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STABLE AND CENTER-STABLE MANIFOLDS OF ADMISSIBLE CLASSES FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the existence of stable and center-stable manifolds of admissible classes for mild solutions to partial functional differential equations of the form $\dot{u}(t) = A(t)u(t) + f(t, u_t)$, $t \geq 0$. These manifolds are constituted by trajectories of the solutions belonging to admissible function spaces which contain wide classes of function spaces like L_p-spaces and many other function spaces occurring in interpolation theory such as the Lorentz spaces $L_{p,q}$. Results in this paper are the generalization and development for our results in **[15]**. The existence of these manifolds obtained in the case that the family of operators $(A(t))_{t\geq 0}$ generate the evolution family $(U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy or trichotomy on the half-line and the nonlinear forcing term f satisfies the φ-Lipschitz condition, i.e., $\|f(t,u_t) - f(t,v_t)\| \leq \varphi(t)\|u_t - v_t\|_{\mathcal{C}}$, where u_t, $v_t \in \mathcal{C} := C([-r,0],X)$, and $\varphi(t)$ belongs to some admissible Banach function space and satisfies certain conditions.

1. Introduction. In this paper, we generalize and develop the results in Huy and Duoc **[15]** regarding the existence of stable and center-stable manifolds for mild solutions to partial functional differential equations of the form

$$\frac{du}{dt} = A(t)u(t) + f(t, u_t), \quad t \in [0, +\infty), \tag{1.1}$$

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where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed t;

$$f : \mathbb{R}_+ \times \mathcal{C} \longrightarrow X$$

is a continuous nonlinear operator, $\mathcal{C} := C([-r,0], X)$ is the Banach space of all continuous functions from $[-r,0]$ into X, equipped with the norm $\|\phi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} \|\phi(\theta)\|$ for $\phi \in \mathcal{C}$, and u_t is the *history function* defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r,0]$.

In the literature on the existence of manifolds the assumption that the linear operators $(A(t))_{t\geq 0}$, which generate the evolution family, having an exponential dichotomy or trichotomy has seen little change. The most popular condition imposed on f is its uniform Lipschitz continuity with a sufficiently small Lipschitz constant, i.e.,

$$\|f(t,\phi) - f(t,\psi)\| \leq q\|\phi - \psi\|_{\mathcal{C}}$$

for q small enough (see [**1**–**6**], [**8, 10, 11, 17, 20**]). However, for equations arising in complicated reaction-diffusion processes, the mapping f represents the source of material or population in many contexts where the Lipschitz coefficient depends on time (see [**21**, Chapter 11], [**22**], [**26**]). Therefore, a natural problem to study is the existence of manifolds when the mapping f has the Lipschitz coefficient dependent on time. Recently, we obtained exciting results in [**12, 13, 15, 16**] for the existence of manifolds based on the notion of admissible Banach function spaces which allow the Lipschitz coefficient of f to depend upon time. More concretely, in [**15**], we have established results on the existence of stable and center-stable manifolds for solutions to partial functional differential equations (1.1), but these manifolds are created by trajectories of bounded solutions.

The purpose of this paper is to show the existence of stable and center-stable manifolds of the $\mathcal{E}$-class for equation (1.1). These manifolds are created by solution trajectories in the admissible Banach space $\mathcal{E}$, see Definitions 2.2 and 2.5. The manifold of the $\mathcal{E}$-class can contain solution trajectories in spaces L_p with $1 \leq p \leq \infty$, Lorentz spaces $L_{p,q}$ or some function spaces occurring in interpolation theory. Thus, the results in [**15**] are only a specific case of $\mathcal{E}$-class manifolds (L_∞).

The difficulties in this paper which we must surmount, as compared to [**15**], include: formulating the definition of invariant manifolds of the $\mathcal{E}$-class such that it contains the existence and uniqueness theorem as a property of the manifold (see Definition 3.3 below), overcoming some proof techniques in the paper **[15]** and imposing suitable conditions for the function φ and the admissible Banach function space E.

The paper is organized as follows. Section 2 recalls some notions on function spaces and redefines the the exponentially E-invariant function. Section 3 proves the existence of the invariant stable manifold of $\mathcal{E}$-class for the mild solutions of equation (1.1): the main results are Theorems 3.6 and 3.7. The last part in Section 3 is Example 3.8, which shows the existence of the invariant stable manifold of $\mathcal{E}$-class which is created by trajectories of the solutions belonging to spaces $L_q(\mathbb{R}_+)$. Section 4 proves the existence of the invariant center-stable manifold of $\mathcal{E}$-class for the mild solutions of equation (1.1), the main result of which is Theorem 4.2, obtained by the method of recalling the evolution family and then applying Theorems 3.6 and 3.7. Finally, Example 4.3 shows the existence of the invariant center-stable manifold of the $\mathcal{E}$-class, which is created by trajectories of the solutions in the Lorentz space $L_{2,1}(\mathbb{R}_+)$.

2. Function spaces and admissibility. We recall some notions on function spaces and refer the reader to Massera and Schäffer **[18]** and Räbiger and Schnaubelt **[23]** for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. The space $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) becomes a Fréchet space with the countable family of seminorms

$$p_n(f) := \int_{J_n} |f(t)|\, dt,$$

where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$, see **[18,** Chapter 2, Section 20].

We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if

(i) E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued

Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ-almost everywhere, then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$;

(ii) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geq 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$;

(iii) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n} \|f\|_E$ for all $f \in E$.

Next, we define Banach space $\mathcal{E}$ corresponding to Banach function space E as follows.

Definition 2.2. Let E be a Banach function space and $\mathcal{C} := C([-r, 0], X)$ a Banach space endowed with the norm $\|\cdot\|_{\mathcal{C}}$. We set

$$\begin{aligned}\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathcal{C}) = \{f : \mathbb{R}_+ \longrightarrow \mathcal{C} \text{ such that}\\ f \text{ is strongly measurable and } \|f(\cdot)\|_{\mathcal{C}} \in E\},\end{aligned}$$

endowed with the norm $\|f\|_{\mathcal{E}} = \|\|f(\cdot)\|_{\mathcal{C}}\|_E$. We can easily see that $\mathcal{E}$ is a Banach space. We call $\mathcal{E}$ the *Banach space corresponding to the Banach function space E.*

Remark 2.3. Note that, in Definition 2.2, we can replace $\mathcal{C}$ by an arbitrary Banach space.

We now introduce the notion of admissibility in the following definition.

Definition 2.4. The Banach function space E is called *admissible* if

(i) there is a constant $M \geq 1$ such that

$$\int_a^b |\varphi(t)|\, dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E$$

for any compact interval $[a, b] \subset \mathbb{R}_+$ and for all $\varphi \in E$.

(ii) For $\varphi \in E$, the function $\Lambda_1\varphi$ defined by

$$\Lambda_1\varphi(t) = \int_t^{t+1} \varphi(\tau)\, d\tau$$

belongs to E;

(iii) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined for $\tau \in \mathbb{R}_+$ by

$$T_\tau^+\varphi(t) = \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0 \\ 0 & \text{for } 0 \leq t < \tau \end{cases}$$
$$T_\tau^-\varphi(t) = \varphi(t+\tau) \qquad \text{for } t \geq 0.$$

(iv) The linear operators T_τ^+ and T_τ^- are uniformly bounded, i.e., there are constants N_1 and N_2 such that

$$\|T_\tau^+\| \leq N_1, \ \|T_\tau^-\| \leq N_2 \quad \text{for all } \tau \in \mathbb{R}_+.$$

Definition 2.5. Let E be an admissible Banach function space. Then, $\mathcal{E}$ is called an *admissible Banach space.*

Example 2.6. The Banach function spaces $L_p(\mathbb{R}_+)$, $1 \leq p \leq \infty$ are admissible. In addition, the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+1} |f(\tau)|\, d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geq 0} \int_t^{t+1} |f(\tau)|\, d\tau$ and many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,\,q}(\mathbb{R}_+)$, $1 < p < \infty$, $1 \leq q \leq \infty$ are also admissible.

Remark 2.7. We can easily see that, if E is an admissible Banach function space, then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [**13,** Proposition 2.6]).

Proposition 2.8. *Let E be an admissible Banach function space. Then, the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is defined as in Definition* 2.4 (ii). *For $\sigma > 0$, we define functions $\Lambda_\sigma\varphi$ and $\overline{\Lambda}_\sigma\varphi$ by*

$$\Lambda_\sigma\varphi(t) = \int_0^t e^{-\sigma(t-s)}\varphi(s)\,ds,$$
$$\overline{\Lambda}_\sigma\varphi(t) = \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then, $\Lambda_\sigma\varphi$ *and* $\overline{\Lambda}_\sigma\varphi$ *belong to* E*. In particular, if* $\sup_{t\geq 0}\int_t^{t+1}|\varphi(\tau)|\,d\tau < \infty$ (*this will be satisfied if* $\varphi \in E$*, see Remark* 2.7), *then* $\Lambda_\sigma\varphi$ *and* $\overline{\Lambda}_\sigma\varphi$ *are bounded. Moreover, with the* ess sup *norm denoted by* $\|\cdot\|_\infty$*, we have*

$$\|\Lambda_\sigma\varphi\|_\infty \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty$$
$$\|\overline{\Lambda}_\sigma\varphi\|_\infty \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty.$$

(b) E *contains exponentially decaying functions* $\psi(t) = e^{-\alpha t}$ *for* $t \geq 0$ *and any fixed constant* $\alpha > 0$.

