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THE CAUCHY PROBLEM FOR THE HOMOGENEOUS MONGE–AMPÈRE EQUATION, I. TOEPLITZ QUANTIZATION

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Abstract

The Cauchy problem for the homogeneous real/complex Monge–Ampère equation (HRMA/HCMA) arises from the initial value problem for geodesics in the space of Kähler metrics equipped with the Mabuchi metric. This Cauchy problem is believed to be ill-posed and a basic problem is to characterize initial data of (weak) solutions which exist up to time T. In this article, we use a quantization method to construct a subsolution of the HCMA on a general projective variety, and we conjecture that it solves the equation for as long as the unique solution exists. The subsolution, called the "quantum analytic continuation potential," is obtained by (i) Toeplitz quantizing the Hamiltonian flow determined by the Cauchy data, (ii) analytically continuing the quantization, and (iii) taking a certain logarithmic classical limit. We then prove that in the case of torus invariant metrics (where the HCMA reduces to the HRMA) the quantum analytic continuation potential coincides with the well-known Legendre transform potential, and hence solves the equation as long as it is smooth. In the sequel [**29**], it is proved that the Legendre transform potential ceases to solve the HCMA once it ceases to be smooth. The results here and in the sequels in particular characterize the initial data of smooth geodesic rays.

1. Introduction

This article is the first in a series whose aim is to study existence, uniqueness, and regularity of solutions of the initial value problem (IVP) for geodesics in the space of Kähler metrics in a fixed class. It is a special case of the Cauchy problem for the HCMA (homogeneous complex Monge–Ampère equation). Unlike the much-studied Dirichlet problem (going back to [**3**]), little has been proven for the Cauchy problem for the Monge–Ampère equation, and there is currently no known method to solve it for smooth Cauchy data. Indeed, it is believed to be an ill-posed problem and one does not expect global in time solutions to exist

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for 'most' initial data $(\omega_{\varphi_0}, \dot{\varphi}_0)$. The goal is thus to determine which initial data give rise to global solutions, especially those of relevance in geometry ('geodesic rays'), to construct the solutions, and to determine the lifespan T_{span} of solutions for general initial data.

In this article, we define a *quantum analytic continuation potential* $\varphi_\infty(s,z)$ on any polarized Kähler manifold (M, ω_{φ_0}) by taking the logarithmic limit of a canonical sequence $\varphi_N(s,z)$ of subsolutions of the HCMA determined by the Cauchy data $(\omega_{\varphi_0}, \dot{\varphi}_0)$ (see Definition 2.1 and (6)). The sequence $\varphi_N(s,z)$ is the quantum analogue of Semmes and Donaldson's formal solution of the Cauchy problem by analytic continuation in time of the Hamilton flow determined by the Cauchy data (3) [**13**, **31**], and $\varphi_\infty(s,z)$ is its classical limit. For any $(M, \omega_{\varphi_0}, \dot{\varphi}_0)$, $\varphi_\infty(s,z)$ is a subsolution of the HCMA. We conjecture that it is the solution of the Cauchy problem for the HCMA as long as a solution exists. The main result of this article (Theorem 1), together with further results, in the sequels [**29**, **30**] comes close to confirming this conjecture when the Kähler manifold (M,ω) has an $(S^1)^n$ symmetry with $n = \dim M$. Examples include toric Kähler manifolds and Abelian varieties. In such cases, the HCMA reduces to the HRMA (homogeneous real Monge–Ampère equation).

One of our principal goals in this series is to determine the lifespan of solutions of the Cauchy problem for HCMA, and to determine the Cauchy data for which the solution has an infinite lifespan ('geodesic rays'). In this article, we prove that in the $(S^1)^n$-invariant case, the quantum analytic continuation potential is a smooth solution of the HRMA until the *convex lifespan* $T^{\text{cvx}}_{\text{span}}$ of the problem (see Definition 2.5). In doing so, we also construct the quantization of the Hamiltonian flow and the quantum analytic continuation potential on a general projective manifold. In the sequel [**29**], we show that the quantum analytic continuation potential fails to solve the equation even in a weak sense after the convex lifespan. This leaves open the possibility that there exist other weak solutions with longer lifespans. But in [**30**], we characterize the smooth lifespan of the HCMA. In particular, for the HRMA, we show that the smooth lifespan T^{∞}_{span} (see Definition 2.2) of the Cauchy problem equals the convex lifespan. It follows that the directions of smooth geodesic rays in the $(S^1)^n$-symmetric case are those with infinite convex lifespan. It would be interesting to generalize this condition to non-symmetric situations.

The Cauchy problem for the HCMA arises from the initial value problem for geodesics in the infinite-dimensional space

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) \,:\, \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \tag{1}$$

of Kähler metrics in a fixed Kähler class equipped with the weak Riemannian metric

$$g_{\mathrm{M}}(\zeta,\eta)_\varphi := \int_M \zeta\eta\,\omega_\varphi^m, \quad \varphi\in\mathcal{H}_\omega, \quad \zeta,\eta\in T_\varphi\mathcal{H}_\omega\cong C^\infty(M). \tag{2}$$

As is well known, when φ is regular enough, the geodesic equation is equivalent to the homogeneous complex Monge–Ampère (HCMA) equation on the product of a Riemann surface with M [**22**, **31**, **13**]. Thus, the initial value problem is the problem of defining the exponential map of $\mathcal{H}_\omega$. As observed by Semmes and Donaldson, $\mathcal{H}_\omega$ is formally an infinite dimensional symmetric space of the type $G^{\mathbb{C}}/G$ where G is the group of Hamiltonian diffeomorphisms of (M,ω). Hence its geodesics should be given by certain 1 PS (one-parameter subgroups) of $G^{\mathbb{C}}$. To a large extent, the Kähler quantization method of this article is an attempt to put these formal arguments on a rigorous basis. The quantum analytic continuation potential gives a rigorous construction of such infinite dimensional 1 PS, to the extent possible, as limits of finite dimensional 1 PS. We refer to [**1**, **9**, **11**, **12**, **13**, **22**, **24**, **25**, **31**, **36**] for further background.

The article is organized as follows. In Section 2 we describe our approach to the IVP using an analytic continuation of Toeplitz quantization. Our main results are stated in Section 3, and in Section 4 we recall some background. In Section 5 we construct the quantization of the Hamiltonian flow. The results in this section hold for an arbitrary projective Kähler manifold. In Section 6 we specialize to the setting of a toric or Abelian variety where we construct a second quantization of the Hamiltonian flow and compare the two quantizations and their analytic continuations. In Section 7 we complete the proof of our main result (Theorem 1), showing that the analytic continuations of the quantizations converge to the Legendre transform potential and hence solve the Cauchy problem until the convex lifespan.

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2. A quantum mechanical approach to Monge–Ampère

In this section we define the *quantum analytic continuation potential* and state the general conjecture that it solves the IVP for geodesics in $(\mathcal{H}_\omega, g_{\mathrm{M}})$, to the extent possible, in the case of projective Kähler manifolds. The definition is inspired by two prior constructions and is largely aimed at reconciling them.

The first is a heuristic analytic continuation argument due to Semmes and Donaldson [**31**, **13**]: Let $\dot\varphi_0$ be a smooth function on M, considered as a tangent vector in $T_{\varphi_0}\mathcal{H}_\omega$. Let $X_{\dot\varphi_0}^{\omega_{\varphi_0}} \equiv X_{\dot\varphi_0}$ denote the Hamiltonian vector field associated to $\dot\varphi_0$ and (M, ω_{φ_0}) and let $\exp tX_{\dot\varphi_0}$ denote the associated Hamiltonian flow. Then let $\exp -\sqrt{-1}sX_{\dot\varphi_0}$ "be" its analytic continuation in time to the Hamiltonian flow at "imaginary" time $\sqrt{-1}s$. Then "define" the *classical analytic continuation potential* φ_s with initial data $(\varphi_0, \dot\varphi_0)$ by

$$((\exp -\sqrt{-1}sX_{\dot\varphi_0})^{-1})^\star\omega_0 - \omega_0 = -\sqrt{-1}\partial\bar\partial\varphi_s. \tag{3}$$

Then φ_s "is" the solution of the initial value problem. We use quotes since there is no obvious reason why $\exp tX_{\dot\varphi_0}$, a rather arbitrary smooth Hamiltonian flow, should admit an analytic continuation in t for any length of time, nor why $\exp -\sqrt{-1}sX_{\dot\varphi_0}$ should be invertible in case such an analytic continuation exists. When the analytic continuation does exist, e.g., if ω_{φ_0} and $\dot\varphi_0$ are real analytic, then φ_s solves the initial value problem for the Monge–Ampère equation for s in some (usually) small time interval [**22**, **31**, **13**, **30**].

The second construction uses finite dimensional approximations deriving from Kähler quantization to solve the endpoint problem for geodesics. The idea is to approximate the space $\mathcal{H}_\omega$ by finite-dimensional spaces of Bergman (or Fubini–Study) metrics induced by holomorphic embeddings of M into $\mathbb{P}^N$ using bases of holomorphic sections $s \in H^0(M, L^k)$ of high powers of a polarizing line bundle. Following an original idea of Yau and Tian, such embeddings were used in [**37**, **8**, **40**] to approximate individual metrics. Phong–Sturm [**24**, **25**] then introduced a Kähler quantization method to approximate geodesic segments with fixed endpoints by geodesics in the space of Bergman metrics. They also used the method to define geodesic rays from test configurations. Further work on Bergman approximations to geodesics, as well as more general harmonic maps, is due to Berndtsson, Chen–Sun, Feng, and others [**4**, **5**, **10**, **15**, **28**, **34**, **35**].

Our approach to the IVP combines the two as follows: we define the analytic continuation of $\exp tX_{\dot\varphi_0}$ by quantizing this Hamiltonian flow, by analytically continuing the quantum flow, and then by taking a kind of logarithmic classical limit of its Schwartz kernel.

Consider the Hilbert spaces of sections $L^2(M, L^N), N \in \mathbb{N}$, associated to powers of a Hermitian line bundle (L, h_0) polarizing (M, ω_{φ_0}), and the corresponding orthogonal projection operators $\Pi_N \equiv \Pi_{N,\varphi_0} : L^2(M, L^N) \to H^0(M, L^N)$, onto the Hilbert subspaces $H^0(M, L^N)$ of holomorphic sections. In order to quantize the Hamiltonian flow of $X_{\dot\varphi_0}$ on (M, ω_{φ_0}), we use the method of Toeplitz quantization (see §4.1 for background and §5 for the main construction). Namely, we consider the self-adjoint zeroth-order Hermitian Toeplitz operators $\Pi_N\dot\varphi_0\Pi_N$, where

$\dot{\varphi}_0$ denotes the operator of multiplication by $\dot{\varphi}_0$. Define the associated one-parameter subgroups of unitary operators on $H^0(M, L^N)$

$$U_N(t) := \Pi_N e^{\sqrt{-1}tN\Pi_N\dot{\varphi}_0\Pi_N}\Pi_N. \tag{4}$$

A key point is that $U_N(t)$ is a semi-classical Fourier integral operator with complex phase (see [**23**, **20**, **39**] for background). For simplicity of expression, we refer to Fourier integral operators with complex phase as 'complex Fourier integral operators'.

