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ISOMORPHISM THEOREMS FOR GYROGROUPS AND L-SUBGYROGROUPS

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Abstract. We extend well-known results in group theory to gyrogroups, especially the isomorphism theorems. We prove that an arbitrary gyrogroup G induces the gyrogroup structure on the symmetric group of G so that Cayley's Theorem is obtained. Introducing the notion of L-subgyrogroups, we show that an L-subgyrogroup partitions G into left cosets. Consequently, if H is an L-subgyrogroup of a finite gyrogroup G, then the order of H divides the order of G.

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1. Introduction

Let c be a positive constant representing the speed of light in vacuum and let $\mathbb{R}^3_c$ denote the c-ball of relativistically admissible velocities, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3;\ \|\mathbf{v}\| < c\}$. In [13], Einstein velocity addition $\oplus_E$ in the c-ball is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} \right\}$$

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where $\gamma_{\mathbf{u}}$ is the Lorentz factor given by $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}}\cdot$

The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. Nevertheless, Ungar showed that $(\mathbb{R}^3_c, \oplus_E)$ is rich in structure and encodes a group-like structure, namely the gyrogroup structure. He introduced space rotations $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, called *gyroautomorphisms*, to repair the breakdown of associativity in $(\mathbb{R}^3_c, \oplus_E)$

$$\begin{aligned}\mathbf{u}\oplus_E(\mathbf{v}\oplus_E\mathbf{w}) &= (\mathbf{u}\oplus_E\mathbf{v})\oplus_E \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}\\ (\mathbf{u}\oplus_E\mathbf{v})\oplus_E\mathbf{w} &= \mathbf{u}\oplus_E(\mathbf{v}\oplus_E \mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w})\end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3_c$. The resulting system forms a gyrocommutative gyrogroup, called the *Einstein gyrogroup*, which has been intensively studied in [3,5,9,11,13, 14,16,17].

There are close connections between the Einstein gyrogroup and the Lorentz transformations, as described in [14, Chapter 11] and [12]. A Lorentz transformation without rotation is called a *Lorentz boost*. Let $L(\mathbf{u})$ and $L(\mathbf{v})$ denote Lorentz boosts parameterized by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3_c$. The composite of two Lorentz boosts is not a pure Lorentz boost, but a Lorentz boost followed by a space rotation

$$L(\mathbf{u})\circ L(\mathbf{v}) = L(\mathbf{u}\oplus_E\mathbf{v})\circ \mathrm{Gyr}[\mathbf{u},\mathbf{v}] \tag{1}$$

where $\mathrm{Gyr}[\mathbf{u},\mathbf{v}]$ is a rotation of spacetime coordinates induced by the Einstein gyroautomorphism $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$. In this paper, we present an abstract version of the composition law (1) of Lorentz boosts.

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk $\mathbb{D}=\{z\in\mathbb{C};\ |z|<1\}$ with Möbius addition

$$a\oplus_M b = \frac{a+b}{1+\bar{a}b} \tag{2}$$

for $a,b\in\mathbb{D}$. The Möbius gyroautomorphisms are given by

$$\mathrm{gyr}[a,b]z = \frac{1+a\bar{b}}{1+\bar{a}b}z, \qquad z\in\mathbb{D}. \tag{3}$$

Let $\mathbb{B}$ denote the open unit ball of n-dimensional Euclidean space $\mathbb{R}^n$ (or more generally of a real inner product space). In [15], Ungar extended Möbius addition from the complex unit disk to the unit ball

$$\mathbf{u}\oplus_M\mathbf{v} = \frac{(1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{4}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. The unit ball together with Möbius addition forms a gyrocommutative gyrogroup, which has been intensively studied in [1,4,6,7,14–16].

The factorization of Möbius gyrogroups was comprehensively studied by Ferreira and Ren in [1,4], in which they showed that any Möbius subgyrogroup partitions the Möbius gyrogroup into left cosets. The fact that any subgyrogroup of an arbitrary gyrogroup partitions the gyrogroup is not stated in the literature, and this is indeed the case, as shown in Theorem 15. This result leads to the introduction of *L-subgyrogroups*. We prove that an L-subgyrogroup partitions the gyrogroup into left cosets and consequently obtain a portion of *Lagrange's Theorem*: if H is an L-subgyrogroup of a finite gyrogroup G, then the order of H divides the order of G. We also prove the isomorphism theorems for gyrogroups, in full analogy with their group counterparts.

2. Basic Properties of Gyrogroups

A pair $(G, \oplus)$ consisting of a nonempty set G and a binary operation $\oplus$ on G is called a *magma*. Let $(G, \oplus)$ be a magma. A bijection from G to itself is called an *automorphism* of G if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. The set of all automorphisms of G is denoted by $\mathrm{Aut}\,(G, \oplus)$. Ungar formulated the formal definition of a gyrogroup as follows.

