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SIGNATURE DEFECTS OF CUSPS OF HILBERT MODULAR VARIETIES AND VALUES OF L-SERIES AT $s = 1$

WERNER MÜLLER

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0. Introduction

Investigating Hilbert modular surfaces, Hirzebruch found a very interesting relation between the signature defect associated to a cusp of a Hilbert modular surface and the value at $s = 1$ of a certain L-series [24, §3]. Hirzebruch's result is interesting since it gives a topological meaning to these values of L-series. However, Hirzebruch's proof is based on very explicit calculations and gives no deeper explanation of this connection between these topological and arithmetic invariants associated to a real quadratic field. He uses his beautiful explicit resolution of the cusp singularities of the compactified surface to compute the signature defect of the cusps. On the other hand, C. Meyer [28] has calculated the value at $s = 1$ of the corresponding L-series and it turns out that this value coincides with the formula for the signature defect of the cusp given by Hirzebruch. Guided by this result, Hirzebruch conjectured that for all Hilbert modular varieties associated with a totally real number field of arbitrary degree the signature defects of the cusp singularities are still given by values at $s = 1$ of certain L-series associated with the corresponding cusp [24, p. 230]. Actually, Hirzebruch's conjecture is more general. It is related to

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"cusps", which may not occur as cusp singularities of any Hilbert modular variety (cf. [24, p. 230]). The L-series in question have been studied by Shimizu [38]. Now, for higher dimensional Hilbert modular varieties the geometry of the cusp singularities is much more complicated than in the two-dimensional case, and there is no hope of getting explicit formulas. The attempt to prove Hirzebruch's conjecture was one of the main motivations for the work of Atiyah, Patodi and Singer on spectral asymmetry [3]. Their work was an attempt to understand the significance of Hirzebruch's result in the wider context of Riemannian geometry. In their paper they extended Hirzebruch's signature theorem to the case of manifolds with boundary. The main result of [3] is that for a compact oriented Riemannian manifold with boundary Y, which, near Y, is isometric to the product $Y \times [0,1]$, the differential geometric signature defect $\delta(Y)$ is a nonlocal spectral invariant of Y. This is the so-called Eta-invariant $\eta(0)$. A proof of Hirzebruch's conjecture along these lines has been developed by Atiyah, Donnelly and Singer [2]. The idea is to apply the results of [3] to the boundary Y of a neighborhood of a cusp.

The purpose of this paper is to understand Hirzebruch's result from a different point of view. It turns out that Hirzebruch's conjecture is a consequence of a certain L^2-index theorem. To explain the main idea we consider the Hilbert modular group. Let $F/\mathbf{Q}$ be a totally real number field of degree n and class number 1. Let $\mathcal{O}_F$ be the ring of integers and consider the Hilbert modular group $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$. Γ acts properly discontinuously on the product $\mathbf{H}^n$ of upper half-planes and $\Gamma \backslash \mathbf{H}^n$ has only one cusp. Let $\mathsf{M} = \mathcal{O}_F$ and $\mathsf{V} = \mathcal{O}_F^{*2}$. The cusp ∞ is of type (M, V) in the sense of Hirzebruch [24]. Let $\mathscr{H}^*_{(2)}(\Gamma \backslash \mathbf{H}^n)$ be the space of Γ-invariant harmonic forms on $\mathbf{H}^n$, which are square integrable mod Γ and let $\mathscr{H}^*_{(2),\pm}(\Gamma \backslash \mathbf{H}^n)$ be the ± 1-eigenspaces of the involution τ defined by the $*$-operator. Then, using results of Harder [19], one can show that

$$\operatorname{Sign}(\Gamma \backslash \mathbf{H}^n) = \dim \mathscr{H}^*_{(2),+}(\Gamma \backslash \mathbf{H}^n) - \dim \mathscr{H}^*_{(2),-}(\Gamma \backslash \mathbf{H}^n).$$

Let $\Lambda^* = \Lambda^*(\Gamma \backslash \mathbf{H}^n)$ be the space of Γ-invariant differential forms on $\mathbf{H}^n$ and let $\Lambda^*_\pm = \Lambda^*_\pm(\Gamma \backslash \mathbf{H}^n)$ be the ± 1-eigenspace of τ. Consider the signature operator $D = d + d^*\colon \Lambda^*_+ \to \Lambda^*_-$. It has a well-defined L^2-index, which is given by

$$\operatorname{Ind}_{L^2} D = \dim \mathscr{H}^*_{(2),+}(\Gamma \backslash \mathbf{H}^n) - \dim \mathscr{H}^*_{(2),-}(\Gamma \backslash \mathbf{H}^n).$$

Thus $\operatorname{Sign}(\Gamma \backslash \mathbf{H}^n) = \operatorname{Ind}_{L^2} D$. Now, one can use the method of the heat equation as in the compact case to compute the L^2-index. Let $\Delta_+ = D^*D$ and $\Delta_- = DD^*$ be the Laplacians on $\Lambda^*_\pm$. The restriction of $\Delta_\pm$ to the space of compactly supported differential forms has a unique self-adjoint extension $\bar{\Delta}_\pm$

to an operator in $L^2\Lambda^*_\pm$. Using the theory of Eisenstein series [20] one proves that $L^2\Lambda^*_\pm$ admits an orthogonal decomposition

$$L^2\Lambda^*_\pm = L^2_d\Lambda^*_\pm \oplus L^2_c\Lambda^*_\pm$$

such that $\overline{\Delta}_\pm$ decomposes discretely in $L^2_d\Lambda^*_\pm$, and $L^2_c\Lambda^*_\pm$ is the subspace of absolute continuity of $\overline{\Delta}_\pm$. Let $\Delta^d_\pm$ be the restriction of $\overline{\Delta}_\pm$ to $L^2_d\Lambda^*_\pm$ and consider the corresponding heat operators $\exp(-t\Delta^d_\pm)$. $L^2_d\Lambda^*_\pm$ contains the space of cusp forms $L^2_0\Lambda^*_\pm$ (see the end of §3 for its definition). Using some results related to Selberg's trace formula, one can show that $\exp(-t\Delta^d_\pm)$ restricted to the subspace $L^2_0\Lambda^*_\pm \subset L^2_d\Lambda^*_\pm$ is of the trace class for each $t > 0$. Moreover, by analysing the constant terms of the Eisenstein series, it turns out that the orthogonal complement of $L^2_0\Lambda^*_\pm$ in $L^2_d\Lambda^*_\pm$ is finite dimensional. Thus $\exp(-t\Delta^d_\pm)$ are trace class operators for each $t > 0$ and

$$\operatorname{Ind}_{L^2} D = \operatorname{Tr}\left(\exp\left(-t\Delta^d_+\right)\right) - \operatorname{Tr}\left(\exp\left(-t\Delta^d_-\right)\right).$$

As in the compact case there is a smooth kernel $K^\pm(z, z', t)$, which represents the heat operator $\exp(-t\Delta^d_\pm)$ and its trace is given by the integral of $\operatorname{tr} K^\pm(z, z, t)$ over $\Gamma \setminus \mathbf{H}^n$. Selberg's trace formula tells us how to compute this integral. Each conjugacy class of Γ makes a certain contribution to the trace of $\exp(-t\Delta^d_\pm)$. A careful analysis of the different conjugacy classes shows that only elliptic and parabolic conjugacy classes give a nonzero contribution to

$$\int_{\Gamma\setminus\mathbf{H}^n} \operatorname{tr} K^+(z, z, t) - \int_{\Gamma\setminus\mathbf{H}^n} \operatorname{tr} K^-(z, z, t).$$

If $z \in \mathbf{H}^n$ is an elliptic fixed point of Γ, then the contribution of the elliptic conjugacy classes with fixed point z is precisely the cotangent sum $\delta(z)$ associated with the quotient singularity z [24, §3.3]. Let $\mathsf{M} = \mathcal{O}_F$, $\mathsf{V} = \mathcal{O}_F^{*2}$ and let $L(\mathsf{M}, \mathsf{V}, s)$ be the L-series associated to (M, V) (see (5.53) for its definition). Then the parabolic contribution turns out to be

$$\frac{i^n}{\pi^n} d(\mathsf{M}) L(\mathsf{M}, \mathsf{V}, 1), \quad \text{where } d(\mathsf{M}) = \left(D_{F/\mathbf{Q}}\right)^{1/2}.$$

Thus, if $z_1, \cdots, z_m \in \mathbf{H}^n$ represent the quotient singularities of $\Gamma \setminus \mathbf{H}^n$, then

$$\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n) = \operatorname{Ind}_{L^2} D = \sum_{j=1}^m \delta(z_j) + \frac{i^n}{\pi^n} d(\mathsf{M}) L(\mathsf{M}, \mathsf{V}, 1).$$

There is another formula for $\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n)$, proved by Hirzebruch [24, §3, (20)]. The contribution of the elliptic fixed points is the same as above, but the contribution of the cusp has to be replaced by the signature defect $\delta(\infty)$

associated to the cusp ∞. Thus, comparing these two formulas, we get

$$\delta(\infty) = \frac{i^n}{\pi^n} d(\mathsf{M}) L(\mathsf{M}, \mathsf{V}, 1),$$

which is Hirzebruch's conjecture in this particular case.

In the same way one can compute the L^2-index of other classical operators. If we consider the Dolbeault operator $\bar\partial + \bar\partial^*\colon \sum_q \Lambda^{p,2q} \to \sum_q \Lambda^{p,2q+1}$, it turns out that its index is related to the dimension of $\mathscr{H}^{p,n-p}_{\text{cus}}(\Gamma \setminus \mathbf{H}^n)$—the space of harmonic cusp forms of bidegree $(p, n-p)$ on $\Gamma \setminus \mathbf{H}^n$. Using Selberg's trace formula, we compute the index of the Dolbeault operator. In this way we get a formula for the dimension of $\mathscr{H}^{p,n-p}_{\text{cus}}(\Gamma \setminus \mathbf{H}^n)$.

Of course, everything that we described for the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_F)$ of a totally real number field F of class number one, can be extended to an arbitrary irreducible discrete subgroup $\Gamma \subset (\mathrm{SL}(2, \mathbf{R}))^n$ of finite co-volume. However, if Γ has several Γ-inequivalent parabolic fixed points $x_1, \cdots, x_h$, then by this method we only get

$$\sum_{\iota=1}^{h} \delta(x_\iota) = \frac{i^n}{\pi^n} \sum_{\iota=1}^{h} d(\mathsf{M}_\iota) L(\mathsf{M}_\iota, \mathsf{V}_\iota, 1),$$

where $\delta(x_\iota)$ is the signature defect associated with x_ι and x_ι is of type $(\mathsf{M}_\iota, \mathsf{V}_\iota)$ [24]. To overcome this difficulty, it is natural to consider Riemannian manifolds X, which are obtained by taking a single cusp, chopped off near infinity, and gluing it together with a compact Riemannian manifold, which has the same boundary. Each cusp of $\Gamma \setminus \mathbf{H}^n$ can be described by a lattice M in a certain totally real number field F of degree n and a subgroup $\mathsf{V} \subset U^+_{\mathsf{M}}$ of finite index [24], [38]. The Riemannian manifold X has a decomposition $X = X_0 \cup X_1$, where X_0 is compact and X_1 is isometric to a cusp of type (M, V) for some M and V as above. We call X a Riemannian manifold with a cusp of type (M, V). Thus, the attempt to prove Hirzebruch's conjecture by the methods described above, leads very naturally to the problem of extending the results concerning the spectral resolution of the Laplacian of the locally symmetric space $\Gamma \setminus \mathbf{H}^n$ to manifolds X with a cusp of type (M, V). This problem has been considered by the author for manifolds which are natural generalizations of the $\mathbb{R}$-rank one case [30], [31]. In principle the same methods can be used in our situation because the "analysis near infinity", i.e. analysis on the cusp, reduces to harmonic analysis. Selberg's trace formula has to be replaced by the asymptotic expansion of the heat kernel. Then, one can compute the L^2-index of the signature operator as above. There are no quotient singularities, but there will be the contribution $\int_X L(p)$, where $L(p)$ is the Hirzebruch polynomial in the Pontrjagin forms. On the other hand, there is a formula for Sign(X), which is analogous to Hirzebruch's formula [24, §3, (20)]. The proof of Hirzebruch's conjecture will be a consequence of these calculations.

We are not going to carry out this program in the present paper. The locally symmetric case which we shall consider in this paper illustrates that the principle, which relates signature defects of cusps and values of L-series at $s = 1$, is essentially based on a L^2-index theorem.

The paper is organized as follows. In §§1 and 2 we recall some facts about homogeneous vector bundles, invariant differential and integral operators and harmonic analysis. §3 collects the pertinent results from the theory of Eisenstein series and the spectral resolution of the regular representation. We explicate these results for our situation. In §4 we discuss Selberg's trace formula. It turns out that for $G = (\mathrm{SL}(2,\mathbb{R}))^n$ and $\Gamma \subset G$ any lattice, the restriction of the operator $R_{\Gamma\backslash G}(f)$ to the discrete spectrum $L^2_d(\Gamma \backslash G)$ is of the trace class for all $f \in \mathscr{C}^1(G)$. Therefore, we can use the version of the trace formula established by Osborne and Warner for a rank one lattice [32]. For the applications we have in mind it is necessary to evaluate the different terms occurring in the trace formula explicitly. We do this up to a stage which is sufficient for our purpose. The case when $f \in C_0^\infty(G)$ is bi-invariant under $K = (\mathrm{SO}(2))^n$ has been treated by P. Sograf [39] for $n = 2$ and by I. Efrat [15] in general. In this case the trace formula has been brought to a final form. The trace formula has been used by I. Efrat to establish Weyl's law for the asymptotic distribution of eigenvalues for any lattice in $(\mathrm{SL}(2,\mathbb{R}))^n$ with $n \geqslant 2$.

In §5 we use the trace formula to compute the index of the signature and the Dolbeault operator. In this way we get our main result, Theorem 5.71. Theorem 5.82 gives our formula for the dimension of $\mathscr{H}^{p,n-p}_{\mathrm{cus}}(\Gamma \backslash \mathbf{H}^n)$. This generalizes parts of the results of Matsushima and Shimura [27] to the case of nonuniform lattices.

Finally, in §6 we discuss briefly our approach to prove Hirzebruch's conjecture in general.

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1. Preliminaries

Let $G = (\mathrm{SL}(2,\mathbb{R}))^n$ and $K = (\mathrm{SO}(2))^n$. K is a maximal compact subgroup of G and we have $G/K = \mathbf{H}^n$, where $\mathbf{H}$ is the upper half-plane. Let $\mathfrak{g} \supset \mathfrak{k}$ be the corresponding Lie algebras and let B be the Killing form of $\mathfrak{g}$. The orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B will be denoted by $\mathfrak{p}$. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. If $n = 1$, we denote the corresponding objects by G_0, K_0, $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. Let $x_0 \in \mathbf{H}^n$ be the coset eK. The projection $G \to G/K$ induces an isomorphism $\mathfrak{p} \to T_{x_0}\mathbf{H}^n$. Let σ: $K \to \mathrm{GL}(V)$ be a finite dimensional representation. To σ corresponds a homogeneous vector bundle $E(\sigma) \to \mathbf{H}^n$. Let $C^\infty(E)$ be the space of C^∞-sections of E. $C^\infty(E)$ can be identified with the space of K-invariants $(C^\infty(G) \otimes E)^K$ of $C^\infty(G) \otimes E$ with respect to the action $k \mapsto R(k) \otimes \sigma(k)$ of K, where R is the right regular representation of G. Similarly, the space of L^2- sections of E will be identified with $(L^2(G) \otimes E)^K$. The tangent bundle $T\mathbf{H}^n$ is associated to the adjoint representation $\mathrm{Ad}_{\mathfrak{p}}$: $K \to \mathrm{GL}(\mathfrak{p})$. Therefore, a C^∞-vector field on $\mathbf{H}^n$ can be identified with a C^∞-map φ: $G \to \mathfrak{p}_{\mathbf{C}}$ such that $\varphi(gk^{-1}) = \mathrm{Ad}_{\mathfrak{p}}(k)\varphi(g)$, $k \in K$, $g \in G$. Correspondingly, a p-form $\omega \in \Lambda^p(\mathbf{H}^n)$ is a C^∞-map ω: $G \to \Lambda^p \mathfrak{p}_{\mathbf{C}}$, which satisfies $\omega(gk^{-1}) = \Lambda^p \mathrm{Ad}_{\mathfrak{p}}^*(k)\omega(g)$. Here $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes_{\mathbf{R}} \mathbf{C}$. Let $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ be the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbf{C}})$ of $\mathfrak{g}_{\mathbf{C}}$ and let $\Omega \in \mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ be the Casimir operator. If Δ_p is the Laplacian on $\Lambda^p(\mathbf{H}^n)$ with respect to the invariant metric, then we have Kuga's Lemma

$$\Omega\omega = -\Delta_p\omega, \qquad \omega \in \Lambda^p(\mathbf{H}^n).$$

Consider the following elements of $\mathfrak{sl}(2, \mathbf{R})$: $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\mathfrak{k}_0 = \mathbf{R}W$, $\mathfrak{p}_0 = \mathbf{R}H \oplus \mathbf{R}Y$ and, if B_0 is the Killing form of $\mathfrak{g}_0$, then $B_0(H, Y) = 0$. Let H_j and Y_j, $j = 1, \cdots, n$, be the elements of $\mathfrak{g}$ with jth component equal to H and Y respectively and the others equal to zero. $\mathrm{Ad}_{\mathfrak{p}}$: $K \to \mathrm{GL}(\mathfrak{p}_{\mathbf{C}})$ can be diagonalized. Eigenvectors are

$$E_j^{\pm} = H_j \pm iY_j, \qquad j = 1, \cdots, n, \tag{1.1}$$

where

$$\mathrm{Ad}(k)E_j^{\pm} = \exp(\pm 2i\theta_j)E_j^{\pm},$$

$$k_j = \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}, \qquad j = 1, \cdots, n.$$

Thus, if we choose $\mathfrak{h}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}}$ as the Cartan algebra of $\mathfrak{g}_{\mathbf{C}}$, then the vectors (1.1) are the nonzero root vectors. Let Φ be the set of roots. We choose the system of positive roots $\Psi = \{\alpha_1, \cdots, \alpha_n\}$ such that $\mathfrak{g}_{\mathbf{C}}^{\alpha_j} = \mathbf{C}E_j^+$. Note that each root is noncompact. Let W be the Weyl group of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. We have $W = \{\pm 1\}^n$.