(c) E *does not contain exponentially growing functions* $f(t) = e^{bt}$ *for* $t \geq 0$ *and any constant* $b > 0$.

We next define the associate spaces of Banach function spaces, as follows.

Definition 2.9. Let E be an admissible Banach function space and denote by $S(E)$ the unit sphere in E. Recall that $L_1 = \{g : \mathbb{R}_+ \to \mathbb{R} \mid g$ is measurable and $\int_0^\infty |g(t)|\,dt < \infty\}$. Then, we consider the set E' of all measurable real-valued functions ψ on $\mathbb{R}_+$ such that

$$\varphi\psi \in L_1, \qquad \int_0^\infty |\varphi(t)\psi(t)|\,dt \leq k \quad \text{for all } \varphi \in S(E),$$

where k depends only upon ψ. Then, E' is a normed space with the norm given by (see [**18,** Chapter 2, 22.M])

$$\|\psi\|_{E'} := \sup\left\{\int_0^\infty |\varphi(t)\psi(t)|\,dt : \varphi \in S(E)\right\} \quad \text{for } \psi \in E'.$$

We call E' the *associated space* of E.

Remark 2.10. Let E be an admissible Banach function space and E' its associated space. Then, from [**18,** Chapter 2, 22.M], we also have that the following "Hölder's inequality" holds:

$$\int_0^\infty |\varphi(t)\psi(t)|\,dt \le \|\varphi\|_E\|\psi\|_{E'} \quad \text{for all } \varphi \in E,\ \psi \in E'. \tag{2.1}$$

In order to study the existence of stable and center-stable manifolds of the $\mathcal{E}$-class for partial functional differential equations, in this paper, we will consider the admissible Banach function space E such that its associate space E' is also an admissible Banach function space. We give the following definition of an exponentially E-invariant function.

Definition 2.11. Let E be an admissible Banach function space and E' its associated space. A positive function $\varphi \in E'$ is called *exponentially E-invariant* if, for any fixed $\nu > 0$, the function h_ν, defined by

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \quad \text{for } t \ge 0,$$

belongs to E.

We also give here some examples of admissible Banach function spaces and their associated function spaces which satisfy the above definition. Note that the functions $\varphi(t) = \beta e^{-\alpha t}$ for $t \ge 0$ and fixed $\beta, \alpha > 0$ are exponentially E-invariant to any admissible Banach function space E.

Example 2.12. $L_p'(\mathbb{R}_+) = L_q(\mathbb{R}_+)$ for $1/p + 1/q = 1$, $1 \le p \le \infty$ and the Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ have $L_{p,q}'(\mathbb{R}_+) = L_{p',q'}(\mathbb{R}_+)$ with $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, $1 < p < \infty$, $1 \le q \le \infty$. In addition to the functions of the form $\varphi(t) = \beta e^{-\alpha t}$, it can be seen that the functions of the form $\varphi = c\chi_{[a,b]}$ for any fixed constant $c > 0$ and any finite interval $[a, b] \subset \mathbb{R}_+$ are also exponentially E-invariant functions, in which E are the spaces $L_p(\mathbb{R}_+)$ or $L_{p,q}(\mathbb{R}_+)$.

3. Exponential dichotomy and stable manifold of $\mathcal{E}$-class. In this section, we prove the existence of the stable manifold of the $\mathcal{E}$-class, see Definition 3.3, for the mild solutions of equation (1.1). Throughout this section, we assume that the evolution family $(U(t, s))_{t\ge s\ge 0}$ has an exponential dichotomy on $\mathbb{R}_+$.

We now make precise the notion of exponential dichotomy in the following definition.

Definition 3.1. An evolution family $(U(t,s))_{t\ge s\ge 0}$ on the Banach space X is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t\ge 0$, on X and positive constants $N,\ \nu$ such that

(a) $U(t,s)P(s)=P(t)U(t,s)$, $t\ge s\ge 0$;
(b) the restriction $U(t,s)_{|}:\operatorname{Ker}P(s)\to\operatorname{Ker}P(t)$, $t\ge s\ge 0$, is an isomorphism, and we denote its inverse by $U(s,t)_{|}:=(U(t,s)_{|})^{-1}$, $0\le s\le t$;
(c) $\|U(t,s)x\|\le Ne^{-\nu(t-s)}\|x\|$ for $x\in P(s)X$, $t\ge s\ge 0$;
(d) $\|U(s,t)_{|}x\|\le Ne^{-\nu(t-s)}\|x\|$ for $x\in\operatorname{Ker}P(t)$, $t\ge s\ge 0$.

The projections $P(t)$, $t\ge 0$, are called the *dichotomy projections*, and the constants $N,\ \nu$ the *dichotomy constants.*

When the evolution family $(U(t,s))_{t\ge s\ge 0}$ on the Banach space X has an exponential dichotomy on $[0,\infty)$, we can define the family of operators $(\widetilde{P}(t))_{t\ge 0}$ on $\mathcal{C}$ as follows.

$$
\begin{aligned}
&\widetilde{P}(t):\mathcal{C}\longrightarrow\mathcal{C}\\
&(\widetilde{P}(t)\phi)(\theta)=U(t-\theta,t)P(t)\phi(0)\quad\text{for all }\theta\in[-r,0].
\end{aligned}
\tag{3.1}
$$

Then, we have that $(\widetilde{P}(t))^2=\widetilde{P}(t)$, and therefore, the operators $\widetilde{P}(t)$, $t\ge 0$, are projections on $\mathcal{C}$. Moreover,

$$
\begin{aligned}
\mathrm{Im}\widetilde{P}(t)=\{\phi\in\mathcal{C}:\phi(\theta)=U(t-\theta,t)\nu_0&\\
\text{for all }\theta\in[-r,0]\text{ for some }\nu_0\in\mathrm{Im}P(t)\}.&
\end{aligned}
$$

Next, we provide the notion of the φ-Lipschitz of the nonlinear term f.

Definition 3.2. Let E be an admissible Banach function space and φ a positive function belonging to E. A function $f:[0,\infty)\times\mathcal{C}\to X$ is said to be *φ-Lipschitz* if f satisfies:

(i) $f(t,0)=0$ for all $t\in\mathbb{R}_+$;

(ii) $\|f(t,\phi_1) - f(t,\phi_2)\| \le \varphi(t)\|\phi_1 - \phi_2\|_{\mathcal{C}}$ for all $t \in \mathbb{R}_+$ and all $\phi_1,\ \phi_2 \in \mathcal{C}$.

Note that, if $f(t,\phi)$ is φ-Lipschitz, then $\|f(t,\phi)\| \le \varphi(t)\|\phi\|_{\mathcal{C}}$ for all $\phi \in \mathcal{C}$ and $t \ge 0$.

The stable manifold of the $\mathcal{E}$-class is constituted by mild solutions of equation (1.1), that is, solutions of the following integral equation

$$\begin{cases} u(t) = U(t,s)u(s) + \displaystyle\int_s^t U(t,\xi)f(\xi,u_\xi)\,d\xi \quad \text{for } t \ge s \ge 0, \\ u_s = \phi \in \mathcal{C}, \end{cases} \tag{3.2}$$

with $(U(t,s))_{t\ge s\ge 0}$ a given evolution family. We note that, if the evolution family $(U(t,s))_{t\ge s\ge 0}$ arises from the well-posed Cauchy problem, then the function $u : [s-r,\infty) \to X$, which satisfies equation (3.2) for some given function f, is called a mild solution of the semilinear problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t,u_t), \quad t \ge s \ge 0, \\ u_s = \phi \in \mathcal{C}. \end{cases}$$

Now, let $\mathcal{E} := \mathcal{E}(\mathbb{R}_+,\mathcal{C})$ be the admissible Banach space corresponding to admissible Banach function space E. Next, we give the definition of a stable manifold of $\mathcal{E}$-class for the solutions of equation (3.2).

Definition 3.3. A set $S \subset \mathbb{R}_+ \times \mathcal{C}$ is said to be an *invariant stable manifold of $\mathcal{E}$-class* for the solutions to equation (3.2) if, for every $t \in \mathbb{R}_+$, the phase space $\mathcal{C}$ splits into a direct sum $\mathcal{C} = \widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with corresponding projections $\widetilde{P}(t)$, i.e., $\widetilde{X}_0(t) = \mathrm{Im}\widetilde{P}(t)$ and $\widetilde{X}_1(t) = \mathrm{Ker}\,\widetilde{P}(t)$, such that

$$\sup_{t\ge 0}\|\widetilde{P}(t)\| < \infty,$$

and there exists a family of Lipschitz mappings

$$\Phi_t : \widetilde{X}_0(t) \longrightarrow \widetilde{X}_1(t), \quad t \in \mathbb{R}_+,$$

with the Lipschitz constants independent of t such that:

(i) $S = \{(t, \psi + \Phi_t(\psi)) \in \mathbb{R}_+ \times (\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)) \,|\, t \in \mathbb{R}_+,\, \psi \in \widetilde{X}_0(t)\}$, and we denote

$$S_t := \{\psi + \Phi_t(\psi) : (t, \psi + \Phi_t(\psi)) \in S\};$$

(ii) S_t is homeomorphic to $\widetilde{X}_0(t)$ for all $t \geq 0$;

(iii) to each $\phi \in S_s$, there corresponds one and only one solution $u(t)$ to equation (3.2) on $[s-r, \infty)$ satisfying the initial condition $u_s = \phi$ and the function

$$z(t) = \begin{cases} u_t & \text{for } t \geq s, \\ 0 & \text{for } 0 \leq t < s \end{cases}$$

belongs to $\mathcal{E}$. Moreover, any two solutions $u(t)$ and $v(t)$ of equation (3.2) corresponding to different functions $\phi_1,\ \phi_2 \in S_s$ attract each other exponentially in the sense that there exist positive constants μ and C_μ independent of $s \geq 0$ such that

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{-\mu(t-s)} \|(\widetilde{P}(s)\phi_1)(0) - (\widetilde{P}(s)\phi_2)(0)\| \quad \text{for } t \geq s; \tag{3.3}$$

(iv) S is positively invariant under equation (3.2), i.e., if $u(t)$, $t \geq s - r$, is a solution to equation (3.2) such that initial conditions $u_s \in S_s$ and $z(t) \in \mathcal{E}$, then we have $u_t \in S_t$ for all $t \geq s$.

