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A NOTE ON SOLUTIONS OF INTERVAL-VALUED VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this paper we consider the interval-valued Volterra integral equations (IVIEs). We study the problem of existence and uniqueness of solutions for IVIEs. Finally, we give some examples for IVIEs.

1. Introduction. Set-valued differential and integral equations are an important part of the theory of set-valued analysis, and they play an important role in the theory and application of control theory. They were first studied in 1969 by De Blasi and Iervolino [**4**]. Recently, set-valued differential equations have been studied by many scientists due to their applications in many areas. For the basic theory on set-valued differential and integral equations, readers can be referred to the following books and papers (see [**1, 2, 5–8, 11–15, 18**] and the references therein).

Interval-valued analysis and interval differential equations (IDEs) are special cases of the set-valued analysis and set-valued differential equations, respectively. In many situations, when modeling real world phenomena, information about the behavior of a dynamic system is uncertain and one has to consider these uncertainties to gain better meaning with full models. Use of the interval-valued differential equation is a natural way to model dynamic systems subject to uncertainties. Recently, much work has been done by several authors in the theory of interval-valued differential equations (see, e.g., [**3, 9, 10, 17, 19**]).

Existence theorems for Volterra integral equations have been studied extensively in view of their applications to predator-prey models and

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medical diagnosis. In this paper, we generalize such existence theorems to interval-valued mappings using the concept of interval-valued functions and the integral due to Stefanini and Bede [**18**]. Similarly, as in classical analysis, we would like to introduce four main types of interval-valued integral equations: their names appear in the list below. Suppose that $F : [t_0, T] \to \mathbf{I}$, $K : [t_0, T] \times [t_0, T] \to \mathbf{I}$ are continuous interval-valued functions, $k : [t_0, T] \to \mathbf{R}$ is a continuous real-valued function and λ is a constant. We classify the interval-valued integral equations as follows:

(i) Interval-valued non-homogeneous Volterra integral equation

$$X(t) = F(t) + \int_{t_0}^{t} k(t,s)X(s)\,ds$$

(ii) Interval-valued homogeneous Volterra integral equation

$$X(t) = \int_{t_0}^{t} k(t,s)X(s)\,ds$$

(iii) Interval-valued non-homogeneous Fredholm integral equation

$$X(t) = F(t) + \lambda \int_{t_0}^{T} k(t,s)X(s)\,ds$$

(iv) Interval-valued homogeneous Fredholm integral equation

$$X(t) = \lambda \int_{t_0}^{T} k(t,s)X(s)ds,$$

where $t \in [t_0, T]$. We have noted that the interval-valued Volterra equation can be considered as a special case of the interval-valued Fredholm equation when $k(t,s) = 0$ for $s > t$ in $[t_0, T]$. The function $k(t,s)$ appearing in the above four equations is called the *kernel* of the interval-valued integral equation. Such a kernel is symmetric if $k(t,s) = k(s,t)$, for all $t, s \in [t_0, T]$.

In this paper, we shall prove the existence and uniqueness theorem of solutions for interval-valued Volterra integral equations of the following form

$$X(t) = F(t) + \int_{t_0}^{t} K(t,s,X(s))\,ds. \tag{1.1}$$

This paper is organized as follows. In Section 2, we recall some basic concepts and notations about interval-valued analysis and interval-valued differential equations. In Section 3, we present the existence and uniqueness theorem of a solution to the interval-valued Volterra integral equations. Finally, we give some examples for IVIEs.