Definition 1 ([14]) *A magma $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms*

(G1) *there is an element 0 in G such that for all elements a in G,* $0 \oplus a = a$

(G2) *for each element a in G, there is an element b in G such that,* $b \oplus a = 0$

(G3) *For all a,b in G, there is* $\mathrm{gyr}[a, b]$ *in Aut*$(G, +)$ *such that for each c in G*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$$

(G4) *for all* $a, b \in G$, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

The axioms in Definition 1 imply the right counterparts.

Theorem 2 ([14]) *A magma $(G, \oplus)$ forms a gyrogroup if and only if it satisfies the following properties*:

(g1) *there is an element 0 in G such that for every a in G,* $0 \oplus a = a$ *and* $a \oplus 0 = a$ (two-sided identity)

(g2) or each a in G, there is an element b in G such that $b \oplus a = 0$ *and* $a \oplus b = 0$ (two-sided inverse)
For $a, b, c \in G$, *define*

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)) \qquad \text{(gyrator identity)}$$

then

(g3) $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$ (gyroautomorphism)

(g3a) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$ (left gyroassociative law)

(g3b) $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$ (right gyroassociative law)

(g4a) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ (left loop property)

(g4b) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$ (right loop property).

The map $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism generated by the group elements* a *and* b. By Theorem 2, any gyroautomorphism is completely determined by its generators via the *gyrator identity*. A gyrogroup G having the additional property that

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a *gyrocommutative gyrogroup*.

Many of group theoretic theorems are generalized to the gyrogroup case with the aid of gyroautomorphisms, see [11, 14] for more details. Some theorems are listed here for easy reference. To shorten notation, we write $a \ominus b$ instead of $a \oplus (\ominus b)$.

Theorem 3 ([11], Theorem 2.11) *Let* G *be a gyrogroup. Then*

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \tag{5}$$

for all $a, b, c \in G$.

Theorem 4 ([11], Theorem 2.25) *For any two elements* a *and* b *of a gyrogroup*

$$\ominus(a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a). \tag{6}$$

Theorem 5 ([11], Theorem 2.27) *The gyroautomorphisms of any gyrogroup* G *are even*

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a, b] \tag{7}$$

and inversive symmetric

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a] \tag{8}$$

for all $a, b \in G$.

Using Theorem 5, one can prove the following proposition.

Proposition 6. *Let G be a gyrogroup and let $X \subseteq G$. Then the following are equivalent*

1) $\mathrm{gyr}[a,b](X) \subseteq X$ *for all* $a, b \in G$

2) $\mathrm{gyr}[a,b](X) = X$ *for all* $a, b \in G$.

The *gyrogroup cooperation* $\boxplus$ is defined by the equation

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b, \qquad a, b \in G. \tag{9}$$

Like groups, every linear equation in a gyrogroup G has a unique solution in G.

Theorem 7 ([11], Theorem 2.15) *Let G be a gyrogroup and let $a, b \in G$. The unique solution of the equation $a \oplus x = b$ in G for the unknown x is $x = \ominus a \oplus b$, and the unique solution of the equation $x \oplus a = b$ in G for the unknown x is $x = b \boxplus (\ominus a)$.*

The following cancellation laws in gyrogroups are derived as a consequence of Theorem 7.

Theorem 8 ([11]) *Let G be a gyrogroup. For all $a, b, c \in G$*

1) $a \oplus b = a \oplus c$ *implies* $b = c$ (general left cancellation law)

2) $\ominus a \oplus (a \oplus b) = b$ (left cancellation law)

3) $(b \ominus a) \boxplus a = b$ (right cancellation law I)

4) $(b \boxplus (\ominus a)) \oplus a = b$ (right cancellation law II).

It is known in the literature that every gyrogroup forms a left Bol loop with the A_ℓ-property, where the gyroautomorphisms correspond to *left inner mappings* or *precession maps*. In fact, gyrogroups and left Bol loops with the A_ℓ-property are equivalent, see for instance [8].

To prove an analog of Cayley's theorem for gyrogroups, we will make use of the following theorem

Theorem 9 ([2], Theorem 1) *Let G be a gyrogroup, let X be an arbitrary set, and let $\phi \colon X \to G$ be a bijection. Then X endowed with the induced operation $a \oplus_X b := \phi^{-1}(\phi(a) \oplus \phi(b))$ for $a, b \in X$ becomes a gyrogroup.*

3. Cayley's Theorem

Recall that for $a \in \mathbb{D}$, the map τ_a that sends a complex number z to $a \oplus_M z$ defines a Möbius transformation or conformal mapping on $\mathbb{D}$, known as a *Möbius translation*. In the literature, the following composition law of Möbius translations is known

$$\tau_a \circ \tau_b = \tau_{a \oplus_M b} \circ \operatorname{gyr}[a, b] \tag{10}$$

for all $a, b \in \mathbb{D}$. In this section, we extend the composition law (10) to an arbitrary gyrogroup G. We also show that the symmetric group of G admits the gyrogroup structure induced by G, thus obtaining an analog of Cayley's theorem for gyrogroups.

