aligned}$$

We have $(\alpha'^2+\beta')!\ j'! \le \alpha'^2!\ C_1^{-j'-|\beta'|}(\operatorname{Im}\xi_n)^{j'+|\beta'|}$ on $A'_{j+|\alpha'|}(C_1)$, and $|s-y_1| \le |x_1-y_1|$ on $\Gamma_\varepsilon(y_1,x_1)$. Therefore we obtain (37), which means $Z \in \mathscr{R}'_{\kappa_1}(C_1)$. The uniqueness is trivial. $\square$

Corollary. *If $X=\sum_j X_j(x,y_1,\xi')$, $Y=\sum_j Y_j(x',y_1,\xi') \in \mathscr{N}'_{\kappa_1}(C)^{m'\times m'}$, then there uniquely exists $Z=\sum_j Z_j(x,y_1,\xi') \in \mathscr{N}'_{\kappa_1}(C_1)^{m'\times m'}$ for $C_1 \gg C$, satisfying* (35).

Proof. There exists $Z=\sum_j Z_j \in \mathscr{R}'_{\kappa_1}(C_1)$ which satisfies (35). Let $Z'_j = Z_1+\cdots+Z_j$ and $Z'=\sum Z'_j$. We define $X'=\sum X'_j$, $Y'=\sum Y'_j$ similarly. Then we have $X',Y' \in \mathscr{R}'_{\kappa_1}(C)^{m'\times m'}$ by definition, and Z'_j automatically satisfies (36) replacing X,Y,Z by X',Y',Z'. This means $Z' \in \mathscr{R}'_{\kappa_1}(C_1)^{m'\times m'}$, and thus $Z \in \mathscr{N}'_{\kappa_1}(C_1)^{m'\times m'}$. $\square$

Proof of Proposition 1. By Proposition 3 we have $W \in \mathscr{S}'_{\kappa_1}(C)^{m'\times m'}$. Lemma 12 means $\partial_{x_1}(V-W)+M \mathbin{\bar{\circ}} (V-W) \in \mathscr{N}'_{\kappa_1}(C)^{m'\times m'}$. By (22) we have $W|_{x_1=y_1}=V|_{x_1=y_1}$. Therefore the above Corollary means $V-W \in \mathscr{N}'_{\kappa_1}(C)^{m'\times m'}$. $\square$

We finally prove the following result:

Proposition 4. *$V \in \mathscr{S}_{\kappa_1}(C)^{m'\times m'}$ for $C \gg 0$.*

Proof. By Proposition 1 we have

$$|V_0+\cdots+V_j| \le C'(\operatorname{Im}\xi_n)^{C'} \exp\left(C(\operatorname{Im}\xi_n)^{\kappa_1}\right) \times \left\{\exp(C\psi(x,y_1,\xi'))+R^j\exp(C|x_1-y_1|\operatorname{Im}\xi_n)\right\} \tag{39}$$

on $A'_j(C)$, and we need to prove it on $A_j(C)$. Since V_j is holomorphic on the whole $A_j(C)$, this is true on the closure set of $A'_j(C)$, which contains $A''_j=\{(x,y_1,\xi')\in A_j(C);\ |x_1|>(\operatorname{Im}\xi_n)^{-1},\ |y_1|>(\operatorname{Im}\xi_n)^{-1}\}$. Let $C_1 \gg C$. We define

$$A'''_{j+k}(C_1)=\left\{(x,y_1,\xi')\in A_{j+k}(C_1);\ |x_1|>2(\operatorname{Im}\xi_n)^{-1},\ |y_1|>2(\operatorname{Im}\xi_n)^{-1}\right\}.$$

Assume that $(x,y_1,\xi')\in A'''_{j+k}(C_1)$ and $z',\zeta' \in \boldsymbol{C}^{n-1}$ satisfies $|z'|\le C_1^{1/2}(k+1)/\operatorname{Im}\xi_n$, $|\zeta'|\le C_1^{1/2}(k+1)$. It is easy to see that $(x_1,x'+z',y_1,\xi'+\zeta')\in A''_j(C)$. Therefore we have

$$\begin{aligned}&\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}(V_0+\cdots+V_j)(x,y_1,\xi')\\ &\quad=\alpha'!\beta'!\iint\frac{(V_0+\cdots+V_j)(x_1,x'+z',y_1,\xi'+\zeta')dz'd\zeta'}{\Big(\prod\limits_{2\le k\le n}2\pi\sqrt{-1}z_k^{\alpha_k+1}\Big)\Big(\prod\limits_{2\le k\le n}2\pi\sqrt{-1}\zeta_k^{\beta_k+1}\Big)}\end{aligned}$$

for $|\alpha'|,|\beta'|\le k$ on $A'''_{j+k}(C_1)$. Here the integration is taken for

$$|z_k|=C_1^{1/2}(k+1)/n\operatorname{Im}\xi_n,\ |\zeta_k|=C_1^{1/2}(k+1)/n,\ 2\le k\le n.$$