2. Preliminaries and notation. Let $\mathcal{K}_c^n$ be the space of non empty compact and convex sets of $\mathbf{R}^n$. The set of real intervals will be denoted by $\mathbf{I}$ where $\mathbf{I} = \mathcal{K}_c^1$. The set interval $[\varnothing, \varnothing]$ is a singleton which contains a single element: $\varnothing \in [\varnothing, \varnothing]$. $\varnothing = \{0\} = [0, 0]$. The addition and scalar multiplication in $\mathbf{I}$ is defined as usual, i.e., for $A, B \in \mathbf{I}$, $A = [a^-, a^+]$, $B = [b^-, b^+]$, where $a^- \le a^+$, $b^- \le b^+$, and $\lambda \ge 0$, then we have

$$A + B = [a^- + b^-, a^+ + b^+], \quad \lambda A = [\lambda a^-, \lambda a^+], \quad (-\lambda A = [\lambda a^+, \lambda a^-]).$$

Furthermore, let $A \in \mathbf{I}$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R}$ and $\lambda_3 \lambda_4 \ge 0$. Then we have $\lambda_1(\lambda_2 A) = (\lambda_1 \lambda_2)A$ and $(\lambda_3 + \lambda_4)A = \lambda_3 A + \lambda_4 A$. Let $A, B \in \mathbf{I}$ as above; then the Hausdorff metric H in $\mathbf{I}$ is defined as follows:

$$H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}. \tag{2.1}$$

We notice that $(\mathbf{I}, H)$ is a complete, separable and locally compact metric space.

We define the magnitude and the length of $A \in \mathbf{I}$ by:

$$H(A, \{0\}) = \|A\| = \max\{|a^-|, |a^+|\}, \quad \operatorname{len}(A) = a^+ - a^-,$$

respectively.

The Hausdorff metric (2.1) satisfies some of the properties below:

$$\begin{aligned}
&H(A + C, B + C) = H(A, B) \quad \text{and} \quad H(A, B) = H(B, A), \\
&H(A + B, C + D) \le H(A, C) + H(B, D), \\
&H(\lambda A, \lambda B) = |\lambda| H(A, B), \\
&H(A, B) \le H(A, C) + H(C, B),
\end{aligned}$$

for all $A, B, C, D \in \mathbf{I}$ and $\lambda \in \mathbf{R}$. Let $A, B \in \mathbf{I}$. If there exists an interval $C \in \mathbf{I}$ such that $A = B + C$, then we call C the Hukuhara difference of A and B. The interval C is denoted by $A \ominus B$. Note that $A \ominus B \neq A + (-)B$. It is known that $A \ominus B$ exists in the case $\text{len}\,(A) \geq \text{len}\,(B)$. Besides that, we can see the following properties for $A, B, C, D \in \mathbf{I}$ (see [**9, 10**]):

- if $A \ominus B$, $A \ominus C$ exist, then $H(A \ominus B, A \ominus C) = H(B, C)$;
- if $A \ominus B$, $C \ominus D$ exist, then $H(A \ominus B, C \ominus D) = H(A + D, B + C)$;
- if $A \ominus B$, $A \ominus (B + C)$ exist, then there exist $(A \ominus B) \ominus C$ and $(A \ominus B) \ominus C = A \ominus (B + C)$;
- if $A \ominus B$, $A \ominus C$, $C \ominus B$ exist, then there exist $(A \ominus B) \ominus (A \ominus C)$ and $(A \ominus B) \ominus (A \ominus C) = C \ominus B$.

Definition 2.1 [**9**]. We say that the interval-valued mapping $F : [t_0, T] \to \mathbf{I}$ is continuous at the point $t \in [t_0, T]$ if, for every $\varepsilon > 0$, there exists $\delta = \delta(t, \varepsilon) > 0$ such that, for all $s \in [t_0, T]$ such that $|t - s| < \delta$, one has $H(F(t), F(s)) \leq \varepsilon$.

The strongly generalized differentiability was introduced in [**18**] and studied in [**3, 9–15**].

