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EXISTENCE AND STABILITY OF NONLINEAR, FRACTIONAL ORDER RIEMANN-LIOUVILLE VOLTERRA-STIELTJES MULTI-DELAY INTEGRAL EQUATIONS

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ABSTRACT. We study the existence and the stability of solutions for Riemann-Liouville, Volterra-Stieltjes quadratic delay integral equations of fractional order. Our results are obtained by applying some fixed point theorems.

1. Introduction. Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as differential calculus since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has developed up to the present day. Integral equations are a useful mathematical tool in both pure and applied analysis [**12, 13**], in the kinetic theory of gases [**18**], transport theory [**19**], theory of radiative transfer [**10, 11**], etc. Fractional integral equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering and other applied sciences. Significant recent developments in ordinary and partial fractional differential and integral equations are described in the monographs of Abbas et al. [**4–6**], Baleanu et al. [**8**], Diethelm [**15**], Hilfer [**17**], Kilbas et al. [**20**], Lakshmikantham et al. [**21**], Podlubny [**24**], Samko et al. [**28**] and Tarasov [**30**], and in the papers by Abbas et al. [**1–3, 7**], Qian et al. [**25–27**] and Vityuk and Golushkov [**31**]. In particular, interesting results on the stability of solutions of various classes of integral equations of fractional order have been obtained by Abbas et al. [**4–6**], Banaś and Zając [**9**], Darwish et al. [**14**], and the references therein.

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In [**14, 25–27**], the authors used the technique associated with a certain measure of noncompactness for the study of the existence of solutions for some integral equations. Recently, in [**2–5**], Abbas et al. proved some results concerning the existence and stability of solutions for various classes of nonlinear fractional order integral equations.

Motivated by these papers, we establish here some sufficient conditions for the existence and the stability of solutions of the delay fractional order Riemann-Liouville, Volterra-Stieltjes quadratic integral equation

$$(1)\quad \begin{aligned} u(t,x) = \mu(t,x) + f\big(t,x,u(t,x)\big)\int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ h\big(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m)\big)\,dy\,d_s g(t,s), \end{aligned}$$

for $(t,x)\in J := \mathbf{R}_+\times[0,b]$, with

$$(2)\quad u(t,x)=\Phi(t,x) \text{ for } (t,x)\in \widetilde{J} := \big([-T,\infty)\times[-\Xi,b]\big)\backslash\big((0,\infty)\times(0,b]\big).$$

Here, u is the unknown solution, and we are given the various constants: $b>0$; r_1, $r_2\in(0,\infty)$; $\tau_i\geq 0$ and $\xi_i\geq 0$ for $i=1,\ldots,m$;

$$T=\max(\tau_1,\ldots,\tau_m) \quad\text{and}\quad \Xi=\max(\xi_1,\ldots,\xi_m).$$

Also given are the continuous functions

$$\mu: J\longrightarrow \mathbf{R},\ g:\mathbf{R}_+\times\mathbf{R}_+\longrightarrow\mathbf{R},\ f:J\times\mathbf{R}\longrightarrow\mathbf{R},\ h:J'\times\mathbf{R}^m\longrightarrow\mathbf{R},$$

where $J'=\{(t,x,s,y)\in J^2 : s\leq t, y\leq x\}$, as well as the continuous function $\Phi:\widetilde{J}\to\mathbf{R}$ satisfying

$$\Phi(t,0)=\mu(t,0)\quad \text{for } t\in\mathbf{R}_+$$

and

$$\Phi(0,x)=\mu(0,x)\quad\text{for } x\in[0,b].$$

Recall that Euler's Gamma function is defined by $\Gamma(r)=\int_0^\infty t^{r-1}e^{-t}\,dt$ for $r>0$.

We use the Schauder fixed point theorem to show the existence of solutions of the problem (1)–(2), and we prove that all solutions are globally asymptotically stable. This paper initiates the question of global existence and stability for this class of integral equations.