A key observation is that there is no obstruction to analytically continuing the quantization: each $U_N(t)$ admits an analytic continuation in time t and induces the imaginary time subgroup

$$U_N(-\sqrt{-1}s) : H^0(M, L^N) \to H^0(M, L^N), \tag{5}$$

with $U_N(-\sqrt{-1}s) \in GL(H^0(M, L^N), \mathbb{C})$. The main idea of this article is that the analytic continuation of $\exp tX_{\dot{\varphi}_0}$ can be constructed by taking a non-standard logarithmic classical limit of the analytic continuation of its quantization. We do this by considering the Schwartz kernel $U_N(-\sqrt{-1}s)(z, w)$ of this operator with respect to the volume form $(N\omega_{\varphi_0})^n$. Set

$$\varphi_N(s, z) := \frac{1}{N}\log U_N(-\sqrt{-1}s, z, z). \tag{6}$$

Definition **2.1.** *Call* $\varphi_\infty(s, z) := \lim_{l\to\infty}(\sup_{N\geq l}\varphi_N)_{\mathrm{reg}}(s, z)$ *the quantum analytic continuation potential, where* u_{reg} *denotes the upper semicontinuous regularization of* u*.*

The limit φ_∞ is constructed out of the quantized potentials φ_N similarly to the geodesic rays constructed by Phong–Sturm [**25**], by using upper envelopes. Note however that our construction of the φ_N for the Cauchy problem involves a new idea, since it comes from analytic continuation of a kernel of a dynamical Toeplitz operator. This limit is also quite different from the semi-classical limits studied in Toeplitz quantization, because the analytic continuation in time may destroy the Toeplitz (i.e., complex Fourier integral operator) structure of the kernel. Moreover, the logarithmic asymptotics of the Schwartz kernel are quite unrelated to symbol asymptotics. One may think of it as extracting an analytic continuation of the 'phase function' of the Toeplitz operator; the 'symbol' of the Toeplitz operator is irrelevant.

Denote $S_T := [0, T]\times\mathbb{R}$. The IVP for geodesics is equivalent to the following Cauchy problem for the homogeneous complex Monge–Ampère equation:

$$\begin{aligned} &(\pi_2^\star\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0 && \text{on } S_T \times M, \\ &\varphi(0, s, \cdot\,) = \varphi_0(\,\cdot\,), \quad \partial_s\varphi(0, s, \cdot\,) = \dot{\varphi}_0(\,\cdot\,) && \text{on } \{0\}\times\mathbb{R}\times M. \end{aligned} \tag{7}$$

Definition **2.2.** *The smooth lifespan (respectively, lifespan) of the Cauchy problem* (7) *is the supremum over all* $T \geq 0$ *such that* (7) *admits a smooth (respectively $\pi_2^\star\omega$-psh) solution. We denote the smooth lifespan (respectively, lifespan) for the Cauchy data* $(\omega_{\varphi_0}, \dot\varphi_0)$ *by* $T^\infty_{\mathrm{span}} \equiv T^\infty_{\mathrm{span}}(\omega_{\varphi_0}, \dot\varphi_0)$ *(respectively,* $T_{\mathrm{span}} \equiv T_{\mathrm{span}}(\omega_{\varphi_0}, \dot\varphi_0)$*).*

Definition **2.3.** *The quantum lifespan* T^Q_{span} *of the Cauchy problem* (7) *is the supremum over all* $T \geq 0$ *such that* φ_∞ *of Definition* 2.1 *solves the HCMA* (7)*.*

We pose the following conjecture, which would give a general method to solve the Cauchy problem for the HCMA to the extent possible.

Conjecture **2.4.** *The quantum analytic continuation potential* φ_∞ *solves the HCMA* (7) *for as long as it admits a solution. In other words,* $T^Q_{\mathrm{span}} = T_{\mathrm{span}}$*.*

2.1. HRMA and the convex lifespan. We now specialize to (M, J, ω) with $(S^1)^n$-symmetry, where the HCMA reduces to the HRMA. In this case, one can linearize the HRMA and define a second (well-known) potential, the *Legendre transform potential.* We need to recall its definition for the statement of our main result; we only give the details for toric Kähler manifolds, but as in [**15**], the same methods apply to Abelian varieties.

We recall that a toric Kähler manifold is a Kähler manifold (M, J, ω) that admits a holomorphic action of a complex torus $(\mathbb{C}^\star)^n$ with an open dense orbit, and for which the Kähler form ω is toric, i.e., invariant under the action of the real torus $\mathbf{T} := (S^1)^n$. See §4.2 for background. We assume that the Cauchy data $(\omega_{\varphi_0}, \dot\varphi_0)$ is toric, and consider the IVP for geodesics in the space of torus-invariant Kähler metrics. Over the open orbit $M_{\mathrm{o}} \cong (\mathbb{C}^\star)^n \cong \mathbb{R}^n \times \mathbf{T}$, the Kähler form ω_{φ_0} is exact and $\mathbf{T}$-invariant, and so we let ψ_0 be a smooth strictly convex function on $\mathbb{R}^n$ satisfying

$$\omega_{\varphi_0}|_{M_{\mathrm{o}}} = \sqrt{-1}\partial\bar\partial\psi_0. \tag{8}$$

Here $[\omega]$ is any integral Kähler class in $H^2(M, \mathbb{Z})$. The initial velocity $\dot\varphi_0$ is also $\mathbf{T}$-invariant, and so it induces, by restriction to the open orbit, a smooth bounded function on $\mathbb{R}^n$ that we denote by $\dot\psi_0$. Analytically, the IVP is then equivalent to studying the following HRMA for a convex function ψ on $[0, T] \times \mathbb{R}^n$,

$$\mathrm{MA}\,\psi = 0, \text{ on } [0,T]\times\mathbb{R}^n,\ \psi(0,\,\cdot\,) = \psi_0(\,\cdot\,),\ \partial_s\psi(0,\,\cdot\,) = \dot\psi_0(\,\cdot\,), \text{ on } \mathbb{R}^n. \tag{9}$$

Here, MA denotes the real Monge–Ampère operator defined on convex functions

$$\mathrm{MA}\, f := d\partial_{x^1} f \wedge \cdots \wedge d\partial_{x^{n+1}} f,$$

as a Borel measure, and equals $\det \nabla^2 f\, dx^1 \wedge \cdots \wedge dx^{n+1}$ on C^2 functions [**26**].

Let $P := \overline{\mathrm{Im}\nabla\psi_0} \subset \mathbb{R}^n$. Recall that on a symplectic toric manifold the Legendre transform $f \mapsto f^\star$ is a bijection between the set of $\mathbf{T}$-invariant Kähler potentials on the open orbit $M_o \cong (\mathbb{C}^n)^\star$ of the (complex) torus action

$$\mathcal{H}(\mathbf{T}) := \{\psi \in C^\infty(\mathbb{R}^n) \,:\, \sqrt{-1}\partial\bar\partial\psi = \omega_\varphi|_{M_o} \text{ with } \varphi \in \mathcal{H}_\omega \\ \text{and } \overline{\mathrm{Im}\nabla\psi} = P\},$$

and the set of symplectic potentials on the moment polytope $P \subset \mathbb{R}^n$

$$\mathcal{LH}(\mathbf{T}) := \{u \in C^\infty(P \setminus \partial P) \cap C^0(P) \,:\, u = \psi^\star \text{ with } \psi \in \mathcal{H}(\mathbf{T})\}. \tag{10}$$

When the latter space is equipped with the standard $L^2(P)$ metric, this map is in fact an isometry and transforms the IVP geodesic equation to the linear equation

$$\ddot u = 0, \quad u_0 = \psi_0^\star, \quad \dot u_0 = -\dot\psi_0 \circ (\nabla\psi_0)^{-1}, \tag{11}$$

whose solution is given by $u_s := u_0 + s\dot u_0$.

Definition **2.5.** *Define the convex lifespan of the Cauchy problem* (9) *as*

$$T^{\mathrm{cvx}}_{\mathrm{span}}(\psi_0, \dot\psi_0) := \sup\{\, s \,:\, \psi_0^\star - s\dot\psi_0 \circ (\nabla\psi_0)^{-1} \textit{ is convex on } P\,\}.$$

We note that $T^{\mathrm{cvx}}_{\mathrm{span}}$ is independent of the choice of ψ_0 satisfying (8).

At least as long as $s < T^{\mathrm{cvx}}_{\mathrm{span}}$, i.e., u_s is strictly convex and hence belongs to $\mathcal{LH}(\mathbf{T})$, it is well known that the IVP for geodesics has an explicit solution,

$$\psi(s,x) = \psi_s(x) := (u_0 + s\dot u_0)^\star(x), \quad s \in [0, T_{\mathrm{span}}),\ x \in \mathbb{R}^n. \tag{12}$$

For a review of this fact and references, we refer to [**29**]. We call ψ the *Legendre transform potential*. What is less transparent is what happens when $s > T^{\mathrm{cvx}}_{\mathrm{span}}$. Firstly, it should be pointed out that, as defined in (12), ψ_s is finite for each $x \in \mathbb{R}^n$. Hence, it is necessarily Lipschitz. Moreover, as we show in [**29**], ψ_s is strictly convex, but not everywhere differentiable.

Denote by $\mathcal{H}^{0,1}(\mathbf{T})$ the closure of $\mathcal{H}(\mathbf{T})$ with respect to the $C^{0,1}$-norm (this space contains also convex functions that are not strictly convex). The corresponding space of ω-psh (plurisubharmonic) functions will be denoted by $\mathcal{H}^{0,1}_\omega$. According to the previous paragraph, one has $\psi_s \in \mathcal{H}^{0,1}(\mathbf{T})$ for all $s > 0$. It therefore makes sense to consider ψ as an infinite ray in the interior of $\mathcal{H}^{0,1}(\mathbf{T})$.

3. Statement of results

The main result in this article is that the sequence of level N quantum analytic continuation potentials φ_N defined by (6) converges uniformly to the Legendre transform potential ψ for all time, and therefore the quantum analytic continuation potential φ_∞ of Definition 2.1 solves the HCMA for $T < T^{\mathrm{cvx}}_{\mathrm{span}}$.