Throughout this section, G and H are arbitrary gyrogroups.

For each $a \in G$, the *left gyrotranslation by* a and the *right gyrotranslation by* a are defined on G by

$$L_a \colon x \mapsto a \oplus x \quad \text{and} \quad R_a \colon x \mapsto x \oplus a. \tag{11}$$

Theorem 10. *Let G be a gyrogroup.*

(1) *The left gyrotranslations are permutations of G.*

(2) *Denote the set of all left gyrotranslations of G by $\overline{G}$. The map $\psi \colon G \to \overline{G}$ defined by $\psi(a) = L_a$ is bijective. The inverse map $\phi := \psi^{-1}$ fulfills the condition in* Theorem 9. *In this case, the induced operation $\oplus_{\overline{G}}$ is given by*

$$L_a \oplus_{\overline{G}} L_b = L_{a \oplus b}$$

for all $a, b \in G$.

(3) *For all $a, b, c \in G$*

$$L_a \circ L_b = L_{a \oplus b} \circ \operatorname{gyr}[a, b] \tag{12}$$

and

$$\operatorname{gyr}_{\overline{G}}[L_a, L_b] L_c = L_{\operatorname{gyr}[a,b]c}. \tag{13}$$

Proof: Let $a, b \in G$.

(1) That L_a is injective follows from the general left cancellation law. That L_a is surjective follows from Theorem 7.

(2) That ψ is bijective is clear. By Theorem 9, the induced operation is given by

$$L_a \oplus_{\overline{G}} L_b = \psi(\psi^{-1}(L_a) \oplus \psi^{-1}(L_b)) = \psi(a \oplus b) = L_{a \oplus b}.$$

(3) By the left cancellation law, $L_a^{-1} = L_{\ominus a}$. By the gyrator identity, $\mathrm{gyr}[a,b] = L_{\ominus(a\oplus b)} \circ L_a \circ L_b$ and hence $\mathrm{gyr}[a,b] = L_{a\oplus b}^{-1} \circ L_a \circ L_b$. It follows that $L_a \circ L_b = L_{a\oplus b} \circ \mathrm{gyr}[a,b]$. Equation (13) follows from the gyrator identity. ■

Let $\mathrm{Stab}\,(0)$ denote the set of permutations of G leaving the gyrogroup identity fixed

$$\mathrm{Stab}\,(0) = \{\rho \in \mathrm{Sym}\,(G)\,;\ \rho(0) = 0\}.$$

It is clear that $\mathrm{Stab}\,(0)$ is a subgroup of the symmetric group, $\mathrm{Sym}\,(G)$, and we have the following inclusions

$$\{\mathrm{gyr}[a,b]\,;\ a,b \in G\} \subseteq \mathrm{Aut}\,(G) \leqslant \mathrm{Stab}\,(0) \leqslant \mathrm{Sym}\,(G).$$

The next theorem enables us to introduce a binary operation $\oplus$ on the symmetric group of G so that $\mathrm{Sym}\,(G)$ equipped with $\oplus$ becomes a gyrogroup containing an isomorphic copy of G.

Theorem 11. *For each $\sigma \in \mathrm{Sym}\,(G)$, σ can be written uniquely as $\sigma = L_a \circ \rho$, where $a \in G$ and $\rho \in \mathrm{Stab}\,(0)$.*

Proof: Suppose that $L_a \circ \rho = L_b \circ \eta$, where $a, b \in G$ and $\rho, \eta \in \mathrm{Stab}\,(0)$. Then $a = (L_a \circ \rho)(0) = (L_b \circ \eta)(0) = b$, which implies $L_a = L_b$ and so $\rho = \eta$. This proves the uniqueness of factorization. Let σ be an arbitrary permutation of G. Choose $a = \sigma(0)$ and set $\rho = L_{\ominus a} \circ \sigma$. Note that $\rho(0) = L_{\ominus a}(a) = \ominus a \oplus a = 0$. Hence, $\rho \in \mathrm{Stab}\,(0)$. Since $L_{\ominus a} = L_a^{-1}$, $\sigma = L_a \circ \rho$. This proves the existence of factorization. ■

The following *commutation relation* determines how to commute a left gyrotranslation and an automorphism of G

$$\rho \circ L_a = L_{\rho(a)} \circ \rho \tag{14}$$

whenever ρ is an automorphism of G.