Definition 2.2 [**18**]. Let $X : [t_0, T] \to \mathbf{I}$ and $t \in [t_0, T]$. We say that X is strongly generalized differentiable of the first-order differential at t, if there exists $X'(t) \in \mathbf{I}$, such that

(i) for all $h > 0$ sufficiently small, there exist $X(t + h) \ominus X(t)$, $X(t) \ominus X(t - h)$ and

$$\lim_{h \searrow 0} H\left(\frac{X(t+h) \ominus X(t)}{h}, X'(t)\right) = 0,$$
$$\lim_{h \searrow 0} H\left(\frac{X(t) \ominus X(t-h)}{h}, X'(t)\right) = 0$$

or

(ii) for all $h > 0$ sufficiently small, there exist $X(t) \ominus X(t + h)$, $X(t - h) \ominus X(t)$ and

$$\lim_{h \searrow 0} H\left(\frac{X(t) \ominus X(t+h)}{-h}, X'(t)\right) = 0,$$
$$\lim_{h \searrow 0} H\left(\frac{X(t-h) \ominus X(t)}{-h}, X'(t)\right) = 0$$

or

(iii) for all $h > 0$ sufficiently small, there exist $X(t+h) \ominus X(t)$, $X(t-h) \ominus X(t)$ and

$$\lim_{h \searrow 0} H\left(\frac{X(t+h) \ominus X(t)}{h}, X'(t)\right) = 0,$$
$$\lim_{h \searrow 0} H\left(\frac{X(t-h) \ominus X(t)}{-h}, X'(t)\right) = 0$$

or

(iv) for all $h > 0$ sufficiently small, there exist $X(t) \ominus X(t+h)$, $X(t) \ominus X(t-h)$ and the limits

$$\lim_{h \searrow 0} H\left(\frac{X(t) \ominus X(t+h)}{-h}, X'(t)\right) = 0,$$
$$\lim_{h \searrow 0} H\left(\frac{X(t) \ominus X(t-h)}{h}, X'(t)\right) = 0.$$

(h in the denominators means $1/h$). In this definition, case (i) ((i)-differentiability for short) corresponds to the classic H-derivative introduced in [**8**], so this differentiability concept is a generalization of the Hukuhara derivative. Case (ii) of this definition corresponds to the second type Hukuhara derivative studied in [**9, 10**].

For an interval-valued function $F : [t_0, T] \to \mathbf{I}$, $F(t) = [F^-(t), F^+(t)]$, one defines the integral by the expression

$$\int_{t_0}^{t} F(s)\, ds = \left[\int_{t_0}^{t} F^-(s)\, ds,\ \int_{t_0}^{t} F^+(s)\, ds\right].$$

By the Newton-Leibniz formula, one can write: if an interval-valued function F is a second type Hukuhara differentiable on $[t_0, T]$, then $F(t_0) = F(t) + (-1)\int_{t_0}^{t} F'(s)\, ds$.

Definition 2.3. A mapping $X : [t_0, T] \to \mathbf{I}$ is bounded, if there exists an element $M > 0$ such that $H(X(t), \{0\}) \le M$, for all $t \in [t_0, T]$.

Corollary 2.1 (see, e.g., [**10**]). *Let $X : [t_0, T] \to \mathbf{I}$ be given. Denote $X(t) = [X^-(t), X^+(t)]$ for $t \in [t_0, T]$, where $X^-, X^+ : [t_0, T] \to \mathbf{R}$.*

(i) *If the mapping* X *is* (i)*-differentiable* (*i.e., classical Hukuhara differentiable*) *at* $t \in [t_0, T]$*, then the real-valued functions* X^-, X^+ *are differentiable at* t *and* $X'(t) = [(X^-)'(t), (X^+)'(t)]$.

(ii) *If the mapping* X *is* (ii)*-differentiable at* $t \in [t_0, T]$*, then the real-valued functions* X^-, X^+ *are differentiable at* t *and* $X'(t) = [(X^+)'(t), (X^-)'(t)]$.

3. Main results. Consider the following interval-valued Volterra integral equations

$$X(t) = F(t) + \int_{t_0}^{t} K(t, s, X(s))\, ds, \tag{3.1}$$

for all $t \in [t_0, T]$, where $F : [t_0, T] \to \mathbf{I}$ and $K : \mathcal{D} \times \mathbf{I} \to \mathbf{I}$, with $\mathcal{D} = \{(t, s) \in [t_0, T] \times [t_0, T] : t_0 \le s \le t < T\}$.