THEOREM **1.** *Let $\varphi := \psi - \psi_0$ be the one-parameter family of Lipschitz continuous ω-psh potentials associated to the Legendre transform potential ψ given by* (12)*, and let φ_N be the quantum analytic continuation potentials given by* (6)*. Then as N tends to infinity, φ_N converges to φ in $C^2([0,T] \times M)$ for $T < T^{\mathrm{cvx}}_{\mathrm{span}}$, and in $C^0([0,T] \times M)$ for $T \geq T^{\mathrm{cvx}}_{\mathrm{span}}$. In particular, the quantum analytic continuation potential coincides with the Legendre transform potential, i.e., $\varphi_\infty = \varphi \in \mathcal{H}^{0,1}_\omega$.*

In order to prove Theorem 1, we first prove in Proposition 5.2 that $U_N(t, z, w)$ is a Toeplitz Fourier integral operator quantizing the Hamilton flow of $\dot\varphi_0$ in the sense of Boutet de Monvel–Guillemin [**6**]. Roughly speaking, this means that the Schwartz kernel $U_N(t, z, w)$ has the kind of asymptotic structure that is similar to the Bergman kernel but quantizes a Hamiltonian flow rather than the identity map. This result holds on any projective Kähler manifold and does not make use of symmetry. The proof is based on the Toeplitz calculus developed by Boutet de Monvel–Sjöstrand [**7**] and Boutet de Monvel–Guillemin [**6**].

However, its analytic continuation $U_N(-\sqrt{-1}s, z, z)$ may lie outside the class of complex Fourier integral operators and, to our knowledge, has not previously been studied. At this time, we have only succeeded in understanding its structure in the $(S^1)^n$-invariant setting. A key step in the analysis is to construct an approximation to $U_N(t, z, w)$ in a second way that we denote by $V_N(t, z, w)$ (see Definition 6.1). This second construction uses the symplectic potential and is only well-defined in the $(S^1)^n$-invariant setting. In Proposition 6.5, we show that the quantum analytic potential may be obtained from the analytic continuations of the simpler kernels $V_N(t, z, w)$.

Once this crucial reduction is made, the logarithmic asymptotics of $U_N(-\sqrt{-1}s, z, z)$ are reduced to the analysis of lattice point sums. The results of [**34**, **41**] then directly imply the C^2 convergence up to $T < T^{\mathrm{cvx}}_{\mathrm{span}}$. Finally, we prove the global C^0 convergence to the Legendre transform subsolution for all times.

3.1. Prior results. While the Dirichlet problem for the HRMA has been extensively studied (see [**26**, **19**] and references therein), the Cauchy problem has not been systematically investigated. In [**2**], uniqueness of C^3 solutions of the Cauchy problem is proved for the more general HCMA. In [**16**, **17**], a sufficient condition on the Cauchy data is given for existence of a smooth short-time solution of HRMA depending on the Cauchy hypersurface, and an explicit formula is given. An explicit formula is also derived in [**38**] for smooth solutions of the 2-dimensional HRMA. For our Cauchy hypersurface, the existence of an explicit smooth short-time solution is not an issue, since it follows independently from the Legendre duality argument.

3.2. Further results. In the sequel, we prove that the quantum analytic continuation potential φ ceases to solve the HCMA (7) for any $T > T^{\mathrm{cvx}}_{\mathrm{span}}$ [**29**]. But at the same time, it does solve the equation on a dense set, whose complement has zero Lebesgue measure. This result, together with Theorem 1, comes close to settling Conjecture 2.4 in the case of toric or Abelian varieties. It leaves open the possibility that there exists an alternative method to solve the HRMA. That possibility is investigated in [**30**], where it is shown that the quantization produces the unique solution whenever a weak C^1 solution exists. It follows that the directions of C^1 geodesic rays in the $(S^1)^n$-symmetric case are characterized as those with infinite convex lifespan, i.e., those for which $\dot{u}_0$ is smooth and convex. In addition, we introduce there the notion of a leafwise subsolution, and show that the Legendre transform is an 'optimal' subsolution in this sense. The results and methods of this series also suggest a general conjecture on the lifespan of solutions on general Riemann surfaces, which we plan to discuss elsewhere.

4. Background

In this section, we review the definitions of Toeplitz operators in the sense of Boutet de Monvel–Guillemin [**6**] and how they are related to harmonic analysis on toric varieties in the $(S^1)^n$-invariant case [**32**]. This material is crucial for the simplification of the analysis of $U_N(-is, z, w)$ to that of the kernel $V_N(-is, z, w)$ in Definition 6.1. We do not expect prior familiarity with the Toeplitz operators of [**6**].

4.1. Kähler quantization and Toeplitz operators. Our setting consists of a polarized Kähler manifold (M, ω) of complex dimension n with $[\omega] \in H^2(M, \mathbb{Z})$. Under this integrality condition, there exists a positive Hermitian holomorphic line bundle $(L, h) \to M$.

Instead of dealing with sequences of Hilbert spaces, observables, and unitary operators on M, it is convenient to lift them to the circle bundle $X = \{\lambda \in L^\star : \|\lambda\|_{h^{-1}} = 1\}$, where $L^\star$ is the dual line bundle to L, and where h^{-1} is the norm on $L^\star$ dual to h. Let us now describe the lifted objects.

Let ρ be the function $||\lambda||_{h^{-1}} - 1$ on $L^\star$. Associated to X is the contact form $\alpha = -\sqrt{-1}\partial\rho|_X = \sqrt{-1}\bar{\partial}\rho|_X$ and the volume form $(d\alpha)^n \wedge \alpha = \pi^\star\omega^n \wedge \alpha$. We let $r_\theta w = e^{\sqrt{-1}\theta} w,\ w \in X$, denote the S^1 action on X, and denote its infinitesimal generator by $\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\theta}$. Holomorphic sections then lift to elements of the Hardy space $H^2(X) \subset L^2(X)$ of square-integrable CR functions on X, i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b := \pi^{0,1} \circ d$ (where $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}\frac{\partial}{\partial\theta}$

and $\pi^{0,1}$ is defined as the projection onto the second factor) and are L^2 with respect to the inner product

$$\langle F_1, F_2 \rangle = \frac{1}{2\pi V} \int_X F_1 \overline{F_2}\, (d\alpha)^n \wedge \alpha, \quad F_1, F_2 \in L^2(X). \tag{13}$$

The S^1 action on X gives a representation of S^1 on $L^2(X)$ with irreducible pieces denoted $L^2_N(X)$. We thus have the Fourier decomposition,

$$L^2(X) = \bigoplus_{N\in\mathbb{Z}} L^2_N(X). \tag{14}$$

We denote by $\mathbf{D}$ the operator on $L^2(X)$ with spectrum $\mathbb{Z}$ and whose N-th eigenspace $L^2_N(X)$ consists of functions transforming by $e^{\sqrt{-1}N\theta}$ under the S^1 action r_θ on X. Thus,

$$\mathbf{D} = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial\theta}. \tag{15}$$

Since $\mathbf{D}$ commutes with $\bar\partial_b$, we have $H^2(X) = \bigoplus_{N=0}^\infty H^2_N(X)$, where

$$H^2_N(X) := \{F \in H^2(X) : F(r_\theta w) = e^{\sqrt{-1}N\theta} F(w)\} = L^2_N(X) \cap \ker \bar\partial_b.$$

A section s_N of L^N determines an equivariant function $\hat{s}_N$ on $L^\star$ by the rule

$$\hat{s}_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right), \quad \lambda \in L^\star_z,\ z \in M, \tag{16}$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. We henceforth restrict $\hat{s}$ to X and then the equivariance property takes the form $\hat{s}_N(r_\theta w) = e^{iN\theta}\hat{s}_N(w)$. Up to a factor of N^n the map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $H^2_N(X)$.

The Szegő kernel of degree N is defined as the Schwartz kernel of the orthogonal projection $\tilde\Pi_N : L^2(X) \to H^2_N(X)$, given by

$$\tilde\Pi_N F(w) = \frac{1}{2\pi V} \int_X \tilde\Pi_N(w, v) F(v)\, (d\alpha)^n \wedge \alpha\, (v), \quad F \in L^2(X). \tag{17}$$

The full Szegő kernel is then $\tilde\Pi = \sum_{N=1}^\infty \tilde\Pi_N$. To simplify notation, we will from now on omit the tilde from the lifted projection operators on X and simply write Π, Π_N. Note also that Π_N depends on h, although we omit that from the notation.

It was proved by Boutet de Monvel and Sjostrand [**7**] (see also the Appendix to [**6**]) that Π is a complex Fourier integral operator (FIO) of positive type, $\Pi \in I^0_c(X \times X, \mathcal{C})$, associated to a positive canonical relation $\mathcal{C}$. For definitions and notation concerning complex FIO, we refer to [**23**, **7**, **6**]. The real points of $\mathcal{C}$ form the diagonal $\Delta_{\Sigma\times\Sigma}$ in the square of the symplectic cone

$$\Sigma := \left\{ \left(w, r\alpha(w)\right) : r > 0,\ w \in X \right\} \subset T^\star X, \tag{18}$$

where α is the connection, or contact, form [**6**, appendix, lemma 4.5]. Let $\omega_{T^\star X}$ denote the canonical symplectic form on $T^\star X$, and let

$$\omega_\Sigma := \omega_{T^\star X}|_\Sigma \tag{19}$$

denote its restriction to Σ, a symplectic form on Σ.

Finally, recall that a Toeplitz operator is an operator of the form $\Pi A \Pi$ where A is a pseudo-differential operator, and a (complex) Toeplitz Fourier integral operator is one where A is allowed to be a (complex) Fourier integral operator. When A is a pseudo-differential operator we denote by s_A its full symbol, and by σ_A its principal symbol. The symbol of $\Pi A \Pi$ is given by $\sigma_A|_\Sigma$ [**6**]. If B is a (complex) Fourier integral operator, we denote its symbol by σ_B.

4.2. Toric Kähler manifolds. We review some geometry and analysis on toric Kähler manifolds. Fuller exposition can be found in [**18**, **28**, **34**, **32**] and references therein.