2. Preliminaries. If $p > 0$ and $q > 0$, then $L^1\big([0,p]\times[0,q]\big)$ denotes the space of Lebesgue-integrable functions $u : [0,p] \times [0,q] \to \mathbf{R}$, and $BC := BC\big([-T,\infty)\times[-\Xi,b]\big)$ denotes the Banach space of all bounded and continuous functions from $[-T,\infty)\times[-\Xi,b]$ into $\mathbf{R}$. We equip these spaces with their respective standard norms,

$$\|u\|_1 = \int_0^p \int_0^q |u(t,x)|\,dx\,dt$$

and

$$\|u\|_{BC} = \sup_{t\in[-T,\infty),x\in[-\Xi,b]} |u(t,x)|,$$

and $B(u_0,\eta)$ denotes the closed ball in BC centered at $u_0 \in BC$ with radius $\eta > 0$.

Definition 2.1 [**28**]. Let $r \in (0,\infty)$ and $u \in L^1\big([0,p] \times [0,q]\big)$. The rth-order partial Riemann-Liouville integral of $u(t,x)$ with respect to t is

$$I^r_{0,t}u(t,x) = \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}\,u(s,x)\,ds$$

for almost all $(t,x) \in [0,p] \times [0,q]$.

Analogously, we define

$$I^r_{0,x}u(t,x) = \frac{1}{\Gamma(r)} \int_0^x \frac{(x-s)^{r-1}}{\Gamma(r)}\,u(t,s)\,ds.$$

Example 2.2. For $\lambda \in (-1,0) \cup (0,\infty)$, $\omega \in (-1,0) \cup (0,\infty)$ and $r > 0$,

$$I^r_{0,t}\,\frac{t^\lambda x^\omega}{\Gamma(1+\lambda)} = \frac{t^{\lambda+r}x^\omega}{\Gamma(1+\lambda+r)}.$$

We will also use the following mixed integral.

Definition 2.3 [**31**]. Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1\big([0, p] \times [0, q]\big)$. The rth-order, left-sided, mixed Riemann-Liouville integral of u is

$$I_\theta^r u(t, x) = \int_0^t \frac{(t-\tau)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-s)^{r_2-1}}{\Gamma(r_2)} u(s, t)\, ds\, d\tau.$$

In particular, $I_\theta^\theta u(t, x) = u(t, x)$ and

$$I_\theta^\sigma u(t, x) = \int_0^t \int_0^x u(\tau, s)\, ds\, d\tau \quad \text{where } \sigma = (1, 1).$$

For instance, $I_\theta^r u$ exists for all r_1, $r_2 \in (0, \infty)$, when $u \in L^1\big([0, p] \times [0, q]\big)$. Note also that, if $u \in C\big([0, p] \times [0, q]\big)$, then $I_\theta^r u \in C\big([0, p] \times [0, q]\big)$; moreover,

$$I_\theta^r u(t, 0) = I_\theta^r u(0, x) = 0 \quad \text{for } t \in [0, p] \text{ and } x \in [0, q].$$

Example 2.4. For $\lambda \in (-1, 0) \cup (0, \infty)$, $\omega \in (-1, 0) \cup (0, \infty)$, $r_1 > 0$ and $r_2 > 0$,

$$I_\theta^r \frac{t^\lambda x^\omega}{\Gamma(1+\lambda)\Gamma(1+\omega)} = \frac{t^{\lambda+r_1} x^{\omega+r_2}}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}.$$

If u is a real function defined on the interval $[a, b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a, b]$. We say that u is of bounded variation on the interval $[a, b]$ whenever $\bigvee_a^b u$ is finite. If $w : [a, b] \times [c, b] \to \mathbf{R}$, then the symbol $\bigvee_{t=p}^q w(t, s)$ indicates the variation of the function $t \to w(t, s)$ on the interval $[p, q] \subset [a, b]$, where s is arbitrarily fixed in $[c, d]$. In the same way, we define $\bigvee_{s=p}^q w(t, s)$. For the properties of functions of bounded variation we refer to Natanson [**23**].

If u and φ are two real functions defined on the interval $[a, b]$, then under some conditions [**23**] we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b u(t)\, d\varphi(t)$$

of function u with respect to φ. In this case, we say that u is Stieltjes integrable on $[a, b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [**23, 29**]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a, b]$.