Note that, if we identify $\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with $\widetilde{X}_0(t) \times \widetilde{X}_1(t)$, then we can write $S_t = \mathrm{graph}(\Phi_t)$. This definition is a natural extension of the definition of stable manifold in the papers **[12, 15, 20]**.

Let the evolution family $(U(t,s))_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N,\ \nu > 0$. Note that the exponential dichotomy of $(U(t,s))_{t \geq s \geq 0}$ implies that $H := \sup_{t \geq 0} \|P(t)\| < \infty$ and the map $t \mapsto P(t)$ is strongly continuous (see [**19,** Lemma 4.2]). We can then define the Green's function on the half-line as:

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)U(t,\tau) & \text{for } t > \tau \geq 0 \\ -U(t,\tau)_|(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases} \tag{3.4}$$

It follows from the exponential dichotomy of $(U(t,s))_{t \geq s \geq 0}$ that

$$\|\mathcal{G}(t,\tau)\| \leq N(1+H)e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau.$$

The next lemma gives the form of the solution to equation (3.2) which belongs to $\mathcal{E}$.

Lemma 3.4. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have exponential dichotomy with the dichotomy projections $P(t)$, $t\geq 0$, and dichotomy constants N, $\nu>0$. Let E be the admissible Banach function space, E' its associated space and $\mathcal{E}:=\mathcal{E}(\mathbb{R}_+,\mathcal{C})$ an admissible Banach space corresponding to E. Suppose that $\varphi\in E'$ is exponentially E-invariant function, defined as in Definition* 2.11. *Let $f:\mathbb{R}_+\times\mathcal{C}\to X$ be φ-Lipschitz and $u(t)$ a solution to equation* (3.2) *such that the function*

$$z(t)=\begin{cases} u_t & \text{for } t\geq s,\\ 0 & \text{for } 0\leq t<s\end{cases}$$

belongs to $\mathcal{E}$ for fixed $s\geq 0$. Then, for $t\geq s$, we can rewrite $u(t)$ in the form

$$\begin{cases} u(t)=U(t,s)\nu_0+\displaystyle\int_s^\infty \mathcal{G}(t,\tau)f(\tau,u_\tau)\,d\tau,\\ u_s=\phi\in\mathcal{C}, \end{cases}\tag{3.5}$$

for some $\nu_0\in X_0(s)=P(s)X$, where $\mathcal{G}(t,\tau)$ is the Green's function defined as in (3.4).

Proof. Set

$$y(t)=\int_s^\infty \mathcal{G}(t,\tau)f(\tau,u_\tau)\,d\tau$$

for $t\geq s$ and $y(t)=0$ for $0\leq t<s$. We have

$$\begin{aligned}\|y(t)\| &\leq N(1+H)\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}\,d\tau\\ &= N(1+H)\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|z(\tau)\|_{\mathcal{C}}\,d\tau.\end{aligned}$$

Using "Hölder's inequality" (2.1), we obtain

$$\|y(t)\|\leq N(1+H)\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}\|z\|_{\mathcal{E}}.\tag{3.6}$$

Note that the function $h_\nu(t)=\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}$ belongs to E. Therefore, by the Banach lattice property, we have that the function $\|y(t)\|\in E$. Similarly, we also have the function $\|u(t)\|\in E$. On the other hand,

$$
\begin{aligned}
U(t,s)y(s) &= -\int_s^t U(t,s)U(s,\tau)_{|}(I-P(\tau))f(\tau,u_\tau)\,d\tau \\
&\quad - \int_t^\infty U(t,s)U(s,\tau)_{|}(I-P(\tau))f(\tau,u_\tau)\,d\tau \\
&= -\int_s^t U(t,\tau)(I-P(\tau))f(\tau,u_\tau)\,d\tau \\
&\quad - \int_t^\infty U(t,\tau)_{|}(I-P(\tau))f(\tau,u_\tau)\,d\tau.
\end{aligned}
$$

Therefore,

$$y(t) = U(t,s)y(s) + \int_s^t U(t,\tau)f(\tau,u_\tau)\,d\tau.$$

Since $u(t)$ is a solution of equation (3.2), we obtain that $u(t) - y(t) = U(t,s)(u(s) - y(s))$. Now, set $\nu_0 = u(s) - y(s)$. From the fact that the functions $\|y(t)\|$ and $\|0u(t)\|$ belong to E, it is implied that $\nu_0 \in X_0(s)$ and $P(s)u(s) = P(s)\phi(0) = \nu_0$. Therefore, $u(t) = U(t,s)\nu_0 + y(t)$ for $t \geq s$. $\square$

Remark 3.5. Equation (3.5) is called the *Lyapunov-Perron equation*. By computing directly, we can see that the converse of Lemma 3.4 is also true. This means that all solutions of the integral equation (3.5) belonging to $\mathcal{E}$ are also solutions of equation (3.2) in the admissible Banach space $\mathcal{E}$ for $t \geq s$.

Using admissibility, we construct the structure of solutions of equation (3.2) in the following theorem.

Theorem 3.6. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and dichotomy constants N, $\nu > 0$. Let E and E' be, respectively, an admissible Banach function space and its associate space. Define the function $h_\nu(t) = \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}$ for $t \geq 0$. Then, we have that, if the function f is φ-Lipschitz with $\varphi \in E'$ being an exponentially E-invariant function and*

$$N(1+H)e^{\nu r}\|h_\nu\|_E < 1,$$

then there corresponds to each $\phi \in \mathrm{Im}\widetilde{P}(s)$ *one and only one solution* $u(t)$ *of equation* (3.2) *on* $[s-r,\infty)$*, satisfying the condition* $\widetilde{P}(s)u_s = \phi$ *and the function*

$$z(t) = \begin{cases} u_t & \text{for } t \geq s, \\ 0 & \text{for } 0 \leq t < s, \end{cases}$$

belong to $\mathcal{E}$*. Moreover, if*

$$N(1+H)e^{\nu r}(N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1,$$

then the following estimate is valid for any two solutions $u(t)$ *and* $v(t)$ *corresponding to different initial functions* $\phi_1,\ \phi_2 \in \mathrm{Im}\widetilde{P}(s)$:

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{-\mu(t-s)}\|\phi_1(0) - \phi_2(0)\| \quad \text{for all } t \geq s \geq 0,$$

where μ *is a positive number satisfying*

$$0 < \mu < \nu + \ln\left(1 - N(1+H)e^{\nu r}(N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)\right),$$

and

$$C_\mu := \frac{Ne^{\nu r}}{1 - [N(1+H)e^{\nu r}]/[1-e^{-(\nu-\mu)}](N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}.$$

Proof. For ease of exposition, the proof is divided into two steps.

Step I. To point out the existence and uniqueness of solution for equation (3.2) with each $\phi \in \mathrm{Im}\widetilde{P}(s)$. Denote by $C([s-r,\infty), X)$ the set of bounded, continuous and X-valued functions defined on $[s-r,\infty)$. Setting $\nu_0 := \phi(0)$, for each $z \in \mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathcal{C})$ then we have the function $Hz \in C([s-r,\infty), X)$, defined as:

$$(Hz)(t) = \begin{cases} U(2s-t,s)\nu_0 + \displaystyle\int_s^\infty \mathcal{G}(2s-t,\tau)f(\tau, z(\tau))\,d\tau & t \in [s-r, s] \\ U(t,s)\nu_0 + \displaystyle\int_s^\infty \mathcal{G}(t,\tau)f(\tau, z(\tau))\,d\tau & t \geq s. \end{cases}$$

We will prove that the transformation T, defined by

$$(Tz)(t) = \begin{cases} (Hz)_t & \text{for } t \geq s, \\ 0 & \text{for } 0 \leq t < s \end{cases}$$

acts from $\mathcal{E}$ into $\mathcal{E}$ and is a contraction mapping.

In fact, we have

$$\|(Hz)(t)\| \leq \begin{cases} Ne^{-\nu(s-t)}\|\nu_0\| + N(1+H) \\ \quad \displaystyle\int_s^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|z(\tau)\|_{\mathcal{C}}\, d\tau & \text{for } s-r\leq t\leq s, \\ Ne^{-\nu(t-s)}\|\nu_0\| + N(1+H) \\ \quad \displaystyle\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|z(\tau)\|_{\mathcal{C}}\, d\tau & \text{for } t\geq s. \end{cases}$$

Therefore, for $t \geq s$, then

$$\begin{aligned}\|(Hz)_t\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} \|(Hz)(t+\theta)\| &\leq Ne^{\nu r}e^{-\nu(t-s)}\|\nu_0\| \\ &+ N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|z(\tau)\|_{\mathcal{C}}\, d\tau.\end{aligned}$$

Using "Hölder's inequality" (2.1), we obtain

$$\|(Hz)_t\|_{\mathcal{C}} \leq Ne^{\nu r}T_s^+ e_\nu(t)\|\nu_0\| + N(1+H)e^{\nu r}h_\nu(t)\|z\|_{\mathcal{E}},$$

where $\|z\|_{\mathcal{E}} = \|\|z(\tau)\|_{\mathcal{C}}\|_E$, $h_\nu(t) = \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}$ and

$$T_s^+ e_\nu(t) = \begin{cases} e^{-\nu(t-s)} & \text{for } t\geq s, \\ 0 & \text{for } 0\leq t < s. \end{cases}$$

Since the functions $T_s^+ e_\nu(t)$, $h_\nu(t) \in E$, by the Banach lattice property of E, therefore, function $\|(Tz)(t)\|_{\mathcal{C}} \in E$. Thus, $Tz \in \mathcal{E}$.