We will work with coordinates on the open dense orbit of the complex torus,

$$z = e^{x/2+\sqrt{-1}\theta}, \quad (x,\theta) \in \mathbb{R}^n \times (S^1)^n \cong M_{\mathrm{o}} \cong (\mathbb{C}^\star)^n. \tag{20}$$

Let $\omega|_{M_{\mathrm{o}}} = \sqrt{-1}\partial\bar\partial\psi$. The work of Atiyah and Guillemin-Sternberg implies that the image of the moment map $\nabla\psi$ is a convex polytope $P \subset \mathbb{R}^n$ and depends only on $[\omega]$. We further assume that this is a lattice polytope. Being a lattice Delzant polytope means that: (i) at each vertex meet exactly n edges, (ii) each edge is the set of points $\{p+tu_{p,j} : t \geq 0\}$ with $p \in \mathbb{Z}^n$ a vertex, $u_{p,j} \in \mathbb{Z}^n$ and $\operatorname{span}\{u_{p,1},\dots,u_{p,n}\} = \mathbb{Z}^n$. Equivalently, there exist outward pointing normal vectors $\{v_j\}_{j=1}^d \subset \mathbb{Z}^n$, with v_j normal to the j-th $(n-1)$-dimensional face of P (also called a facet), that are primitive (i.e., their components have no common factor), and P may be written as

$$P = \{y \in \mathbb{R}^n : l_j(y) := \langle y, v_j\rangle - \lambda_j \geq 0, \quad j = 1,\dots,d\},$$

with $\lambda_j = \langle p, v_j\rangle \in \mathbb{Z}$ with p any vertex on the j-th facet, and y the coordinate on $\mathbb{R}^n$. Note that the main results in this article extend to orbifold toric varieties, since we only make essential use of (i).

The Kähler form ω is the curvature $(1,1)$ form of a line bundle $L \to M$. A basis for the space $H^0(M,L)$ of holomorphic sections is given by the monomials $\chi_\alpha(z) = z^\alpha$ with $\alpha \in P$. More generally, $H^0(M,L)$ generates the coordinate ring $\oplus_{N=1}^\infty H^0(M,L^N)$, and each lattice point γ in NP corresponds to a section χ_γ of $L^N \to M$ defined by

$$\chi_\gamma = \chi_{\beta_1} \otimes \cdots \otimes \chi_{\beta_N}, \tag{21}$$

where $\beta_1,\dots,\beta_N \in P$ such that $\gamma = \beta_1 + \cdots + \beta_N$ (see [**32**]).

We now consider the homogenization (lift to X) of toric Kähler manifolds. The lattice points in NP for each $N \in \mathbb{N}$ correspond in X to the 'homogenized' lattice points $\widehat{NP} \subset \mathbb{Z}^{n+1}$ of the form

$$\widehat{\alpha}^N = \widehat{\alpha} := (\alpha_1, \dots, \alpha_n, Np - |\alpha|), \quad \alpha = (\alpha_1, \dots, \alpha_n) \in NP \cap \mathbb{Z}^n,$$

where $p = \max_{\beta \in P \cap \mathbb{Z}^n} |\beta|$, and $|\alpha| := \sum \alpha_i$. For simplicity, we generally assume henceforth that $p = 1$. We also define the cone $\Lambda_P := \bigcup_{N=1}^{\infty} \widehat{NP}$. Rays $\mathbb{N}\widehat{\alpha}$ in this cone define the semiclassical limit.

The monomials χ_α lift to the CR monomials $\widehat{\chi}_{\widehat{\alpha}}(w) \equiv \widehat{\chi}_\alpha(w)$, $w \in X$ (see (16)), for $\widehat{\alpha} \in \Lambda_P$. They are joint eigenfunctions of a quantized torus action on X. Let $\xi_j := \frac{\partial}{\partial \theta_j}, 1 \leq j \leq n$, denote the Hamiltonian vector fields generating the $\mathbf{T}$ action on M. The horizontal lifts ξ_j^h of the Hamiltonian vector fields ξ_j are defined by

$$\pi_* \xi_j^h = \xi_j, \quad \alpha(\xi_j^h) = 0, \quad 1 \leq j \leq n. \tag{22}$$

Let $\xi_j^* \in \mathbb{R}^n$ denote the element of the Lie algebra of $\mathbf{T}$ which acts as ξ_j on M. We then define the vector fields Ξ_j by

$$\Xi_j := \xi_j^h + 2\pi\sqrt{-1}\langle \nabla\psi \circ \pi, \xi_j^* \rangle \partial_\theta = \xi_j^h + 2\pi\sqrt{-1}(\nabla\psi \circ \pi)_j \, \partial_\theta, \quad 1 \leq j \leq n. \tag{23}$$

Finally, we define the differential operators (lifted action operators),

$$\hat{I}_j := \Xi_j, \;\; j = 1, \dots, n, \quad \hat{I}_{n+1} := -\sqrt{-1}\partial_\theta - \textstyle\sum_{j=1}^n \Xi_j. \tag{24}$$

The monomials $\widehat{\chi}_{\widehat{\alpha}}$ are the joint CR eigenfunctions of $(\hat{I}_1, \dots, \hat{I}_{n+1})$ for the joint eigenvalues $\widehat{\alpha} \in \Lambda_P$, i.e., $\hat{I}_j \widehat{\chi}_{\widehat{\alpha}} = \hat{\alpha}_j \widehat{\chi}_{\widehat{\alpha}}, \;\; \widehat{\alpha} \in \Lambda_P, \;\; \bar{\partial}_b \widehat{\chi}_{\widehat{\alpha}} = 0, \;\; j = 1, \dots, n+1$. For simplicity of notation, we denote by $D_{\hat{I}}$ the vector of first-order operators

$$D_{\hat{I}} := -2\pi\sqrt{-1}\big(\hat{I}_1, \dots, \hat{I}_n\big), \tag{25}$$

and use the same notation for the quantized torus action on $H^0(M, L^N)$ and on X.

Although we are primarily concerned with holomorphic sections over M and their lifts as CR holomorphic functions on X, we need to consider non-CR holomorphic eigenfunctions of the action operators as well. We thus need to consider the anti-Hardy space $\overline{\mathcal{H}}^2(X)$ of anti-CR functions, i.e., solutions of $\partial_b f = 0$. A Hilbert basis is given by the complex-conjugate monomials $\overline{\widehat{\chi}}_{\hat{\alpha}}$. Products of eigenfunctions are also eigenfunctions. Hence, the orthonormal mixed monomials $\hat{\chi}_{\hat{\alpha},\hat{\beta}}(x) = \hat{\chi}_{\hat{\alpha}} \overline{\hat{\chi}}_{\hat{\beta}}$ are eigenfunctions of eigenvalue $\hat{\alpha} - \hat{\beta}$ for $\{\hat{I}_1, \dots, \hat{I}_{n+1}\}$. It can be shown [**32**] that

$$\begin{aligned} L^2(X) = \bigoplus_{\hat{\alpha}, \hat{\beta} \in \Lambda_P} \mathbb{C}\hat{\chi}_{\hat{\alpha},\hat{\beta}}, \text{ and } \operatorname{Spec}|_{L^2(X)} \, (\hat{I}_1, \dots, \hat{I}_{n+1}) \\ = \Lambda_P - \Lambda_P = \mathbb{Z}^{n+1}. \end{aligned} \tag{26}$$

4.3. Symplectic potential and convex analysis. Here we define some basic notation related to convex functions. For general background on Legendre duality and convexity we refer the reader to [**27**].

A vector $v \in (\mathbb{R}^n)^\star$ is said to be a subgradient of a function f at a point x if $f(z) \geq f(x) + \langle v, z - x\rangle$ for all z. The set of all subgradients of f at x is called the subdifferential of f at x, denoted $\partial f(x)$. The conjugate of a continuous function $f = f(x)$ on $\mathbb{R}^n$ is defined by $f^\star(y) := \sup_{x\in\mathbb{R}^n}\big(\langle x, y\rangle - f(x)\big)$. For simplicity, we will refer to $f^\star$ sometimes as the Legendre dual, or just dual, of f. An open-orbit Kähler potential $\psi \in \mathcal{H}(\mathbf{T})$ is a smooth strictly convex function on $\mathbb{R}^n$ in logarithmic coordinates. Therefore its gradient $\nabla\psi$ is one-to-one onto $P = \mathrm{Im}\nabla\psi$ and one has the following explicit expression for its Legendre dual,

$$u(y) = \psi^\star(y) = \langle y, (\nabla\psi)^{-1}(y)\rangle - \psi \circ (\nabla\psi)^{-1}(y), \tag{27}$$

which is a smooth strictly convex function on P, satisfying

$$\nabla u(y) = (\nabla\psi)^{-1}(y). \tag{28}$$

Following Guillemin [**18**], the function u is called the symplectic potential of $\sqrt{-1}\partial\bar\partial\psi$. The space of all symplectic potentials is denoted by $\mathcal{LH}(\mathbf{T})$. Put

$$u_G := \textstyle\sum_{k=1}^d l_k \log l_k. \tag{29}$$

A result of Guillemin [**18**] states that for any symplectic potential u the difference $u - u_G$ is a smooth function on P (that is, up to the boundary). In other words, (10) may be rewritten as (note here that P denotes the *closed* polytope)

$$\mathcal{LH}(\mathbf{T}) = \{u \in C^\infty(P \setminus \partial P) \,:\, u = u_G + F, \quad \text{with } F \in C^\infty(P)\}. \tag{30}$$

5. Quantizing the Hamiltonian flow of $\dot\varphi_0$

In this section (M, ω) is an arbitrary projective Kähler manifold. The first step in defining the analytic continuation of $\exp tX_{\dot\varphi_0}$ is to quantize this Hamiltonian flow. We use the method of Toeplitz quantization [**6**, **39**]. We may state the result either in terms of one homogeneous Fourier integral operator on $L^2(X)$ or as a semi-classical Fourier integral operator on each of the spaces $L^2_N(X)$ in the decomposition (14).

To quantize the classical Hamiltonian, we first quantize the Hamiltonian as the zeroth order Toeplitz operator $\Pi\dot\varphi_0\Pi$ on $H^2(X)$.

Definition **5.1.** *Define the one-parameter subgroup $U(t)$ of unitary operators on $L^2(X)$ by (cf. (15))*

$$U(t) = \Pi e^{\sqrt{-1}t\Pi\mathbf{D}\dot\varphi_0\Pi}\Pi.$$

Its Fourier components are given by $U_N(t) = \Pi_N e^{\sqrt{-1}tN\Pi_N\dot\varphi_0\Pi_N}\Pi_N$.

It should be noted that the quantization we use is not unique, i.e., there exists more than one unitary group of Toeplitz Fourier integral operators with underlying canonical flow equal to the Hamiltonian flow of $\dot\varphi_0$. Indeed, for any unitary pseudo-differential operator $V = e^{\sqrt{-1}A}$ obtained by exponentiating a self-adjoint pseudo-differential operator A of degree zero, and any quantization $U(t)$ of $\exp tX_{\dot\varphi_0}$, the operator $V^*U(t)V$ is another quantization with the same principal symbol. This lack of uniqueness will be seen below in the fact that we have more than one version of the quantization. They are closely related and differ by lower order terms.