Let $G = UAK$ be the Iwasawa decomposition of G. Every $g \in \mathrm{SL}(2, \mathbf{R})$ can be uniquely written as

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.2}$$

In this way we introduce coordinates $(x, y, \theta) \in \mathbb{R}^n \times (\mathbb{R}^+)^n \times [0, 2\pi)^n$ on G. Consider the C^∞-functions $\varphi_j^\pm\colon G \to \mathfrak{p}_{\mathbf{C}}$ defined by $\varphi_j^\pm(nak) = \exp(\mp 2i\theta(k_j))E_j^\pm$, $j = 1,\cdots,n$. Since $\varphi_j^\pm(gk^{-1}) = \mathrm{Ad}(k)\varphi_j^\pm(g)$, $\varphi_j^\pm$ corresponds to a vector field $Z_j^\pm$ on $\mathbf{H}^n$ and a calculation shows that $Z_j^+ = 4iy_j\,\partial/\partial z_j$ and $Z_j^- = -4iy_j\partial/\partial \bar{z}_j$, where $z_j = x_j + iy_j$. Finally, note that the Casimir operator on G is given by

$$\Omega = \sum_{j=1}^{n} \frac{1}{4}\left(H_j^2 + Y_j^2 - W_j^2\right).$$

We add some remarks about invariant differential and integral operators. If π is any representation of G on a topological vector space V, we denote by V^K the space of K-invariant vectors in V and by V^∞ the space of C^∞-vectors. Let $\sigma_i\colon K \to U(E_i)$, $i = 1, 2$, be two finite dimensional unitary representations of K. K acts on $\mathfrak{U}(\mathfrak{g}_{\mathbf{C}}) \otimes \mathrm{End}(E_1, E_2)$ by $\mathrm{Ad} \otimes \sigma_2 \otimes \sigma_1^{-1}$, where $\mathrm{End}(E_1, E_2) \cong E_2 \otimes E_1^*$. Let $(\mathfrak{U}(\mathfrak{g}_{\mathbf{C}}) \otimes \mathrm{End}(E_1, E_2))^K$ be the space of K-invariants with respect to this action and let $D = \sum_i Z_i \otimes C_i$ be an element of $(\mathfrak{U}(\mathfrak{g}_{\mathbf{C}}) \otimes \mathrm{End}(E_1, E_2))^K$. Let π be any unitary representation of G on a Hilbert space $\mathscr{H}_\pi$. Then we let $\pi(D)$ be the operator from $(\mathscr{H}_\pi \otimes E_1)^K$ to $(\mathscr{H}_\pi \otimes E_2)^K$ with domain $(\mathscr{H}_\pi^\infty \otimes E_1)^K$, which is defined by

$$\pi(D) = \sum_i \pi(Z_i) \otimes C_i. \tag{1.3}$$

Let $E \to \mathbf{H}^n$ be a homogeneous vector bundle defined by the isotropy representation $\sigma\colon K \to \mathrm{GL}(V)$. Let $L\colon C^\infty(E) \to C^\infty(E)$ be a G-invariant integral operator. If we identify $C^\infty(E)$ with $(C^\infty(G) \otimes V)^K$, then the kernel e of L will be an element of $L^2(G \times G) \otimes \mathrm{End}(V)$, which satisfies

$$\begin{aligned} &\text{(i)} \quad e(gg_1, gg_2) = e(g_1, g_2),\\ &\text{(ii)} \quad e(g_1k_1, g_2k_2) = \sigma(k_1^{-1}) \circ e(g_1, g_2) \circ \sigma(k_2),\\ &\qquad\qquad \text{for all } g, g_1, g_2 \in G, k_1, k_2 \in K. \end{aligned}$$

If L is symmetric, then e satisfies symmetry

$$e(g_1, g_2) = e^*(g_2, g_1),$$

where $*$ denotes the adjoint operation in $\mathrm{End}(V)$. Let $h(g) = e(1, g)$. Then $h\colon G \to \mathrm{End}(V)$ and $e(g_1, g_2) = h(g_1^{-1}g_2)$. Moreover, by (ii) we get

$$h(k_1gk_2) = \sigma(k_1) \circ h(g) \circ \sigma(k_2). \tag{1.4}$$

The space of all C^∞-functions $f\colon G \to \mathrm{End}(V)$ which satisfy (1.4) will be denoted by $L_\sigma(G)$. We introduce some spaces of functions: $\mathscr{C}^p(G, \sigma)$, $0 < p < \infty$, is Harish-Chandra's space of p-integrable rapidly decreasing functions of type σ. It is defined as follows. For $D_1, D_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbf{C}})$ and $f \in L_\sigma(G)$ let

$f(D_1; g; D_2)$ have the usual meaning [40, p. 104]. If $r \in \mathbb{R}$, $D_1, D_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, $f \in L_\sigma(G)$ and $0 < p < \infty$, define

$$(1.5) \qquad {}_{D_1}\nu^p_{D_2,r}(f) = \sup_{g \in G} \|f(D_1; g; D_2)\| \left(\frac{(1 + \delta(g))^r}{(\Xi(g))^2} \right)^{1/p},$$

where $\delta(g) = d(x_0, gx_0)$, d the geodesic distance on $\mathbf{H}^n$, and

$$\Xi(g) = \int_K \exp(-\rho(H(gk)))\, dk$$

with the usual notations. Then

$$(1.6) \qquad \begin{aligned} &\mathscr{C}^p(G, \sigma) \\ &\quad = \left\{ f \in L_\sigma(G) \,|\, {}_{D_1}\nu^p_{D_2,r}(f) < \infty, \text{ for all } D_1, D_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}), r \in \mathbb{R} \right\}. \end{aligned}$$

Finally, if $\Gamma \subset G$ is a discrete subgroup, we denote by $R_{\Gamma \backslash G}$ the regular representation of G on $L^2(\Gamma \backslash G)$ or $C^\infty(\Gamma \backslash G)$. Let σ: $K \to \mathrm{GL}(V)$ be a finite-dimensional representation. Then we set

$$(1.7) \qquad \begin{aligned} C^\infty(\Gamma \backslash G, \sigma) &= \left(C^\infty(\Gamma \backslash G) \otimes V\right)^K, \\ L^2(\Gamma \backslash G, \sigma) &= \left(L^2(\Gamma \backslash G) \otimes V\right)^K, \end{aligned}$$

where K acts via $R_{\Gamma \backslash G} \otimes \sigma$.

2. Harmonic analysis on G

In this section we collect some facts about unitary representations of $G = \mathrm{SL}(2, \mathbf{R}))^n$. Let π be any irreducible unitary representation of G. There exist irreducible unitary representations π_i of $\mathrm{SL}(2, \mathbb{R})$ such that $\pi = \otimes_{i=1}^n \pi_i$ [13, Proposition 13.1.8]. Let Θ_π be the character of π. If $\varphi \in C_0^\infty(G)$ is a product

$$\varphi(g) = \prod_{i=1}^n \varphi_i(g_i), \quad \varphi_i \in C_0^\infty(\mathrm{SL}(2, \mathbb{R})),$$

then

$$\Theta_\pi(\varphi) = \prod_{i=1}^n \operatorname{tr} \pi_i(\varphi_i) = \prod_{i=1}^n \Theta_{\pi_i}(\varphi_i)$$

where Θ_{π_i} is the character of π_i. This reduces harmonic analysis on G to that on $\mathrm{SL}(2, \mathbb{R})$. Now, consider $G_0 = \mathrm{SL}(2, \mathbb{R})$, $K_0 = \mathrm{SO}(2)$. We denote by $\Theta_\lambda^\pm$, $\lambda \in \mathbb{R}$, the character of the principal series representation $\pi_{i\lambda}^\pm$ and by $\Theta_n^\pm$, $n \in \mathbb{N}$, the character of the discrete series representation $\pi_n^\pm$. $\Theta_\lambda^\pm$ and $\Theta_n^\pm$ are tempered distributions. This follows from [40, Theorem 8.3.8.2] and the explicit character formulas for $\Theta_\lambda^\pm$ and $\Theta_n^\pm$ [25]. Note that the complementary series representations of G_0 do not have tempered characters [40, p. 174]. The matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_0$$

will be denoted by $k(\theta)$. Let (π, H_π) be an irreducible unitary representation of G_0 and let $H_n \subset H\pi$, $n \in \mathbf{Z}$, be the subspace $\{v \in H\pi | \pi(k(\theta))v = \exp(in\theta)v\}$. Then $\dim H_n \leqslant 1$ (cf. [25]) and the restriction of π to K_0 has a direct sum decomposition $H\pi = \hat{\oplus}_{n\in\mathbf{Z}} H_n$. If $H_n \neq 0$, we choose $v_n \in H_n$ with $\|v_n\| = 1$ and set

$$\Phi_{\pi,n}(g) = \langle \pi(g)v_n, v_n\rangle. \tag{2.1}$$

The spherical trace function $\Phi_{\pi,n}$ satisfies $\Phi_{\pi,n}(1) = 1$ and

$$\Phi_{\pi,n}(k(\theta_1)gk(\theta_2)) = \exp(in(\theta_1 + \theta_2))\Phi_{\pi,n}(g).$$

Moreover, if $f \in C_0^\infty(G_0)$, then

$$\Theta_\pi(f) = \sum_{n\in\mathbf{Z}} \int_{G_0} f(g)\Phi_{\pi,n}(g)\, dg.$$

Assume that $f \in C_0^\infty(G_0)$ satisfies $f(k(\theta_1)gk(\theta_2)) = \exp(-im(\theta_1 + \theta_2))f(g)$. If we use the Cartan decomposition $G_0 = K_0A_0^+K_0$ to calculate $\int_{G_0} f(g)\,\Phi_{\pi,n}(g)\,dg$, we get

$$\Theta_\pi(f) = \int_0^\infty f(a_t)\Phi_{\pi,m}(a_t)\operatorname{sh}(2t)\, dt, \tag{2.2}$$

where $a_t \in A_0^+$. If $\pi = \pi_{i\lambda}^\pm$, $\lambda \in \mathbb{R}$, we set $\Phi_{\lambda,n}^\pm = \Phi_{\pi,n}$ and if $\pi = \pi_m^\pm$, $m \in \mathbb{N}$, $m \geqslant 2$, we set $\Phi_{m,n}^\pm = \Phi_{\pi,n}$. The Casimir operator Ω acts on a principal series representation $\pi_s^\pm$, $s \in i\mathbb{R}$, of G_0 by $\pi_s^\pm(\Omega) = (s^2 - 1)I/4$ and on a discrete series representation $\pi_k^\pm$, $k \in \mathbb{N}$, $k \geqslant 2$, by $\pi_k^\pm(\Omega) = k(k-2)I/4$ [17]. Therefore

$$\begin{aligned} R(\Omega)\Phi_{\lambda,n}^\pm(g) &= -\frac{1+\lambda^2}{4}\Phi_{\lambda,n}^\pm(g), \\ R(\Omega)\Phi_{m,n}^\pm(g) &= \frac{m(m-2)}{4}\Phi_{m,n}^\pm(g). \end{aligned} \tag{2.3}$$

Let $r(z, z')$ be the hyperbolic distance on $\mathbf{H}$. For $g \in G$ let $\theta(g) \in [0, 2\pi)$ be determined by $g = nak(\theta(g))$. By an easy calculation one can show that

$$\Phi_{\pi,n}(g)e^{-in\theta(g)}\left(\frac{gi+i}{i-\overline{gi}}\right)^{-n/2}$$

is bi-invariant under K. The map $\dot{g} \in K\backslash G/K \to \operatorname{ch} r(gi, i) \in [1, \infty)$ is a diffeomorphism. Therefore, there exists $\varphi_{\pi,n} \in C^\infty([1,\infty))$ such that

$$\Phi_{\pi,n}(g) = e^{in\theta(g)}\left(\frac{gi+i}{i-\overline{gi}}\right)^{n/2}\varphi_{\pi,n}(\operatorname{ch} r(gi, i)). \tag{2.4}$$

Note, that $\varphi_{\pi,n}$ satisfies $\varphi_{\pi,n}(1) = 1$. We rewrite (2.3) in terms of $\varphi_{\pi,n}$. It follows as in [16] that $\varphi_{\pi,n}$ satisfies

$$\left(\frac{d^2}{du^2} + \frac{2u}{u^2-1}\frac{d}{du} + \frac{n^2}{2(u+1)(u^2-1)} + \frac{\lambda_\pi}{u^2-1}\right)\varphi_{\pi,n} = 0,$$

where

$$\lambda_\pi = \begin{cases} (1-s^2)/4, & \text{if } \pi = \pi_s^{\pm}, s \in i\mathbb{R}, \\ k(2-k)/4, & \text{if } \pi = \pi_k^{\pm}, k \in \mathbb{N}, k \geqslant 2. \end{cases}$$

Let $\lambda = (1-s^2)/4$, $s \in \mathbb{C}$, and consider this differential equation with λ_π replaced by λ. The unique solution φ which satisfies $\varphi(1) = 1$ is the Legendre function

$$P_{s,n}(u) = \left(\frac{2}{1+u}\right)^{(s+1)/2} F\left(\frac{s+1}{2} - \frac{n}{2}, \frac{s+1}{2} + \frac{n}{2}, 1; \frac{u-1}{u+1}\right). \tag{2.5}$$

F denotes the hypergeometric series. Note that $P_{s,n}$ satisfies $P_{s,n} = P_{-s,n}$ and $P_{s,n} = P_{s,-n}$.

3. Eisenstein series and the spectral resolution

We start by recalling some facts about discrete subgroups of $(\mathrm{SL}(2,\mathbb{R}))^n$. Let $F/\mathbb{Q}$ be a totally real number field of degree n. The ring of integers of F will be denoted by $\mathcal{O}_F$. Let $\mathcal{G}_0$ be the algebraic group $\mathrm{SL}(2)/F$ defined over F and let $\mathcal{G} = R_{F/\mathbb{Q}}\mathcal{G}_0$ be the algebraic group obtained from $\mathcal{G}_0$ by restriction of scalars à la Weil [42]. $\mathcal{G}$ is defined over $\mathbb{Q}$ and has $\mathbb{Q}$-rank one. Let $G = \mathcal{G}(\mathbb{R})$ be the group of real points of $\mathcal{G}$. G is isomorphic to $(\mathrm{SL}(2,\mathbb{R}))^n$. Moreover note that $\mathcal{G}(\mathbb{Q}) \cong \mathrm{SL}(2, F)$ and $\mathcal{G}(\mathbb{Z}) \cong \mathrm{SL}(2, \mathcal{O}_F)$, the Hilbert modular group of the field F. If we identify G with $(\mathrm{SL}(2,\mathbb{R}))^n$, then $\mathrm{SL}(2, F)$ corresponds to a subgroup of $(\mathrm{SL}(2,\mathbb{R}))^n$. This subgroup is obtained by sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, F)$ to

$$\left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix}\right) \in (\mathrm{SL}(2,\mathbb{R}))^n,$$

where $x \mapsto x^{(i)}$ is the ith embedding of F in $\mathbb{R}$. Therefore $\mathrm{SL}(2, \mathcal{O}_F) \cong \mathcal{G}(\mathbb{Z})$ is a discrete subgroup of $(\mathrm{SL}(2,\mathbb{R}))^n$.