Since we have $\psi(x_1,x'+z',\xi'+\zeta')\le 2\psi(x,y_1,\zeta')+12C_1^{-1/2}(k+1)$, it follows that

$$
\begin{aligned}
&\big|\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}(V_0+\cdots+V_j)\big| \\
&\quad\le 2C'(\operatorname{Im}\xi_n)^{C'}\big(2nC_1^{-1/2}\operatorname{Im}\xi_n\big)^{|\alpha'|}\big(2nC_1^{-1/2}\big)^{|\beta'|}\exp\big(2C(\operatorname{Im}\xi_n)^{\kappa_1}\big) \\
&\qquad\times\big\{\exp(2C\psi(x,y_1,\xi'))+R^j\exp(2C|x_1-y_1|\operatorname{Im}\xi_n)\big\}
\end{aligned}
\tag{40}
$$

for $|\alpha'|,|\beta'|\le k$ on $A'''_{j+k}(C_1)$ (for $\exists C'>0$, $\exists R\in(0,1)$). From the beginning we have $\sum V_j\in\mathscr{R}_{\kappa_1}(C)$, and we can similarly prove

$$
\begin{aligned}
\big|\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}V_j\big| &\le C'R^jC^{2|\alpha'+\beta'|}(\operatorname{Im}\xi_n)^{C'-|\beta'|}\alpha'!\beta'! \\
&\quad\times\exp\big(2C|x_1-y_1|\operatorname{Im}\xi_n+2C(\operatorname{Im}\xi_n)^{\kappa_1}\big)
\end{aligned}
\tag{41}
$$

on $A_j(C)$.

Now let $(x,y_1,\xi')\in A_j(6C_1)$ and let us prove (39). We consider the following four cases separately:

$$
\begin{aligned}
&\text{(a)}\quad |x_1|>2/\operatorname{Im}\xi_n,\ |y_1|>2/\operatorname{Im}\xi_n,\\
&\text{(b)}\quad |x_1|<4/\operatorname{Im}\xi_n,\ |y_1|<4/\operatorname{Im}\xi_n,\\
&\text{(c)}\quad |x_1|>3/\operatorname{Im}\xi_n,\ |y_1|<3/\operatorname{Im}\xi_n,\\
&\text{(d)}\quad |x_1|<3/\operatorname{Im}\xi_n,\ |y_1|>3/\operatorname{Im}\xi_n,
\end{aligned}
$$

In case (a), we have $(x,y_1,\xi')\in A''_j(C_1)$, and (39) is true. Next we consider case (b). We have $|x_1-y_1|\operatorname{Im}\xi_n<8$, and from (41) it follows that $|V_0+\cdots+V_j|\le e^{8C}C'(\operatorname{Im}\xi_n)^{C'}\exp(C(\operatorname{Im}\xi_n)^{\kappa_1})/(1-R)$, which means (39). Let us consider case (c). Let $z_1=3/\xi_n$. We have $(x,z_1,\xi')\in A'''_j(C_1)$, $(z_1,x',y_1,\xi')\in A_j(C_1)$, and $V(x,y_1,\xi')=V(x,z_1,\xi')\mathbin{\bar\circ}V(z_1,x',y_1,\xi')$. From (40) and (41) we obtain

$$
\begin{aligned}
&\big|(V_0+\cdots+V_j)\big|(x,y_1,\xi') \\
&\quad\le\sum_{k+l+|\alpha'|=j}\frac{1}{\alpha'!}\big|\partial_{\xi'}^{\alpha'}(V_0+\cdots+V_k)(x,z_1,\xi')\big|\cdot\big|\partial_{x'}^{\alpha'}V_l(z_1,x',y_1,\xi')\big| \\
&\quad\le 2C'(\operatorname{Im}\xi_n)^{C'}\exp\big(2C(\operatorname{Im}\xi_n)^{\kappa_1}\big)\big(2nC_1^{-1/2}\big)^{|\alpha'|} \\
&\qquad\times\big\{\exp(2C\psi(x,z_1,\xi'))+R^k\exp(2C|x_1-z_1|\operatorname{Im}\xi_n)\big\} \\
&\qquad\times C'R^lC^{2|\alpha'+\beta'|}(\operatorname{Im}\xi_n)^{C'}\exp\big(2C|z_1-y_1|\operatorname{Im}\xi_n+2C(\operatorname{Im}\xi_n)^{\kappa_1}\big).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\psi(x,z_1,\xi')\le\psi(x,y_1,\xi')+|y_1-z_1|\operatorname{Im}\xi_n\le\psi(x,y_1,\xi')+7,\\
&|x_1-z_1|\operatorname{Im}\xi_n\le|x_1-y_1|\operatorname{Im}\xi_n+7,\\
&|z_1-y_1|\operatorname{Im}\xi_n\le 7.
\end{aligned}
$$

It follows that

$$|\,(V_0+\cdots+V_j)\,|(x,y_1,\xi') \le \frac{2C'^2e^{28C}}{1-\sqrt{R}}(\operatorname{Im}\xi_n)^{2C'}\exp\big(4C(\operatorname{Im}\xi_n)^{\kappa_1}\big) \times\big\{\exp(2C\psi(x,y_1,\xi'))+R_1^j\exp(2C|\,x_1-y_1\,|\operatorname{Im}\xi_n)\big\}$$

with $R_1=\max(\sqrt{R},C_1^{-1/3})$. This means (39) replacing C, C' and R by new constants. Similarly we can prove (39) for the last case (d). Therefore (39) is true on $A_j(6C_1)$. $\square$

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