Let σ and τ be permutations of G. By Theorem 11, σ and τ have factorizations $\sigma = L_a \circ \gamma$ and $\tau = L_b \circ \delta$, where $a, b \in G$ and $\gamma, \delta \in \mathrm{Stab}\,(0)$. Define an operation $\oplus$ on $\mathrm{Sym}\,(G)$ by

$$\sigma \oplus \tau = L_{a\oplus b} \circ (\gamma \circ \delta). \tag{15}$$

Because of the uniqueness of factorization, $\oplus$ is a binary operation on $\mathrm{Sym}\,(G)$. In fact, $(\mathrm{Sym}\,(G), \oplus)$ forms a gyrogroup.

Theorem 12. $\mathrm{Sym}\,(G)$ *is a gyrogroup under the operation defined by (15), and*

$$L_a \oplus L_b = L_a \oplus_{\overline{G}} L_b = L_{a\oplus b}$$

for all $a, b \in G$. In particular, the map $a \mapsto L_a$ defines an injective gyrogroup homomorphism from G into $\mathrm{Sym}\,(G)$.

Proof: Suppose that $\sigma = L_a \circ \gamma$, $\tau = L_b \circ \delta$ and $\rho = L_c \circ \lambda$, where $a, b, c \in G$ and $\gamma, \delta, \lambda \in \mathrm{Stab}\,(0)$. The identity map $\mathrm{Id}\,(G)$ acts as a left identity of $\mathrm{Sym}\,(G)$ and $L_{\ominus a} \circ \gamma^{-1}$ is a left inverse of σ with respect to $\oplus$. The gyroautomorphisms of $\mathrm{Sym}\,(G)$ are given by

$$\mathrm{gyr}[\sigma, \tau]\rho = (\mathrm{gyr}[L_a, L_b]L_c) \circ \lambda = L_{\mathrm{gyr}[a,b]c} \circ \lambda.$$

Since G satisfies the left gyroassociative law and the left loop property, so does $\mathrm{Sym}\,(G)$. ∎

By Theorem 12, the following version of Cayley's theorem for gyrogroups is immediate.

Corollary 13 (Cayley's Theorem) *Every gyrogroup is isomorphic to a subgyrogroup of the gyrogroup of permutations.*

Proof: The map $a \mapsto L_a$ defines a gyrogroup isomorphism from G onto $\overline{G}$ and $\overline{G}$ is a subgyrogroup of $\mathrm{Sym}\,(G)$. ∎

4. L-Subgyrogroups

Throughout this section, G is an arbitrary gyrogroup.

A nonempty subset H of G is a *subgyrogroup* if H forms a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$. If H is a subgyrogroup of G, then we write $H \leqslant G$ as in the group case.

Proposition 14 (The Subgyrogroup Criterion) *A nonempty subset H of G is a subgyrogroup if and only if $\ominus a \in H$ and $a \oplus b \in H$ for all $a, b \in H$.*

Proof: Axioms (G1), (G2), (G4) hold trivially. Let $a, b \in H$. By the gyrator identity, $\mathrm{gyr}[a, b](H) \subseteq H$. Since the gyroautomorphisms are inversive symmetric (Theorem 5), we also have the reverse inclusion. Thus, the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H and so axiom (G3) holds. ∎

Let H be a subgyrogroup of G. In contrast to groups, the relation

$$a \sim b \quad \text{if and only if} \quad \ominus a \oplus b \in H \tag{16}$$

does not, in general, define an equivalence relation on G. Nevertheless, we can modify (16) to obtain an equivalence relation on G. From this point of view, any subgyrogroup of G partitions G. This leads to the introduction of L-subgyrogroups.

Let H be a subgyrogroup of G. Define a relation $\sim_H$ on G by letting

$$a \sim_H b \quad \text{if and only if} \quad \ominus a \oplus b \in H \text{ and } \operatorname{gyr}[\ominus a, b](H) = H. \tag{17}$$

Theorem 15. *The relation $\sim_H$ defined by* (17) *is an equivalence relation on G.*

Proof: Let $a, b, c \in G$. Since $\ominus a \oplus a = 0 \in H$ and $\operatorname{gyr}[\ominus a, a] = \operatorname{Id}(G)$, $a \sim_H a$. Hence, $\sim_H$ is reflexive. Suppose that $a \sim_H b$. By Theorem 4 $\operatorname{gyr}[\ominus a, b](\ominus b \oplus a) = \ominus(\ominus a \oplus b)$. Hence, $\ominus b \oplus a = \operatorname{gyr}^{-1}[\ominus a, b](\ominus(\ominus a \oplus b))$, which implies $\ominus b \oplus a \in H$ since $\operatorname{gyr}^{-1}[\ominus a, b](H) = H$. By Theorem 5

$$\operatorname{gyr}[\ominus a, b] = \operatorname{gyr}[\ominus a, \ominus(\ominus b)] = \operatorname{gyr}[a, \ominus b] = \operatorname{gyr}^{-1}[\ominus b, a].$$