A subgroup $\Gamma \subset G$ is called arithmetic if: (1) $\Gamma \subset \mathcal{G}(\mathbb{Q})$ and (2) Γ is commensurable with $\mathcal{G}(\mathbb{Z})$.

Thus, Γ is a subgroup of $\mathrm{SL}(2, F)$ which is commensurable with the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_F)$. An arithmetic subgroup $\Gamma \subset G$ has the following properties:

(1) Γ is a discrete irreducible subgroup of G.

(2) $\mathrm{Vol}(\Gamma \setminus G) < \infty$.

(3) Γ has at least one parabolic fixed point on $\overline{\mathbf{H}}^n$.

Moreover, $\mathrm{rank}(\Gamma) = 1$ (see [32] for the definition of $\mathrm{rank}(\Gamma)$). On the other hand, Selberg's rigidity theorem [37] states that any subgroup $\Gamma \subset G$ which satisfies (1)–(3) is conjugate in G to a group commensurable with the Hilbert modular group of some totally real number field F of degree n.

Let $\Gamma \subset G$ be an arithmetic subgroup. We discuss some aspects of the spectral resolution of the regular representation of G on $L^2(\Gamma \setminus G)$. We start with the theory of Eisenstein series. The basic references are [20], [26] and [33]. For all details we refer the reader to these references. Since Γ is arithmetic, we can use Harish-Chandra's approach [20]. Eisenstein series are associated with the Γ-cuspidal parabolic subgroups $P \subset G$ [26], [33]. In our case one can describe the Γ-cuspidal subgroups $P \subset G$ as follows. Let $\mathscr{B} \subset \mathrm{SL}(2)$ be a Borel subgroup defined over F and let $\mathscr{P} = R_{F/\mathbf{Q}}\mathscr{B}$. $\mathscr{P} \subset \mathscr{G}$ is a Borel subgroup defined over $\mathbf{Q}$. Set $P = \mathscr{P}(\mathbb{R})$. Then $P \subset G$ is a Γ-cuspidal subgroup and all Γ-cuspidal parabolic subgroups arise in this way (cf. [20]). Since $\mathrm{rank}_{\mathbf{Q}} \mathscr{G} = 1$, all Γ-cuspidal subgroups of G are Γ-percuspidal (cf. [32]). We denote the unipotent radical of $\mathscr{P}$ by $\mathscr{U}$. Let $\mathscr{T} \subset \mathscr{P}$ be a maximal torus of $\mathscr{P}$. $\mathscr{T}$ is defined over $\mathbf{Q}$ and $\mathscr{P} = \mathscr{U} \cdot \mathscr{T}$. Let $\mathscr{A} \subset \mathscr{T}$ be the $\mathbf{Q}$-split component of $\mathscr{T}$ and $\mathscr{M} \subset \mathscr{T}$ the anisotropic subtorus. Then $\mathscr{T} = \mathscr{A} \cdot \mathscr{M}$ and $\mathscr{P} = \mathscr{U} \cdot \mathscr{A} \cdot \mathscr{M}$. Consider the corresponding groups of real points $P = \mathscr{P}(\mathbb{R})$, $M = \mathscr{M}(\mathbb{R}), \cdots$. Then $P = UAM$. U is the unipotent radical of P. We call this decomposition of P Langlands decomposition of P over $\mathbf{Q}$. The group UM has the following alternative description. Let α: $\mathscr{B} \to \mathbb{G}_m$ be the positive root. α induces a homomorphism

$$\alpha_\infty : P = \mathscr{B}(F \otimes_{\mathbf{Q}} \mathbb{R}) \to \mathbb{G}_m(F \otimes_{\mathbf{Q}} \mathbb{R})$$

and if we compose α_∞ with the norm homomorphism

$$\nu : (F \otimes_{\mathbf{Q}} \mathbb{R})^\times \to (\mathbb{R}^+)^\times$$

we get a homomorphism

$$|\alpha| : P \to (\mathbb{R}^+)^\times .$$

Let ${}^0P = \{ p \in P \,|\, |\alpha|(p) = 1\}$. Then $UM = {}^0P$ and $U \backslash {}^0P = U \backslash UM = M$. Therefore, we get a natural homomorphism $\pi_{P|M}$: ${}^0P \to M$. Let $K_M = \pi_{P|M}(K \cap {}^0P)$, where $K = (\mathrm{SO}(2))^n$. K_M is a maximal compact subgroup of M

and $\pi_{P|M}$ is an isomorphism of $K \cap {}^0P$ onto K_M. Since T splits over $\mathbb{R}$, it follows that K_M is finite. In particular $(K_M)^0 = \{1\}$. Let $X_M = M/K_M$. Then $X_M = M^0/(K_M)^0 = M^0$. Further note that $\Gamma \cap P = \Gamma \cap {}^0P$ and $\Gamma \cap U \backslash U$ is compact. In our case U is abelian and isomorphic to $\mathbb{R}^n$. $\Gamma \cap U$ can be considered as a discrete subgroup of translations of $\mathbb{R}^n$. Therefore $\Gamma \cap U \backslash U$ is an n-dimensional torus $(S^1)^n$. Let

$$\Gamma_M = \pi_{P|M}(\Gamma \cap {}^0P).$$

Γ_M is an arithmetic subgroup of M. Since $\mathscr{M}$ has $\mathbb{Q}$-rank zero, it follows that $\Gamma_M \backslash M$ is compact. Let

$$\Gamma_M^0 = (\Gamma_M \cdot K_M) \cap M^0. \tag{3.1}$$

Since M is commutative, we get $\Gamma_M \backslash X_M = \Gamma_M^0 \backslash M^0$. Therefore, $\Gamma_M \backslash X_M$ is a torus of dimension $n - 1$. We have an exact sequence

$$1 \to \Gamma \cap U \to \Gamma \cap P \to \Gamma_M \to 1.$$

Let ${}^0X = {}^0Px_0$, where $x_0 \in \mathbf{H}^n$ is the coset eK. ${}^0X \subset \mathbf{H}^n$ is a subspace of codimension one. If we use the above remarks, it follows that

$$\pi_{P|M}\colon \Gamma \cap P \backslash {}^0X \to \Gamma_M \backslash X_M$$

is a locally trivial fibration over $\Gamma_M \backslash X_M \cong (S^1)^{n-1}$ with fibre $\Gamma \cap U \backslash U \cong (S^1)^n$. This fibration has a description in terms of the number field F [38]. We describe Γ_M and $\Gamma \cap U$. There exists a unique $x \in (P_1(\mathbb{R}))^n$ such that P is the stabilizer of x in G. x is a parabolic fixed point of Γ and $\Gamma \cap P$ is the stabilizer of x in Γ. Let P_∞ be the stabilizer of $\infty \in (P_1(\mathbb{R}))^n$ in G. There exists $\rho \in \mathrm{SL}(2, F)$ such that ${}^\rho P = P_\infty$, where ${}^\rho P$ denotes conjugation with ρ. Thus $\rho x = \infty$ and $x \in P_1(F)$. The group ${}^\rho\Gamma$ is again arithmetic and commensurable with Γ. Further ${}^\rho(\Gamma_x) = ({}^\rho\Gamma)_\infty$.

Lemma 3.2. *Let $\rho \in \mathrm{SL}(2, F)$ be such that ${}^\rho P = P_\infty$. There exists a subgroup $V_1 \subset \mathcal{O}_F^*$ of finite index such that*

$$\rho\Gamma_M\rho^{-1} = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \middle| \varepsilon \in V_1 \right\}.$$

Proof. Let $U_\infty A_\infty M_\infty = P_\infty$ be the Langlands decomposition of P_∞. Then ${}^\rho(\Gamma_M) = ({}^\rho\Gamma)_{M_\infty}$. Since ${}^\rho\Gamma$ is arithmetic, we can assume that ${}^\rho\Gamma = \Gamma$ and $P = P_\infty$. Let $\delta \in \Gamma_M$, $\delta = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. Since $\Gamma_M = M \cap (\Gamma U)$, there exists $\gamma \in \Gamma$ such that $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a, b \in F$. Since Γ is arithmetic, $\Gamma/\Gamma \cap \mathrm{SL}(2, \mathcal{O}_F)$ is finite. Therefore, there exists $n \in \mathbb{N}$ such that $\gamma^n = \left(\begin{smallmatrix} a^n & * \\ 0 & a^{-n} \end{smallmatrix}\right)$ is in $\Gamma \cap \mathrm{SL}(2, \mathcal{O}_F)$, i.e., $a^n, a^{-n} \in \mathcal{O}_F$. Hence, a and a^{-1} are algebraic integers. Since $a \in F$, we get

$a \in \mathcal{O}_F^*$. Let

$$V_1 = \left\{ \varepsilon \in \mathcal{O}_F^* \middle| \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \in \Gamma_M \right\}.$$

Since $\Gamma_M \backslash M$ is compact, Γ_M has rank $n-1$. Therefore, it follows from Dirichlet's unit theorem that $V_1 \subset \mathcal{O}_F^*$ is a subgroup of finite index. Note that V_1 is independent of the particular choice of ρ. q.e.d.

The group ${}^{\rho}(\Gamma \cap U) = ({}^{\rho}\Gamma) \cap U_\infty$ consists of matrices $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ with $\mu \in F$. Since $\Gamma \cap U \backslash U$ is compact, the set of all such $\mu \in F$ forms an additive subgroup $\mathsf{M} \subset F$ of rank n, i.e. $\mathsf{M} \subset F$ is a lattice. The lattice M depends on the choice of ρ, but the strict equivalence class of M is uniquely determined (M and M' are called strictly equivalent, if there exists $\alpha \in F$, α totally positive, such that $\mathsf{M} = \alpha \mathsf{M}'$). Thus we have

Lemma 3.3. *For each Γ-cuspidal parabolic subgroup $P \subset G$ there exists $\rho \in \mathrm{SL}(2, F)$, a lattice $\mathsf{M} \subset F$ and a subgroup $V_1 \subset \mathcal{O}_F^*$ of finite index such that ${}^{\rho}P = P_\infty$ and ${}^{\rho}(\Gamma \cap P)$ is an extension of M by V_1*

$$0 \to \mathsf{M} \to {}^{\rho}(\Gamma \cap P) \to V_1 \to 1,$$

where

$$\mu \in \mathsf{M} \mapsto \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in {}^{\rho}(\Gamma \cap P)$$

and

$$\begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} \in {}^{\rho}(\Gamma \cap P) \mapsto \varepsilon \in V_1.$$

Since $\Gamma \cap P$ is the normalizer of $\Gamma \cap U$ in Γ, we get an action of Γ_M on $\Gamma \cap U$. This action corresponds to the action of V_1 on M which is given by $\mu \mapsto \varepsilon^2 \mu$, $\varepsilon \in V_1$, $\mu \in \mathsf{M}$. Let U_{M}^+ be the group of all totally positive units ε of $\mathcal{O}_F$ such that $\varepsilon \mathsf{M} = \mathsf{M}$. The group U_{M}^+ is abelian of rank $n-1$ [11] and $(V_1)^2 \subset U_{\mathsf{M}}^+$ is a subgroup of finite index. In general, ${}^{\rho}(\Gamma \cap P)$ is not the semidirect product of M and V_1 with respect to the action of V_1 on M defined above. However $H^2(V_1, \mathsf{M})$ is finite.

The fibration $\Gamma \cap P \backslash {}^0X \to \Gamma_M \backslash X_M$ is equivalent to the fibration

$$(3.4) \qquad {}^{\rho}(\Gamma \cap P) \backslash {}^{\rho}({}^0X) \to {}^{\rho}(\Gamma_M) \backslash {}^{\rho}(X_M).$$

If $P_\infty = U_\infty A_\infty M_\infty$ is the Langlands decomposition of P_∞ over $\mathbf{Q}$, then ${}^{\rho}(X_M) = X_{M_\infty}$, ${}^{\rho}(\Gamma_M) = ({}^{\rho}\Gamma)_{M_\infty}$ and ${}^{\rho}({}^0X)$ is the orbit of x_0 under $U_\infty M_\infty$. This is the subspace

$$W = \left\{ z \in \mathbf{H}^n \middle| \prod_{i=1}^{n} \mathrm{Im}(z_i) = 1 \right\}.$$

Let

$$Y = \left\{ y \in (\mathbb{R}^+)^n \,\middle|\, \prod_{i=1}^{n} y_i = 1 \right\}.$$

$(M_\infty)^0$ is isomorphic to Y. This isomorphism is given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in (M_\infty)^0 \mapsto (\lambda_1^2, \cdots, \lambda_n^2) \in Y.$$

Thus $X_{M_\infty} \cong (M_\infty)^0 \cong Y$. The projection $W \to Y$ is given by $z_j \mapsto \operatorname{Im} z_j$. The group of units V_1 acts on Y by $\varepsilon \cdot y = ((\varepsilon^{(1)})^2 y_1, \cdots, (\varepsilon^{(n)})^2 y_n)$, $\varepsilon \in V_1$. Let $\mathsf{V} = (V_1)^2$. V can be identified with a discrete subgroup of Y and ${}^{\rho}(\Gamma_M) \backslash X_{M_\infty} \cong \mathsf{V} \backslash Y$. U_∞ is isomorphic to $\mathbb{R}^n$. By sending $\mu \in \mathsf{M}$ to $(\mu^{(1)}, \cdots, \mu^{(n)}) \in \mathbb{R}^n$, the lattice M is mapped isomorphically to a lattice in $\mathbb{R}^n$ which we also denote by M. The exact sequence of Lemma 3.3 shows that the fibration (3.4) is equivalent to the fibration

$$(3.5) \qquad {}^{\rho}(\Gamma \cap P) \backslash W \to \mathsf{V} \backslash Y$$

with fibre $\mathsf{M} \backslash \mathbb{R}^n$. Let $\tilde{\Gamma}$ be the image of Γ in $(\mathrm{PL}_2^+(\mathbb{R}))^n$. Then we have a corresponding extension

$$(3.6) \qquad 0 \to \mathsf{M} \to {}^{\rho}(\widetilde{\Gamma \cap P}) \to \mathsf{V} \to 1,$$

where

$$\mu \in \mathsf{M} \mapsto \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in {}^{\rho}(\widetilde{\Gamma \cap P})$$

and

$$\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \in {}^{\rho}(\widetilde{\Gamma \cap P}) \mapsto \varepsilon \in \mathsf{V}.$$

This is the description given in [24].

Now we turn to the theory of Eisenstein series on $\Gamma \backslash G$. We recall the general context in which Eisenstein series are defined. Let $\sigma\colon K \to \mathrm{GL}(V)$ be a finite dimensional representation. Let $P \subset G$ be a Γ-cuspidal parabolic subgroup as above with Langlands decomposition $P = UAM$ over $\mathbb{Q}$. Let $\mathfrak{Z}_M$ be the center of the universal enveloping algebra of the Lie algebra $\mathfrak{m}$ of M and consider a representation

$$\chi\colon \mathfrak{Z}_M \to \operatorname{Hom}_{K_M}(V, V)$$

of $\mathfrak{Z}_M$. We consider the vector space

$$\mathscr{A}(\Gamma_M \backslash M, \sigma, \chi) = \{ \psi\colon \Gamma_M \backslash M \to V \mid \psi \in C^\infty, \psi(mk^{-1}) = \sigma(k)\psi(m),$$
$$k \in K_M, (Z\psi)(m) = \chi(Z)(\psi(m)), Z \in \mathfrak{Z}_M \}.$$

Since $\Gamma_M \backslash M$ is compact, we need no growth condition and $\mathscr{A}$ is a space of automorphic forms in the sense of Langlands [20]. $\mathscr{A}$ is finite dimensional [20] and coincides with the space $L^2(\Gamma_M \backslash M, \sigma, \chi)$ of cusp forms of type (σ, χ). We extend $\Phi \in \mathscr{A}(\Gamma_M \backslash M, \sigma, \chi)$ to a function

(3.7) $$\Phi_s \colon \Gamma \cap P \backslash G \to V,$$

depending on $s \in \mathbf{C}$, by $\Phi_s(uamk^{-1}) = \sigma(k)\Phi(m)e^{(s+1)\ln a}$, where $u \in U$, $a \in A^+$, $m \in M$, $k \in K$ and $\ln\colon A^+ \xrightarrow{\sim} \mathrm{Lie}(A) = \mathbf{R}$. Φ_s is in $C^\infty(\Gamma \cap P \backslash G, \sigma)$. The Eisenstein series attached to P and Φ is defined as

(3.8) $$E(P, \Phi, s, g) = \sum_{\Gamma \cap P \backslash \Gamma} \Phi_s(\gamma g)$$

for $\mathrm{Re}(s) > 1$. $E(P, \Phi, s, g)$ has a meromorphic continuation onto the entire s-plane. As a function of g it belongs to $C^\infty\,(\Gamma \backslash G, \sigma)$ and it is slowly increasing on any Siegel domain [20]. Let $\mathfrak{Z}_1 = \mathfrak{Z}(\mathfrak{a}_{\mathbf{C}}\mathfrak{m}_{\mathbf{C}})$. In our case we have $\mathfrak{Z}_1 = S(\mathfrak{a}_{\mathbf{C}})S(\mathfrak{m}_{\mathbf{C}})$. Let $\mu\colon \mathfrak{Z} \to \mathfrak{Z}_1$ be the Harish-Chandra homomorphism [20, I, §6]. If $Z \in \mathfrak{Z}$ let $\mu(Z) = \sum_{i=1}^r \zeta_i q_i$ with $\zeta_i \in S(\mathfrak{m}_{\mathbf{C}})$ and $q_i \in S(\mathfrak{a}_{\mathbf{C}})$. For $s \in \mathbf{C}$ let

$$\mu_s(Z) = \sum_{i=1}^r q_i(s)\zeta_i.$$

Then the Eisenstein series satisfies

(3.9) $$R(Z)E(P, \Phi, s, g) = \chi(\mu_s(Z))E(P, \Phi, s, g)$$

for each $Z \in \mathfrak{Z}$ [20, II, §2].