In what follows, we will use a few properties of the Stieltjes integral contained in the lemmas given below:

Lemma 2.5 [**22**]. *If u is Stieltjes integrable on the interval $[a, b]$ with respect to a function ψ of bounded variation, then*

$$\left|\int_a^b u(t)\, d\psi(t)\right| \le \int_a^b |u(t)|\, d_t\Psi(t), \quad \textit{where } \Psi(t) = \bigvee_a^t \psi.$$

Lemma 2.6 [**22**]. *Let u and v be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function ψ, such that $u(t) \le v(t)$ for $t \in [a, b]$. Then*

$$\int_a^b u(t)\, d\psi(t) \le \int_a^b v(t)\, d\psi(t).$$

In the sequel, we will also consider Stieltjes integrals of the form

$$\int_a^b u(t)\, d_s g(t, s),$$

and Riemann-Liouville fractional Stieltjes integrals of the form

$$\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}\, u(s)\, d_s g(t, s),$$

where $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, $r \in (0, \infty)$ and the symbol d_s indicates that the integration variable is s. From Lemma 2.5, we get

$$\left| \int_0^t u(t,s)\, d_s g(t,s) \right| \le \int_0^t |u(t,s)|\, d_s G(t,s),$$

$$\text{where } G(t,s) = \bigvee_{\sigma=0}^{s} g(t,\sigma).$$

Let $\varnothing \neq \Omega \subset BC$ and $F : \Omega \to \Omega$, and consider the equation

$$u(t,x) = (Fu)(t,x). \tag{3}$$

We introduce the following concept of attractivity for solutions of (3).

Definition 2.7. Solutions of equation (3) are *locally attractive* if there exists a ball $B(u_0, \eta)$ in the space BC such that, for $v = v(t,x)$ and $w = w(t,x)$ satisfying (3) and belonging to $B(u_0, \eta) \cap \Omega$, we have that

$$\lim_{t\to\infty} \big[v(t,x) - w(t,x)\big] = 0 \quad \text{for each } x \in [0,b]. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (3) are said to be *uniformly locally attractive* or, equivalently, solutions of (3) are *locally asymptotically stable*.

Definition 2.8. A solution $v = v(t)$ of equation (3) is said to be *globally attractive* if (4) holds for each solution $w = w(t,x)$ of (3). If condition (4) is satisfied uniformly with respect to the set Ω, solutions of (3) are said to be *globally asymptotically stable* (or *uniformly globally attractive*).

Lemma 2.9 [**12**, page 62]. *Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold*:

(a) *D is uniformly bounded in BC*;

(b) *The functions belonging to D are almost equicontinuous on $\mathbf{R}_+ \times [0,b]$, i.e., equicontinuous on every compact subset of $\mathbf{R}_+ \times [0,b]$*;

(c) *The functions from D are equiconvergent, that is, given $\varepsilon > 0$ and $x \in [0, b]$ there corresponds a $T(\varepsilon, x) > 0$ such that*

$$\big|u(t,x) - \lim_{t\to\infty} u(t,x)\big| < \varepsilon \quad \textit{for all } t \geq T(\varepsilon, x) \textit{ and } u \in D.$$

3. Existence and stability of solutions. We will use the following hypotheses.

H1. The functions μ and Φ are in BC. Moreover,

$$\lim_{t\to\infty} \mu(t,x) = 0 \quad \text{for } x \in [0, b].$$

H2. There exists a positive function $P \in BC$ such that

$$\big|f(t,x,u) - f(t,x,v)\big| \leq P(t,x)|u-v| \quad \text{for } (t,x) \in J \text{ and } u, v \in \mathbf{R}.$$

Moreover, $\lim_{t\to\infty} f(t,x,0) = 0$ for $x \in [0, b]$ and, if B is any bounded subset of BC, then the set of functions $\{(t,x) \mapsto f(t,x,u) : u \in B\}$ is equicontinuous on $\mathbf{R}_+ \times [0, b]$.

H3. For all t_1, $t_2 \in \mathbf{R}_+$ such that $t_1 < t_2$, the function $s \mapsto g(t_2, s) - g(t_1, s)$ is nondecreasing on $\mathbf{R}_+$.

H4. The function $s \mapsto g(0, s)$ is nondecreasing on $\mathbf{R}_+$.

H5. The functions $s \mapsto g(t, s)$ and $t \mapsto g(t, s)$ are continuous on $\mathbf{R}_+$ for each fixed $t \in \mathbf{R}_+$ or $s \in \mathbf{R}_+$, respectively.