Next, we prove that, if $N(1+H)e^{\nu r}\|h_\nu\|_E < 1$, then T is the contraction. For $x, z \in \mathcal{E}$, we have estimate

$$\|(Hx)_t - (Hz)_t\|_{\mathcal{C}} \leq N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|x(\tau)-z(\tau)\|_{\mathcal{C}}\, d\tau$$

for $t \geq s$. By "Hölder's inequality" (2.1), we obtain

$$\|(Hx)_t - (Hz)_t\|_{\mathcal{C}} \leq N(1+H)e^{\nu r}h_\nu(t)\|x-z\|_{\mathcal{E}}.$$

From the Banach lattice property of E, we have

$$\|Tx - Tz\|_{\mathcal{E}} \leq N(1+H)e^{\nu r}\|h_\nu\|_E\|x-z\|_{\mathcal{E}}.$$

Thus, T is a contractive mapping. Hence, there exists a unique $z \in \mathcal{E}$ such that $Tz = z$. Therefore, we have

$$z(t) = \begin{cases} (Hz)_t & \text{for } t \geq s, \\ 0 & \text{for } 0 \leq t < s. \end{cases}$$

This yields

$$(Hz)(t) = \begin{cases} U(2s-t,s)\nu_0 \\ \quad + \int_s^\infty \mathcal{G}(2s-t,\tau) f(\tau,(Hz)_\tau)\, d\tau & t \in [s-r,s] \\ U(t,s)\nu_0 + \int_s^\infty \mathcal{G}(t,\tau) f(\tau,(Hz)_\tau)\, d\tau & t \geq s. \end{cases}$$

Hence, $u(t) = (Hz)(t)$ is unique solution of equation (3.5). By Lemma 3.4 and Remark 3.5, then $u(t)$ is also a unique solution of equation (3.2) with

$$u_s(\theta) = U(s-\theta,s)\nu_0 + \int_s^\infty \mathcal{G}(s-\theta,\tau) f(\tau,u_\tau)\, d\tau$$

for $\theta \in [-r,0]$, and $P(s)u(s) = \nu_0 = \phi(0)$. Therefore, $\widetilde{P}(s)u_s = \phi$ by the definition of $\widetilde{P}(s)$, see equality (3.1).

Step II. Show that any two solutions have the property of exponential attraction. Let $u(t), v(t)$ be two solutions to equation (3.5), corresponding to different initial functions $\phi_1,\ \phi_2 \in \mathrm{Im}\widetilde{P}(s)$, respectively. Setting $\nu_1 := \phi_1(0)$ and $\nu_2 := \phi_2(0)$, we have that

$$\begin{aligned} &\|u(t)-v(t)\| \\ &\leq \begin{cases} N(1+H)\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \\ \quad +Ne^{-\nu(t-s)}\|\nu_1-\nu_2\| & \text{if } t \geq s, \\ N(1+H)\int_s^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \\ \quad +Ne^{-\nu(s-t)}\|\nu_1-\nu_2\| & \text{if } s-r \leq t \leq s. \end{cases} \end{aligned}$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in [s, \infty)$ and $\theta \in [-r, 0]$, we obtain

$$\begin{aligned}\|u_t - v_t\|_{\mathcal{C}} &\le N e^{\nu r} e^{-\nu(t-s)} \|\nu_1 - \nu_2\| \\ &\quad + N(1+H) e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau, \quad t \ge s.\end{aligned}$$

Set

$$h(t) = \begin{cases} \|u_t - v_t\|_{\mathcal{C}} & \text{for } t \ge s, \\ 0 & \text{for } 0 \le t < s. \end{cases}$$

Then, $h(t) \in E$, and

$$\begin{aligned}h(t) &\le N e^{\nu r} e^{-\nu(t-s)} \|\nu_1 - \nu_2\| \\ &\quad + N(1+H) e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) h(\tau)\, d\tau, \quad t \ge s.\end{aligned} \tag{3.7}$$

We will use the cone inequality theorem (see [**13,** Theorem 2.8]) for admissible Banach function space E and the cone $\mathcal{K}$ as the set of all nonnegative functions. We then consider the linear operator A, defined for $g \in E$, by

$$(Ag)(t) = \begin{cases} N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) g(\tau)\, d\tau & \text{for } t \ge s, \\ 0 & \text{for } 0 \le t < s. \end{cases}$$

By "Hölder's inequality" (2.1),

$$|(Ag)(t)| \le N(1+H) e^{\nu r} h_\nu(t) \|g\|_E.$$

From the Banach lattice property of E, we have $\|Ag\|_E \le N(1+H)e^{\nu r}\|h_\nu\|_E \|g\|_E$. Therefore, $A \in \mathcal{L}(E)$ and $\|A\| \le N(1+H)e^{\nu r}\|h_\nu\|_E < 1$. Obviously, the cone $\mathcal{K}$ is invariant under the operator A. Inequality (3.7) can now be rewritten as

$$h \le Ah + z \quad \text{for } z(t) = N e^{\nu r} e^{-\nu(t-s)} \|\nu_1 - \nu_2\|.$$

From the cone inequality theorem [**13,** Theorem 2.8], we obtain that $h \le g$, where g is a solution in E of the equation $g = Ag + z$, which can be rewritten as:

$$g(t) = N e^{\nu r} e^{-\nu(t-s)} \|\nu_1 - \nu_2\| + N(1+H) e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) g(\tau)\, d\tau$$

for $t \geq s$. We set $w(t) = e^{\mu(t-s)} g(t)$ for $t \geq s$. Then, we obtain that w is solution of the equation

$$w(t) = Ne^{\nu r} e^{-(\nu-\mu)(t-s)} \|\nu_1 - \nu_2\| + N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)} \varphi(\tau) w(\tau)\, d\tau. \tag{3.8}$$

We find w in $L_\infty[s,\infty)$, which is a space of real-valued functions, defined and essentially bounded on $[s,\infty)$ (endowed with the sup-norm denoted by $\|\cdot\|_\infty$). We consider the linear operator K, defined on $L_\infty[s,\infty)$

$$(K\phi)(t) = N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)} \varphi(\tau)\phi(\tau)\, d\tau \quad \text{for all } t \geq s.$$

By Proposition 2.8, it can easily be seen that $K \in \mathcal{L}(L_\infty[s,\infty))$ and

$$\|K\| \leq \frac{N(1+H)e^{\nu r}}{1-e^{-(\nu-\mu)}} (N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty).$$

Equation (3.8) can be rewritten as:

$$w = Kw + \widetilde{z} \quad \text{for } \widetilde{z}(t) = Ne^{\nu r} e^{-(\nu-\mu)(t-s)} \|\nu_1 - \nu_2\|.$$

We have $\|K\| < 1$ if

$$0 < \mu < \nu + \ln\left(1 - N(1+H)e^{\nu r}(N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty)\right).$$

Under this condition, the equation $w = Kw + \widetilde{z}$ has the unique solution $w \in L_\infty[s,\infty)$ and $w = (I-K)^{-1}\widetilde{z}$. Hence, we obtain that

$$\begin{aligned}
\|w\|_\infty = \|(I-K)^{-1}\widetilde{z}\|_\infty &\leq \frac{Ne^{\nu r}}{1-\|K\|} \|\nu_1 - \nu_2\| \\
&\leq \frac{Ne^{\nu r}\|\nu_1 - \nu_2\|}{1 - [N(1+H)e^{\nu r}]/[1-e^{-(\nu-\mu)}](N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty)} \\
&:= C_\mu \|\nu_1 - \nu_2\|.
\end{aligned}$$

This yields that $w(t) \leq C_\mu \|\nu_1 - \nu_2\|$ for $t \geq s$. Hence,

$$h(t) = \|u_t - v_t\|_{\mathcal{C}} \leq g(t) = e^{-\mu(t-s)} w(t) \leq C_\mu e^{-\mu(t-s)} \|\nu_1 - \nu_2\| \quad \text{for } t \geq s.$$

□

We now prove our main result of this section.

Theorem 3.7. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have exponential dichotomy with the dichotomy projections $P(t)$, $t\geq 0$, and dichotomy constants N, $\nu>0$. Let E and E' be, respectively, an admissible Banach function space and its associated space. Define the functions $e_\nu(t)=e^{-\nu t}$ and $h_\nu(t)=\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}$ for $t\geq 0$. Suppose that $\varphi\in E'$ is an exponentially E-invariant function, defined as in Definition* 2.11*. Then, if the function f is φ-Lipschitz with φ satisfying*

$$\max\{N(1+H)e^{\nu r}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty+N_2\|\Lambda_1\varphi\|_\infty),\\ N(1+H)e^{\nu r}(\|h_\nu\|_E+NN_1\|e_\nu\|_E\|\varphi\|_{E'})\}<1,$$

then there exists an invariant stable manifold S of $\mathcal{E}$-class for the solutions to equation (3.2).