Let $P_i \subset G$, $i = 1, 2$, be two Γ-cuspidal parabolic subgroups with Langlands decomposition $P_i = U_i A_i M_i$ defined over $\mathbf{Q}$. Let $\Phi \in L^1(\Gamma_{M_1} \backslash M_1, \sigma, \chi)$ and consider the Eisenstein series $E(P_1, \Phi, s, g)$ attached to P_1 and Φ. The constant term of $E(P_1, \Phi, s, g)$ along P_2 is defined as

(3.10) $$E^{P_2}(P_1, \Phi, s, g) = \int_{\Gamma \cap U_2 \backslash U_2} E(P_1, \Phi, s, u_2 g)\, du_2,$$

where the Haar measure on U_2 is normalized by the condition $\mathrm{Vol}_{du_2}(\Gamma \cap U_2 \backslash U_2) = 1$. For all facts concerning the theory of the constant term and the functional equations satisfied by the Eisenstein series we refer to [20], [26]. For simplicity we shall assume in the sequel that all Γ-cuspidal parabolic subgroups are Γ-conjugate. Thus, we can restrict ourselves to the case $P_1 = P_2 = P$. Since P is fixed, we shall write $E(\Phi, s, g)$ instead of $E(P, \Phi, s, g)$. The Weyl group $W(A)$ of (G, A) operates in a natural manner on the group $\hat{\mathfrak{Z}}_M$ of characters of $\mathfrak{Z}_M$. Let $w \in W(A)$ be the nontrivial element. There exists a linear map

(3.11) $$C(\chi, \sigma, s)\colon L^2(\Gamma_M \backslash M, \sigma, \chi) \to L^2(\Gamma_M \backslash M, \sigma, {}^w\chi),$$

which is meromorphic in $s \in \mathbf{C}$, such that

$$E^P(\Phi, s, g) = \Phi_s(g) + (C(\chi, \sigma, s)\Phi)_{-s}(g) \tag{3.12}$$

for each $\Phi \in L^2(\Gamma_M \backslash M, \sigma, \chi)$. There is an orthogonal sum decomposition

$$L^2(\Gamma_M \backslash M) = \bigoplus_{\tau \in \hat{K}} m(\tau, \Gamma_M) H_\tau, \tag{3.13}$$

where $m(\tau, \Gamma_M) \leqslant 1$ and $H_\tau = \mathbf{C}\tau$. Since $M = M^0 \times K_M$, a character τ is uniquely determined by $d\tau$ and $\tau|_{K_M}$. From this one can conclude that $\dim L^2(\Gamma_M \backslash M, \sigma, \chi) \leqslant 1$ and $L^2(\Gamma_M \backslash M, \sigma, \chi)$ coincides with one of the spaces H_τ. If $w \in W(A)$ is the nontrivial element, then we have $w^{-1}mw = m^{-1}$, $m \in M$. Therefore, ${}^w\chi \neq \chi$ if $\chi \neq 0$. Thus, all characters $\chi \neq 0$ are unramified. This is an important observation because it implies

Lemma 3.14. *Let $\sigma \in \hat{K}$, $\chi \in \hat{\mathfrak{Z}}_M$ and $\Phi \in L^2(\Gamma_M \backslash M, \sigma, \chi)$. If $\chi \neq 0$, then $E(\Phi, s, g)$ is holomorphic in the half-plane* $\mathrm{Re}(s) > 0$.

Proof. The poles of $E(\Phi, s, g)$ and $C(\chi, \sigma, s)$ coincide. If so, $\mathrm{Re}(s_0) > 0$, is a pole of $C(\chi, \sigma, s)$, then s_0 is simple and $s_0 \in (0, 1]$, [20, IV]. Let

$$\gamma(s_0) = -2\pi \operatorname*{Res}_{s=s_0}\left(C(\chi, \sigma, s) \oplus C({}^w\chi, \sigma, s)\right)$$

and consider $\gamma(s_0)$ as an operator in

$$L^2((\Gamma_M \backslash M, \sigma, \chi) \oplus L^2(\Gamma_M \backslash M, \sigma, {}^w\chi).$$

$\gamma(s_0)$ is a positive semidefinite operator (see [1]). The proof of this fact is similar to the proof of Lemma 2.1 in [41]. On the other hand, $\gamma(s_0)$ maps $L^2(\Gamma_M \backslash M, \sigma, \chi)$ into $L^2((\Gamma_M \backslash M, \sigma, {}^w\chi)$ and vice versa. If ${}^w\chi \neq \chi$, then $L^2(\Gamma_M \backslash M, \sigma, \chi)$ and $L^2(\Gamma_M \backslash M, \sigma, {}^w\chi)$ are orthogonal subspaces of $L^2(\Gamma_M \backslash M)$. Thus $\mathrm{Tr}(\gamma(s_0)) = 0$. But $\gamma(s_0)$ is positive semidefinite. Hence $\gamma(s_0) = 0$. There is another way to see this by using (3.9). If $s_0 \in (0, 1]$ is a pole of $E(\Phi, s, g)$, then it follows from (3.9) and (3.12) that $\mathrm{Res}_{s=s_0} E(\Phi, s, g)$ is a nonzero L^2-eigenfunction of $\mathfrak{Z}$ with character $\chi(\mu_{s_0}(\cdot))$. Let Ω_j be the Casimir element of the jth component of G. If we appeal to Corollary 1.2 of [29], it follows that $R(\Omega_j), j = 1, \cdots, n$, are self-adjoint operators in $L^2(\Gamma \backslash G, \sigma)$. On the other hand, an easy computation shows that $\chi(\mu_{s_0}(\Omega_j))$ is real for all j iff $\chi = 0$. q.e.d.

We shall discuss now the Eisenstein series which occur in our situation. Let P_∞ be the stabilizer of the cusp ∞. We shall describe only the Eisenstein series which are associated to P_∞. The others can easily be related to these. For simplicity we delete the index and write P instead of P_∞. Let $P = UAM$ be the Langlands decomposition over $\mathbf{Q}$. The basic representation is $\Lambda^* \mathrm{Ad}\, \mathfrak{p}$: $K \to \mathrm{GL}(\Lambda^* \mathfrak{p}_{\mathbf{C}})$. If $\sigma \in \hat{K}$ occurs in $\Lambda^* \mathrm{Ad}_{\mathfrak{p}}$, then it follows from (1.1) that $\sigma|_{K_M} = 1$. Therefore, we shall restrict ourselves to characters $\sigma \in \hat{K}$ with

$\sigma|_{K_M} = 1$. In this case we have

$$L^2(\Gamma_M \backslash M, \sigma, \chi) = L^2(\Gamma_M^0 \backslash M^0, \chi),$$

where Γ_M^0 is defined by (3.1). Note that $K_{(M^0)} = \{1\}$ and $\Gamma_M K_M \backslash M \cong \Gamma_M^0 \backslash M^0$. M^0 is isomorphic to $\Lambda = \{\lambda \in (\mathbb{R}^+)^n | \prod_{i=1}^n \lambda_i = 1\}$, where $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right) \mapsto (\lambda_1, \cdots, \lambda_n)$. According to Lemma 3.2 there exists a subgroup $V_1 \subset \mathcal{O}_F^*$ of finite index such that

$$\Gamma_M = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \middle| \varepsilon \in V_1 \right\}.$$

Thus, Γ_M^0 corresponds to the subgroup $V_1' = \{(|\varepsilon^{(1)}|, \cdots, |\varepsilon^{(n)}|) | \varepsilon \in V_1\}$ of Λ. V_1' is free abelian of rank $n-1$ by Dirichlet's unit theorem. Let $H \subset \mathbb{R}^{n-1}$ be the hyperplane $\sum x_i = 0$ and let $L \subset H$ be the additive subgroup of rank $n-1$ which corresponds to V_1' under the map $\lambda_j \mapsto \log \lambda_j$. Then $\Gamma_M^0 \backslash M^0 \cong L \backslash H \cong (S^1)^{n-1}$ and

$$L^2(\Gamma_M^0 \backslash M^0) = \bigoplus_{\tau \in \hat{M}^0} m(\tau, \Gamma_M^0) H_\tau,$$

where τ is a character, $H_\tau = \mathbb{C}\tau$ and $m(\tau, \Gamma_M^0) \leqslant 1$. Each τ is an eigenfunction of the Laplacian Δ_M of the torus $\Gamma_M^0 \backslash M^0$. The characters τ with $m(\tau, \Gamma_M^0) \neq 0$ can be described as follows.

Let $\{\varepsilon_1, \cdots, \varepsilon_{n-1}\}$ be a system of independent generators of the free part of V_1 and let $l_{ij} = \log|\varepsilon_i^{(j)}|$, $1 \leqslant i \leqslant n-1$, $l_{nj} = 1/n$, $j = 1, \cdots, n$. The matrix (l_{ij}) has rank n. A fundamental domain of $L \subset H$ is the set

$$\left\{ x \in \mathbb{R}^n | x_j = \sum_{i=1}^{n-1} l_{ij} u_i, 0 < u_i < 1 \right\}.$$

Let

$$B = (l_{ij})^{-1}. \tag{3.15}$$

The isomorphism $L \backslash H \cong (S^1)^{n-1}$ is given by

$$u_k = \log\left(\sum_{j=1}^n (\lambda_j)^{b_{jk}} \right).$$

If $\omega \in \mathbb{Z}^{n-1}$ let $(B\omega)_j$ be the jth component of $B\left[\begin{smallmatrix}\omega\\0\end{smallmatrix}\right]$. Then

$$e^{2\pi i \langle \omega, u \rangle} = \prod_{j=1}^n (\lambda_j)^{2\pi i (B\omega)_j}.$$

We define the character $\tau_\omega \colon M \to \mathbb{C}^\times$ by

$$\tau_\omega\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right) = \prod_{j=1}^n |\lambda_j|^{2\pi i (B\omega)_j}. \tag{3.16}$$

Then $\tau_\omega(\Gamma_M K_M) = 1$. Thus $m(\tau_\omega, \Gamma_M^0) = 1$ and each character τ with $m(\tau, \Gamma_M^0) \neq 0$ is of this form. Let $\sigma \in \hat{K}$, $\sigma|_{K_M} = 1$ and $\omega \in \mathbf{Z}^{n-1}$. τ_ω determines a character $\chi_\omega \in \hat{\mathfrak{Z}}_M$ and $\tau_\omega \in L^2(\Gamma_M \backslash M, \sigma, \chi_\omega)$. We shall denote the Eisenstein series which is associated to (τ_ω, σ) by $E_\omega(\sigma, s, g)$. The constant term of $E_\omega(\sigma, s, g)$ along P is of the form

$$E_\omega^P(\sigma, s, g) = \Phi_{\omega,s}(g) + C_\omega(\sigma, s)\Phi_{-\omega,-s}(g), \tag{3.17}$$

where $\Phi_{\omega,s}$ is the function which is defined by $\Phi_{\omega,s}(uamk) = \sigma(k)^{-1}\tau_\omega(m)e^{(s+1)\ln a}$ and $C_\omega(\sigma, s)$ is a meromorphic function of $s \in \mathbf{C}$. Let $\Omega \in \mathfrak{Z}$ be the Casimir element. It follows from (3.9) by an easy computation that

$$R(\Omega)E_\omega(\sigma, s, g) = \left(n\frac{s^2-1}{4} - \pi^2 \sum_{j=1}^{n} (B\omega)_j^2\right) E_\omega(\sigma, s, g). \tag{3.18}$$

Let $E_j^\pm$, $j = 1,\cdots,n$, be the basis of $\mathfrak{p}_{\mathbf{C}}$ defined by (1.1). We set $E_j = -iE_j^+/4, j = 1,\cdots,n$. Then $\bar{E}_j = E_j^-/4$. We identify $\mathfrak{p}_{\mathbf{C}}$ with its dual $\mathfrak{p}_{\mathbf{C}}^*$ via the Killing form and we introduce the following notations: By $I, J, \cdots$ we denote subsets $\{i_1,\cdots,i_p\}$ of $\{1,\cdots,n\}$ with $i_1 < i_2 < \cdots < i_p$. The cardinality of I will be denoted by $|I|$. For I, J as above we set

$$v_{I,J} = E_{i_1} \wedge \cdots \wedge E_{i_p} \wedge \bar{E}_{j_1} \wedge \cdots \wedge \bar{E}_{j_q}. \tag{3.19}$$

Let χ: SO(2) $\to \mathbf{C}^\times$ be defined by $\chi(k(\theta)) = e^{2\pi i\theta}$ and put

$$\sigma_{I,J}(k) = \prod_{\iota\in I}\chi(k_\iota)\prod_{j\in J}\bar{\chi}(k_j). \tag{3.20}$$

The set $\{v_{I,J}|I, J \subset \{1,\cdots,n\}\}$ forms a basis of $\Lambda^*\mathfrak{p}_{\mathbf{C}}^*$ which consists of common eigenvectors of $\{\Lambda^* \operatorname{Ad}_{\mathfrak{p}}^*(k)|k \in K\}$ with $\Lambda^* \operatorname{Ad}_{\mathfrak{p}}^*(k)v_{I,J} = \sigma_{I,J}(k)v_{I,J}$. The Eisenstein series $E_\omega(\sigma_{I,J}, s, g)$ corresponds to a Γ-invariant differential form on $\mathbf{H}^n$ of bidegree (p, q), $p = |I|$, $q = |J|$. For $\gamma \in G$, and $z \in \mathbf{H}^n$ let

$$j_{I,J}(\gamma, z) = \prod_{i\in I} j(\gamma_i, z_i)\prod_{\iota\in J} j(\gamma_\iota, z_\iota)^{-1},$$

where $j(\gamma, z) = (cz + d)/(c\bar{z} + d)$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and $z \in \mathbf{H}$. Then

$$E_\omega(\sigma_{I,J}, s, z) = f_{\omega,I,J}(s, z)\frac{dz^I}{y_I} \wedge \frac{d\bar{z}^J}{y_J},$$

where $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, $y_I = \prod_{i\in I} y_i$ and the function $f_{\omega,I,J}$ satisfies

$$f_{\omega,I,J}(\gamma z) = j_{I,J}(\gamma, z)f_{\omega,I,J}(z)$$

for $\gamma \in \Gamma$. To describe $f_{\omega,I,J}$ we use the coordinates (x, y, θ) on G introduced by (1.2). It is easy to see that

$$f_{\omega,I,J}(s, z) = \sum_{\Gamma\cap P\backslash\Gamma} j_{I,J}(z, \gamma)^{-1} \prod_{j=1}^{n} y_j(\gamma z)^{\pi i(B\omega)_j + (s+1)/2}.$$

We calculate the constant term of $E_\omega(\sigma_{I,J}, s, g)$ along P. Let $\Phi_{\omega,s}$ be defined by

$$\Phi_{\omega,s}(uamk) = \sigma_{I,J}(k)^{-1}\tau_\omega(m)e^{(s+1)\ln a}$$

(for simplicity we suppress the indices I, J). The constant term is given by

$$E^P_\omega(\sigma_{I,J}, s, g) = \sum_{\Gamma\cap P\backslash\Gamma/\Gamma\cap U} \Phi_{s,\gamma}(g),$$

where

$$\Phi_{s,\gamma}(g) = \int_{(\Gamma\cap U\cap {}^\gamma P)\backslash U} \Phi_{\omega,s}(\gamma ug)\, du$$

[20, II]. If $(\Gamma \cap P)\gamma(\Gamma \cap U)$ is the trivial double coset, then we get $\Phi_{s,\gamma}(g) = \Phi_{\omega,s}(g)$. If γ represents a nontrivial double coset, then $U \cap {}^\gamma P = \{1\}$ and $\Phi_{s,\gamma}(g) = \int_U \Phi_{\omega,s}(\gamma ug)\, du$. We have to insert the explicit expression for $\Phi_{\omega,s}$ and compute the resulting integral. To describe the final result, we introduce some notation. Let $S = I \cup J - (I \cap J)$, $\bar{S} = \{1,\cdots,n\} - S$ and $d = |S|$. For $\omega \in \mathbb{Z}^{n-1}$ we define the Γ-factor

$$\Gamma_{\omega,I,J}(s) = \prod_{j\in S} 2^{-2\pi i(B\omega)_j} \frac{\Gamma(s + 2\pi i(B\omega)_j)}{\Gamma((s-1)/2 + \pi i(B\omega)_j)\Gamma((s+3)/2 + \pi i(B\omega)_j)}$$

$$\cdot \prod_{k\in\bar{S}} \frac{\Gamma(s/2 + \pi i(B\omega)_j)}{\Gamma((s+1)/2 + \pi i(B\omega)_k)}. \tag{3.21}$$

For $\mu \in F^\times$ let

$$\chi_\omega(\mu) = \prod_{j=1}^{n} |\mu^{(j)}|^{-2\pi i(B\omega)_j}. \tag{3.22}$$

If Γ is the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_F)$ of a field F with class number one, then the cusp at infinity is of type $(\mathcal{O}_F, \mathcal{O}_F^{*2})$. In this case χ_ω is a Grössencharacter of the field F as defined by Hecke [21]. Now, let

$$C_\omega(\sigma_{I,J}, s) = \frac{(-1)^d 2^{(1-s)d}\pi^{(n+d)/2}}{\mathrm{Vol}(\Gamma\cap U\backslash U)}\Gamma_{\omega,I,J}(s) \sum_{\gamma\in\Gamma\cap P\backslash\Gamma/\Gamma\cap U} \chi_\omega(c)|N(c)|^{-(s+1)}, \tag{3.23}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $N(c) = \prod_{j=1}^{n} c^{(j)}$. Then the constant term is given by

$$E_\omega^P(\sigma_{I,J}, s, g) = \Phi_{\omega,s}(g) + C_\omega(\sigma_{I,J}, s)\Phi_{-\omega,-s}(g).$$

By similar calculations one can determine the constant term along any other Γ-cuspidal parabolic subgroup $P \subset G$.