Theorem 3.1. *Let* N, M *and* L *be positive numbers. Assume that* F, K *satisfy the following conditions*:

(i) $F : [t_0, T] \to \mathbf{I}$ *is continuous and satisfies* $H(X(t), F(t)) \le N$;

(ii) $K : \mathcal{D} \times \mathbf{I} \to \mathbf{I}$ *is continuous and satisfies the Lipschitz condition with respect to* X*, i.e.,*

$$H(K(t, s, X), K(t, s, Y)) \le L H(X, Y),$$

for $(t, s, X), (t, s, Y) \in \mathcal{D} \times \mathbf{I}$.

In addition, if $H(K(t, s, X), \{0\}) \le M$*, then there is a unique solution of* IVIE (3.1) *on* $[t_0, \mathbf{T}]$ *where* $\mathbf{T} = \min\{T - t_0, (N/M)\}$.

Proof. In the proof of this theorem, we apply the method of successive approximations to construct a sequence of continuous functions $X_n : [t_0, \mathbf{T}] \to \mathbf{I}$ as follows:

$$X_0(t) = F(t), \qquad X_n(t) = F(t) + \int_{t_0}^{t} K(t, s, X_{n-1}(s))\, ds, \quad n \ge 1. \tag{3.2}$$

By the mathematical induction method, we can see that all $\{X_n\}_{n \ge 0}$ are continuous mappings on $[t_0, \mathbf{T}]$. Further $H(X_n, F(t)) \le M(t - t_0) \le N$, for all $n \ge 1$. Then, for $t \in [t_0, \mathbf{T}]$, we have

$$H(X_1(t), X_0(t)) = H\left(\int_{t_0}^{t} K(t, s, X_0(s))\, ds, \{0\} \right) \leq M(t - t_0) \leq N.$$

By (ii) and (3.2), we find that

$$\begin{aligned} &H(X_n(t), X_{n-1}(t)) \\ &\qquad \leq H\left(\int_{t_0}^{t} K(t, s, X_{n-1}(s))\, ds,\ \int_{t_0}^{t} K(t, s, X_{n-2}(s))\, ds \right) \\ &\qquad \leq L \int_{t_0}^{t} H(X_{n-1}(s), X_{n-2}(s))\, ds. \end{aligned}$$

In particular,

$$\begin{aligned} H(X_2(t), X_1(t)) &\leq L \int_{t_0}^{t} H(X_1(s), X_0(s))\, ds \\ &\leq L \int_{t_0}^{t} M(s - t_0)\, ds = \frac{ML(t - t_0)^2}{2!}, \quad t \in [t_0, \mathbf{T}]. \end{aligned}$$

Furthermore, if we assume that

$$H(X_{n-1}(t), X_{n-2}(t)) \leq \frac{M}{L} \frac{[L(t - t_0)]^{n-1}}{(n-1)!}, \quad t \in [t_0, \mathbf{T}], \tag{3.3}$$

then we have

$$\begin{aligned} H(X_n(t), X_{n-1}(t)) &\leq L \int_{t_0}^{t} \frac{M}{L} \frac{[L(s - t_0)]^{n-1}}{(n-1)!}\, ds \\ &= \frac{M}{L} \frac{[L(t - t_0)]^{n}}{n!}, \quad t \in [t_0, \mathbf{T}]. \end{aligned}$$