Hence, $\operatorname{gyr}[\ominus b, a] = \operatorname{gyr}^{-1}[\ominus a, b]$. Since $\operatorname{gyr}[\ominus a, b](H) = H$, $\operatorname{gyr}[\ominus b, a](H) = H$ as well. This proves $b \sim_H a$ and so $\sim_H$ is symmetric. Suppose that $a \sim_H b$ and $b \sim_H c$. By Theorem 3, $\ominus a \oplus c = (\ominus a \oplus b) \oplus \operatorname{gyr}[\ominus a, b](\ominus b \oplus c)$ and so $\ominus a \oplus c \in H$. Using the composition law (12) and the commutation relation (14), we have $\operatorname{gyr}[\ominus a, c] = \operatorname{gyr}[\ominus a \oplus b, \operatorname{gyr}[\ominus a, b](\ominus b \oplus c)] \circ \operatorname{gyr}[\ominus a, b] \circ \operatorname{gyr}[\ominus b, c]$. This implies $\operatorname{gyr}[\ominus a, c](H) = H$ and so $a \sim_H c$. This proves $\sim_H$ is transitive. ■

Let $a \in G$. Let $[a]$ denote the equivalence class of a determined by $\sim_H$. Theorem 15 says that $\{[a];\, a \in G\}$ is a partition of G. Set $a \oplus H := \{a \oplus h;\, h \in H\}$, called the *left coset of H induced by a.*

Proposition 16. *For each $a \in G$, $[a] \subseteq a \oplus H$.*

Proof: If $x \in [a]$, by (17), $\ominus a \oplus x \in H$. Hence, $x = a \oplus (\ominus a \oplus x) \in a \oplus H$. ■

Proposition 16 leads to the notion of L-subgyrogroups

Definition 17. *A subgyrogroup H of G is said to be an L-subgyrogroup, denoted by $H \leqslant_L G$, if* $\operatorname{gyr}[a, h](H) = H$ *for all $a \in G$ and $h \in H$.*

Example 18. *In* [10, p. 41], *Ungar exhibited the gyrogroup K_{16} whose addition table is presented in* Table 1. *In K_{16}, there is only one nonidentity gyroautomorphism, denoted by A, whose transformation is given in cyclic notation by*

$$A = (8\ 9)(10\ 11)(12\ 13)(14\ 15). \tag{18}$$

The gyration table for K_{16} is presented in Table 2. *According to* (18), *$H_1 = \{0, 1\}$, $H_2 = \{0, 1, 2, 3\}$, and $H_3 = \{0, 1, \ldots, 7\}$ are easily seen to be L-subgyrogroups* *of K_{16}. In contrast, $H_4 = \{0,8\}$ forms a non-L-subgyrogroup of K_{16} since* $\mathrm{gyr}[4,8](H_4) \neq H_4$.

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	1	0	6	7	5	4	11	10	8	9	15	14	12	13
3	3	2	0	1	7	6	4	5	10	11	9	8	14	15	13	12
4	4	5	6	7	3	2	0	1	15	14	12	13	9	8	11	10
5	5	4	7	6	2	3	1	0	14	15	13	12	8	9	10	11
6	6	7	5	4	0	1	2	3	13	12	15	14	10	11	9	8
7	7	6	4	5	1	0	3	2	12	13	14	15	11	10	8	9
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	9	8	14	15	13	12	3	2	0	1	7	6	4	5
11	11	10	8	9	15	14	12	13	2	3	1	0	6	7	5	4
12	12	13	14	15	11	10	8	9	6	7	5	4	0	1	2	3
13	13	12	15	14	10	11	9	8	7	6	4	5	1	0	3	2
14	14	15	13	12	8	9	10	11	4	5	6	7	3	2	0	1
15	15	14	12	13	9	8	11	10	5	4	7	6	2	3	1	0

Table 1. Addition table for the gyrogroup K_{16}, (cf [10]).

The importance of L-subgyrogroups lies in the following results.

Proposition 19. *If $H \leqslant_L G$, then $[a] = a \oplus H$ for all $a \in G$.*

Proof: Assume that $H \leqslant_L G$. By Proposition 16, $[a] \subseteq a \oplus H$. If $x = a \oplus h$ for some $h \in H$, then $\ominus a \oplus x = h$ is in H. The left and right loop properties together imply $\mathrm{gyr}[\ominus a, x] = \mathrm{gyr}[h, a] = \mathrm{gyr}^{-1}[a, h]$. By assumption, $\mathrm{gyr}[a, h](H) = H$, which implies $\mathrm{gyr}[\ominus a, x](H) = \mathrm{gyr}^{-1}[a, h](H) = H$. Hence, $a \sim_H x$ and so $x \in [a]$. This establishes the reverse inclusion. ■

Theorem 20. *If H is an L-subgyrogroup of a gyrogroup G, then the set*

$$\{a \oplus H \,;\, a \in G\}$$

forms a disjoint partition of G.

gyr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
1	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
2	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
3	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
4	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
5	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
6	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
7	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
8	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
9	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
10	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
11	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
12	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
13	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
14	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
15	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I

Table 2. Gyration table for K_{16}. Here, A is given by (18) and I stands for the identity transformation, (cf [10]).