If Γ is a principal congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$, then one can describe the intertwining operator $C(s)$ explicitly in terms of L-series associated with the field F. We consider the simplest example. Assume that $F/\mathbf{Q}$ is a totally real number field with class number one and let $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$. Then there exists only one Γ-conjugacy class of Γ-cuspidal parabolic subgroups of G. The stabilizer of ∞ in Γ is

$$\Gamma \cap P_\infty = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} \middle| \varepsilon \in \mathcal{O}_F^*, \mu \in \mathcal{O}_F \right\}.$$

The cusp ∞ is of type $(\mathcal{O}_F, \mathcal{O}_F^{*2})$ in the sense of [24] and the characters χ_ω, $\omega \in \mathbf{Z}^{n-1}$, defined by (3.22), coincide with Hecke's Grössencharacters [21]. Let us denote the infinite sum in (3.23) by $\xi(s)$. An easy calculation gives

$$\xi(s) = \frac{L(s, \chi_\omega)}{L(s+1, \chi_\omega)},$$

where

$$L(s, \chi_\omega) = \sum_{\mu \in (\mathcal{O}_F - 0)/\mathcal{O}_F^*} \chi_\omega(\mu) |N(\mu)|^{-s}.$$

Finally note that $\mathrm{Vol}(\Gamma \cap U \setminus U) = (D_{F/\mathbf{Q}})^{1/2}$, where $D_{F/\mathbf{Q}}$ is the discriminant of the field F. By these remarks we obtain

$$(3.24) \quad C_\omega(\sigma_{I,J}, s) = \frac{(-1)^d 2^{(1-s)d} \pi^{(n+d)/2}}{(D_{F/\mathbf{Q}})^{1/2}} \Gamma_{\omega,I,J}(s) \frac{L(s, \chi_\omega)}{L(s+1, \chi_\omega)}.$$

Let $R_{\Gamma\setminus G}$ be the right regular representation of G on the Hilbert space $L^2(\Gamma \setminus G)$. Using the theory of Eisenstein series one gets an orthogonal sum decomposition

$$(3.25) \quad L^2(\Gamma \setminus G) = L^2_d(\Gamma \setminus G) \oplus L^2_c(\Gamma \setminus G),$$

where $L^2_d(\Gamma \setminus G)$ and $L^2_c(\Gamma \setminus G)$ are invariant subspaces in which $R_{\Gamma\setminus G}$ decomposes discretely and continuously respectively. $L^2_d(\Gamma \setminus G)$ contains the invariant subspaces $L^2_0(\Gamma \setminus G)$ of cusp forms. Recall that $f \in L^2(\Gamma \setminus G)$ is a cusp form if for each Γ-cuspidal parabolic subgroup $P \subset G$ with unipotent radical U one has $\int_{\Gamma \cap U \setminus U} f(ug)\, du = 0$ for almost all $g \in G$. Let $L^2_{\mathrm{res}}(\Gamma \setminus G)$ be the orthogonal complement of $L^2_0(\Gamma \setminus G)$ in $L^2_d(\Gamma \setminus G)$. $L^2_{\mathrm{res}}(\Gamma \setminus G)$ is generated by the residues of all Eisenstein series with respect to poles in $(0, 1]$

[1]. The proof of this fact is essentially the same as the one given in [1]. Let $R^0_{\Gamma\backslash G}$, $R^d_{\Gamma\backslash G}$, $R^{\mathrm{res}}_{\Gamma\backslash G}$ and $R^c_{\Gamma\backslash G}$ be the restriction of $R_{\Gamma\backslash G}$ to the corresponding invariant subspaces $L^2_0(\Gamma\setminus G),\cdots$. Let $\alpha\in\mathscr{C}^1(G)$. Then $R^0_{\Gamma\backslash G}(\alpha)$ is a trace class operator [33, Theorem 8.2]. In our case we have

Theorem 3.26. *Let $\alpha\in\mathscr{C}^1(G)$ be right K-finite. Then the operator $R^d_{\Gamma\backslash G}(\alpha)$ is of the trace class.*

Proof. We know that $R^d_{\Gamma\backslash G}(\alpha)$ restricted to the space of cusp forms is of the trace class. Since α is right K-finite there exist $\sigma_1,\cdots,\sigma_r\in\hat{K}$ such that $R^d_{\Gamma\backslash G}(\alpha)$ restricted to the orthogonal complement of $\oplus_{i=1}^r L^2_{\mathrm{res}}(\Gamma\setminus G,\sigma_i)$ in $L^2_{\mathrm{res}}(\Gamma\setminus G)$ is zero. Each $L^2_{\mathrm{res}}(\Gamma\setminus G,\sigma_i)$ is generated by the residues of the poles, which lie in $\mathrm{Re}(s)>0$, of all Eisenstein series $E(\Phi,s,g)$ with $\Phi\in L^2(\Gamma_M\setminus M,\sigma_i,\chi)$ and χ runs over $\hat{\mathfrak{Z}}_M$. Let $\sigma\in\hat{K}$ be given. It follows from Lemma 3.14 that the space of those Eisenstein series, which are associated to σ and $\Phi\in L^2(\Gamma_M\setminus M,\sigma,\chi)$ and which can have poles in $\mathrm{Re}(s)>0$, is finite dimensional. Since each Eisenstein series can have only finitely many poles in the half-plane $\mathrm{Re}(s)>0$ [20, IV, §7], it follows that $\dim L^2_{\mathrm{res}}(\Gamma\setminus G,\sigma_i)<\infty$, $i=1,\cdots,r$. Thus $R^d_{\Gamma\backslash G}(\alpha)$, restricted to $L^2_{\mathrm{res}}(\Gamma\setminus G)$, is of finite rank. Therefore, $R^d_{\Gamma\backslash G}(\alpha)$ is a trace class operator.

4. The Selberg trace formula

Let $\Gamma\subset G$ be as in §3. In our computation of the index of the signature operator we are going to use the Selberg trace formula developed by Osborne and Warner [32] for a lattice of rank one. In this section we explain some facts connected with the trace formula. For all details regarding the trace formula the reader is referred to [1], [32]. The situation which we consider is much simpler than the general case of a rank one lattice treated in [32]. The contribution of the various conjugacy classes to the trace formula can be computed rather explicitly. The trace formula for the case where $f\in C_0^\infty(G)$ is K-bi-invariant has been established by P. Sograf [39] for $n=2$ and by I. Efrat [15] in general.

Let $\sigma\colon K\to\mathrm{GL}(V)$ be a finite-dimensional representation and let $\mathscr{C}^1(G,\sigma)$ be defined by (1.6). For any $f\in\mathscr{C}^1(G,\sigma)$ we set

$$R_{\Gamma\backslash G}(f)=\int_G R_{\Gamma\backslash G}(g)\otimes f(g)\,dg. \tag{4.1}$$

$R_{\Gamma\backslash G}(f)$ is a bounded operator on the Hilbert space $L^2(\Gamma\setminus G)\otimes V$. Let

$$P_\sigma=\int_K R_{\Gamma\backslash G}(k)\otimes\sigma(k)\,dk \tag{4.2}$$

be the orthogonal projection of $L^2(\Gamma \setminus G) \otimes V$ onto its K-invariant part $L^2(\Gamma \setminus G, \sigma)$. Since $f \in \mathscr{C}^1(G, \sigma)$, it follows from (1.4) that

$$P_\sigma R_{\Gamma\setminus G}(f) = R_{\Gamma\setminus G}(f) P_\sigma = R_{\Gamma\setminus G}(f).$$

Thus, relative to the splitting

$$L^2(\Gamma \setminus G) \otimes V = L^2(\Gamma \setminus G, \sigma) \oplus L^2(\Gamma \setminus G, \sigma)^\perp,$$

$R_{\Gamma\setminus G}(f)$ has the form

$$R_{\Gamma\setminus G}(f) = \begin{pmatrix} R_\sigma(f) & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

with $R_\sigma(f)$ acting on $L^2(\Gamma \setminus G, \sigma)$. $R_{\Gamma\setminus G}(f)$ is an integral operator whose kernel is given by

$$K(g, g') = \sum_{\gamma \in \Gamma} f(g^{-1}\gamma g'). \tag{4.4}$$

The series converges uniformly on compact subsets. The Casimir operator $\Omega \in \mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ induces an operator Δ_σ on $C_0^\infty(\Gamma \setminus G, \sigma) = (C_0^\infty(\Gamma \setminus G) \otimes V)^K$ which we call Laplacian. By [29, Corollary 1.2] Δ_σ has a unique selfadjoint extension $\overline{\Delta}_\sigma$ to an unbounded operator in $L^2(\Gamma \setminus G, \sigma)$. Let

$$L_d^2(\Gamma \setminus G, \sigma) = \left(L_d^2(\Gamma \setminus G) \otimes V\right)^K$$

and

$$L_c^2(\Gamma \setminus G, \sigma) = \left(L_c^2(\Gamma \setminus G) \otimes V\right)^K.$$

From (3.25) we get a decomposition

$$L^2(\Gamma \setminus G, \sigma) = L_d^2(\Gamma \setminus G, \sigma) \oplus L_c^2(\Gamma \setminus G, \sigma). \tag{4.5}$$

$\overline{\Delta}_\sigma$ decomposes discretely in $L_d^2(\Gamma \setminus G, \sigma)$ and continuously in $L_c^2(\Gamma \setminus G, \sigma)$. This decomposition is invariant under $R_\sigma(f)$. Let $R_\sigma^d(f)$ and $R_\sigma^c(f)$ be the restrictions of $R_\sigma(f)$ to the corresponding subspaces. These operators are integral operators. We denote by tr the trace in $\operatorname{End}(V)$.

Proposition 4.6. *Let $\sigma\colon K \to \mathrm{GL}(V)$ be a finite-dimensional unitary representation and let $f \in \mathscr{C}^1(G, \sigma)$. Then $R_{\Gamma\setminus G}^d(f)$ and $R_\sigma^d(f)$ are trace class operators. Moreover, $\operatorname{tr} f \in \mathscr{C}^1(G)$, $\operatorname{tr} f$ is right K-finite and*

$$\operatorname{Tr} R_\sigma^d(f) = \operatorname{Tr} R_{\Gamma\setminus G}^d(f) = \operatorname{Tr} R_{\Gamma\setminus G}^d(\operatorname{tr} f).$$

Proof. σ splits into characters $\sigma = \oplus_{i=1}^r \tau_i$, $\tau_i \in \hat{K}$. Let $v_1, \cdots, v_r \in V$ be an orthonormal basis such that $\sigma(k)v_i = \tau_i(k)v_i$ and let $f_{ij}(g) = \langle f(g)v_i, v_j \rangle$. With respect to the basis $v_1, \cdots, v_r$, $R_{\Gamma\setminus G}^d(f)$ is represented by the matrix $(R_{\Gamma\setminus G}^d(f_{ij}))$. The f_{ij} are right K-finite functions in $\mathscr{C}^1(G)$. Thus, by Theorem

3.26, $R^d_{\Gamma\backslash G}(f_{ij})$ are trace class operators and this implies that $R^d_{\Gamma\backslash G}(f)$ is of the trace class. (4.3) shows that $R^d_\sigma(f)$ is also a trace class operator and $\operatorname{Tr} R^d_\sigma(f) = \operatorname{Tr} R^d_{\Gamma\backslash G}(f)$. Since $\operatorname{tr} f = \sum_{i=1}^r f_{ii}$, $\operatorname{tr} f$ is a right K-finite element of $\mathscr{C}^1(G)$ and it is obvious that

$$\operatorname{Tr} R^d_{\Gamma\backslash G}(f) = \sum_{i=1}^r \operatorname{Tr} R^d_{\Gamma\backslash G}(f_{ii}) = \operatorname{Tr} R^d_{\Gamma\backslash G}(\operatorname{tr} f). \quad \text{q.e.d.}$$

Let $K_0(g, g')$ be the kernel of $R^d_{\Gamma\backslash G}(f)$. Then one has

$$\operatorname{Tr} R^d_{\Gamma\backslash G}(f) = \int_{\Gamma\backslash G} \operatorname{tr} K_0(g, g)\, dg. \tag{4.7}$$

The integral on the right-hand side can be calculated along Selberg's path [1], [32]. By Proposition 4.6 it is sufficient to consider the case where $\sigma \in \hat{K}$. Before stating the trace formula, we have to discuss the classification of elements of Γ and to introduce some notation. All details concerning the classification of elements of Γ can be found in [32, §5], [38, §1].

Given $\gamma \in \Gamma$, we denote by G_γ (resp. Γ_γ) its G-centralizer (resp. Γ-centralizer). We write $\{\gamma\}_G$ (resp. $\{\gamma\}_\Gamma$) for the conjugacy class of γ in G (resp. Γ). Let Z_Γ be the center of Γ. By our assumption, $\Gamma \subset G$ is an irreducible discrete subgroup. An equivalent condition is that Γ contains no element $\gamma = (\gamma_1, \cdots, \gamma_n)$ with $\gamma \neq 1$ and $\gamma_i = 1$ for some i. The only possible central elements of Γ are ± 1. This follows from the assumption $\Gamma \subset \mathrm{SL}(2, F)$ and [38, §1]. An element $\gamma \in \Gamma$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, is called elliptic, parabolic or hyperbolic if all components γ_i as elements of $\mathrm{SL}(2, \mathbb{R})$ are of the corresponding type. Every element of Γ which is not central and which is different from all types above, is called mixed. If γ is mixed, then its components are either elliptic or hyperbolic. The hyperbolic elements are divided in two classes. The first class consists of those hyperbolic elements such that none of its fixed points on $\overline{\mathbf{H}}^n$ is a parabolic fixed point of Γ. The remaining elements are in the second class. If $\gamma \neq \pm 1$, then $\Gamma_\gamma \backslash G_\gamma$ is compact, except in the case where γ is a hyperbolic element of type II. If γ is hyperbolic of type II, then $\Gamma_\gamma \backslash G_\gamma$ is isomorphic to the product of $\mathbb{R}$ and a compact group. For a rank one lattice Γ, Osborne and Warner defined in [32, §5] a decomposition $\Gamma = Z_\Gamma \cup \Gamma_S \cup \Gamma_P$ (disjoint union). Z_Γ is the center, Γ_S consists of elements $\gamma \in \Gamma$ with the property that $\{\gamma\}_\Gamma \cap P = \varnothing$ for all Γ-percuspidal parabolic subgroups $P \subset G$ and Γ_P is the complement of $Z_\Gamma \cup \Gamma_S$ in Γ. Γ_P has an additional decomposition $\Gamma_P = \Gamma_P(r) \cup \Gamma_P(s)$ in "regular" and "singular" elements (see [32, §5] for the definition). In our case we have $Z_\Gamma \subset \{\pm 1\}$, Γ_S is the union of the elliptic, hyperbolic type I and mixed elements, $\Gamma_P(r)$ is the set of hyperbolic type II elements and $\Gamma_P(s)$ is the set of parabolic elements.