We note that $U(t)$ is not quite the same as $\Pi e^{\sqrt{-1}t\mathbf{D}\dot\varphi_0}\Pi$, which is manifestly the composition of complex Fourier integral operators. However, $\Pi_N\dot\varphi_0\Pi_N$ is the quantization of $\dot\varphi_0$. We compose $e^{\sqrt{-1}tN\Pi_N\dot\varphi_0\Pi_N}$ with Π_N to make the operator preserve $H^0(M, L^N)$. Note that $U(t) = \Pi e^{\sqrt{-1}t\Pi\mathbf{D}\dot\varphi_0\Pi} = e^{\sqrt{-1}t\Pi\mathbf{D}\dot\varphi_0\Pi}\Pi$.

We now verify that $U(t)$ is a complex Fourier integral operator with underlying canonical relation equal to the graph of the Hamiltonian flow at time t of $r\dot\varphi_0$ on (Σ, ω_Σ), where r and (Σ, ω_Σ) are defined in (18)–(19). This is the content of saying that $U_N(t)$ is a quantization of the Hamiltonian flow of $\dot\varphi_0$ on (M, ω_{φ_0}).

Proposition **5.2.** *$U(t)$ is a group of complex Toeplitz Fourier integral operators on $L^2(X)$ whose underlying canonical relation is the graph of the time t Hamiltonian flow of $r\dot\varphi_0$ on the symplectic cone (Σ, ω_Σ).*

Proof. We first observe that $U(t)$ is characterized as the unique solution of the ordinary differential equation

$$\frac{d}{dt}U(t) = \left(\sqrt{-1}\Pi\mathbf{D}\dot\varphi_0\Pi\right)U(t), \quad U(0) = \Pi.$$

We use the following result of Boutet de Monvel–Guillemin.

Lemma **5.3.** [**6**, proposition 2.13] *Let T be a Toeplitz operator on Σ of order p. Then there exists a pseudo-differential operator Q of order p on X such that $[Q, \Pi] = 0$ and $T = \Pi Q \Pi$.*

We apply Lemma 5.3 to $T = \Pi\dot\varphi_0\Pi$. Thus, there exists a zeroth order pseudo-differential operator Q on X with $\sigma_Q|_\Sigma = \dot\varphi_0|_\Sigma$ (see [**6**], theorem 2.9 and proposition 2.13, for background). Note that here we identify $\dot\varphi_0$ with its lift to $\Sigma \subset T^\star X$.

Since $\Pi e^{\sqrt{-1}t\Pi\mathbf{D}Q\Pi}\Pi$ and $\Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$ satisfy the same differential equation

$$\frac{d}{dt}W(t) = \sqrt{-1}\Pi\mathbf{D}Q\Pi W(t)$$

and have the same initial condition, we have $U(t) = \Pi e^{\sqrt{-1}t\Pi\mathbf{D}Q\Pi}\Pi = \Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$. Here, we use that $\Pi^2 = \Pi$, hence $\Pi Q = \Pi Q \Pi$, and that Π and $\mathbf{D}$ commute.

Now $e^{\sqrt{-1}t\mathbf{D}Q}$ is the exponential of a real principal type pseudodifferential operator of order one on $L^2(X)$ and hence is a unitary group of Fourier integral operators on $L^2(X)$ quantizing the Hamiltonian flow of $\sigma_{\mathbf{D}Q}$ on $T^\star X$. Since Π is a complex Fourier integral operator whose real canonical relation is the diagonal in $\Sigma\times\Sigma$ [**7**], $U(t)$ is also a complex Fourier integral operator. To complete the proof of the proposition, it suffices to prove that the canonical relation of $U(t)$ is the graph of the time t Hamiltonian flow of $r\dot{\varphi}_0$ on (Σ,ω_Σ).

Let Ψ_t denote the time t Hamiltonian flow of $\sigma_{\mathbf{D}}\sigma_Q$ on $(T^\star X,\omega_{T^\star X})$. By the composition theorem for complex Fourier integral operators [**23**], the operator $\Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$ is a complex Fourier integral operator whose canonical relation is the set-theoretic composition

$$\begin{aligned}&\{(v,v)\,:\,v\in\Sigma\}\circ\{(p,\Psi_t(p)\,:\,p\in T^*X\}\circ\{(q,q)\,:\,q\in\Sigma\}\\ &\qquad\qquad=\{(m,\Psi_t(m))\,:\,m\in\Sigma\}\cap\Sigma\times\Sigma.\end{aligned}\tag{31}$$

Here we make use of the fact that the symbol of Π is nowhere vanishing on Σ and that of $e^{\sqrt{-1}t\mathbf{D}Q}$ is nowhere vanishing on the graph of Ψ_t. It only remains to equate (31) with the graph of the time t Hamiltonian flow of $r\dot{\varphi}_0$ on (Σ,ω_Σ).

Since $[\Pi,Q]=0$, we have $\Pi e^{\sqrt{-1}t\mathbf{D}Q}=\Pi e^{\sqrt{-1}t\mathbf{D}Q}\Pi$. This implies that the canonical relations of both sides in this equation must be equal. The canonical relation of the left hand side is

$$\{(v,v)\,:\,v\in\Sigma\}\circ\{(p,\Psi_t(p)\,:\,p\in T^*X\}=\{(q,\Psi_t(q)\,:\,q\in\Sigma\}.$$

Equating this to (31), it follows that Ψ_t preserves Σ. Hence, the Hamiltonian vector field $X^{T^*X}_{\sigma_{\mathbf{D}}\sigma_Q}$ of $\sigma_{\mathbf{D}}\sigma_Q$ with respect to ω_{T^*X} is tangent to the symplectic sub-cone Σ.

We note that the Clairaut integral $\sigma_{\mathbf{D}}(x,\xi)=\langle\xi,\frac{\partial}{\partial\theta}\rangle$ is the symbol of $\mathbf{D}$. Since $\alpha\big(\frac{\partial}{\partial\theta}\big)=1$, it follows from (18) that $\sigma_{\mathbf{D}}|_\Sigma=r$. Recall also that $\sigma_Q|_\Sigma=\dot{\varphi}_0|_\Sigma$. Thus, to complete the proof, it remains to show that the restriction of Ψ_t to Σ is the Hamiltonian flow of

$$\sigma_{\mathbf{D}}\sigma_Q|_\Sigma=r\dot{\varphi}_0\tag{32}$$

on (Σ,ω_Σ). Let $X^\Sigma_{r\dot{\varphi}_0}$ be the Hamiltonian vector field of $\sigma_{\mathbf{D}}\sigma_Q|_\Sigma$ with respect to ω_Σ. At a point of Σ, we have

$$\omega_{T^*X}(X^{T^*X}_{\sigma_{\mathbf{D}}\sigma_Q},\,\cdot\,)=d(\sigma_{\mathbf{D}}\sigma_Q),\qquad\omega_\Sigma(X^\Sigma_{r\dot{\varphi}_0},\,\cdot\,)=d(r\dot{\varphi}_0).$$

Evaluating these 1-forms on all tangent vectors $Y\in T\Sigma$, and using equations (19), (32), and that $X_{\sigma_{\mathbf{D}}\sigma_Q}$ is tangent to Σ, we conclude that $X^\Sigma_{r\dot{\varphi}_0}=X^{T^*X}_{\sigma_{\mathbf{D}}\sigma_Q}|_\Sigma$. q.e.d.

5.1. The level N quantum analytic continuation potentials in the toric setting. In this subsection we specialize the construction from a general projective manifold to a toric manifold. Recall from §4.2 that the toric monomials $\{\chi_\alpha(z) := z^\alpha\}_{\alpha\in NP\cap\mathbb{Z}^n}$ are an orthogonal basis of $H^0(M, L^N)$ with respect to any toric-induced Hilbert space structure on this vector space. Hence any such toric inner product is completely determined by the L^2 norms (up to N^n/V), or "norming constants," of the toric monomials—

$$\mathcal{Q}_{h^N}(\alpha) := ||\chi_\alpha||^2_{h^N} = \int_{(\mathbb{C}^*)^n} |z^\alpha|^2 e^{-N\psi}\omega_h^n. \tag{33}$$

Here we let $h = e^{-\psi}$ with $\psi \in \mathcal{H}(\mathbf{T})$, and put $\mathcal{P}_{h^N}(\alpha, z) := \frac{|\chi_\alpha(z)|^2_{h^N}}{||\chi_\alpha||^2_{h^N}}$.

We then consider the one-parameter subgroup $U(t)$ given by Definition 5.1 on a toric manifold. The first observation is that since $\dot\varphi_0$ is torus-invariant, the multiplication operator $\dot\varphi_0$ preserves the block decomposition (14). Therefore the toric monomials diagonalize the Toeplitz operators $\Pi_N\dot\varphi_0\Pi_N$, that is, $\Pi_N\dot\varphi_0\Pi_N\chi_\alpha = \mu_{N,\alpha}\chi_\alpha$, for some real numbers $\{\mu_{N,\alpha}\}_{\alpha\in NP\cap\mathbb{Z}^n}$. Since $\{\chi_\alpha\}_{\alpha\in NP\cap\mathbb{Z}^n}$ are orthogonal with respect to a toric inner product, we have $\mu_{N,\alpha} = \frac{1}{\mathcal{Q}_{h_0^N}(\alpha)}\int_M \dot\varphi_0|\chi_\alpha|^2_{h_0^N}\omega^n_{\varphi_0}$. Hence we have the following expression for the level N quantum analytic continuation potential induced by $U(\sqrt{-1}s)$:

$$\begin{aligned}\varphi_N(s,z) &= N^{-1}\log U_N(-\sqrt{-1}s, z, z)\\ &= N^{-1}\log \sum_{\alpha\in NP\cap\mathbb{Z}^n} e^{sN\mu_{N,\alpha}}\frac{|\chi_\alpha(z)|^2_{h_0^N}}{\mathcal{Q}_{h_0^N}(\alpha)}.\end{aligned} \tag{34}$$

6. Simplification of $U_N(-\sqrt{-1}s, z, z)$ on a toric manifold

We would like to determine the asymptotics of the potentials φ_N defined in (34) on a toric variety. The main problem is to determine the eigenvalues $\mu_{N,\alpha}$ with sufficient precision so that we can obtain asymptotics of lattice sums. In principle, one could evaluate the eigenvalues directly by pushing forward the eigenvalue integral to P,

$$\mu_{N,\alpha} = \int_P -\dot u_0(y)e^{N(u_0(y)+\langle\frac{\alpha}{N}-y,\nabla u_0(y)\rangle)}dy/\mathcal{Q}_{h_0^N}(\alpha).$$

Integrals similar to this one are calculated asymptotically in [**33**] for $M = \mathbb{CP}^1$ and $P = [0,1]$. However, the analysis becomes difficult and laborious near the boundary, and for higher dimensional M, P such integrals were avoided in [**34**] in favor of torus averages. In this section, we make a substantial simplification by showing that $U_N(t, z, w)$ is asymptotically the same as an operator with kernel $V_N(t, z, w)$ (Definition 6.1) that makes use of the quantum integrability of the toric setting, in particular the operators of (25). These results will then be applied in Section 7 to complete the proof of Theorem 1. We remark that the situation here is similar to that for the Bergman kernel itself: if one only used lattice point sums, it would not be clear why there exists a nice expansion around the divisor at infinity, i.e., around lattice points near the boundary. Yet we know that one does exist using the parametrix formula. In a generalized sense, this is how the Toeplitz calculus gets around the complications of the boundary.