H6. There exists a continuous function $Q : J' \to \mathbf{R}_+$ such that

$$\left(1 + \sum_{i=1}^{m} |u_i|\right)|h(t,x,s,y,u_1,\ldots,u_m)| \leq Q(t,x,s,y);$$

for each $(t,x,s,y) \in J'$ and all $u_i \in \mathbf{R}$ for $i = 1, \ldots m$. Moreover,

$$\lim_{t\to\infty} \int_0^t (t-s)^{r-1} Q(t,x,s,y)\, d_s G(t,s) = 0$$

$$\text{where } G(t,s) = \bigvee_{\sigma=0}^{s} g(t,\sigma).$$

Remark 3.1. Set

$$\mu^* := \sup_{(t,x)\in J} \mu(t,x), \qquad \Phi^* := \sup_{(t,x)\in \tilde{J}} \Phi(t,x),$$
$$f^* := \sup_{(t,x)\in J} f(t,x,0), \quad p^* := \sup_{(t,x)\in J} P(t,x),$$

and

$$q^* := \sup_{(t,x)\in J} \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} Q(t,x,s,y)\, dy\, d_s G(t,s).$$

From the hypotheses above, we infer that μ^*, Φ^*, f^*, p^* and q^* are finite.

We can now prove the existence and stability of a solution to the problem (1)–(2).

Theorem 3.2. *Assume that hypotheses* **H1–H6** *hold. If*

$$p^* q^* < 1, \tag{5}$$

then the problem (1)–(2) *has at least one solution in the space BC. Moreover, solutions of the problem* (1)–(2) *are globally asymptotically stable.*

Proof. Define the operator N such that $N(u)(t,x)$ equals the right-hand side of (1) for $(t,x) \in J$, and equals $\Phi(t,x)$ for $(t,x) \in \widetilde{J}$, so that the problem (1)–(2) is equivalent to $u = N(u)$. Also, for brevity, define

$$\mathsf{h}(t,x,s,y,u) = h\big(t,x,s,y,u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m)\big).$$

Operator N maps BC into BC. Indeed, the function $N(u)$ is continuous on J for any $u \in BC$ and, to see that $N(u)$ is bounded, we write

$$(Nu)(t,x) = \mathrm{I} + \mathrm{II} + \mathrm{III} \quad \text{for } (t,x) \in J, \tag{6}$$

where $\mathrm{I} = \mu(t,x)$,

$$\mathrm{II} = f(t,x,0) \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \mathsf{h}(t,x,s,y,u)\, dy\, d_s g(t,s),$$

and

$$\mathrm{III} = \big[f\big(t,x,u(t,x)\big) - f(t,x,0)\big] \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ \mathsf{h}(t,x,s,y,u)\,dy\,d_s g(t,s).$$

Here, $|\mathrm{I}| \le \mu^*$. Hypotheses **H4** and **H6** imply

$$|\mathrm{II}| \le f^* \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ Q(t,x,s,y)\,dy\,d_s G(t,s) \le f^* q^*,$$

and Hypotheses **H2** and **H6** imply

$$|\mathrm{III}| \le P(x,t)|u(t,x)| \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ Q(t,x,s,y)\,dy\,d_s G(t,s),$$

so $|(Nu)(t,x)| \le \mu^* + q^*(f^* + p^*\|u\|_{BC})$ for $(t,x) \in J$. Also, if $(t,x) \in \widetilde{J}$, then $|N(u)(t,x)| = |\Phi(t,x)| \le \Phi^*$, showing that $N(u) \in BC$ with

$$\|N(u)\|_{BC} \le \max\big\{\Phi^*, \mu^* + q^*(f^* + p^*\|u\|_{BC})\big\}, \tag{7}$$

and proving that $N : BC \to BC$. Moreover, inequalities (5) and (7) imply that N transforms the ball $B_\eta := B(0,\eta)$ into itself if

$$\eta = \max\{\Phi^*, \eta^*\} \quad \text{and} \quad \eta^* > \frac{\mu^* + q^* f^*}{1 - p^* q^*}.$$

We shall now prove, in several steps, that $N : B_\eta \to B_\eta$ satisfies the assumptions of the Schauder fixed point theorem [**16**].