It follows by mathematical induction that (3.3) holds for any $n \geq 1$. Consequently, the series $\sum_{n=1}^{\infty} H(X_n(t), X_{n-1}(t))$ is uniformly convergent on $[t_0, \mathbf{T}]$, and hence the sequence $\{X_n\}_{n \geq 0}$ is uniformly convergent. It follows that there exists a continuous function $X : [t_0, \mathbf{T}] \to \mathbf{I}$ such that $H(X_n(t), X(t)) \to 0$ as $n \to \infty$. Since

$$H(K(t, s, X_n(s)), K(t, s, X_n(s))) \leq LH(X_n(t), X(t)) \longrightarrow 0$$

on $[t_0, \mathbf{T}]$ as $n \to \infty$ and since

$$H\left(\int_{t_0}^{t} K(t, s, X_n(s))\, ds, \int_{t_0}^{t} K(t, s, X(s))\, ds \right) \\ \leq \int_{t_0}^{t} H(K(t, s, X_n(s), K(t, s, X(s))))\, ds,$$

it follows that

$$\lim_{n\to\infty} \int_{t_0}^{t} K\left(t, s, X_n(s)\right)\, ds = K\left(t, s, X(s)\right), \quad t \in [t_0, \mathbf{T}].$$

By (3.2), we obtain that

$$X(t) = F(t) + \int_{t_0}^{t} K(t, s, X(s))\, ds$$

and so X satisfies (3.1).

To prove the uniqueness, let $Y : [t_0, \mathbf{T}] \to \mathbf{I}$ be a second solution for (3.1), and

$$Y(t) = F(t) + \int_{t_0}^{t} K(t, s, Y(s))\, ds.$$

Then, for every $t \in [t_0, \mathbf{T}]$, we get

$$H(X(t), Y(t)) \leq \int_{t_0}^{t} LH(X(s), Y(s))\, ds.$$

By applying the Gronwall inequality, we get $H(X(t), Y(t)) \leq 0$, which completes the proof. $\square$

Remark 3.1. With the assumptions of Theorem 3.1, the existence and uniqueness of the solution for problem (3.1) can be obtained by using the contraction principle. Let $\mathbf{S}$ be the space of continuous functions from $[t_0, \mathbf{T}]$ into $(\mathbf{I}, H)$ with $H_0(Z, F) \leq N$, i.e., $\mathbf{S} = \{Z \mid Z : [t_0, \mathbf{T}] \to \mathbf{I}$ continuous and $H_0(Z, F) \leq N\}$ where $H_0(Z, F) = \sup_{t\in[t_0,\mathbf{T}]} H(Z(t), F(t))$. We define an operator $\mathbf{A} : \mathbf{S} \to \mathbf{S}$ by

$$\mathbf{A}Z(t) = F(t) + \int_{t_0}^{t} K(t, s, Z(s))\, ds.$$

Now we shall prove the operator $\mathbf{A}$ is a contraction mapping with the assumptions of Theorem 3.1.

Proof. *Step* 1. To prove that $\mathbf{A} : \mathbf{S} \to \mathbf{S}$, we have to prove that $\mathbf{AZ}$ is continuous and $\mathbf{AZ} \subset \mathbf{S}$. For each $t \in [t_0, \mathbf{T}]$, we get

$$\begin{aligned}
(3.4)\quad & H(\mathbf{AZ}(t+h), \mathbf{AZ}(t)) \\
&= H\Big(F(t+h) + \int_{t_0}^{t+h} K(t+h, s, Z(s))\, ds, F(t) + \int_{t_0}^{t} K(t, s, Z(s))\, ds\Big) \\
&\le H(F(t+h), F(t)) + \int_{t_0}^{t} H\Big(K(t+h, s, Z(s)), K(t, s, Z(s))\Big)\, ds \\
&\quad + \int_{t}^{t+h} H(K(t+h, s, Z(s)), \{0\})\, ds.
\end{aligned}$$

By using assumptions (i) and (ii), we obtain that the right hand side of (3.4) tends to zero as $h \to 0$. So $\mathbf{AZ}$ is continuous. Let $Z \in \mathbf{S}$. We have

$$\begin{aligned}
H_0(\mathbf{AZ}, F) &= \sup_{t \in [t_0, T]} H(\mathbf{AZ}(t), F(t)) \\
&= \sup_{t \in [t_0, T]} H\Big(F(t) + \int_{t_0}^{t} K(t, s, Z(s))\, ds, F(t)\Big) \\
&\le \sup_{t \in [t_0, T]} \int_{t_0}^{t} H(K(t, s, Z(s)), \{0\})\, ds \\
&\le M(t - t_0) \le N.
\end{aligned}$$

Therefore, $\mathbf{AZ} \in \mathbf{S}$.