6.1. Simplification of $U_N(t,z,w)$ to $V_N(t,z,w)$. In this section, we define the simpler kernel $V_N(t,x,y)$, whose eigenvalues are special values of the velocity of the symplectic potential. In effect, it is an explicit construction of the operator Q in Lemma 5.3, at least to leading order (which is sufficient for our purposes).

DEFINITION **6.1.** *Let*

$$V(t) = \Pi e^{-\sqrt{-1}t\mathbf{D}\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})}\Pi$$

be the one-parameter subgroup of unitary operators on $L^2(X)$, with components $V_N(t) = \Pi_N e^{-\sqrt{-1}tN\dot{u}_0(N^{-1}D_{\hat{I}})}\Pi_N$.

This is a simpler kernel because its exponent is directly defined in terms of the commuting operators $D_{\hat{I}}$ and the symplectic potential. Below we use it to define a new sequence of potentials $\tilde{\varphi}_N$. In order to relate them to the quantum analytic continuation potentials, φ_N the following fact is crucial.

PROPOSITION **6.2.** *The sequence of unitary operators $\{V_N(t)\}_{N\geq 1}$ is a semi-classical complex Toeplitz Fourier integral operator quantizing the time t Hamiltonian flow of $\dot{\varphi}_0$ on (M,ω_{φ_0}).*

Proof. It is convenient to lift to the circle bundle X and use the full spectral theory of the action operators of §4.2. Observe that $\Pi\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})\Pi$ is defined by the Spectral Theorem to be the operator on

$$H^2_{>0}(X) := \bigoplus_{N\in\mathbb{N}} H^2_N(X),$$

whose eigenfunctions are the same as the joint eigenfunctions of the quantum torus action, i.e., the lifted monomials $\{\hat{\chi}_{\hat{\alpha}} : \hat{\alpha} \in \Lambda_P\}$, and whose corresponding eigenvalues are $\{\dot{u}_0(\alpha/N) : N \in \mathbb{N},\ \alpha \in NP \cap \mathbb{Z}^n\}$. However, in order to apply classical results concerning operators of the form $e^{\sqrt{-1}tP}$ where P is a real first-order pseudo-differential operator of principal type, we need to replace $\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})$ with an operator defined on all of $L^2(X)$. Yet, since eventually we pre- and post-compose with Π, we are ultimately only interested in the restriction to $H^2_{>0}(X)$ of the extended operator. Hence we would like the extended operator to coincide with $\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})$ on $H^2_{>0}(X)$. This is the purpose of the following lemma.

Lemma **6.3.** *There exists a pseudo-differential operator R of order zero on $L^2(X)$ such that*

$$R|_{H^2_{>0}(X)} = \dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})|_{H^2_{>0}(X)}. \tag{35}$$

Proof. There are two obstacles to defining $\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})$ on all of $L^2(X)$. First, according to (26) we need to define $\dot{u}_0$ on $\mathbb{R}^n$, while originally it is only defined on P. Second, the operator $\mathbf{D}^{-1}$ is only defined on the orthocomplement of the invariant functions on X for the S^1 action. The non-constant CR functions are orthogonal to the invariant functions, so $\Pi\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})\Pi$ is well-defined on $H^2_{>0}(X)$. But we wish to extend $\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})$ outside the Hardy space.

To deal with the first point, note that since $\dot{u}_0$ is smooth up to the boundary of P, we may assume it is defined in some neighborhood of P in $\mathbb{R}^n$, and then multiply it by a smooth cutoff function η equal to 1 in a neighborhood of P and with compact support in $\mathbb{R}^n$. Then $\eta\dot{u}_0$ is a smooth function of compact support in $\mathbb{R}^n$, and therefore $\eta\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1}) \equiv (\eta\dot{u}_0)(D_{\hat{I}}\mathbf{D}^{-1})$ is well-defined on $(\ker\mathbf{D})^\perp \subset L^2(X)$. As noted above, $H^2_{>0}(X) \subset (\ker\mathbf{D})^\perp$, and since $\operatorname{Spec} D_{\hat{I}}\mathbf{D}^{-1}|_{H^2_{>0}(X)} \subset P$, we have

$$\eta\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})|_{H^2_{>0}(X)} = \dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})|_{H^2_{>0}(X)}. \tag{36}$$

We now turn to the second point. For any $\epsilon > 0$, we denote by $\gamma_\epsilon = \gamma_\epsilon(\sigma_{\hat{I}}, \sigma_{\mathbf{D}}) \in C^\infty(T^\star X \setminus \{0\})$ a homogeneous frequency cut-off, equal to 1 in an open conic neighborhood of the set $\{\sigma_{\mathbf{D}} = 0\}$:

$$\big\{(x,\xi) \in T^\star X \setminus \{0\} \,:\, |\sigma_{\mathbf{D}}| < \epsilon\big(|\sigma_{D_{\hat{I}}}|^2 + \sigma_{\mathbf{D}}^2\big)^{1/2}\big\}, \tag{37}$$

and vanishing on $\{(x,\xi) \in T^\star X \setminus \{0\} \,:\, |\sigma_{\mathbf{D}}| > 2\epsilon\big(|\sigma_{D_{\hat{I}}}|^2 + \sigma_{\mathbf{D}}^2\big)^{1/2}\}$ (note that $n+1$ of the vertical directions in $T^\star X$ are not involved). Let $\beta \in \mathbb{Z}^{n+1}$, and let $\chi_\beta \in L^2(X)$ be the associated monomial. Denote by $\gamma_\epsilon(D_{\hat{I}}, \mathbf{D})$ the Fourier multiplier associated to γ_ϵ, namely such that $\gamma_\epsilon(D_{\hat{I}}, \mathbf{D})\chi_\beta(w) = \gamma_\epsilon(\beta)\chi_\beta(w)$. This defines $\gamma_\epsilon(D_{\hat{I}}, \mathbf{D})$ on $L^2(X)$ (see (26)). Let I denote the identity operator on $L^2(X)$. Then $I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D})$ is a pseudo-differential operator with $\ker\mathbf{D} \subset \ker(I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D}))$, and $R_\epsilon := \eta\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D}))$ is a pseudo-differential operator of order zero, defined on all of $L^2(X)$.

To complete the proof of the lemma, we will prove that (35) holds for $R := R_\epsilon$, for any $\epsilon > 0$ small enough. Let $\hat{\alpha} \in \Lambda_P$ with $\chi_\alpha \in H^0(M, L^N)$, $\alpha \in NP \cap \mathbb{Z}^n$. We claim that for small enough $\epsilon > 0$ in (37), we have $\gamma_\epsilon(D_{\hat{I}}, \mathbf{D})\hat{\chi}_{\hat{\alpha}}(w) = \gamma_\epsilon(\alpha, N)\hat{\chi}_{\hat{\alpha}}(w) = 0$. For the second equality, note that for $(w, r\alpha(w)) \in \Sigma$, we have $\gamma_\epsilon(w, r\alpha(w)) = 0$ unless

$$r \le 2\epsilon r(|\nabla\psi_0 \circ \pi(w)|^2 + 1)^{1/2}, \tag{38}$$

where $\pi : X \to M$ is the bundle projection map. For $r > 0$, equation (38) cannot hold if we take ϵ such that

$$0 < \epsilon < \epsilon_0 := \big(\sup_{y \in P} |y|^2 + 1\big)^{-1}/2, \tag{39}$$

since $\nabla\psi_0 \circ \pi(w) \in P$ (and P is a bounded set in $\mathbb{R}^n$). This proves the claim, for $\epsilon \in (0, \epsilon_0)$ (note that in the proof of the last assertion, instead of working in 'homogeneous' notation, we could have replaced $r > 0$ by $N \in \mathbb{N}$ and $r\nabla\psi_0 \circ \pi(w)$ by $\alpha \in NP$). It follows that $I = I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D})$, on $H^2_{>0}(X)$. Together with (36), this proves that for each $\epsilon \in (0, \epsilon_0)$, $\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})\Pi = \eta\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D}))\Pi$, as desired. q.e.d.

The following lemma is the concrete realization of Lemma 5.3 in our setting.

Lemma **6.4.** *Let $\epsilon \in (0, \epsilon_0)$ and $R := \eta\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})(I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D}))$, with ϵ_0 given by* (39)*. The operator $\Pi R \Pi$ is a Toeplitz operator of order zero and its symbol is given by*

$$\sigma_R(w, \xi) = \dot{u}_0 \circ \nabla\psi_0 \circ \pi(w) = -\dot{\varphi}_0 \circ \pi(w), \quad (w, \xi) \in \Sigma,$$

where $\pi : X \to M$ is the projection onto the base.

Proof. As noted in the proof of Lemma 6.3, the symbol of $I - \gamma_\epsilon(D_{\hat{I}}, \mathbf{D})$ equals one on Σ. In addition, when restricting to Σ, the operator $\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})$ has a well-defined symbol, equal to the symbol of $\eta\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})$, restricted to Σ. On Σ, the symbols of the vector fields ξ^h_j (see (22)) are the Clairaut integrals $\sigma_{\xi^h_j}(w, r\alpha(w)) = \alpha_w(\xi^h_j) = 0$. Hence, on Σ, the symbol of $\hat{I}_j, 1 \le j \le n$, is that of the second term in (23): $2\pi\sqrt{-1}r(\nabla\psi_0 \circ \pi)_j$. Thus, by (25) $\sigma_{D_{\hat{I}}}(w, r\alpha(w)) = r\nabla\psi_0 \circ \pi(w)$. Since $\sigma_{\mathbf{D}^{-1}}(w, r\alpha(w)) = 1/r$ (see the proof of Proposition 5.2), it follows that the symbol of $\dot{u_0}(D_{\hat{I}}\mathbf{D}^{-1})$, restricted to Σ, is $\dot{u_0}(\pi^\star\nabla\psi_0)$ and thus equals the stated Hamiltonian σ_R. It is the lift of the Hamiltonian $H(z) = \dot{u}_0 \circ \nabla\psi_0(z)$ to the cone $\Sigma = \Sigma_{h_0}$. q.e.d.