Proof. Since $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy, we have that, for each $t\geq 0$, the phase space $\mathcal{C}$ splits into the direct sum $\mathcal{C}=\mathrm{Im}\widetilde{P}(t)\oplus\mathrm{Ker}\,\widetilde{P}(t)$ where the projections $\widetilde{P}(t)$, $t\geq 0$, are defined as in equality (3.1). Clearly, $\sup_{t\geq 0}\|\widetilde{P}(t)\|<\infty$. We now construct a stable manifold $S=\{(t,S_t)\}_{t\geq 0}$ for the solutions to equation (3.2). In order to do this, we determine the surface S_t for $t\geq 0$ by the formula:

$$S_t:=\{\phi+\Phi_t(\phi):\phi\in\mathrm{Im}\widetilde{P}(t)\}\subset\mathcal{C},$$

where the operator Φ_t is defined for each $t\geq 0$ by

$$\Phi_t(\phi)(\theta)=\int_t^\infty \mathcal{G}(t-\theta,\tau)f(\tau,u_\tau)\,d\tau\quad\text{for all }\theta\in[-r,0];$$

here, $u(\cdot)$ is the unique solution of equation (3.2) on $[-r+t,\infty)$, satisfying $\widetilde{P}(t)u_t=\phi$ (note that the existence and uniqueness of $u(\cdot)$ is guaranteed by Theorem 3.6). On the other hand, by the definition of Green's function $\mathcal{G}$, see equation (3.4), we have that $\Phi_t(\phi)\in\mathrm{Ker}\,\widetilde{P}(t)$.

We next show that the stable manifold S satisfies the conditions of Definition 3.3. Firstly, we prove that Φ_{t_0} is of Lipschitz continuity with the Lipschitz constant independent of t_0. Indeed, for ϕ_1 and ϕ_2

belonging to $\mathrm{Im}\widetilde{P}(t_0)$, we have

$$\begin{aligned} \|\Phi_{t_0}(\phi_1)(\theta) - \Phi_{t_0}(\phi_2)(\theta)\| & \\ &\le N(1+H)\int_{t_0}^{\infty} e^{-\nu|t_0-\theta-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \\ &\le N(1+H)\int_{t_0}^{\infty} \varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \\ &\le N(1+H)\|\varphi\|_{E'}\|u-v\|_{\mathcal{E}}. \end{aligned} \tag{3.9}$$

Moreover, by the Lyapunov-Perron equation for $u(\cdot)$ and $v(\cdot)$, see equation (3.5), we have

$$\|u_t - v_t\|_{\mathcal{C}} \le Ne^{\nu r}e^{-\nu(t-t_0)}\|\phi_1-\phi_2\|_{\mathcal{C}} + N(1+H)e^{\nu r}h_\nu(t)\|u-v\|_{\mathcal{E}}$$

for $t \ge t_0$. By the Banach lattice property of E and $e^{-\nu(t-t_0)} = T^+_{t_0}e_\nu(t)$, we obtain

$$\begin{aligned} \|u-v\|_{\mathcal{E}} &\le Ne^{\nu r}\|T^+_{t_0}e_\nu\|_E\|\phi_1-\phi_2\|_{\mathcal{C}} + N(1+H)e^{\nu r}\|h_\nu\|_E\|u-v\|_{\mathcal{E}} \\ &\le NN_1e^{\nu r}\|e_\nu\|_E\|\phi_1-\phi_2\|_{\mathcal{C}} + N(1+H)e^{\nu r}\|h_\nu\|_E\|u-v\|_{\mathcal{E}}. \end{aligned}$$

It follows that

$$\|u-v\|_{\mathcal{E}} \le \frac{NN_1e^{\nu r}\|e_\nu\|_E}{1-N(1+H)e^{\nu r}\|h_\nu\|_E}\,\|\phi_1-\phi_2\|_{\mathcal{C}}.$$

Substituting this inequality for (3.9), we obtain that

$$\|\Phi_{t_0}(\phi_1) - \Phi_{t_0}(\phi_2)\|_{\mathcal{C}} \le \frac{N^2(1+H)N_1e^{\nu r}\|e_\nu\|_E\|\varphi\|_{E'}}{1-N(1+H)e^{\nu r}\|h_\nu\|_E}\,\|\phi_1-\phi_2\|_{\mathcal{C}}.$$

Therefore, Φ_{t_0} is of Lipschitz continuity with the Lipschitz constant

$$k := \frac{N^2(1+H)N_1e^{\nu r}\|e_\nu\|_E\|\varphi\|_{E'}}{1-N(1+H)e^{\nu r}\|h_\nu\|_E} < 1$$

independent of t_0.

In order to show that S_{t_0} is homeomorphic to $\mathrm{Im}\widetilde{P}(t_0)$, we define the transformation

$$\mathbf{F} : \mathrm{Im}\widetilde{P}(t_0) \longrightarrow S_{t_0}$$

by $\mathbf{F}\phi := \phi + \Phi_{t_0}(\phi)$ for all $\phi \in \mathrm{Im}\widetilde{P}(t_0)$. Obviously, $\mathbf{F}$ is surjective and of continuous mapping. If $\mathbf{F}\phi_1 = \mathbf{F}\phi_2$, then $\phi_1 - \phi_2 = \Phi_{t_0}(\phi_2) - \Phi_{t_0}(\phi_1)$. Since Φ_{t_0} is of Lipschitz mapping with Lipschitz constant $k < 1$, so

$\phi_1 = \phi_2$. Thus, $\mathbf{F}$ is a bijective. We also have

$$\begin{aligned}\|\phi_1 - \phi_2\|_{\mathcal{C}} &\le \|\mathbf{F}\phi_1 - \mathbf{F}\phi_2\|_{\mathcal{C}} + \|\Phi_{t_0}(\phi_2) - \Phi_{t_0}(\phi_1)\|_{\mathcal{C}} \\ &\le \|\mathbf{F}\phi_1 - \mathbf{F}\phi_2\|_{\mathcal{C}} + k\|\phi_1 - \phi_2\|_{\mathcal{C}}.\end{aligned}$$

Therefore, $\|\phi_1 - \phi_2\|_{\mathcal{C}} \le 1/(1-k)\|\mathbf{F}\phi_1 - \mathbf{F}\phi_2\|_{\mathcal{C}}$. This yields continuity of inverse map $\mathbf{F}^{-1}$. Hence, $\mathbf{F}$ is a homeomorphism. Therefore, condition (ii) in Definition 3.3 is satisfied.

Condition (iii) in Definition 3.3 follows from Theorem 3.6. We shall now prove that condition (iv) of Definition 3.3 is satisfied. Indeed, let $u(\cdot)$ be a solution of equation (3.2) such that the function $u_s(\theta) \in S_s$. Then, by Lemma 3.4, the solution $u(t)$ for $t \in [s, \infty)$ can be rewritten in the form:

$$u(t) = U(t,s)\nu_0 + \int_s^\infty \mathcal{G}(t,\tau) f(\tau, u_\tau)\, d\tau \quad \text{for some } \nu_0 \in \mathrm{Im}P(s).$$

Thus, for $t \ge s$ and $\theta \in [-r, 0]$, we have

$$\begin{aligned}u(t-\theta) &= U(t-\theta, s)\nu_0 + \int_s^\infty \mathcal{G}(t-\theta,\tau) f(\tau, u_\tau)\, d\tau \\ &= U(t-\theta, s)\nu_0 + \int_s^t \mathcal{G}(t-\theta,\tau) f(\tau, u_\tau)\, d\tau \\ &\quad + \int_t^\infty \mathcal{G}(t-\theta,\tau) f(\tau, u_\tau)\, d\tau \\ &= U(t-\theta, s)\nu_0 + \int_s^t U(t-\theta,\tau)P(\tau) f(\tau, u_\tau)\, d\tau \\ &\quad + \int_t^\infty \mathcal{G}(t-\theta,\tau) f(\tau, u_\tau)\, d\tau \\ &= U(t-\theta, t)\left[U(t,s)\nu_0 + \int_s^t U(t,\tau)P(\tau) f(\tau, u_\tau)\, d\tau\right] \\ &\quad + \int_t^\infty \mathcal{G}(t-\theta,\tau) f(\tau, u_\tau)\, d\tau.\end{aligned}$$

Set

$$\mu_0 = U(t,s)\nu_0 + \int_s^t U(t,\tau)P(\tau) f(\tau, u_\tau)\, d\tau.$$

We have $P(t)\mu_0 = \mu_0$; hence, $\mu_0 \in \mathrm{Im}P(t)$. We thus obtain that $U(t-\theta,t)\mu_0$ belongs to $\mathrm{Im}\widetilde{P}(t)$ and

$$u(t-\theta) = U(t-\theta,t)\mu_0 + \int_t^\infty \mathcal{G}(t-\theta,\tau)f(\tau,u_\tau)\,d\tau.$$

By the uniqueness of $u(\cdot)$ on $[s-r,\infty)$ as in the proof of Theorem 3.6, we have that equation (3.2) has a unique solution $u(\cdot)$ on $[-r+t,\infty)$ satisfying $(\widetilde{P}(t)u_t)(\theta) = U(t-\theta,t)\mu_0$ and

$$u(\xi) = U(2t-\xi,t)\mu_0 + \int_t^\infty \mathcal{G}(2t-\xi,\tau)f(\tau,u_\tau)\,d\tau$$

for $\xi \in [-r+t,t]$. Therefore, the history function u_t can be viewed as:

$$u_t(\theta) = u(t+\theta) = U(t-\theta,t)\mu_0 + \int_t^\infty \mathcal{G}(t-\theta,\tau)f(\tau,u_\tau)\,d\tau = \phi(\theta) + \Phi_t(\phi)(\theta).$$

Hence, $u_t \in S_t$ for $t \geq s$. $\square$

Finally we give an illustrative example. This example was considered in **[15]**; thus, work verifying the φ-Lipschitz condition for the nonlinear part is a repetition. However, the computation is significant for the conditions on the functions φ and h_ν since we take E, E' to be a concrete admissible Banach function space.