Proof: This follows directly from Theorem 15 and Proposition 19. ∎

In light of Theorem 20, we derive the following version of Lagrange's theorem for L-subgyrogroups.

Theorem 21 (Lagrange's Theorem for L-Subgyrogroups) *In a finite gyrogroup G, if $H \leqslant_L G$, then $|H|$ divides $|G|$.*

Proof: Being a finite gyrogroup, G has a finite number of left cosets, namely $a_1 \oplus H, a_2 \oplus H, \dots, a_n \oplus H$. Since $|a_i \oplus H| = |H|$ for $i = 1, 2, \dots, n$, it follows that $|G| = \left|\bigcup_{i=1}^{n} a_i \oplus H\right| = \sum_{i=1}^{n} |a_i \oplus H| = n|H|$, which completes the proof. ∎

Let us denote by $[G\colon H]$ the number of left cosets of H in G.

Corollary 22. *In a finite gyrogroup G, if $H \leqslant_L G$, then $|G| = [G\colon H]|H|$.*

For a *non*-L-subgyrogroup K of G, it is no longer true that the left cosets of K partition G. Moreover, the formula $|G| = [G\colon K]|K|$ is not true, in general.

5. Isomorphism Theorems

A map $\varphi\colon G \to H$ between gyrogroups is called a *gyrogroup homomorphism* if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. We say that G and H are *isomorphic gyrogroups*, written $G \cong H$, if there exists a gyrogroup isomorphism from G to H. The next proposition lists basic properties of gyrogroup homomorphisms.

Proposition 23. *Let $\varphi\colon G \to H$ be a homomorphism of gyrogroups.*

(1) $\varphi(0) = 0$

(2) $\varphi(\ominus a) = \ominus\varphi(a)$ *for all* $a \in G$

(3) $\varphi(\mathrm{gyr}[a, b]c) = \mathrm{gyr}[\varphi(a), \varphi(b)]\varphi(c)$ *for all* $a, b, c \in G$

(4) $\varphi(a \boxplus b) = \varphi(a) \boxplus \varphi(b)$ *for all* $a, b \in G$.

The proof of the following two propositions is routine, using the subgyrogroup criterion and the definition of an L-subgyrogroup.

Proposition 24. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant G$, then $\varphi(K) \leqslant H$. If $K \leqslant_L G$ and if φ is surjective, then $\varphi(K) \leqslant_L H$.*

Proposition 25. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant H$, then $\varphi^{-1}(K) \leqslant G$. If $K \leqslant_L H$, then $\varphi^{-1}(K) \leqslant_L G$.*

Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. The kernel of φ is defined to be the inverse image of the trivial subgyrogroup $\{0\}$ under φ, hence is a subgyrogroup. The kernel of φ is invariant under the gyroautomorphisms of G, that is, $\mathrm{gyr}[a, b](\ker\varphi) \subseteq \ker\varphi$ for all $a, b \in G$. By Proposition 6, $\mathrm{gyr}[a, b](\ker\varphi) = \ker\varphi$ for all $a, b \in G$ and so $\ker\varphi$ is an L-subgyrogroup of G. From this the relation (17) becomes

$$a \sim_{\ker\varphi} b \quad \text{if and only if} \quad \ominus a \oplus b \in \ker\varphi \tag{19}$$

for all $a, b \in G$. More precisely, we have the following proposition.

Proposition 26. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. For all $a, b \in G$, the following are equivalent*

1) $a \sim_{\ker\varphi} b$

2) $\ominus a \oplus b \in \ker\varphi$

3) $\varphi(a) = \varphi(b)$

4) $a \oplus \ker\varphi = b \oplus \ker\varphi$.

In view of Proposition 26, we define a binary operation on the set $G/\ker\varphi$ of left cosets of $\ker\varphi$ in the following natural way

$$(a \oplus \ker\varphi) \oplus (b \oplus \ker\varphi) = (a \oplus b) \oplus \ker\varphi, \qquad a, b \in G. \tag{20}$$

The resulting system forms a gyrogroup, called a *quotient gyrogroup*.