We may now conclude the proof of Proposition 6.2. Indeed, from Lemma 6.3 we have that $V(t) = \Pi e^{-\sqrt{-1}t\mathbf{D}R}\Pi$. Since $\mathbf{D}R$ is a real principal type pseudo-differential operator of order 1, it follows that $e^{\sqrt{-1}t\mathbf{D}R}$ is a unitary Fourier integral operator whose canonical relation is given by

$$\begin{aligned} C = \{((w, \xi), (v, \zeta)) \,:\, & (w, \xi), (v, \zeta) \in T^\star X \setminus \{0\}, (w, \xi) \\ & = \exp tX^{T^\star X}_{\sigma_{\mathbf{D}R}}(v, \zeta)\} \end{aligned}$$

(see, e.g., [**14**], theorem 1.1, or [**20**], theorem 29.1.1; note that ellipticity is not essential). It follows then from Lemma 6.4 that the canonical relation of $V(t)$ is given by the time t flow-out of Σ under the flow of the Hamiltonian $-\sigma_{\mathbf{D}R} = r\pi^\star\dot{\varphi}_0$ with respect to $(T^\star X, \omega_{T^\star x})$. As shown in the proof of Proposition 5.2, this coincides with the time t flow of Σ under the flow of the same Hamiltonian with respect to (Σ, ω_Σ). Finally, the corresponding statement for the operators $V_N(t)$ asserted in the proposition follows by 'de-homogenization', since when restricting to $H^0(M, L^N), N \in \mathbb{N}$, the operator $\mathbf{D}$ simply acts by multiplication by N, and so we may replace r by the constant N, concluding the proof. q.e.d.

6.2. The sequence of potentials $\tilde{\varphi}_N(s,z)$. We now introduce the level N quantum analytic continuation potential induced by $V(\sqrt{-1}s)$. Put:

$$\begin{aligned}\tilde{\varphi}_N(s,z) &:= \frac{1}{N}\log V_N(-\sqrt{-1}s, z, z)\\ &= \frac{1}{N}\log \sum_{\alpha\in NP\cap\mathbb{Z}^n} e^{-sN\dot{u}_0(\alpha/N)}|\chi_\alpha(z)|^2_{h_0^N}/\mathcal{Q}_{h_0^N}(\alpha).\end{aligned} \tag{40}$$

These potentials are simpler than the $\varphi_N(s,z)$ defined in (34) in terms of U_N, since the eigenvalues $\dot{u}(\alpha/N)$ are explicitly given in terms of the symplectic potential. The following proposition shows that the limit quantum analytic potential can be defined in terms of them:

PROPOSITION **6.5.** *There exists a constant $C > 0$ independent of N or z such that*

$$|\tilde{\varphi}_N(s,z) - \varphi_N(s,z)| \leq Cs\log N/N.$$

Proof. The first step is to prove:

LEMMA **6.6.** *We have $\mu_{N,\alpha} = -\dot{u}_0(\alpha/N) + O(1/N)$. More precisely, there exists $C > 0$ independent of N or $\alpha \in NP$ such that*

$$|\mu_{N,\alpha} + \dot{u}_0(\alpha/N)| \leq C/N.$$

By Lemma 6.4, $\Pi_N\dot{\varphi}_0\Pi_N$ and $-\Pi_N\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})\Pi_N$ are zeroth order Toeplitz operators with the same principal symbols. Hence they differ by a Toeplitz operator of order -1. Let $\chi_\alpha \in H^0(M, L^N)$. It follows that $\mu_{N,\alpha}$ equals

$$\begin{aligned}\langle \Pi_N\dot{\varphi}_0\Pi_N\chi_\alpha, \chi_\alpha\rangle/Q_{h_0^N}(\alpha) = &-\langle \Pi_N\dot{u}_0(D_{\hat{I}}\mathbf{D}^{-1})\Pi_N\chi_\alpha, \chi_\alpha\rangle/Q_{h_0^N}(\alpha)\\ &+ O(1/N),\end{aligned}$$

proving the lemma.

We now complete the proof of the proposition. By Lemma 6.6, we have for some uniformly bounded function $R(N,\alpha)$ that

$$\begin{aligned}\varphi_N(s,z) &= \frac{1}{N}\log\sum_\alpha e^{sN\mu_{N,\alpha}}\frac{|\chi_\alpha|^2_{h_0^N}}{\mathcal{Q}_{h_0^N}(\alpha)}\\ &= \frac{1}{N}\log\sum_\alpha e^{-sN\dot{u}_0(\alpha/N)+sR(N,\alpha)}\frac{|\chi_\alpha|^2_{h_0^N}}{\mathcal{Q}_{h_0^N}(\alpha)}.\end{aligned} \tag{41}$$

The result now follows by comparing this with the expression (40) for $\tilde{\varphi}_N(s,z)$. q.e.d.

Equation (41) leads to a heuristic proof of Theorem 1: According to [**34**, Propositions. 3.1, 6.1],

$$\mathcal{Q}_{h_0^N}(\alpha) = F(\alpha,N)e^{Nu_0(\alpha/N)}/N^{C(\alpha,n)}, \tag{42}$$

where $C(\alpha,n)$ and $F(\alpha,N)$ are some uniformly bounded functions. Substituting this into (41), we obtain

$$\varphi_N(s,z) = N^{-1}\log\sum e^{N(\langle x,\alpha/N\rangle-\psi_0(x)-u_s(\alpha/N))+sR(N,\alpha)} + O(\log N/N). \tag{43}$$

Intuitively, the leading order logarithmic asymptotics are given by the value of the principal part of the exponent, $\langle x,\alpha/N\rangle-\psi_0(x)-u_s(\alpha/N)$, at its maximum (over $\alpha\in NP\cap\mathbb{Z}^n$). But this value is $u_s^\star(x)-\psi_0(x)$, as stated in Theorem 1. In the next section we give a rigorous proof.

7. Completion of the proof of Theorem 1

We now complete the proof of Theorem 1 by an argument related to those in [**34**, **35**]. Let $h_s = e^{-\varphi_s}h_0$. By Proposition 6.5, in order to prove convergence of $\varphi_N(s,z)$ to $\varphi_s(z)$, it will be enough to consider the difference

$$E_N(s,z) := \tilde{\varphi}_N(s,z) - \varphi_s(z) = \frac{1}{N}\log\sum_{\alpha\in NP\cap\mathbb{Z}^n} e^{-sN\dot{u}_0(\alpha/N)}\frac{|\chi_\alpha(z)|^2_{h_s^N}}{\mathcal{Q}_{h_0^N}(\alpha)} \tag{44}$$

Theorem 1 will then follow from the following result.

Lemma **7.1.** *For every $T>0$, we have*

$$\lim_{N\to\infty}\sup_{s\in[0,T]}||E_N(s,z)||_{C^0(M)} = 0.$$

Proof. Whenever $T<T^{\mathrm{cvx}}_{\mathrm{span}}$ the result follows directly from (42) and the asymptotic expansion of the Bergman kernel: applying (42) to h_0, using the explicit formula for u_s and then applying (42) to h_s, we obtain

$$E_N(s,z) = \frac{1}{N}\log\sum_{\alpha\in NP\cap\mathbb{Z}^n}\frac{|\chi_\alpha(z)|^2_{h_s^N}}{\mathcal{Q}_{h_s^N}(\alpha)} + O(\log N/N),$$

and this is $O(\log N/N)$ by the asymptotic expansion of the Bergman kernel [**40**]. Here by $O(\log N/N)$ we mean a quantity that is bounded from above and below by $\pm C\frac{\log N}{N}$ where C may depend on the Cauchy data and on T.

Assume now that $T\geq T^{\mathrm{cvx}}_{\mathrm{span}}$. First, we have (recall that $|z|^2=e^x$)

$$e^{-sN\dot{u}_0(\alpha/N)}|\chi_\alpha(z)|^2_{h_s^N}/\mathcal{Q}_{h_0^N}(\alpha) = e^{-sN\dot{u}_0(\alpha/N)}e^{\langle x,\alpha\rangle-N\psi_s}/\mathcal{Q}_{h_0^N}(\alpha). \tag{45}$$

From the definition of the Legendre transform, we obtain that this is bounded from above by $e^{-sN\dot{u}_0(\alpha/N)+\langle x,\alpha\rangle+Nu_s(\alpha/N)-N\langle x,\alpha/N\rangle}/\mathcal{Q}_{h_0^N}(\alpha) = e^{Nu_0(\alpha/N)}/\mathcal{Q}_{h_0^N}(\alpha)$. Applying (42) to h_0 and using the fact that the dimension of $H^0(M,L^N)$ is polynomial in N, we obtain that

$$E_N(s,z) \le O(\log N/N).$$

We now turn to proving a lower bound for $E_N(s,z)$ when $T \ge T^{\mathrm{cvx}}_{\mathrm{span}}$. Rewrite (45) as $N^C e^{\langle x,\alpha\rangle - N\psi_s - Nu_s(\alpha/N)}/F(\alpha,N)$. A lower bound for $E_N(s,z)$ will follow once we find one summand in (44) that is not decaying to zero too fast. More precisely, we will seek $\tilde{N}=\tilde{N}(s,x)$ and one $\alpha=\alpha(N,s,x)\in NP\cap\mathbb{Z}^n$ for each $N>\tilde{N}$, for which

$$e^{\langle x,\alpha\rangle - N\psi_s - Nu_s(\alpha/N)} \ge e^{-CN^{1-\epsilon}}, \quad \text{for some } \epsilon>0.$$

Fix $x\in\mathbb{R}^n$ (recall $|z|^2=e^x$). The Kähler potential ψ_s is defined on all of $\mathbb{R}^n$ and

$$\psi_s(x) \ge \langle x,y\rangle - u_s(y), \quad \forall y\in P, \tag{46}$$

with equality if and only if $y\in\partial\psi_s(x)$ (see [**27**]). Let $y_1\in P$ satisfy equality in (46). Such a y_1 exists, for P is compact and u_s is bounded, and so the supremum in $\psi_s(x)=\sup_{y\in P}[\langle x,y\rangle - u_s(y)]$ is necessarily achieved and finite. Hence, by convexity of ψ_s we have $\partial\psi_s(x)\ne\emptyset$, and one may choose $y_1\in\partial\psi_s(x)$. Then we need to find $\tilde{N}=\tilde{N}(s,x)$ and $\alpha=\alpha(N,s,x)$ such that

$$e^{N(\langle x,\alpha/N-y_1\rangle + u_s(y_1)-u_s(\alpha/N))} \ge e^{-CN^{1-\epsilon}}, \quad \text{for each } N>\tilde{N}.$$

In fact, we will derive such an estimate where the right hand side is $e^{-C\log N}$.