To simplify notation, we make the following assumption about our discrete subgroup Γ.

Assumption. *There is only one Γ-conjugacy class of Γ-percuspidal parabolic subgroups $P \subset G$.*

In other words, $\Gamma \backslash \mathbf{H}^n$ has a single cusp. We make this assumption to keep the notation in a manageable form. But all our calculations can easily be extended to the case of several cusps. An example of a discrete group, which satisfies our assumption, is the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_F)$ of a field F with class number one.

We denote by P the stabilizer of ∞. Let $P = UAM$ be the Langlands decomposition of P over $\mathbf{Q}$. Note that in our case M is commutative. This will simplify some terms occurring in the trace formula. An element in Γ_M is called regular if its centralizer in U is trivial. Let $\Gamma_M(r) \subset \Gamma_M$ be the set of all regular elements of Γ_M and let $\Gamma_M(s) \subset \Gamma_M$ be the complement of $\Gamma_M(r)$. $\Gamma_M(s)$ are the singular elements of Γ_M. For $\delta \in \Gamma_M$ we let $\iota(\delta) = |\det(\mathrm{Ad}(\delta)|\mathfrak{u} - 1)|$, where $\mathfrak{u}$ denotes the Lie algebra of U.

Consider the orthogonal sum decomposition (3.13). Given $\chi \in \hat{\mathfrak{Z}}_M$, let $L^2(\Gamma_M \backslash M, \chi)$ be the sum of the irreducible subspaces of $L^2(\Gamma_M \backslash M)$ with infinitesimal character χ. The Weyl group $W(A)$ acts on $\hat{\mathfrak{Z}}_M$. If $\vartheta \in W(A) \backslash \hat{\mathfrak{Z}}_M$ let

$$L^2(\Gamma_M \backslash M, \vartheta) = \bigoplus_{\chi \in \vartheta} L^2(\Gamma_M \backslash M, \chi).$$

For $\vartheta \in W(A) \backslash \hat{\mathfrak{Z}}_M$ and $s \in \mathbf{C}$ let $r_{\vartheta,s}$ be the representation of $P = UAM$ on $L^2(\Gamma_M \backslash M, \vartheta)$ which is defined by $r_{\vartheta,s}(uam) = R_{\Gamma_M \backslash M}(m) e^{s \ln a}$ and let $\pi_{\vartheta,s} = \mathrm{Ind}_P^G(r_{\vartheta,s})$. The Hilbert space $\mathcal{H}_{\vartheta,s}$ of $\pi_{\vartheta,s}$ consists of all measurable functions $\Phi\colon G \to L^2(\Gamma_M \backslash M, \vartheta)$ which satisfy

$$\Phi(uamg) = e^{(s+1)\ln a} R_{\Gamma_M \backslash M}(m)(\Phi(g))$$

and which have the property that

$$\|\Phi\|^2 = \int_K \int_{\Gamma_M \backslash M} |\Phi(k)(m)|^2 \, dm \, dk < \infty.$$

$\pi_{\vartheta,s}$ is unitary if s lies on the imaginary axis. For $\sigma \in \hat{K}$ let $\mathcal{H}_{\vartheta,s}(\sigma)$ be the σth-isotypic component of $\mathcal{H}_{\vartheta,s}$. There is a canonical identification $\mathcal{H}_{\vartheta,s}(\sigma) \cong \bigoplus_{\chi \in \vartheta} L^2(\Gamma_M \backslash M, \sigma, \chi)$. The theory of Eisenstein series produces certain intertwining operators

$$C_\vartheta(s)\colon \mathcal{H}_{\vartheta,s} \to \mathcal{H}_{\vartheta,s}, \qquad \vartheta \in W(A) \backslash \hat{\mathfrak{Z}}_M,$$

which are meromorphic in $s \in \mathbf{C}$. $C_\vartheta(s)$ maps $\mathcal{H}_{\vartheta,s}(\sigma)$ into $\mathcal{H}_{\vartheta,-s}(\sigma)$ and it satisfies the functional equation

$$C_\vartheta(s) C_\vartheta(-s) = \mathrm{Id}$$

(see [32], [33]). By Proposition 4.6, we can use the trace formula established in [32].

Theorem 4.8. *Let f be a K-finite function in $\mathscr{C}^p(G)$, $0 < p < 1$. Then $R^d_{\Gamma\backslash G}(f)$ is a trace class operator and* $\operatorname{Tr} R^d_{\Gamma\backslash G}(f)$ *is the sum of the following terms*:

(i) (*central*)

$$\sum_{\gamma \in Z_\Gamma} \operatorname{Vol}(\Gamma \setminus G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg.$$

(ii) (*elliptic, hyperbolic type* I, *mixed*)

$$(S) \sum_{\{\gamma\}_\Gamma} \operatorname{Vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg.$$

The sum runs over all Γ-conjugacy classes of elements of Γ_S.

(iii) (*type* II *hyperbolic*)

$$\frac{1}{2}\operatorname{Vol}(\Gamma_M \setminus M)\left\{(r) \sum_{\{\delta\}_{\Gamma_M}} C_\delta \int_U f(\delta u)\, du \right.$$

$$\left. + (r) \sum_{\{\delta\}_{\Gamma_M}} \iota(\delta) \int_U f(u^{-1}\delta u)\alpha(H(wu))\, du \right\}.$$

The sum runs over all Γ_M-conjugacy classes of elements of $\Gamma_M(r)$. C_δ is a certain constant depending on δ. It is defined in [32, *p.* 69]. *α is the positive root of A, $w \in W(A)$ the nontrivial element of the Weyl group of A and $H(wu) \in \mathfrak{a}$ the unique element which is determined by the Iwasawa decomposition of wu.*

(iv) (*parabolic*)

$$(s) \sum_{\{\delta\}_{\Gamma_M}} \lim_{z\to 0} \frac{d}{dz}(zU_\delta(f, z)),$$

where $(s)\Sigma$ is the sum over all Γ_M-conjugacy classes of elements of $\Gamma_M(s)$ and

$U_\delta(f, z)$

$$= \int_A \int_{\Gamma_M\backslash M} \int_{\Gamma\cap U\backslash U} \sum_{\gamma \in \delta U \cap \Gamma_P(s)} f(m^{-1}a^{-1}u^{-1}\gamma uam)e^{-2(z+1)\ln a}\, du\, da\, dm.$$

(v) (*intertwining*)

$$\frac{1}{4\pi} \sum_{\vartheta} \int_{\operatorname{Re}(s)=0} \operatorname{Tr}\left(\pi_{\vartheta,s}(f)\frac{d}{ds}C_\vartheta(s)C_\vartheta(-s)\right)|ds|.$$

(vi) (*residual*)

$$-\frac{1}{4}\sum_{\vartheta} \operatorname{Tr}\left(\pi_{\vartheta,0}(f)C_\vartheta(0)\right).$$

We shall apply this version of the trace formula to our situation. For this purpose we have to compute some of the terms occurring in the trace formula more explicitly. Let $\sigma \in \hat{K}$ and let $\sigma = \otimes_{j=1}^{n} \sigma_j$, where σ_j: $\mathrm{SO}(2) \to \mathbb{C}^{\times}$ is a character. Moreover, we assume that $f(g) = \prod_{j=1}^{n} f_j(g_j)$, where $f_j \in \mathscr{C}^p(G_0, \sigma_j)$ with $0 < p < 1$ and f_j satisfies $f_j(g) = f_j(-g)$.

(i) *The central term.* Above we have seen that $Z_\Gamma \subset \{\pm 1\}$. Thus, the central contribution is

$$|Z_\Gamma| \mathrm{Vol}(\Gamma \setminus G) f(1).$$

(ii) *The elliptic term.* Let $\gamma \in \Gamma$ be elliptic. We have $\Gamma_\gamma \cong \mathbb{Z}/l\mathbb{Z}$. Let $\gamma_0 \in \Gamma_\gamma$ be a generator of Γ_γ. γ_0 is a primitive elliptic element and $\gamma = \gamma_0^q$, $1 \leqslant q < l$. γ_0 is conjugate in G to an element $k \in K$ with

$$k_j = \begin{pmatrix} \cos \frac{2\pi}{l} r_j & \sin \frac{2\pi}{l} r_j \\ -\sin \frac{2\pi}{l} r_j & \cos \frac{2\pi}{l} r_j \end{pmatrix}, \qquad (r_j, l) = 1; j = 1, \cdots, n.$$

Moreover, $G_k = K$. Hence $\mathrm{Vol}(\Gamma_\gamma \setminus G_\gamma) = (2\pi)^n / l$ and

$$\begin{aligned} \int_{G_\gamma \setminus G} f(g^{-1} \gamma g)\, dg &= \int_{K \setminus G} f(g^{-1} k g)\, dg \\ &= \prod_{j=1}^{n} \int_{K_0 \setminus G_0} f_j(g^{-1} k_j g)\, dg. \end{aligned} \tag{4.9}$$

The corresponding orbit integrals on G_0 are calculated in [17]. We use formula (2) of I, §5.4 in [17]. Let $\varphi \in \mathscr{C}^1(G_0)$ and assume that $\varphi(-g) = \varphi(g)$. Then

$$\begin{aligned} \int_{K_0 \setminus G_0} \varphi(g^{-1} k(\theta) g)\, dg = &-\frac{1}{4\pi i \sin\theta} \Bigg\{ \frac{\Theta_0^+(\varphi) - \Theta_0^-(\varphi)}{2} \\ &+ \sum_{m=2}^{\infty} \left(\Theta_m^+(\varphi) e^{i(m-1)\theta} - \Theta_m^-(\varphi) e^{-i(m-1)\theta} \right) \Bigg\} \\ &+ \frac{1}{16\pi \sin|\theta|} \int_{-\infty}^{\infty} \Theta_\lambda^+(\varphi) \frac{\mathrm{ch}((|\theta| - \pi/2)\lambda)}{\mathrm{ch}(\pi\lambda/2)}\, d\lambda. \end{aligned} \tag{4.10}$$

(iii) *The type* I *hyperbolic term.* Let $\gamma \in \Gamma$ be hyperbolic of type I. Let

$$D_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{R}^{\times} \right\}.$$

γ is conjugate in G to an element $a \in \prod_{j=1}^{n} D_0$ with

$$a_j = \begin{pmatrix} N(\gamma_j) & 0 \\ 0 & N(\gamma_j)^{-1} \end{pmatrix}, \qquad |N(\gamma_j)| > 1,$$

where γ_j denotes the jth component of γ, G_γ is conjugate to $\prod_{j=1}^n D_0$ and

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,dg = \int_{\prod D_0\backslash G} f(g^{-1}ag)\,dg = \prod_{j=1}^{n} \int_{D_0\backslash G_0} f_j\left(g^{-1}a_j g\right)\,dg.$$

The orbit integrals on G_0 can be calculated as in [17]. Let $\varphi \in \mathscr{C}^1(G_0)$ be such that $\varphi(-g) = \varphi(g)$ and let $a = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$, $\alpha \neq \pm 1$. Then formula (2) of I, §5.3 in [17] gives

$$\int_{D_0\backslash G_0} \varphi(g^{-1}ag)\,dg = \frac{1}{4\pi|\alpha - \alpha^{-1}|} \int_{-\infty}^{\infty} \Theta_\lambda^+(\varphi)|\alpha|^{i\lambda}\,d\lambda.$$

Let

$$g_j(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_\lambda^+(f_j)e^{iu\lambda}\,d\lambda.$$

Then we obtain

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,dg = \prod_{j=1}^{n} \frac{g_j\left(\log N(\gamma_j)\right)}{4\mathrm{sh}\left(N(\gamma_j)\right)}. \tag{4.11}$$

(iv) *The mixed term*. Let $\gamma \in \Gamma$ be mixed. Every component γ_j of γ is either elliptic or hyperbolic. The corresponding orbit integral splits again into a product of orbit integrals with respect to G_0 and we can use the same calculations as in (ii) and (iii).

(v) *The type* II *hyperbolic term*. According to Lemma 3.2 there exists a subgroup $V_1 \subset \mathscr{O}_F^*$ of finite index such that

$$\Gamma_M = \left\{ \begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix} \middle| v \in V_1 \right\}.$$

For $v \in V_1$ let $\delta_v = \left(\begin{smallmatrix} v & 0\\ 0 & v^{-1}\end{smallmatrix}\right)$. We determine $\Gamma_M(r)$ and $\Gamma_M(s)$.

Lemma 4.12. $\Gamma_M(s) = Z_\Gamma$.

Proof. Let $\delta_v \in \Gamma_M(s)$. Then there exists $u \in U$, $u \neq 1$, such that $\delta_v u = u\delta_v$. This implies $v = \pm 1$. Thus $\Gamma_M(s) \subset \{\pm 1\}$. Recall that $Z_\Gamma \subset \{\pm 1\}$. If $-1 \in \Gamma$, then $-1 \in \Gamma_M$ and therefore $\Gamma_M(s) = \{\pm 1\} = Z_\Gamma$. Now assume that $-1 \notin \Gamma$, but $-1 \in \Gamma_M$. Then there exists $a \in F$, such that $\gamma = \left(\begin{smallmatrix} -1 & a\\ 0 & -1\end{smallmatrix}\right) \in \Gamma$. Since Γ is commensurable with SL(2, $\mathscr{O}_F$), there exists $k \in \mathbb{N}$ such that

$$\begin{pmatrix} -1 & (2k+1)a \\ 0 & -1 \end{pmatrix} = \gamma^{2k+1} \in \Gamma \cap \mathrm{SL}(2, \mathscr{O}_F).$$

Let $b = (2k+1)a$. Then $\left(\begin{smallmatrix} -1 & b\\ 0 & -1\end{smallmatrix}\right) \in \Gamma \cap \mathrm{SL}(2, \mathscr{O}_F)$. On the other hand, since $\left(\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}(2, \mathscr{O}_F)$ and Γ is commensurable with SL(2, $\mathscr{O}_F$) there exists $m \in N$ such that

$$\begin{pmatrix} 1 & (2m+1)b \\ 0 & 1 \end{pmatrix} \in \Gamma \cap \mathrm{SL}(2, \mathscr{O}_F).$$

This implies

$$-1 = \begin{pmatrix} -1 & (2m+1)b \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & (2m+1)b \\ 0 & 1 \end{pmatrix} \in \Gamma \cap \mathrm{SL}(2, \mathcal{O}_F).$$

Therefore $-1 \notin \Gamma_M$ and $\Gamma_M(s) = \{1\} = Z_\Gamma$. q.e.d.

By Lemma 4.12 we have $\Gamma_M(r) = \Gamma_M - Z_\Gamma$ and the regular conjugacy classes $\{\delta\}_{\Gamma_M}$ can be identified with the set $V - \{\pm 1\}$. The constant $\iota(\delta)$ is defined as $|\det(\mathrm{Ad}(\delta)|\mathfrak{u} - 1)|$. Hence

$$\iota\left(\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}\right) = |N(v^2 - 1)| = |N(v - v^{-1})|.$$

The integrals occurring in the contribution of the type II hyperbolic conjugacy classes can be computed as follows.

$$\int_U f(\delta_v u)\, du = \prod_{j=1}^n \int_{U_0} f_j\left(\begin{pmatrix} v^{(j)} & 0 \\ 0 & (v^{(j)})^{-1} \end{pmatrix} u\right) du.$$

Let

$$a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda \in \mathbb{R}^\times, \lambda \neq \pm 1,$$

and let $\varphi \in \mathscr{C}^1(G_0, \sigma)$, $\sigma \in \hat{K}_0$. If we use formula I.2 of V, §1 in [25] and formula (2) of I, §5.3 in [17], then we get

$$\text{(4.13)} \quad \begin{aligned} \lambda \int_{U_0} \varphi(a_\lambda u)\, du &= |\lambda - \lambda^{-1}| \int_{A_0 \backslash G_0} \varphi(g^{-1} a_\lambda g)\, dg \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \left(\Theta_\rho^+(\varphi)|\lambda|^{i\rho} + \operatorname{sign}\lambda\, \Theta_\rho^-(\varphi)|\lambda|^{i\rho}\right) d\rho\right). \end{aligned}$$

Let

$$g_j(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_\rho^+(f_j) e^{iu\rho}\, d\rho.$$

Since $v^{(1)} \cdots v^{(n)} = 1$, it follows from (4.13) that

$$\int_U f(\delta_v u)\, du = \prod_{j=1}^n g_j(\log|v^{(j)}|).$$

For $f \in \mathscr{C}^1(G_0)$ and $\lambda \neq \pm 1$ we put

$$I(f)(\lambda) = \int_{U_0} f(u^{-1} a_\lambda u)\, \alpha_0(H(w_0 u))\, du,$$

where $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and α_0 is the positive root. Then we get the following expression for the type II hyperbolic contribution:

$$\begin{aligned}(4.14)\quad & \frac{1}{2}\operatorname{Vol}(\Gamma_M \backslash M)\Bigg\{ \sum_{\substack{v \in V_1 \\ v \neq \pm 1}} C_v \prod_{j=1}^{n} g_j\big(\log|v^{(j)}|\big) \\ & + \sum_{\substack{v \in V_1 \\ v \neq \pm 1}} |N(v - v^{-1})| \sum_{j=1}^{n} I(f_j)(v^{(j)}) \prod_{k=1}^{n}{}^{(j)} g_k\big(\log|v^{(k)}|\big)\Bigg\},\end{aligned}$$

where $\prod^{(j)}$ denotes the product with jth factor deleted and C_v is a certain constant depending on v.