Theorem 27. *If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then $G/\ker\varphi$ with operation defined by* (20) *is a gyrogroup.*

Proof: Set $K = \ker\varphi$. The coset $0 \oplus K$ is a left identity of G/K. The coset $(\ominus a) \oplus K$ is a left inverse of $a \oplus K$. For $X = a \oplus K, Y = b \oplus K \in G/K$, the gyroautomorphism generated by X and Y is given by

$$\mathrm{gyr}[X, Y](c \oplus K) = (\mathrm{gyr}[a, b]c) \oplus K$$

for $c \oplus K \in G/K$. ■

The map $\Pi\colon G \to G/\ker\varphi$ given by $\Pi(a) = a \oplus \ker\varphi$ defines a surjective gyrogroup homomorphism, which will be referred to as the *canonical projection*. In light of Theorem 27, the first isomorphism theorem for gyrogroups follows.

Theorem 28 (The First Isomorphism Theorem) *If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then $G/\ker\varphi \cong \varphi(G)$ as gyrogroups.*

Proof: Set $K = \ker\varphi$. Define $\phi\colon G/K \to \varphi(G)$ by $\phi(a \oplus K) = \varphi(a)$. By Proposition 26, ϕ is well defined and injective. A direct computation shows that ϕ is a gyrogroup isomorphism from G/K onto $\varphi(G)$. ■

It is known that a subgroup of a group is normal if and only if it is the kernel of some group homomorphism. This characterization of a normal subgroup allows us to define a normal subgyrogroup in a similar fashion, as follows. A subgyrogroup N of a gyrogroup G is *normal in* G, denoted by $N \trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G.

Lemma 29. *Let G be a gyrogroup. If $A \leqslant G$ and $B \trianglelefteq G$, then*

$$A \oplus B := \{a \oplus b;\, a \in A, b \in B\}$$

forms a subgyrogroup of G.

Proof: By assumption, $B = \ker \phi$, where ϕ is a gyrogroup homomorphism of G. Using Theorem 7, one can prove that $B \oplus a = a \oplus B$ for all $a \in G$.

Let $x = a \oplus b$, with $a \in A$, $b \in B$. Since $\phi(\mathrm{gyr}[a, b]\ominus a) = \mathrm{gyr}[\phi(a), 0]\phi(\ominus a) = \phi(\ominus a)$, we have $\mathrm{gyr}[a, b]\ominus a = \ominus a \oplus b_1$ for some $b_1 \in B$. Set $b_2 = \mathrm{gyr}[a, b]\ominus b$. Since $b_2 \in B$ and $B \oplus (\ominus a) = (\ominus a) \oplus B$, there is an element $b_3 \in B$ for which $b_2 \ominus a = \ominus a \oplus b_3$. The left and right loop properties together imply $\ominus x = \ominus a \oplus (b_3 \oplus \mathrm{gyr}[b_3, \ominus a](\mathrm{gyr}[b_2, \ominus a]b_1))$, whence $\ominus x$ belongs to $A \oplus B$.

For $x, y \in A \oplus B$, we have $x = a \oplus b$ and $y = c \oplus d$ for some $a, c \in A$, $b, d \in B$. Since $\phi(b \oplus \mathrm{gyr}[b, a](c \oplus d)) = \phi(b) \oplus \mathrm{gyr}[\phi(b), \phi(a)](\phi(c) \oplus \phi(d)) = \phi(c)$, we have $b \oplus \mathrm{gyr}[b, a](c \oplus d) = c \oplus b_1$ for some $b_1 \in B$. The left and right loop properties together imply $x \oplus y = (a \oplus c) \oplus \mathrm{gyr}[a, c]b_1$, whence $x \oplus y$ belongs to $A \oplus B$. This proves $A \oplus B \leqslant G$. ■

Theorem 30 (The Second Isomorphism Theorem) *Let G be a gyrogroup and let $A, B \leqslant G$. If $B \trianglelefteq G$, then $A \cap B \trianglelefteq A$ and $(A \oplus B)/B \cong A/(A \cap B)$ as gyrogroups.*

Proof: As in Lemma 29, $B = \ker \phi$. Note that $A \cap B \trianglelefteq A$ since $\ker \phi\big|_A = A \cap B$. Hence, $A/(A \cap B)$ admits the quotient gyrogroup structure.

Define $\varphi\colon A \oplus B \to A/(A \cap B)$ by $\varphi(a \oplus b) = a \oplus (A \cap B)$ for $a \in A$ and $b \in B$. To see that φ is well defined, suppose that $a \oplus b = a_1 \oplus b_1$, where $a, a_1 \in A$ and $b, b_1 \in B$. Note that $b_1 = \ominus a_1 \oplus (a \oplus b) = (\ominus a_1 \oplus a) \oplus \mathrm{gyr}[\ominus a_1, a]b$. Set $b_2 = \ominus \mathrm{gyr}[\ominus a_1, a]b$. Then $b_2 \in B$ and $b_1 = (\ominus a_1 \oplus a) \ominus b_2$. The right cancellation law I gives $\ominus a_1 \oplus a = b_1 \boxplus b_2 = b_1 \oplus \mathrm{gyr}[b_1, \ominus b_2]b_2$, which implies $\ominus a_1 \oplus a \in A \cap B$. By Proposition 26, $a_1 \oplus (A \cap B) = a \oplus (A \cap B)$.