Example 3.8. Consider the following problem.

(3.10)
$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \displaystyle\sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left(a_{kl}(t,x)\frac{\partial}{\partial x_l}u(t,x)\right) & \\ \qquad +\delta u(t,x) + b\,te^{-\alpha t}\displaystyle\int_{-r}^0 \ln(1+|u(t+\theta,x)|\,d\theta & \text{for } t \geq s \geq 0, \\ & x \in \Omega, \\ \displaystyle\sum_{k,l=1}^n n_k(x)a_{kl}(t,x)\frac{\partial}{\partial x_l}u(t,x) = 0 & x \in \partial\Omega, \\ u_s(\theta,x) = u(s+\theta,x) = \phi(\theta,x), \quad \theta \in [-r,0] & x \in \Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ oriented by outer unit normal vector $n(x)$. The coefficients $a_{kl}(t,x) \in C_b^\mu(\mathbb{R}_+, C(\overline{\Omega})) \cap C_b(\mathbb{R}_+, C^1(\overline{\Omega}))$, $1/2 < \mu \leq 1$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^{n} a_{kl}(t,x)v_k v_l \geq \eta |v|^2, \quad \text{for all } x \in \Omega \text{ and for some constant } \eta > 0.$$

Finally, the constants $\alpha > 0$, $b \neq 0$ and $\delta > 0$ are sufficient. We now choose the Hilbert space $X = L_2(\Omega)$ and define the differential operator

$$A(t,x,D) = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kl}(t,x)\frac{\partial}{\partial x_l}\right) + \delta$$

with domain

$$D(A(t)) = \{f \in W^{2,2}(\Omega) : \sum_{k,l=1}^{n} n_k(x) a_{kl}(t,x)\frac{\partial}{\partial x_l} f(x) = 0,\ x \in \partial\Omega\}.$$

Therefore, this problem can be rewritten as an abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t,\cdot) = A(t)u(t,\cdot) + F(t, u_t(\theta,\cdot)) & \text{for } t \geq s \geq 0, \\ u_s(\theta,\cdot) = \phi(\theta,\cdot) \in \mathcal{C} & \text{for } \theta \in [-r,0], \end{cases}$$

where $F : \mathbb{R}_+ \times \mathcal{C} \to X$ is defined by

$$F(t,\phi)(x) = b\, te^{-\alpha t}\int_{-r}^{0} \ln(1 + |(\phi(\theta))(x)|)\, d\theta, \quad x \in \Omega.$$

We have $F(t,\phi)(\cdot) \in X$ since Minkowski's inequality implies that

$$\begin{aligned}
&\left(\int_\Omega |F(t,\phi)(x)|^2 dx\right)^{1/2} \\
&\quad = |b|\, te^{-\alpha t}\left(\int_\Omega \left(\int_{-r}^{0} \ln(1 + |(\phi(\theta))(x)|)\, d\theta\right)^2 dx\right)^{1/2} \\
&\quad \leq |b|\, te^{-\alpha t}\int_{-r}^{0}\left(\int_\Omega \ln^2(1 + |(\phi(\theta))(x)|)\, dx\right)^{1/2} d\theta \\
&\quad \leq |b|\, te^{-\alpha t}\int_{-r}^{0}\left(\int_\Omega |(\phi(\theta))(x)|^2 dx\right)^{1/2} d\theta \\
&\quad = b|\, te^{-\alpha t}\int_{-r}^{0} \|\phi(\theta)\|_2\, d\theta < \infty.
\end{aligned}$$

From [**24,** Theorem 3.3, Example 4.2], the family of operators $(A(t))_{t\geq 0}$ generates an evolution family having an exponential dichotomy, with the dichotomy constants N and ν, provided that the Hölder constants of a_{kl} are sufficiently small. In addition, the dichotomy projections $P(t)$, $t \geq 0$, satisfy $\sup_{t\geq 0} \|P(t)\| \leq N$.

Now, we verify that F is φ-Lipschitz with $\varphi(t) = |b|rte^{-\alpha t} \in E' = L_p(\mathbb{R}_+)$, $p \in (1,\infty)$. Indeed, condition (i) is evident. To verify condition (ii), we use Minkowski's inequality and the fact that $\ln(1+h) \leq h$ for all $h \geq 0$. Then,

$$\begin{aligned}
&\|F(t,\phi_1)(x) - F(t,\phi_2)(x)\|_2 \\
&\quad = |b|\, te^{-\alpha t}\biggl(\int_\Omega \biggl(\int_{-r}^0 \ln\frac{1+|(\phi_1(\theta))(x)|}{1+|(\phi_2(\theta))(x)|}d\theta\biggr)^2 dx\biggr)^{1/2} \\
&\quad \leq |b|\, te^{-\alpha t}\int_{-r}^0 \biggl(\int_\Omega \ln^2\frac{1+|(\phi_1(\theta))(x)|}{1+|(\phi_2(\theta))(x)|}\,dx\biggr)^{1/2} d\theta \\
&\quad = |b|\, te^{-\alpha t}\int_{-r}^0 \biggl(\int_\Omega \ln^2\biggl(1+\frac{|(\phi_1(\theta))(x)|-|(\phi_2(\theta))(x)|}{1+|(\phi_2(\theta))(x)|}\biggr)\,dx\biggr)^{1/2} d\theta \\
&\quad \leq |b|\, te^{-\alpha t}\int_{-r}^0 \biggl(\int_\Omega |(\phi_1(\theta))(x)-(\phi_2(\theta))(x)|^2 dx\biggr)^{1/2} d\theta \\
&\quad = |b|\, te^{-\alpha t}\int_{-r}^0 \|\phi_1(\theta)-\phi_2(\theta)\|_2\, d\theta \\
&\quad \leq |b|\, rte^{-\alpha t}\sup_{\theta\in[-r,0]}\|\phi_1(\theta)-\phi_2(\theta)\|_2.
\end{aligned}$$

Hence, F is φ-Lipschitz with $\varphi \in E' = L_p(\mathbb{R}_+)$, $p \in (1,\infty)$. Therefore, $E = L_q(\mathbb{R}_+)$ with $1/p + 1/q = 1$. In the space $L_p(\mathbb{R}_+)$, the constants N_1 and N_2 in Definition 2.4 are defined by $N_1 = N_2 = 1$. In addition, we have

$$\Lambda_1\varphi(t) = \int_t^{t+1} \varphi(\tau)\,d\tau$$

and

$$\Lambda_1 T_1^+\varphi(t) = \int_{(t-1)_+}^{t} \varphi(\tau)\,d\tau,$$

where $(t-1)_+ = \max\{0, t-1\}$. Thus,

$$\max\{\|\Lambda_1\varphi\|_\infty, \|\Lambda_1 T_1^+\varphi\|_\infty\} < \frac{|b|r(1+e^{-1}-e^{-\alpha})}{\alpha^2}.$$

On the other hand, we have $t^p < e^{pt}$ for all $t \geq 0$. Thus, for $\alpha > \nu+1$, then

$$\begin{aligned}\|\varphi\|_{E'} &= |b|r\bigg(\int_0^\infty \tau^p e^{-\alpha p\tau}\,d\tau\bigg)^{1/p} \leq |b|r\bigg(\int_0^\infty e^{-(\alpha-1)p\tau}\,d\tau\bigg)^{1/p}\\ &= |b|r\bigg(\frac{1}{(\alpha-1)p}\bigg)^{1/p},\end{aligned}$$

and

$$\begin{aligned}h_\nu(t) &= |b|r\bigg(\int_0^\infty e^{-\nu p\,|t-\tau|}\tau^p e^{-\alpha p\tau}\,d\tau\bigg)^{1/p}\\ &\leq |b|r\bigg(\frac{2}{p(\alpha-\nu-1)}\bigg)^{1/p} e^{-\nu t} \quad \text{for } t \geq 0.\end{aligned}$$

Therefore,

$$\|h_\nu\|_E \leq |b|r\bigg(\frac{2}{p(\alpha-\nu-1)}\bigg)^{1/p}\bigg(\frac{1}{\nu q}\bigg)^{1/q}.$$

From Theorem 3.7, we obtain that, if

$$\begin{aligned}\max\Bigg\{&\frac{2|b|r(1+e^{-1}-e^{-\alpha})}{\alpha^2},\\ &|b|r\bigg(\frac{1}{\nu q}\bigg)^{1/q}\bigg[\bigg(\frac{2}{p(\alpha-\nu-1)}\bigg)^{1/p} + N\bigg(\frac{1}{(\alpha-1)p}\bigg)^{1/p}\bigg]\Bigg\}\\ &\leq \frac{e^{-\nu r}}{N(1+N)},\end{aligned}$$

then there is an invariant stable manifold of $\mathcal{E}$-class S for the mild solutions to problem (3.10). The S manifold is created by trajectories of the solutions belonging to $L_q(\mathbb{R}_+)$.