Step 1: *N is continuous.* Let u_n be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in [-T,\infty) \times [-\Xi, b]$,

$$(Nu_n)(t,x) - (Nu)(t,x) = \mathrm{I} + \mathrm{II},$$

where, this time,

$$\mathrm{I} = \big[f\big(t,x,u_n(t,x)\big) - f\big(t,x,u(t,x)\big)\big]\Big| \int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)} \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ \mathsf{h}(t,x,s,y,u_n)\,dy\,d_s g(t,s)$$

and

$$\mathrm{II} = f\big(t,x,u(t,x)\big)\int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ \big[\mathsf{h}(t,x,s,y,u_n)-\mathsf{h}(t,x,s,y,u)\big]\,dy\,d_sg(t,s).$$

Hypotheses **H2** and **H6** imply that

$$|\mathrm{I}| \le p^*q^*|u_n(t,x)-u(t,x)| \le p^*q^*\|u_n-u\|_{BC},$$

so $\mathrm{I}\to 0$ as $n\to\infty$. Likewise, $\big|f\big(t,x,u(t,x)\big)\big| \le f^*+p^*\|u\|_{BC}$ so

$$|\mathrm{II}| \le (f^*q^*+\eta)\int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)} \\ \big|\mathsf{h}(t,x,s,y,u_n)-\mathsf{h}(t,x,s,y,u)\big|\,dy\,d_sG(t,s).$$

This integral tends to 0 uniformly for $(t,x)\in \widetilde{J}\cup\big([0,a]\times[0,b]\big)$ with any fixed $a>0$, because h is continuous. If $(t,x)\in(a,\infty)\times[0,b]$, then

$$|\mathrm{II}|\le(f^*q^*+\eta)\int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_2)}\bigg(\int_0^t \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}2Q(t,x,s,y)\,d_sG(t,s)\bigg)dy,$$

and, by **H6**, the inner integral can be made arbitrarily small with a chosen sufficiently large. Thus, $\|N(u_n)-N(u)\|_{BC}\to 0$.

Step 2: *$N(B_\eta)$ is uniformly bounded.* This is clear since $N(B_\eta)\subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on $[-T,a]\times[-\Xi,b]$ for each $a>0$.* Let (t_1,x_1), $(t_2,x_2)\in[0,a]\times[0,b]$ satisfy $t_1<t_2$ and $x_1<x_2$, let $u\in B_\eta$ and write

$$(Nu)(t_2,x_2)-(Nu)(t_1,x_1)=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where, this time, $\mathrm{I}=\mu(t_2,x_2)-\mu(t_1,x_1)$,

$$\mathrm{II} = \big[f\big(t_2,x_2,u(t_2,x_2)\big)-f\big(t_1,x_1,u(t_1,x_1)\big)\big] \\ \int_0^{t_2} \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\int_0^{x_2} \frac{(x-y)^{r_2-1}}{\Gamma(r_2)}\,\mathsf{h}(t_2,x_2,s,y,u)\,dy\,d_sg(t,s)$$

and

$$\mathrm{III} = f\big(t_1, x_1, u(t_1, x_1)\big)$$
$$\bigg(\int_0^{t_2} \frac{(t_2 - s)^{r_1 - 1}}{\Gamma(r_1)} \int_0^{x_2} \frac{(x_2 - y)^{r_2 - 1}}{\Gamma(r_2)} \mathsf{h}(t_2, x_2, s, y, u)\, dy\, d_s g(t_2, s)$$
$$- \int_0^{t_1} \frac{(t_1 - s)^{r_1 - 1}}{\Gamma(r_1)} \int_0^{x_1} \frac{(x_1 - y)^{r_2 - 1}}{\Gamma(r_2)} \mathsf{h}(t_1, x_1, s, y, u)\, dy\, d_s g(t_1, s)\bigg).$$

Thus,

$$|\mathrm{II}| \le q^* \big| f\big(t_2, x_2, u(t_2, x_2)\big) - f\big(t_1, x_1, u(t_1, x_1)\big|$$

and

$$|\mathrm{III}| \le \frac{f^* + p^* \eta}{\Gamma(r_1)\Gamma(r_2)} \big(|\mathrm{IV}| + |\mathrm{V}| + |\mathrm{VI}|\big)$$