As we computed in the lemma, if $a, c \in A$ and $b, d \in B$, then

$$(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus \mathrm{gyr}[a, c]\tilde{b}$$

for some $\tilde{b} \in B$. Hence, $\varphi((a \oplus b) \oplus (c \oplus d)) = (a \oplus c) \oplus A \cap B = \varphi(a \oplus b) \oplus \varphi(c \oplus d)$. This proves $\varphi\colon A \oplus B \to A/(A \cap B)$ is a surjective gyrogroup homomorphism whose kernel is $\{a \oplus b;\ a \in A, b \in B, a \in A \cap B\} = B$. Thus, $B \trianglelefteq A \oplus B$ and by the first isomorphism theorem, $(A \oplus B)/B \cong A/(A \cap B)$. ■

Theorem 31 (The Third Isomorphism Theorem) *Let G be a gyrogroup and let H, K be normal subgyrogroups of G such that $H \subseteq K$. Then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$ as gyrogroups.*

Proof: Let ϕ and ψ be gyrogroup homomorphisms of G such that $\ker \phi = H$ and $\ker \psi = K$. Define $\varphi\colon G/H \to G/K$ by $\varphi(a \oplus H) = a \oplus K$ for $a \in G$. Note that φ is well defined since $H \subseteq K$. Furthermore, φ is a surjective gyrogroup homomorphism such that $\ker\varphi = K/H$. Hence, $K/H \trianglelefteq G/H$. By the first isomorphism theorem, $(G/H)/(K/H) \cong G/K$. ∎

Theorem 32 (The Lattice Isomorphism Theorem) *Let G be a gyrogroup and let $N \trianglelefteq G$. There is a bijection Φ from the set of subgyrogroups of G containing N onto the set of subgyrogroups of G/N. The bijection Φ has the following properties*

1) $A \subseteq B$ *if and only if* $\Phi(A) \subseteq \Phi(B)$

2) $A \leqslant_L G$ *if and only if* $\Phi(A) \leqslant_L G/N$

3) $A \trianglelefteq G$ *if and only if* $\Phi(A) \trianglelefteq G/N$

for all subgyrogroups A and B of G containing N.

Proof: Set $\mathcal{S} = \{K \subseteq G;\, K \leqslant G \text{ and } N \subseteq K\}$. Let $\mathcal{T}$ denote the set of subgyrogroups of G/N. Define a map Φ by $\Phi(K) = K/N$ for $K \in \mathcal{S}$. By Proposition 24, $\Phi(K) = K/N = \Pi(K)$ is a subgyrogroup of G/N, where $\Pi\colon G \to G/N$ is the canonical projection. Hence, Φ maps $\mathcal{S}$ to $\mathcal{T}$.

Assume that $K_1/N = K_2/N$, with K_1, K_2 in $\mathcal{S}$. For $a \in K_1$, $a \oplus N \in K_2/N$ implies $a \oplus N = b \oplus N$ for some $b \in K_2$. Hence, $\ominus b \oplus a \in N$. Since $N \subseteq K_2$, $\ominus b \oplus a \in K_2$, which implies $a = b \oplus (\ominus b \oplus a) \in K_2$. This proves $K_1 \subseteq K_2$. By interchanging the roles of K_1 and K_2, one obtains similarly that $K_2 \subseteq K_1$. Hence, $K_1 = K_2$ and Φ is injective.

Let Y be an arbitrary subgyrogroup of G/N. By Proposition 25

$$\Pi^{-1}(Y) = \{a \in G;\, a \oplus N \in Y\}$$

is a subgyrogroup of G containing N for $a \in N$ implies $a \oplus N = 0 \oplus N \in Y$. Because $\Phi(\Pi^{-1}(Y)) = Y$, Φ is surjective. This proves Φ defines a bijection from $\mathcal{S}$ onto $\mathcal{T}$.

The proof of Item 32 is straightforward. From Propositions 24 and 25, we have Item 32. To prove Item 32, suppose that $A \trianglelefteq G$. Then $A = \ker\psi$, where $\psi\colon G \to H$ is a gyrogroup homomorphism. Define $\varphi\colon G/N \to H$ by $\varphi(a \oplus N) = \psi(a)$. Since $N \subseteq A$, φ is well defined. Also, φ is a gyrogroup homomorphism. Since $\ker\varphi = A/N$, we have $A/N \trianglelefteq G/N$. Suppose conversely that $\Phi(A) \trianglelefteq G/N$. Then $A/N = \ker\phi$, where ϕ is a gyrogroup homomorphism of G/N. Set $\varphi = \phi \circ \Pi$. Thus, φ is a gyrogroup homomorphism of G with kernel A and hence $A \trianglelefteq G$. ∎

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