Step 2. We have to prove that $\mathbf{S}$ is a complete metric space. Indeed, let $\{Z_n\}$ be a sequence in $\mathbf{S}$ converging to $Z \in C([t_0, \mathbf{T}], \mathbf{I})$. We consider

$$H_0(Z, F) \le H_0(Z_n, Z) + H_0(Z_n, F) \le \varepsilon + N$$

for sufficiently large n and all $\varepsilon > 0$. So $Z \in \mathbf{S}$. Therefore, $\mathbf{S}$ is a complete metric space.

Step 3. We have to prove that $\mathbf{A}$ is a contraction mapping. For $Z, W \in \mathbf{S}$, we get

$$\begin{aligned} H_0(\mathbf{AZ}, \mathbf{AW}) &= \sup_{t\in[t_0,T]} H\bigg(\int_{t_0}^t K(t,s,Z(s))\,ds, \int_{t_0}^t K(t,s,W(s))\,ds\bigg) \\ &= \sup_{t\in[t_0,T]} L\int_{t_0}^t H(Z(s),W(s))\,ds \le L\mathbf{T}H_0(Z,W). \end{aligned}$$

Therefore, $\mathbf{A} : \mathbf{S} \to \mathbf{S}$ is a contraction only if $L\mathbf{T} < 1$. Since $\mathbf{S}$ is a complete metric space and $\mathbf{A}$ is a contracting self-map on $\mathbf{S}$, it has a unique fixed point $Z \in \mathbf{S}$. This fixed point is the required unique solution to IVIE (3.1). $\square$

Theorem 3.2 (Existence and uniqueness). *Assume that* F, K *satisfy the following assumptions*:

(i) $F : [t_0, T] \to \mathbf{I}$ *is continuous and bounded*;

(ii) $K : \mathcal{D} \times \mathbf{I} \to \mathbf{I}$ *is continuous and satisfies the Lipschitz condition with respect to* X, *i.e.*,

$$H(K(t,s,X), K(t,s,Y)) \le LH(X,Y),$$

for $(t,s,X), (t,s,Y) \in \mathcal{D} \times \mathbf{I}$, $L > 0$;

(iii) $K(t,s,\{0\})$ *is bounded on* $[t_0,T]$. *If* $LT < 1$, *then* IVIE (3.1) *has a unique solution* X *on* $[t_0,T]$.

Proof. One can obtain this result easily by using the methods of successive approximations and the contraction principle as in the proof of Theorem 3.1 and Remark 3.1, respectively. $\square$

Finally, we shall give some simple examples of an IVIE. We consider the following homogeneous Volterra integral equation

$$X(t) = F(t) + \lambda\int_0^t k(t,s)X(s)\,ds, \quad t \in [0,T], \tag{3.5}$$

where $F(t), X(t) \in \mathbf{I}$ and $k(t,s) \in \mathbf{R}$.

Corollary 3.1. *Suppose that* $k : [0,T]\times[0,T] \to \mathbf{R}$ *and* $F : [0,T] \to \mathbf{I}$ *are given continuous functions and* λ *is an arbitrary parameter. If* $|k(t,s)| \leq M$ *for all* $0 \leq t,s \leq T$*, then equation* (3.5) *has a unique interval-valued solution.*

Case 1. We assume that $\lambda \cdot k(t,s) > 0$ and $F(t) = [F^-(t), F^+(t)]$, $X(t) = [X^-(t), X^+(t)]$. From equation (3.5), we obtain

$$\begin{cases} X^-(t) = F^-(t) + \int_0^t \lambda \cdot k(t,s)\, X^-(s)\, ds \\ X^+(t) = F^+(t) + \int_0^t \lambda \cdot k(t,s)\, X^+(s)\, ds \\ X^-(0) = F^-(0) \\ X^+(0) = F^+(0). \end{cases}$$