where

$$\mathrm{IV} = \int_0^{t_1} \int_0^{x_1} (t_2 - s)^{r_1 - 1}(x_2 - y)^{r_2 - 1} \mathsf{h}(t_2, x_2, s, y, u)\, dy\, d_s g(t_2, s)$$
$$- \int_0^{t_1} \int_0^{x_1} (t_1 - s)^{r_1 - 1}(x_1 - y)^{r_2 - 1} \mathsf{h}(t_1, x_1, s, y, u)\, dy\, d_s g(t_1, s),$$
$$\mathrm{V} = \int_{t_1}^{t_2} \int_0^{x_2} (t_2 - s)^{r_1 - 1}(x_2 - y)^{r_2 - 1} \mathsf{h}(t_2, x_2, s, y, u)\, dy\, d_s g(t_2, s),$$
$$\mathrm{VI} = \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1 - 1}(x_2 - y)^{r_2 - 1} \mathsf{h}(t_2, x_2, s, y, u)\, dy\, d_s g(t_2, s).$$

Continuity of μ, f, g and h implies that I, II and III all tend to zero as (t_1, x_1) tends to (t_2, x_2). The equicontinuity for cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \le 0 \le t_2$, $x_1 \le 0 \le x_2$ is obvious.

Step 4: *$N(B_\eta)$ is equiconvergent.* Let $(t, x) \in J$ and $u \in B_\eta$, and decompose $N(u)(t, x)$ as in (6). By **H2** and **H6**,

$$|\mathrm{II}| \le \frac{f^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x \frac{(t - s)^{r_1 - 1}(x - y)^{r_2 - 1} Q(t, s, x, y)}{1 + \sum_{i=1}^m |u_i(s - \tau_i, y - \xi_i)|}\, dy\, d_s G(t, s)$$

and

$$|\mathrm{III}| \le \frac{P(t, x)|u(t, x)|}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1}(x - y)^{r_2 - 1} Q(t, x, s, y)\, dy\, d_s G(t, s),$$

so $\big|(Nu)(t,x)\big|$ is bounded by

$$|\mu(t,x)|+\frac{f^*+p^*\eta}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}Q(t,x,s,y)\,dy\,d_sG(t,s).$$

Thus, for each $x\in[0,b]$, we have $|(Nu)(t,x)|\to 0$ as $t\to+\infty$, and hence

$$\big|(Nu)(t,x)-(Nu)(+\infty,x)\big|\longrightarrow 0 \text{ as } t\to+\infty.$$

As a consequence of Steps 1–4 together with Lemma 2.9, we conclude that $N:B_\eta\to B_\eta$ is continuous and compact. Applying Schauder's theorem [**16**], we deduce that N has a fixed point u which is a solution of the problem (1)–(2).

Step 5: *Global asymptotic stability of solutions.* Assume that u and v are two solutions of the problem (1)–(2). Then, for each $(t,x)\in J$,

$$\begin{aligned}|u(t,x)-v(t,x)| &= |(Nu)(t,x)-(Nv)(t,x)|\\ &\le p^*q^*|u(t,x)-v(t,x)|\\ &\quad+\frac{f^*+p^*\|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\\ &\quad\times 2Q(t,x,s,y)\,dy\,d_sG(t,s),\end{aligned}$$

so $|u(t,x)-v(t,x)|$ is bounded by

$$\frac{2(f^*+p^*\|u\|_{BC})}{(1-p^*q^*)\Gamma(r_1)\Gamma(r_2)}\int_0^x(x-y)^{r_2-1}\bigg(\int_0^t(t-s)^{r_1-1}Q(t,x,s,y)\,d_sG(t,s)\bigg)dy.$$

We deduce that

$$\lim_{t\to\infty}|u(t,x)-v(t,x)|=0,$$

and consequently all solutions of (1)–(2) are globally asymptotically stable. $\square$

4. An example. As an application and to illustrate our results, we consider the following problem of fractional order Riemann-Liouville Volterra-Stieltjes quadratic multi-delay integral equations

$$(8)\quad u(t,x)=\mu(t,x)+f\big(t,x,u(t,x)\big)\int_0^t\frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\int_0^x\frac{(x-y)^{r_2-1}}{\Gamma(r_2)}\\ h\big(t,x,s,y,u(s-1,y-2),u(s-\tfrac{1}{2},y-\tfrac{2}{5})\big)\,dy\,d_sg(t,s),$$