Claim **7.2.** *Let $x\in\mathbb{R}^n$ and let $y_1\in\partial\psi_s(x)$. Then $y_1\in P\setminus\partial P$.*

Proof. Note that by duality $x\in\partial u_s(y_1)$ (this holds even though u_s need not be convex; see [**21**], Theorem 1.4.1, p. 47), and in particular $\partial u_s(y_1)\ne\emptyset$. Therefore, it suffices to show that $\lim_{y\to\partial P}|\nabla u_s(y)|=\infty$, since that will imply that $\partial u_s(y)=\emptyset$ whenever $y\in\partial P$. Denote by $\{w_i\}\subset P\setminus\partial P$ a sequence converging to $y\in\partial P$. Assume without loss of generality that $l_1,\ldots,l_n$ provide a coordinate chart in a neighborhood of y in P. Using Guillemin's formula (29), in these coordinates the gradient of u_s takes the form $(\log l_1+h_1,\ldots,\log l_n+h_n)$, where h_j belongs to $C^\infty(P)$ for each $j=1,\ldots,n$. Therefore, $\lim_{y\to\partial P}|\nabla u_s(y)|=\infty$, as desired. q.e.d.

The points $\{\alpha/N\}_{NP\cap\mathbb{Z}^n}$ are C/N-dense in P, where $C>0$ is some uniform constant. Hence, for each of the 2^n orthants in $\mathbb{R}^n$ there exists a point α/N that is C/N-close to y_1 and such that the vector $\alpha/N-y_1$ is contained in that orthant. Now let $\tilde{N}$ be chosen large enough so that

$\mathrm{dist}(y_1, \partial P) > C/\tilde{N}$ (possible by Claim 7.2). Further, let $\tilde{N}$ be such that $\exists \alpha_1 = \alpha_1(\tilde{N})$ such that $\alpha_1/\tilde{N} \in P \setminus \partial P$ and

$$\begin{aligned} &\mathrm{dist}(\alpha_1/\tilde{N}, \partial P) > C/\tilde{N},\ \langle \alpha_1/\tilde{N} - y_1, x\rangle \geq 0, \\ &\text{and} \quad \frac{C}{2\tilde{N}} \leq |\alpha_1/\tilde{N} - y_1| \leq \frac{C}{\tilde{N}}. \end{aligned} \tag{47}$$

Note that y_1 depends only on s and x and so does $\tilde{N}$. Further, for every $N > \tilde{N}$ one may find an $\alpha_1 = \alpha_1(N)$ satisfying the inequalities (47) with $\tilde{N}$ replaced by N.

Applying the mean value theorem to the line segment between α_1/N and y_1, it follows that

$$e^{N(\langle x, \alpha_1/N - y_1\rangle + u_s(y_1) - u_s(\alpha_1/N))} \geq e^{-N|y_1 - \alpha_1/N||\nabla u_s(y_2)|}, \tag{48}$$

where $y_2 \in P \setminus \partial P$ is some point on the line segment between α_1/N and y_1. Hence, $\mathrm{dist}(y_2, \partial P) > C/N$. By Guillemin's formula (29), we therefore have

$$|\nabla u_s(y_2)| < C \log N + s||\dot{u}_0||_{C^1(P)} < C_T \log N,$$

for some constant C_T that depends on T. It follows that

$$\begin{aligned} E_N(s,z) &\geq \frac{1}{N} \log e^{N(\langle x, \alpha_1(N)/N - y_1\rangle + u_s(y_1) - u_s(\alpha_1(N)/N))} \\ &\geq \log e^{-C_T \log N}/N \geq -C_T \log N/N, \end{aligned} \tag{49}$$

and this concludes the proof of Lemma 7.1. q.e.d.

Lemma 7.1 completes the proof of Theorem 1.

References

[1] C. Arezzo & G. Tian, *Infinite geodesic rays in the space of Kähler potentials,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), 617–630, MR 2040638, Zbl 1170.32312.

[2] E. Bedford & D. Burns, *Holomorphic mapping of annuli in $\mathbb{C}^n$ and the associated extremal function,* Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), 381–414, MR 0553791, Zbl 0422.32021.

[3] E. Bedford & B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation,* Inv. Math. **37** (1976), 1–44, MR 0445006, Zbl 0315.31007.

[4] B. Berndtsson, *Positivity of direct image bundles and convexity on the space of Kähler metrics,* J. Diff. Geom. **81** (2009), 457–482, MR 2487599, Zbl 1187.53076.

[5] B. Berndtsson, *Probability measures related to geodesics in the space of Kähler metrics,* preprint, arxiv:0907.1806.

[6] L. Boutet de Monvel & V. Guillemin, *The spectral theory of Toeplitz operators,* Princeton University Press, 1981, MR 0620794, Zbl 0469.47021.

[7] L. Boutet de Monvel & J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö,* Astérisque 34-35 (1976), 123–164, MR 0590106, Zbl 0344.32010.

[8] D. Catlin, *The Bergman kernel and a theorem of Tian,* in: *Analysis and geometry in several complex variables* (G. Komatsu et al., eds.), Birkhäuser, 1999, 1–23, MR 1699887, Zbl 0941.32002.

[9] X.-X. Chen, *The space of Kähler metrics,* J. Diff. Geom. **56** (2000), 189–234, MR 1863016, Zbl 1041.58003.

[10] X.-X. Chen & S. Sun, *Space of Kähler metrics (V)—Kähler quantization,* preprint, arxiv:0902.4149.

[11] X.-X. Chen & Y.-D. Wang, *Test configurations and geodesic rays,* Astérisque **321** (2008), 139–167, MR 2521647, Zbl 1181.53058.

[12] X.-X. Chen & G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs,* Publ. Math. Inst. Hautes Études Sci. **107** (2008), 1–107, MR 2434691, Zbl 1182.32009.

[13] S.K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics,* in: Northern California Symplectic Geometry Seminar (Ya. Eliashberg et al., eds.), Amer. Math. Soc., 1999, pp. 13–33, MR 1736211, Zbl 0972.53025.

[14] J.J. Duistermaat & V.W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics,* Invent. Math. **29** (1975), 39–79 MR 0405514, Zbl 0307.35071.

[15] R. Feng, *Bergman metrics and geodesics in the space of Kähler metrics on principally polarized Abelian varieties,* J. Math. Inst. Jussieu, in press.

[16] R.L. Foote, *Differential geometry of real Monge–Ampère foliations,* Math. Z. **194** (1987), 331–350, MR 0879936, Zbl 0636.53012.

[17] R.L. Foote, *A geometric solution to the Cauchy problem for the homogeneous Monge–Ampère equation,* Proc. Workshops Pure Math. **11** (1991), 31–39, Korean Acad. Council.

[18] V. Guillemin, *Kaehler structures on toric varieties,* J. Diff. Geom. **40** (1994), 285–309, MR 1293656, Zbl 0813.53042.

[19] C.E. Gutiérrez, *The Monge–Ampère equation,* Birkhäuser, 2001, MR 1829162, Zbl 0989.35052.

[20] L. Hörmander, *The analysis of linear partial differential operators,* Vols. I–IV, Springer-Verlag, 1990, MR 1996773, Zbl 1028.35001.

[21] J.-B. Hiriart-Urruty & C. Lemaréchal, *Convex analysis and minimization algorithms II,* Springer, 1993, MR 1295240, Zbl 0795.49002.

[22] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds (I),* Osaka J. Math. **24** (1987), 227–252, MR 0909015, Zbl 0645.53038.

[23] A. Melin & J. Sjöstrand, *Fourier integral operators with complex-valued phase functions,* Lecture Notes in Math. **459** (1975), 120–223, MR 0431289, Zbl 0306.42007.

[24] D.H. Phong & J. Sturm, *The Monge–Ampère operator and geodesics in the space of Kähler potentials,* Invent. Math. **166** (2006), 125–149, MR 2242635, Zbl 1120.32026.

[25] D.H. Phong & J. Sturm, *Test configurations for K-stability and geodesic rays,* J. Symp. Geom. **5** (2007), 221–247, MR 2377252, Zbl 1193.53104.

[26] J. Rauch & B.A. Taylor, *The Dirichlet problem for the multidimensional Monge–Ampère equation,* Rocky Mountain J. Math. **7** (1977), 345–364, MR 0454331, Zbl 0367.35025.

[27] R.T. Rockafellar, *Convex analysis,* Princeton University Press, 1970, MR 0274683, Zbl 0193.18401.

[28] Y.A. Rubinstein & S. Zelditch, *Bergman approximations of harmonic maps into the space of Kähler metrics on toric varieties,* J. Symp. Geom. **8** (2010), 239–265, MR 2684507, Zbl 1205.53067.

[29] Y.A. Rubinstein & S. Zelditch, *The Cauchy problem for the homogeneous Monge–Ampère equation,* II. Legendre transform, Adv. Math., **228** (2011), 2989–3025.

[30] Y.A. Rubinstein & S. Zelditch, *The Cauchy problem for the homogeneous Monge–Ampère equation,* III. Lifespan, preprint, 2011.

[31] S. Semmes, *Complex Monge–Ampère and symplectic manifolds,* Amer. J. Math. **114** (1992), no. 3, 495–550, MR 1165352, Zbl 0790.32017.

[32] B. Shiffman, T. Tate & S. Zelditch, *Harmonic analysis on toric varieties,* in: *Explorations in complex and Riemannian geometry,* American Mathematical Society, 2003, pp. 267–286, MR 2018345, Zbl 1041.32004.

[33] J. Song & S. Zelditch, *Convergence of Bergman geodesics on* $\mathbf{CP}^1$*,* Ann. Inst. Fourier (Grenoble) **57** (2007), 2209–2237, MR 2394541, Zbl 1144.53089.

[34] J. Song & S. Zelditch, *Bergman metrics and geodesics in the space of Kähler metrics on toric varieties,* Analysis & PDE **3** (2010), 295–358, MR 2672796, Zbl 1144.53089.

[35] J. Song & S. Zelditch, *Test configurations, large deviations and geodesic rays on toric varieties,* Adv. in Matn. (in press), arxiv:0712.3599.

[36] S. Sun, *Note on geodesic rays and simple test configurations,* J. Symp. Geom. **8** (2010), 57–65, MR 2609628, Zbl 1198.53083.

[37] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds,* J. Diff. Geom. **32** (1990), 99–130, MR 1064867, Zbl 0706.53036.

[38] V. Ushakov, *The explicit general solution of trivial Monge–Ampère equation,* Comm. Math. Helv. **75** (2000), 125–133, MR 1760498, Zbl 0948.35026.

[39] S. Zelditch, *Index and dynamics of quantized contact transformations,* Ann. Inst. Fourier **47** (1997), 305–363, MR 1437187, Zbl 0865.47018.

[40] S. Zelditch, *Szegö kernels and a theorem of Tian,* Internat. Math. Research Notices **6** (1998), 317–331, MR 1616718, Zbl 0922.58082.

[41] S. Zelditch, *Bernstein polynomials, Bergman kernels and toric Kähler varieties,* J. Symp. Geom. **7** (2009), 51–76, MR 2496414, Zbl 1178.32019.

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