Case 2. We suppose that $\lambda \cdot k(t,s) < 0$. From equation (3.5), we obtain

$$\begin{cases} X^-(t) = F^-(t) + \int_0^t \lambda \cdot k(t,s)\, X^+(s)\, ds \\ X^+(t) = F^+(t) + \int_0^t \lambda \cdot k(t,s)\, X^-(s)\, ds \\ X^-(0) = F^-(0) \\ X^+(0) = F^+(0) \end{cases}$$

$$\begin{bmatrix} X^-(t) \\ X^+(t) \end{bmatrix} = \begin{bmatrix} F^-(t) \\ F^+(t) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & \lambda k(t,s) \\ \lambda \cdot k(t,s) & 0 \end{bmatrix} \begin{bmatrix} X^-(s) \\ X^+(s) \end{bmatrix} ds.$$

Example 3.3. Let us consider the interval-valued Volterra integral equation

$$X(t) = [1,2]\cos(t) + \int_0^t X(s)\, ds. \tag{3.6}$$

We transform problem (3.6) into the following ordinary integral equation system

$$\begin{cases} X^-(t) = \cos(t) + \int_0^t X^-(s)\, ds \\ X^+(t) = 2\cos(t) + \int_0^t X^+(s)\, ds \\ X^-(0) = 1 \\ X^+(0) = 2. \end{cases}$$

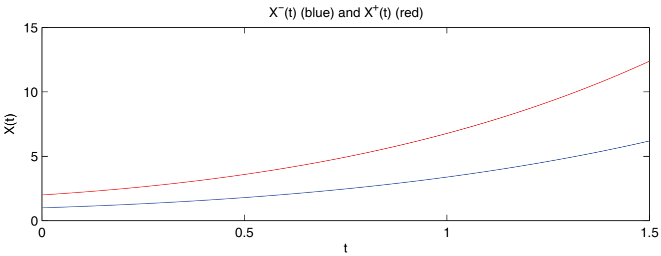

FIGURE 1. Solution of Example 3.3.

We obtain a unique solution to (3.6) defined on $[0, \pi/2]$, and it is of the form

$$X(t) = \left[\frac{-(\sin(t)+\cos(t))+3e^t}{2}, -(\sin(t)+\cos(t))+3e^t\right].$$

This solution is illustrated in Figure 1.

Example 3.4. Let us consider the interval-valued Volterra integral equation

$$X(t) = [t^2, 1+2t^2] + \int_0^t -X(s)\,ds. \tag{3.7}$$

We transform problem (3.7) into the following ordinary integral equation system

$$\begin{bmatrix} X^-(t) \\ X^+(t) \end{bmatrix} = \begin{bmatrix} t^2 \\ 1+2t^2 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} X^-(s) \\ X^+(s) \end{bmatrix} ds.$$

We obtain a unique solution to (3.7) defined on $[0, 2]$, and it is of the form

$$X(t) = \left[-e^t + 2e^{-t} + 2t - 1, e^t + 2e^{-t} + t - 2\right].$$

This solution is illustrated in Figure 2.

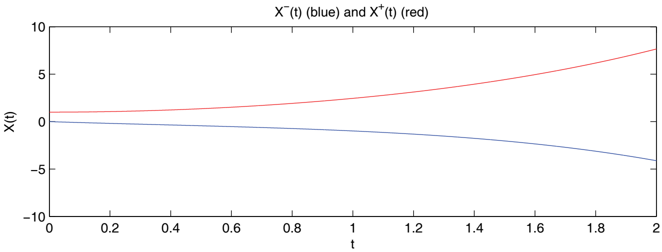

FIGURE 2. Solution of Example 3.4.

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