(vi) *The parabolic term.* According to Lemma 3.3 there exists a lattice $\mathsf{M} \subset F$ such that $\Gamma \cap U = \{\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right) | \mu \in \mathsf{M}\}$. For $\mu \in \mathsf{M}$ let

$$\gamma_\mu = \left(\begin{pmatrix} 1 & \mu^{(1)} \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & \mu^{(n)} \\ 0 & 1 \end{pmatrix}\right).$$

Recall that $\Gamma_P(s)$ is the set of parabolic elements in Γ. If $\delta \in \Gamma_M(s)$, then, by Lemma 4.12, we obtain $(\delta U) \cap \Gamma_P(s) = \delta(U \cap \Gamma_P(s)) = \delta(U \cap \Gamma)$. Using these remarks we can conclude that

$$(s) \sum_{\{\delta\}_{\Gamma_M}} U_\delta(f, z)$$

is equal to

$$\begin{aligned}(4.15)\quad & |Z_\Gamma| \cdot \operatorname{Vol}(\Gamma \cap U \backslash U) \\ & \cdot \int_{\Gamma_M \backslash M} \int_A \sum_{\mu \in \mathsf{M} - 0} f\big(m^{-1}a^{-1}\gamma_\mu a m\big) e^{-2(z+1)\ln a}\, da\, dm.\end{aligned}$$

By Lemma 3.2 there exists a subgroup $V_1 \subset \mathcal{O}_F^*$ of finite index such that $\Gamma_M = \{\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right) | v \in V_1\}$. If $-1 \notin V_1$, then the map $v \in V_1 \mapsto v^2 \in (V_1)^2$ is an isomorphism and if $-1 \in V_1$, its kernel is $\{\pm 1\}$. Let $\mathsf{V} = (V_1)^2$. Then (4.15) is equal to

$$(4.16)\quad \operatorname{Vol}(\Gamma \cap U \backslash U) \int_M \int_A \sum_{\mu \in (\mathsf{M} - 0)/\mathsf{V}} f\big(m^{-1}a^{-1}\gamma_\mu a m\big) e^{-2(z+1)\ln a}\, da\, dm.$$

Let $\varepsilon_\mu^j = \mu^{(j)}/|\mu^{(j)}|$, $j = 1, \cdots, n$, and let $\varepsilon_\mu \in \Gamma \cap U$ be the matrix with jth component

$$\begin{pmatrix} 1 & \varepsilon_\mu^j \\ 0 & 0 \end{pmatrix}.$$

Further, let $a_\mu \in A$ be the element which has all components equal to

$$\begin{pmatrix} |N(\mu)|^{1/2n} & 0 \\ 0 & |N(\mu)|^{-1/2n} \end{pmatrix}$$

and let $m_\mu \in M$ be defined by

$$m_\mu^{(j)} = \begin{pmatrix} \left(\dfrac{|\mu^{(j)}|}{|N(\mu)|^{1/n}} \right)^{1/2} & 0 \\ 0 & \left(\dfrac{|\mu^{(j)}|}{|N(\mu)|^{1/n}} \right)^{-1/2} \end{pmatrix}.$$

Then $\gamma_\mu = m_\mu a_\mu \varepsilon_\mu a_\mu^{-1} m_\mu^{-1}$. If we change variables in (4.16) by $m \mapsto m_\mu m$ and $a \mapsto a_\mu a$, then (4.16) is equal to

$$\begin{aligned} \text{(4.17)} \qquad & \operatorname{Vol}(\Gamma \cap U \backslash U) \\ & \cdot \int_M \int_A \sum_{\mu \in (\mathsf{M}-0)/\mathsf{V}} |N(\mu)|^{-(z+1)} f\left(m^{-1} a^{-1} \varepsilon_\mu a m\right) e^{-2(z+1)\ln a}\, da\, dm. \end{aligned}$$

For $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, with $\varepsilon_i \in \{\pm 1\}$, let $u(\varepsilon) \in U$ be the element with jth component equal to $\left(\begin{smallmatrix} 1 & \varepsilon_j \\ 0 & 1 \end{smallmatrix}\right)$, $j = 1, \cdots, n$. Moreover, for $\operatorname{Re}(s) > 1$, let

$$\text{(4.18)} \qquad \zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s) = \sum_{\substack{\mu \in (\mathsf{M}-0)/\mathsf{V} \\ \varepsilon_j \mu(j) > 0}} |N(\mu)|^{-s}.$$

Then (4.17) can be rewritten as

$$\begin{aligned} \text{(4.19)} \qquad & \operatorname{Vol}(\Gamma \cap U \backslash U) \\ & \cdot \sum_{\varepsilon \in \{\pm 1\}^n} \zeta_\varepsilon(\mathsf{M}, \mathsf{V}, z+1) \int_M \int_A f\left(m^{-1} a^{-1} u(\varepsilon) a m\right) e^{-2(z+1)\ln a}\, da\, dm. \end{aligned}$$

The integral is holomorphic at $z = 0$. Concerning $\zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s)$ we have the following

Lemma 4.20. *For each $\varepsilon \in \{\pm 1\}^n$, $\zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s)$ has an analytic continuation to the entire complex plane with a simple pole at $s = 1$. The residue is independent of ε.*

Proof. For $a \in (\mathbf{Z}/2\mathbf{Z})^n$ let λ_a be the character of $F^\times$ defined by

$$\lambda_a(\mu) = \prod_{j=1}^n \left(\frac{\mu^{(j)}}{|\mu^{(j)}|} \right)^{a_j}$$

for $\mu \in F^\times$. Set

$$L(\mathsf{M}, \mathsf{V}, \lambda_a, s) = \sum_{(\mathsf{M}-0)/\mathsf{V}} \frac{\lambda_a(\mu)}{|N(\mu)|^s}, \qquad \operatorname{Re}(s) > 1.$$

The functions $L(\mathsf{M}, \mathsf{V}, \lambda_a, s)$, $a \in (\mathbb{Z}/2\mathbb{Z})^n$, and $\zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s)$, $\varepsilon \in \{\pm 1\}^n$, are linearly equivalent. Let $\lambda_a(\varepsilon) = \prod_{i=1}^n (\varepsilon_i)^{a_i}$. Then we have

$$2^n \zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s) = \sum_a \lambda_a(\varepsilon) L(\mathsf{M}, \mathsf{V}, \lambda_a, s). \tag{4.21}$$

Using Hecke's method [22], one can show that $L(\mathsf{M}, \mathsf{V}, \lambda_a, s)$ has an analytic continuation to the entire complex plane. If $\lambda_a \neq 1$, then $L(\mathsf{M}, \mathsf{V}, \lambda_a, s)$ is an entire function. $L(\mathsf{M}, \mathsf{V}, 1, s)$ has a simple pole at $s = 1$. This together with (4.21) proves the lemma. q.e.d.

Let

$$\zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s) = \frac{a_{-1}}{s-1} + a_0(\varepsilon) + O(s-1)$$

be the Laurent expansion of $\zeta_\varepsilon(\mathsf{M}, \mathsf{V}, s)$ at $s = 1$. Then it follows from (4.19) and Lemma 4.20 that the parabolic contribution is equal to

$$\begin{aligned} \operatorname{Vol}(\Gamma \cap U \backslash U) \Big\{ &-2a_{-1} \sum_\varepsilon \int_M \int_A \ln(a) f(m^{-1}a^{-1}u(\varepsilon)am) e^{-2\ln a}\, da\, dm \\ &+ \sum_\varepsilon a_0(\varepsilon) \int_M \int_A f(m^{-1}a^{-1}u(\varepsilon)am) e^{-2\ln a}\, da\, dm \Big\}. \end{aligned} \tag{4.22}$$

We compute now the integrals occurring in (4.22). We start with some comments on the choice of the invariant measures. The measure on M has been normalized by the requirement

$$\int_G f(g)\, dg = \iiiint_{U \times A \times M \times K} f(uamk) e^{-2\rho(\ln a)}\, dk\, dm\, da\, du,$$

$f \in C_0(G)$. The measure on K is normalized by the condition $\operatorname{Vol}(K) = 1$ and U has the measure induced from the natural Euclidean structure on $\mathfrak{u}$. On A, the measure is determined as follows. We have $P = \mathscr{B}(F \otimes_{\mathbf{Q}} \mathbb{R})$, where $\mathscr{B} \subset \mathrm{SL}(2)$ is the standard Borel subgroup. The fundamental dominant weight α: $\mathscr{B} \to \mathbb{G}_m$ induces a homomorphism $|\alpha|$: $P \to (\mathbb{R}^+)^\times$, which is the composition of α_∞: $\mathscr{B}(F \otimes_{\mathbf{Q}} \mathbb{R}) \to \mathbb{G}_m(F \otimes_{\mathbf{Q}} \mathbb{R})$ and the norm homomorphism ν: $(F \otimes_{\mathbf{Q}} \mathbb{R}) \to (\mathbb{R}^+)^\times$. The kernel of $|\alpha|$ is UM and $|\alpha|$ induces an isomorphism $|\alpha|$: $A \to (\mathbb{R}^+)^\times$. We choose the measure on A which corresponds to dt/t under $|\alpha|$. Furthermore, we have $M = M^0 \times K_M$, where $K_M = M \cap K$ and $AM^0 = \prod_{j=1}^n A_0$. Let dm_0 be the measure on M^0 so that $da\, dm_0 = \prod_{j=1}^n da_j$. Then the normalized measure on M is $dm = dm_0/2^n$. This implies

$$\int_M \int_A f(m^{-1}a^{-1}u(\varepsilon)am)e^{-2\ln a}\,da\,dm$$

$$(4.23)\qquad = \prod_{j=1}^{n} \int_{A_0} f_j(a^{-1}u(\varepsilon_j)a)e^{-2\ln a}\,da = \prod_{j=1}^{n} \int_{U_0\backslash G_0} f_j(g^{-1}u(\varepsilon_j)g)\,dg,$$

where $u(\varepsilon_j) = \left(\begin{smallmatrix}1 & \varepsilon_j\\ 0 & 1\end{smallmatrix}\right)$. In order to calculate these integrals, we use Theorem 6.7 of [5]. Let $\varphi \in C_0^\infty(G_0)$ and $u_\pm = \left(\begin{smallmatrix}1 & \pm 1\\ 0 & 1\end{smallmatrix}\right)$. Then this theorem states that there exists a constant C, which is independent of φ, such that

$$(4.24)\qquad \lim_{\theta\to 0^+} \theta \int_{K_0\backslash G_0} \varphi(g^{-1}k(\pm\theta)g)\,dg = C\int_{U_0\backslash G_0} \varphi(g^{-1}u_\pm g)\,dg.$$

This theorem can be extended to functions $\varphi \in \mathscr{C}^1(G_0)$. Using formula (2) of I, §5.4 in [17], one can compute the left-hand side of (4.24). The result is

$$(4.25)\qquad \begin{aligned} &\pm \frac{i}{4\pi}\left(\frac{1}{2}(\Theta_0^+(\varphi) - \Theta_0^-(\varphi)) + \sum_{k=2}^{\infty} (\Theta_k^+(\varphi) - \Theta_k^-(\varphi))\right) \\ &\qquad + \frac{1}{16\pi}\left(\int_{-\infty}^{\infty} \Theta_\lambda^+(\varphi)\,d\lambda + \int_{-\infty}^{\infty} \Theta_\lambda^-(\varphi)\,d\lambda\right). \end{aligned}$$

The constant C in (4.24) can be determined as follows. Using formula (2) of I, §5.3 in [17] and formula I.2 of V, §1 in [25], we obtain

$$\begin{aligned} &\int_{A_0} \varphi(a^{-1}u_+a)e^{-2\ln a}\,da + \int_{A_0} \varphi(a^{-1}u_-a)e^{-2\ln a}\,da \\ &\quad = \frac{1}{2}\int_0^\infty \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right) du + \frac{1}{2}\int_0^\infty \varphi\left(\begin{pmatrix}1 & -u\\ 0 & 1\end{pmatrix}\right) du = \frac{1}{2}\int_{U_0} \varphi(u)\,du \\ &\quad = \frac{1}{2}\lim_{\lambda\to 1} \lambda \int_{U_0} \varphi(a_\lambda u)\,du = \frac{1}{2}\lim_{\lambda\to 1} |\lambda - \lambda^{-1}| \int_{A_0\backslash G_0} \varphi(g^{-1}a_\lambda g)\,dg \\ &\quad = \frac{1}{4\pi}\left(\int_{-\infty}^{\infty} \Theta_\rho^+(\varphi)\,d\rho + \int_{-\infty}^{\infty} \Theta_\rho^-(\varphi)\,d\rho\right), \qquad \text{where } a_\lambda = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}. \end{aligned}$$

(Note that our integral is $2I(a_\lambda)$ in the notation of [17, I, §5.3].) Combined with (4.25) this gives $C = 1/2$. Now, let us assume that φ satisfies $\varphi(-g) = \varphi(g)$. If we use (4.24) together with (4.25) we obtain

$$(4.26)\qquad \begin{aligned} \int_{U_0\backslash G_0} \varphi(g^{-1}u_\pm g)\,dg = &\pm \frac{i}{2\pi}\left\{\frac{1}{2}(\Theta_0^+(\varphi) - \Theta_0^-(\varphi))\right. \\ &\left. + \sum_{m=2}^{\infty} (\Theta_m^+(\varphi) - \Theta_m^-(\varphi))\right\} + \frac{1}{8\pi}\int_{-\infty}^{\infty} \Theta_\lambda^+(\varphi)\,d\lambda. \end{aligned}$$

We apply this formula to each $f_j \in \mathscr{C}^p(G_0, \sigma_j)$, $j = 1, \cdots, n$, and compute in this way the second integral occurring in (4.22). Now we turn to the first integral in (4.22). We introduce the following distributions on G_0: For $f \in \mathscr{C}^1(G_0)$ and $u \in U_0$, $u \neq 1$, we set

$$F(f, u) = \int_{A_0} f(a^{-1}ua)e^{-2\ln a}\, da,$$

$$G(f, u) = \int_{A_0} \ln(a) f(a^{-1}ua)e^{-2\ln a}\, da.$$

As above we obtain

$$\begin{aligned}(4.27) \qquad &\int_M \int_A \ln(a) f(m^{-1}a^{-1}u(\varepsilon)am)e^{-2\ln a}\, da\, dm \\ &= \sum_{j=1}^{n} G(f_j, u(\varepsilon_j)) \prod_{k=1}^{n}{}^{(j)} F(f_k, u(\varepsilon_k)),\end{aligned}$$

where $\Pi^{(j)}$ denotes the product with jth factor deleted. By changing variables we get

$$\begin{aligned}F(f, u_+) + F(f, u_-) &= \int_0^\infty f\left(\begin{pmatrix} 1 & \lambda^{-2} \\ 0 & 1 \end{pmatrix}\right)\frac{d\lambda}{\lambda^3} + \int_0^\infty f\left(\begin{pmatrix} 1 & -\lambda^{-2} \\ 0 & 1 \end{pmatrix}\right)\frac{d\lambda}{\lambda^3} \\ &= \frac{1}{2}\int_{U_0} f(u)\, du.\end{aligned}$$

We assume that f satisfies $f(g) = f(-g)$. If we use the calculations by which we pinned down the constant in (4.24) then we get

$$\int_{U_0} f(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_\lambda^+(f)\, d\lambda.$$

By similar arguments one can show that

$$G(f, u_+) + G(f, u_-) = -\frac{1}{2}\int_{-\infty}^{\infty} (\ln|x|) f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx.$$

Let

$$g_j(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_\lambda^+(f_j)e^{iu\lambda}\, d\lambda.$$