4. Exponential trichotomy and center-stable manifolds of $\mathcal{E}$-class. In this section, we will consider the case where the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential trichotomy on $\mathbb{R}_+$. With the same conditions as in Section 3, we will prove that there exists a center-stable manifold of $\mathcal{E}$-class for the solutions to equation (3.2).

We now recall the definition of an evolution family which has exponential trichotomy.

Definition 4.1. A given evolution family $(U(t,s))_{t\ge s\ge 0}$ is said to have an *exponential trichotomy* on the half-line if there are three families of projections $(P_j(t))_{t\ge 0}$, $j=1,2,3$, and positive constants N, α and β such that the following conditions are fulfilled:

(i) $H=\sup_{t\ge 0}\|P_j(t)\|<\infty$, $j=1,2,3$.
(ii) $P_1(t)+P_2(t)+P_3(t)=\mathrm{Id}$ for $t\ge 0$ and $P_j(t)P_i(t)=0$, $j\ne i$.
(iii) $P_j(t)U(t,s)=U(t,s)P_j(s)$ for $t\ge s\ge 0$ and $j=1,2,3$.
(iv) $U(t,s)|_{\mathrm{Im}P_j(s)}$ are isomorphisms from $\mathrm{Im}P_j(s)$ onto $\mathrm{Im}P_j(t)$, for all $t\ge s\ge 0$ and $j=2,3$, respectively; we also denote the inverse of $U(t,s)|_{\mathrm{Im}P_j(s)}$ by $U(s,t)_|$, $0\le s\le t$.
(v) For all $t\ge s\ge 0$ and $x\in X$, the following estimates hold:

$$\|U(t,s)P_1(s)x\|\le Ne^{-\beta(t-s)}\|P_1(s)x\|$$
$$\|U(s,t)_|P_2(t)x\|\le Ne^{-\beta(t-s)}\|P_2(t)x\|$$
$$\|U(t,s)P_3(s)x\|\le Ne^{\alpha(t-s)}\|P_3(s)x\|.$$

The projections $P_j(t)$, $t\ge 0$, $j=1,2,3$, are called the *trichotomy projections*, and the constants N, α and β, the *trichotomy constants*.

Note that, in the above definition, the Banach space X is split into direct sums of three subspaces such that the evolution family $(U(t,s))_{t\ge s\ge 0}$ is an exponential decay on $\mathrm{Im}P_1(s)$, the exponential growth on $\mathrm{Im}P_2(s)$ and exponentially bounded on $\mathrm{Im}P_3(s)$ for each fixed $s\ge 0$. Moreover, the evolution family $(U(t,s))_{t\ge s\ge 0}$ becomes an exponential dichotomy if the family of projections $P_3(t)$ is trivial, i.e., $P_3(t)=0$ for all $t\ge 0$.

Given that the evolution family $(U(t,s))_{t\ge s\ge 0}$ has an exponential trichotomy on the half-line, we can now construct three families of projections $\widetilde{P}_j(t)$, $t\ge 0$, $j=1,2,3$, on $\mathcal{C}$ as follows:

(4.1) $$(\widetilde{P}_j(t)\phi)(\theta)=U(t-\theta,t)P_j(t)\phi(0)\quad\text{for all } \theta\in[-r,0] \text{ and } \phi\in\mathcal{C}.$$

Next, is our second main result. We prove the existence of a center-stable manifold of $\mathcal{E}$-class for solutions of equation (3.2).

Theorem 4.2. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have exponential trichotomy with the trichotomy projections $(P_j(t))_{t\geq 0}$, $j=1,2,3$ and the constants N, α and β given as in Definition 4.1. Let E and E' be, respectively, an admissible Banach function space and its associated space. For each fixed δ such that $\delta>\alpha$, define the functions $e_\nu(t)=e^{-\nu t}$ and $h_\nu(t)=\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}$ for $t\geq 0$ and $\nu=(\delta-\alpha)/2$. Set*

$$q:=\sup\{\|P_j(t)\|: t\geq 0,\ j=1,\,3\},$$

and

$$N_0:=\max\{N,\ 2Nq\}.$$

Suppose that $\varphi\in E'$ is an exponentially E-invariant function and the function f is φ-Lipschitz with φ satisfying

$$\max\{N_0(1+H)e^{\nu r}(N_1\|\Lambda_1T_1^+\varphi\|_\infty+N_2\|\Lambda_1\varphi\|_\infty),\\ N_0(1+H)e^{\nu r}(\|h_\nu\|_E+N_0N_1\|e_\nu\|_E\|\varphi\|_{E'})\}<1.$$

Then, there exists a manifold $S=\{(t,S_t)\}_{t\geq 0}\subset\mathbb{R}_+\times\mathcal{C}$ for the solutions to equation (3.2), called a center-stable manifold *of $\mathcal{E}$-class, that is represented by the graphs of a family of Lipschitz continuous mappings*

$$\Phi_t:\mathrm{Im}(\widetilde{P}_1(t)+\widetilde{P}_3(t))\longrightarrow \mathrm{Im}\widetilde{P}_2(t)$$

with the Lipschitz constants independent of t such that $S_t=\mathrm{graph}(\Phi_t)$ has the following properties:

(i) *S_t is homeomorphic to $\mathrm{Im}(\widetilde{P}_1(t)+\widetilde{P}_3(t))$ for all $t\geq 0$.*

(ii) *To each $\phi\in S_s$, there corresponds one and only one solution $u(t)$ to equation (3.2) on $[s-r,\infty)$, satisfying $e^{-\gamma(s+\theta)}u_s(\theta)=\phi(\theta)$ for $\theta\in[-r,0]$, and the function*

$$z(t)=\begin{cases} e^{-\gamma(t+\cdot)}u_t(\cdot) & \text{for } t\geq s,\\ 0 & \text{for } 0\leq t<s\end{cases}$$

belongs to $\mathcal{E}$, where $\gamma=(\delta+\alpha)/2$. Moreover, for any two solutions $u(t)$ and $v(t)$ to equation (3.2) corresponding to different functions

$\phi_1, \phi_2 \in S_s$, we have the estimate

(4.2)

$$\|u_t - v_t\|_{\mathcal{C}} \le C_\mu e^{(\gamma-\mu)(t-s)} \|(\widetilde{P}(s)\phi_1)(0) - (\widetilde{P}(s)\phi_2)(0)\| \quad \text{for } t \ge s,$$

where μ and C_μ are positive constants independent of $s, u(\cdot)$, and $v(\cdot)$.

(iii) *S is positively invariant under equation* (3.2) *in the sense that, if $u(t)$, $t \ge s-r$, is the solution to equation* (3.2), *satisfying the condition $e^{-\gamma(s+\cdot)}u_s(\cdot) \in S_s$, and*

$$z(t) = \begin{cases} e^{-\gamma(t+\cdot)}u_t(\cdot) & \text{for } t \ge s, \\ 0 & \text{for } 0 \le t < s \end{cases}$$

belongs to $\mathcal{E}$, then the function $e^{-\gamma(t+\cdot)}u_t(\cdot) \in S_t$ for all $t \ge s$.

Proof. Set $P(t) := P_1(t) + P_3(t)$ and $Q(t) := P_2(t) = \mathrm{Id} - P(t)$ for $t \ge 0$. We have that $P(t)$ and $Q(t)$ are projections complementary to each other on X. We then define the families of projections $\widetilde{P}_j(t)$, $t \ge 0$, $j = 1, 2, 3$, on $\mathcal{C}$ as in equality (4.1). Setting $\widetilde{P}(t) = \widetilde{P}_1(t) + \widetilde{P}_3(t)$ and $\widetilde{Q}(t) = \widetilde{P}_2(t)$, $t \ge 0$, we obtain that $\widetilde{P}(t)$ and $\widetilde{Q}(t)$ are complementary projections on $\mathcal{C}$ for each $t \ge 0$.

We consider the following rescaling evolution family

$$\widetilde{U}(t,s) = e^{-\gamma(t-s)}U(t,s) \quad \text{for all } t \ge s \ge 0.$$

Now, we prove that the evolution family $\widetilde{U}(t,s)$ has an exponential dichotomy with dichotomy projections $P(t)$, $t \ge 0$. Indeed,

$$\begin{aligned} P(t)\widetilde{U}(t,s) &= e^{-\gamma(t-s)}(P_1(t) + P_3(t))U(t,s) \\ &= e^{-\gamma(t-s)}U(t,s)(P_1(s) + P_3(s)) = \widetilde{U}(t,s)P(s). \end{aligned}$$

Since $U(t,s)|_{\mathrm{Im}P_2(s)}$ is a isomorphism from $\mathrm{Im}P_2(s)$ onto $\mathrm{Im}P_2(t)$ and $\mathrm{Im}P_2(t) = \operatorname{Ker} P(t)$ for all $t \ge 0$, thus, $\widetilde{U}(t,s)|_{\operatorname{Ker} P(s)}$ is also an isomorphism from $\operatorname{Ker} P(s)$ onto $\operatorname{Ker} P(t)$, and we denote $\widetilde{U}(s,t)_| := (\widetilde{U}(t,s)|_{\operatorname{Ker} P(s)})^{-1}$ for $0 \le s \le t$. By the definition of exponential trichotomy we have

$$\|\widetilde{U}(s,t)_| Q(t)x\| \le e^{-(\beta+\gamma)(t-s)}\|Q(t)x\| \quad \text{for all } t \ge s \ge 0.$$

On the other hand,

$$\begin{aligned}
&\|\widetilde{U}(t,s)P(s)x\| \\
&\quad = e^{-\gamma(t-s)}\|U(t,s)(P_1(s)+P_3(s))x\| \\
&\quad \le Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)x\| + e^{\alpha(t-s)}\|P_3(s)x\|) \\
&\quad = Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)P(s)x\| + e^{\alpha(t-s)}\|P_3(s)P(s)x\|)
\end{aligned}$$

for all $t \ge s \ge 0$ and $x \in X$. Setting $q := \sup\{\|P_j(t)\|,\ t \ge 0,\ j = 1, 3\}$, we finally obtain the following estimate:

$$\|\widetilde{U}(t,s)P(s)x\| \le 2Nqe^{-(\delta-\alpha)/2(t-s)}\|P(s)x\|.$$

Therefore, $\widetilde{U}(t,s)$ has an exponential dichotomy with the dichotomy projections $P(t)$, $t \ge 0$, and dichotomy constants $N_0 := \max\{N, 2Nq\}$, $\nu := (\delta - \alpha)/2$.