for $(t,x) \in J := \mathbf{R}_+ \times [0,1]$, with

$$u(t,x) = \frac{2}{(2+t^2)(2+x^2)} \tag{9}$$

for $(t,x) \in \widetilde{J} := \big([-1,\infty)\times[-2,1]\big)\backslash\big((0,\infty)\times(0,1]\big)$. This problem has the form (1)–(2) with

$$\begin{aligned} m &= 2, \quad \tau_1 = 1, \qquad\quad \tau_2 = \frac{1}{2}, \\ \xi_1 &= 2, \quad \xi_2 = \frac{2}{5}, \quad \Phi(t,x) = \frac{2}{(2+t^2)(2+x^2)}, \end{aligned}$$

so $T = 1$ and $\Xi = 2$. We choose

$$\begin{aligned} r_1 = \frac{1}{4}, \quad r_2 = \frac{1}{2}, \qquad\quad \mu(t,x) &= \frac{1}{2+t^2+x^2}, \\ f(t,x,u) = \frac{e^{x-t}|u|}{1+|u|}, \quad g(t,s) &= s, \end{aligned}$$

with

$$h(t,x,s,y,u_1,u_2) = \frac{cxs^{-3/4}|u|\sin\sqrt{t}\,\sin s}{(1+t^2+y^2)(2+|u_1|+|u_2|)},$$
$$c = \frac{\pi}{16e\Gamma(\frac{1}{4})},$$

for $(t,x,s,y) \in J'$, $s \neq 0$ and $u_1, u_2 \in \mathbf{R}$, and zero otherwise.

We can see that $1/(2+t^2+x^2) \to 0$ as $t \to \infty$, for each $x \in [0,1]$, and the assumption **H1** is satisfied with $\mu^* = \Phi^* = 1/2$. The function f is continuous, $f^* = 0$, and

$$\big|f(t,x,u) - f(t,x,v)\big| \le e^{x-t}|u-v|$$
$$\text{for } (t,x) \in J \text{ and } u, v \in \mathbf{R},$$

so assumption **H2** holds with $P(t,x) = e^{x-t}$ and $p^* = e$. The function g satisfies hypotheses **H3–H5**, and the function h satisfies assumption **H6**. Indeed, h is continuous and

$$\big|h(t,x,s,y,u_1,u_2)\big| \le \frac{Q(t,x,s,y)}{1+|u_1|+|u_2|}$$
$$\text{for } (t,x,s,y) \in J' \text{ and } u_1, u_2 \in \mathbf{R},$$

where

$$Q(t,x,s,y)=\frac{cxs^{-3/4}\sin\sqrt{t}\,\sin s}{1+t^2+y^2}\quad \text{for } (t,x,s,y)\in J' \text{ and } s\neq 0,$$

with $Q(t,x,0,y)=0$ for $(t,x)\in J$ and $y\in[0,1]$. Then,

$$\begin{aligned}\left|\int_0^t (t-s)^{r-1}Q(t,x,s,y)d_sg(t,s)\right| &\le \int_0^t \frac{|cx\sin\sqrt{t}\,\sin s|}{(t-s)^{3/4}s^{3/4}}\,d_sG(t,s)\\ &\le cx|\sin\sqrt{t}|\int_0^t (t-s)^{-3/4}s^{-3/4}\,ds\\ &\le \frac{cx\Gamma(1/4)^2}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| \le \frac{cx\Gamma(1/4)^2}{\sqrt{\pi t}},\end{aligned}$$

which tends to zero as $t\to\infty$, and

$$\begin{aligned}q^* &:= \sup_{(t,x)\in J}\int_0^t\int_0^x \frac{(t-s)^{r_1-1}}{\Gamma(r_1)}\frac{(x-y)^{r_2-1}}{\Gamma(r_2)}Q(t,x,s,y)\,dy\,d_sG(t,s)\\ &\le \sup_{(t,x)\in J}\frac{1}{\Gamma(\frac{3}{2})}\frac{cx\Gamma(\frac14)}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| = \frac{2c\Gamma(\frac14)}{\pi}=\frac{1}{8e}.\end{aligned}$$

Finally, we can see that $p^*q^*\le 1/8<1$ so inequality (5) is satisfied. Consequently, by Theorem 3.2, problem (8)–(9) has at least one solution in the space $BC([-1,\infty)\times[-2,1])$, and all solutions are globally asymptotically stable.

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