Set $\widetilde{x}(t) = e^{-\gamma t}x(t)$, and define the mapping $F : \mathbb{R}_+ \times \mathcal{C} \to X$ as

$$F(t,\phi) = e^{-\gamma t} f(t, e^{\gamma(t+\cdot)}\phi(\cdot)) \quad \text{for } (t,\phi) \in \mathbb{R}_+ \times \mathcal{C}.$$

Obviously, F is also φ-Lipschitz. Thus, we can rewrite equation (3.2) in the new form

$$\begin{cases} \widetilde{x}(t) = \widetilde{U}(t,s)\widetilde{x}(s) + \int_s^t \widetilde{U}(t,\xi)F(\xi,\widetilde{x}_\xi)\,d\xi & \text{for all } t \ge s \ge 0, \\ \widetilde{x}_s(\cdot) = e^{-\gamma(s+\cdot)}\phi(\cdot) \in \mathcal{C}. \end{cases} \tag{4.3}$$

Hence, by Theorem 3.7, we obtain that, if

$$\max\{N_0(1+H)e^{\nu r}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty), \\ N_0(1+H)e^{\nu r}(\|h_\nu\|_E + N_0N_1\|e_\nu\|_E\|\varphi\|_{E'})\} < 1$$

then there exists an invariant stable manifold of $\mathcal{E}$-class S for the solutions to equation (4.3). Returning to equation (3.2) by using the relation $x(t) := e^{\gamma t}\widetilde{x}(t)$ and Theorems 3.6 and 3.7, we can easily verify the properties of S which are stated in (i), (ii) and (iii). Thus, S is a center-stable manifold of $\mathcal{E}$-class for the solutions of equation (3.2). $\square$

Next, we give an example in which the center-stable manifold is created by trajectories of solutions in the Lorentz space $L_{2,1}(\mathbb{R}_+)$. In order to simplify the computations, the linear operator generates an analytic semigroup and the nonlinear part is as in Example 3.8, but the Lipschitz coefficient is different.

Example 4.3. For fixed $n \in \mathbb{N}\setminus\{0,1\}$, consider the following problem:

$$
(4.4)\quad \begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + n^2 u(t,x) & \\ \quad +\varphi(t)\displaystyle\int_{-r}^{0} \ln(1+|u(t+\theta,x)|)\,d\theta & \text{for } t > s \geq 0,\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0 & t \in \mathbb{R}_+, \\ u_s(\theta,x) = u(s+\theta,x) = \phi(\theta,x) & \theta \in [-r,0],\ x \in [0,\pi], \end{cases}
$$

where

$$
\varphi(t) = \sum_{j=1}^{\infty} b\sqrt{j}\,\chi_{(1/(j+1),1/j]}(t) + \sum_{j=1}^{\infty} be^{-2j}\chi_{(j,j+1]}(t), \quad b \neq 0.
$$

Choose $X = L_2([0,\pi])$, the operator $A : D(A) \subset X \to X$ defined by

$$
\begin{aligned} Au &= \frac{\partial^2}{\partial x^2}u + n^2 u, \\ D(A) &= \{u \in H^2([0,\pi]) : u(0) = u(\pi) = 0\}. \end{aligned}
$$

Then, equation (4.4) can be rewritten as an abstract Cauchy problem

$$
\begin{cases} \dfrac{d}{dt}u(t,\cdot) = Au(t,\cdot) + F(t,u_t(\theta,\cdot)) & \text{for } t > s \geq 0, \\ u_s(\theta,\cdot) = \phi(\theta,\cdot) \in \mathcal{C} & \text{for } \theta \in [-r,0], \end{cases}
$$

where $F : \mathbb{R}_+ \times \mathcal{C} \to X$ is defined by

$$
F(t,\phi)(x) = \varphi(t)\int_{-r}^{0} \ln(1+|(\phi(\theta))(x)|)\,d\theta, \quad x \in [0,\pi].
$$

It can be seen (see **[9,** Chapter II]) that A generates an analytic semigroup $(T(t))_{t\geq 0}$. On the other hand, the spectrum of A is defined as

$$
\sigma(A) = \{-1^2+n^2, -2^2+n^2, \ldots, -(n-1)^2+n^2, 0, -(n+1)^2+n^2, \ldots\}.
$$

Choose $t_0 > 0$, by the spectral mapping theorem for the analytic semigroup which yields that $\sigma(T(t_0))$ consists of three disjoint compact sets $\sigma_1, \sigma_2, \sigma_3$, where

$$\sigma_1 \subset \{|z| < 1\}, \qquad \sigma_2 \subset \{|z| > 1\} \quad \text{and} \quad \sigma_3 \subset \{|z| = 1\}.$$

Hence, the trichotomy projections P_j, $j = \overline{1,3}$, are the Riesz projections for $T(t_0)$ corresponding to spectral sets σ_1, σ_2, σ_3. Since σ_2 and σ_3 consist of finitely many points, so $T(t)|_{\mathrm{Im}P_j}$ are isomorphisms on $\mathrm{Im}P_j$ for all $t \geq 0$ and $j = 2, 3$. Therefore, the evolution family $(U(t,s))_{t\geq s\geq 0}$, defined by $U(t,s) = T(t-s)$, has an exponential trichotomy with the trichotomy constants N, α, β.

As in Example 3.8, F is φ-Lipschitz with $\varphi(t) \in E' = L_{2,\infty}(\mathbb{R}_+)$. Therefore, $E = L_{2,1}(\mathbb{R}_+)$. Obviously, the function $\varphi(t)$ cannot belong to $L_p(\mathbb{R}_+)$ spaces with $p \geq 2$. In the Lorentz space $L_{2,\infty}(\mathbb{R}_+)$, the constants N_1 and N_2 in Definition 2.4 equal 1. We also have

$$\|\Lambda_1\varphi\|_\infty \leq 2|b|, \qquad \|\Lambda_1 T_1^+\varphi\|_\infty \leq 2|b| \quad \text{and} \quad \|\varphi\|_{E'} \leq |b|.$$

Let $\nu \in (0, 1/2]$, i.e, $\delta \in (\alpha, 2\alpha]$. Set $g_t(\tau) = e^{-\nu|t-\tau|}\varphi(\tau)$. This yields the estimate for the nonincreasing rearrangement function g_t^* of g_t, as follows:

- If $t \in (1/(k+1), 1/k]$, $k \geq 1$, then

$$\begin{aligned} g_t^*(\tau) \leq & \sum_{j=1}^{k-1} |b| e^{\nu(t-1/(j+1))}\sqrt{j}\,\chi_{[1/(j+1),1/j)}(\tau) \\ & + |b|\sqrt{k}\,\chi_{[1/(k+1),1/k)}(\tau) \\ & + \sum_{j=k+1}^{\infty} |b| e^{\nu(1/j-t)}\sqrt{j}\,\chi_{[1/(j+1),1/j)}(\tau) \\ & + \sum_{j=1}^{\infty} |b| e^{\nu(t-j)-2j}\chi_{[j,j+1)}(\tau). \end{aligned}$$

- If $t \in (k, k+1]$, $k \geq 1$, then

$$\begin{aligned} g_t^*(\tau) \leq & \sum_{j=1}^{\infty} |b| e^{\nu(1/j-t)}\sqrt{j}\,\chi_{[1/(j+1),1/j)}(\tau) + \sum_{j=1}^{k-1} |b| e^{\nu(j+1-t)-2j}\chi_{[j,j+1)}(\tau) \\ & + \quad |b| e^{-2k}\chi_{[k,k+1)}(\tau) + \sum_{j=k+1}^{\infty} |b| e^{\nu(t-j)-2j}\chi_{[j,j+1)}(\tau). \end{aligned}$$

Therefore,

$$h_\nu(t) = \|g_t\|_{E'} = \sup_{\tau\geq 0} \sqrt{\tau} g_t^*(\tau) \leq \begin{cases} |b| & \text{if } t \in [0,1], \\ |b|e^{-\nu(t-1)} & \text{if } t > 1. \end{cases}$$

Hence,

$$\|h_\nu\|_E \leq |b|\left(2 + \frac{1}{\nu}\right).$$

On the other hand, we have

$$\|e_\nu\|_E \leq 2 + \frac{1}{\nu e^\nu}.$$

From Theorem 4.2, we obtain that, if

$$|b|\left(2 + \frac{1}{\nu}\right) \leq \frac{e^{-\nu r}}{N_0(1+N_0)(1+H)},$$

then there exists an invariant center-stable manifold of $\mathcal{E}$-class for mild solutions to the problem (4.4). The above condition will be satisfied when b is sufficiently small. Moreover, this center-stable manifold is created by trajectories of solutions in the Lorentz space $L_{2,1}(\mathbb{R}_+)$.

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