If we sum over all $\varepsilon \in \{\pm 1\}^n$ in (4.27), then we get the following expression for the first sum in (4.22):

$$(4.28) \qquad \frac{a_{-1}}{2^{n-1}} \operatorname{Vol}(\Gamma \cap U \setminus U) \sum_{j=1}^{n} \int_{-\infty}^{\infty} (\ln|x|) f_j\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \prod_{k=1}^{n}{}^{(j)} g_k(0).$$

(vii) *The intertwining and the residual term.* By our assumption we have $f \in \mathscr{C}^p(G, \sigma)$, where $\sigma|_{K_M} = 1$. Let $\mathscr{H}_{\vartheta,s}(\sigma)$ be the σth-isotypic subspace. Then $\pi_{\vartheta,s}(f)$ maps $\mathscr{H}_{\vartheta,s}$ into $\mathscr{H}_{\vartheta,s}(\sigma)$. Since $\sigma|_{K_M} = 1$, we have by §3 that

$$\mathscr{H}_{\vartheta,s}(\sigma) = \bigoplus_{\chi\in\vartheta} L^2(\Gamma_M\backslash M, \sigma, \chi) = \bigoplus_{\chi\in\vartheta} L^2(\Gamma_M^0\backslash M^0, \chi).$$

The restriction of the intertwining operator $C_\vartheta(s)$ to $\mathscr{H}_{\vartheta,s}(\sigma)$ coincides with the operator $\oplus_{\chi\in\vartheta} C(\chi, \sigma, s)$, where $C(\chi, \sigma, s)$ is the operator (3.11). There exists $\omega \in \mathbf{Z}^{n-1}$ such that $L^2(\Gamma_M^0\backslash M^0, \chi) = \mathbf{C}\tau_\omega$, where τ_ω is the character (3.16). We introduce the quasicharacter $\chi_{\omega,s}\colon P \to \mathbf{C}^\times$ by $\chi_{\omega,s}(uam) = \tau_\omega(m)e^{s\ln a}$. Let $\pi_{\omega,s} = \operatorname{Ind}_P^G(\chi_{\omega,s})$ and let $\Theta_{\omega,s}$ be the character of $\pi_{\omega,s}$. Then the trace of $\pi_{\vartheta,s}(f)(d/ds)C_\vartheta(s)C_\vartheta(-s)$ coincides with the trace of this operator restricted to $\mathscr{H}_{\vartheta,s}(\sigma)$ and this trace is equal to

$$\Theta_{\omega,s}(f)\frac{d}{ds}C_\omega(\sigma, s)C_\omega(\sigma, -s) + \Theta_{-\omega,s}(f)\frac{d}{ds}C_{-\omega}(\sigma, s)C_{-\omega}(\sigma, -s)$$

if $\omega \neq 0$, and

$$\Theta_{\omega,s}(f)\frac{d}{ds}C_\omega(\sigma, s)C_\omega(\sigma, -s)$$

if $\omega = 0$. Thus, the intertwining term is

$$\frac{1}{4\pi}\sum_{\omega\in\mathbf{Z}^{n-1}}\int_{\operatorname{Re}(s)=0}\Theta_{\omega,s}(f)\frac{d}{ds}C_\omega(\sigma, s)C_\omega(\sigma, -s)\,|ds|. \tag{4.29}$$

In the same way one can show that the residual term is given by

$$-\tfrac{1}{4}\Theta_{0,0}(f)C_0(\sigma, 0). \tag{4.30}$$

5. The index of the signature and the Dolbeault operator

Let $\Gamma \subset G$ be as in §3 and assume that $n = 2p$, $p \in \mathbf{N}$. We shall now investigate the L^2-index of the signature operator on $\Gamma\backslash\mathbf{H}^n$ by using Selberg's trace formula. Since Γ may have elements of finite order, we have to modify the usual definition of the signature operator. Let $\Lambda^*(\Gamma\backslash\mathbf{H}^n)$ be the space of C^∞-differential forms on $\mathbf{H}^n$ which are Γ-invariant. According to §1, $\Lambda^*(\Gamma\backslash\mathbf{H}^n)$ can be identified with the space $C^\infty(\Gamma\backslash G, \Lambda^*\operatorname{Ad}_{\mathfrak{p}}^*)$. By $\Lambda_0^*(\Gamma\backslash\mathbf{H}^n)$ we denote the subspace of $\Lambda^*(\Gamma\backslash\mathbf{H}^n)$, consisting of forms with compact supports mod Γ. Since Γ acts by isometries on $\mathbf{H}^n$, it follows that $\Lambda^*(\Gamma\backslash\mathbf{H}^n)$ is invariant under the Hodge $*$-operator. Let τ be the involution on $\Lambda^*(\Gamma\backslash\mathbf{H}^n)$ defined by $\tau\Phi = i^{p(p-1)+n} * \Phi$ for $\Phi \in \Lambda^p(\Gamma\backslash\mathbf{H}^n)$. The

± 1-eigenspaces of τ are denoted by $\Lambda^*_\pm = \Lambda^*_\pm(\Gamma \setminus \mathbf{H}^n)$. Let δ be the codifferential on $\Lambda^*(\Gamma \setminus \mathbf{H}^n)$. $d + \delta$ anti-commutes with τ and its restriction to $\Lambda^*_+(\Gamma \setminus \mathbf{H}^n)$ is by definition the signature operator

$$D\colon \Lambda^*_+(\Gamma \setminus \mathbf{H}^n) \to \Lambda^*_-(\Gamma \setminus \mathbf{H}^n). \tag{5.1}$$

Let $\mathscr{D} \subset \mathbf{H}^n$ be a fundamental domain of Γ. The L^2-norm of $\Phi \in \Lambda^*(\Gamma \setminus \mathbf{H}^n)$ is defined as

$$\|\Phi\|^2 = \int_{\mathscr{D}} \Phi \wedge *\overline{\Phi}.$$

We denote by $L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$ the Hilbert space of L^2-forms. $L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$ can be identified with $L^2(\Gamma \setminus G, \Lambda^* \operatorname{Ad}^*_{\mathfrak{p}})$. By $L^2\Lambda^*_\pm(\Gamma \setminus \mathbf{H}^n)$ we denote the ± 1-eigenspaces of τ, acting on $L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$. Now, consider D with domain $\Lambda^*_{0,+}(\Gamma \setminus \mathbf{H}^n)$, where $\Lambda^*_{0,\pm}(\Gamma \setminus \mathbf{H}^n)$ are the ± 1-eigenspaces of τ restricted to $\Lambda^*_0(\Gamma \setminus \mathbf{H}^n)$, and let $\overline{D}$ be its closure in L^2. The L^2-index of the signature operator is by definition

$$\operatorname{Ind}_{L^2} D = \dim \ker \overline{D} - \dim \operatorname{coker} \overline{D}.$$

We have to show that this number exists. Let $D^*\colon \Lambda^*_-(\Gamma \setminus \mathbf{H}^n) \to \Lambda^*_+(\Gamma \setminus \mathbf{H}^n)$ be the formal adjoint operator to D and let $\overline{D}^*$ be the closure in L^2 of D^* acting on $\Lambda^*_{0,-}(\Gamma \setminus \mathbf{H}^n)$. Note that D^* is the restriction of $d + \delta$ to $\Lambda^*_-(\Gamma \setminus \mathbf{H}^n)$. $\overline{D}^*$ is the adjoint operator to $\overline{D}$. If Γ has no elements of finite order, then $\Gamma \setminus \mathbf{H}^n$ is a complete Riemannian manifold and the assertion is a consequence of the results of [12]. In general, Γ has a normal subgroup Γ_1 of finite index which contains no elements of finite order [36]. Since $\Gamma_1 \setminus \Gamma$ is finite, we have

$$L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n) = L^2\Lambda^*(\Gamma_1 \setminus \mathbf{H}^n)^{\Gamma_1 \setminus \Gamma} \subset L^2\Lambda^*(\Gamma_1 \setminus \mathbf{H}^n)$$

and this reduces our problem to the torsion free case. Let

$$\Delta^+ = D^*D \quad \text{and} \quad \Delta^- = DD^*.$$

$\Delta^\pm$ are the Laplacians on $\Lambda^*_\pm(\Gamma \setminus \mathbf{H}^n)$. By using the same arguments as above, we obtain that $\Delta^\pm$, acting on $\Lambda^*_{0,\pm}(\Gamma \setminus \mathbf{H}^n)$, is essentially selfadjoint. We use the same notation $\Delta^\pm$ for the unique selfadjoint extension to $L^2\Lambda^*_\pm(\Gamma \setminus \mathbf{H}^n)$. Then $\ker \Delta^+ = \ker \overline{D}$ and $\ker \Delta^- = \ker \overline{D}^*$. Let

$$\mathscr{H}^*_{(2)}(\Gamma \setminus \mathbf{H}^n) = \left\{\Phi \in \Lambda^*(\Gamma \setminus \mathbf{H}^n) \,|\, \Delta\Phi = 0,\ \|\Phi\|_{L^2} < \infty\right\}.$$

This is the space of Γ-invariant square integrable harmonic forms on $\mathbf{H}^n$. The involution acts on $\mathscr{H}^*_{(2)}(\Gamma \setminus \mathbf{H}^n)$ and we denote by $\mathscr{H}^*_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ the corresponding ± 1-eigenspaces. Then $\ker \Delta^\pm = \mathscr{H}^*_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$.

Proposition 5.2. *The spaces $\mathscr{H}^*_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ are finite dimensional and the L^2-index of the signature operator D is given by*

$$\operatorname{Ind}_{L^2} D = \dim \mathscr{H}^*_{(2),+}(\Gamma \setminus \mathbf{H}^n) - \dim \mathscr{H}^*_{(2),-}(\Gamma \setminus \mathbf{H}^n).$$

Proof. It follows from Theorem 5.5 in [10] that $\mathscr{H}^{*}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ are finite dimensional. Above we have seen that $\ker \overline{D} = \mathscr{H}^{*}_{(2),+}(\Gamma \setminus \mathbf{H}^n)$ and $\ker \overline{D}^* = \mathscr{H}^{*}_{(2),-}(\Gamma \setminus \mathbf{H}^n)$. This proves the second statement. q.e.d.

Let $\mathscr{H}^{p}_{(2)}(\Gamma \setminus \mathbf{H}^n)$ be the space of Γ-invariant square integrable harmonic p-forms on $\mathbf{H}^n$. Since $\tau\colon \mathscr{H}^{p}_{(2)}(\Gamma \setminus \mathbf{H}^n) \to \mathscr{H}^{2n-p}_{(2)}(\Gamma \setminus \mathbf{H}^n)$, it follows that $\mathscr{H}^{p}_{(2)}(\Gamma \setminus \mathbf{H}^n) \oplus \mathscr{H}^{2n-p}_{(2)}(\Gamma \setminus \mathbf{H}^n)$, $0 \leqslant p < n$, and $\mathscr{H}^{n}_{(2)}(\Gamma \setminus \mathbf{H}^n)$ are invariant under τ. Let $\mathscr{H}^{p}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ and $\mathscr{H}^{n}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ be the corresponding ± 1-eigenspaces of τ. Then

$$\mathscr{H}^{*}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n) = \bigoplus_{0 \leqslant p \leqslant n} \mathscr{H}^{p}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n).$$

Moreover, if $0 \leqslant p < n$, then

$$\mathscr{H}^{p}_{(2),\pm}(\Gamma \setminus \mathbf{H}^n) = \left\{ \Phi \pm \tau\Phi \,|\, \Phi \in \mathscr{H}^{p}_{(2)}(\Gamma \setminus \mathbf{H}^n) \right\}.$$

Thus $\dim \mathscr{H}^{p}_{(2),+}(\Gamma \setminus \mathbf{H}^n) = \dim \mathscr{H}^{p}_{(2),-}(\Gamma \setminus \mathbf{H}^n)$ for $p < n$ and therefore

$$\text{(5.3)} \qquad \operatorname{Ind}_{L^2} D = \dim \mathscr{H}^{n}_{(2),+}(\Gamma \setminus \mathbf{H}^n) - \dim \mathscr{H}^{n}_{(2),-}(\Gamma \setminus \mathbf{H}^n).$$

We can continue now as in the compact case. Let $L^2_d\Lambda^*_{\pm}(\Gamma \setminus \mathbf{H}^n) \subset L^2\Lambda^*_{\pm}(\Gamma \setminus \mathbf{H}^n)$ be the subspace which is spanned by the eigenforms of $\Delta^{\pm}$. By Theorem 5.5 of [10], the eigenspaces of $\Delta^{\pm}$ are all finite dimensional. D carries eigenforms into eigenforms with the same eigenvalue and it defines an isomorphism on the eigenspaces which correspond to nonzero eigenvalues. Let $\Delta^{\pm}_d$ be the restriction of $\Delta^{\pm}$ to $L^2_d\Lambda^*_{\pm}(\Gamma \setminus \mathbf{H}^n)$ and consider the corresponding heat operators $\exp(-t\Delta^{\pm}_d)$, $t > 0$. We will show that $\exp(-t\Delta^{\pm}_d)$ are trace class operators. Thus

$$\text{(5.4)} \qquad \operatorname{Ind}_{L^2} D = \operatorname{Tr}\left(\exp(-t\Delta^{+}_d)\right) - \operatorname{Tr}\left(\exp(-t\Delta^{-}_d)\right).$$

As in the compact case there are kernels which represent the heat operators $\exp(-t\Delta^{\pm}_d)$. They are obtained from the kernels of the heat operators $\exp(-t\Delta^{\pm})$ by subtracting the continuous part. The heat kernels we are considering are closely related to the spinor heat kernels studied by Barbasch and Moscovici in [6]. We shall use the results of [6] to determine the relevant properties of our heat kernels.

Let $\tau\colon \Lambda^*\mathfrak{p}_{\mathbf{C}} \to \Lambda^*\mathfrak{p}_{\mathbf{C}}$ be the involultion defined by $\tau X = i^{p(p-1)+n} * X$, if $X \in \Lambda^p\mathfrak{p}_{\mathbf{C}}$, and let $\Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}$ be the ± 1-eigenspaces of τ. $\Lambda^* \operatorname{Ad}^*_{\mathfrak{p}}$ decomposes into two representations

$$\text{(5.5)} \qquad \sigma^{\pm}\colon K \to \operatorname{GL}(\Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}).$$

Let $\tilde{\Delta}^{\pm}$ be the Laplacians on $\Lambda^*_{\pm}(\mathbf{H}^n) = (C^{\infty}(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})^K$. $\tilde{\Delta}^{\pm}$ is the restriction of $-R(\Omega) \otimes \operatorname{Id}_{\Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}}$ to the K-invariant part of $C^{\infty}(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}$. If we restrict $\tilde{\Delta}^{\pm}$ to $(C^{\infty}_0(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})^K$, then it has a unique selfadjoint extension to an unbounded operator in $L^2\Lambda^*_{\pm}(\mathbf{H}^n) = (L^2(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})^K$, for which we use the same notation $\tilde{\Delta}^{\pm}$ (see [29, Corollary 1.2]). For each $t > 0$, the heat operator $\exp(-t\tilde{\Delta}^{\pm})$ is a G-invariant smoothing operator. Therefore, there exists

$$h_t^{\pm}: G \to \operatorname{End}(\Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})$$

which is in $C^\infty \cap L^2$ and which satisfies (1.4) with respect to $\sigma^{\pm}$, such that

$$\exp(-t\tilde{\Delta}^{\pm})\Phi(g) = \int_G h_t^{\pm}(g^{-1}g')\Phi(g')\,dg' \tag{5.6}$$

for $\Phi \in (L^2(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})^K$. We have to show that $h_t^{\pm} \in \mathscr{C}^p(G, \sigma^{\pm})$, $0 < p < 1$, where $\sigma^{\pm}$ is the representation (5.5), and that $\exp(-t\Delta_d^{\pm}) = R^d_{\Gamma\backslash G}(h_t^{\pm})$. For this purpose we consider the Dirac operator. We choose $\mathfrak{h}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}}$ as a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. The vectors $E_j^{\pm} \in \mathfrak{p}_{\mathbf{C}}$, $j = 1, \cdots, n$, defined by (1.1), are the nonzero root vectors and all roots are noncompact. The system of positive roots ψ is chosen as in §1. Moreover, $\rho = \frac{1}{2}\sum_{\alpha \in \psi} \alpha$ and W is the Weyl group of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Now, let

$$s^{\pm}: \operatorname{Spin}(\mathfrak{p}) \to \operatorname{GL}(S^{\pm})$$

be the half-spin representations [34] and let $s^{\pm}: \mathfrak{so}(\mathfrak{p}_{\mathbf{C}}) \to \operatorname{End}(S^{\pm})$ be the differential. Via ad, $\mathfrak{k}_{\mathbf{C}}$ operates on $\mathfrak{p}_{\mathbf{C}}$. When $\mathfrak{p}_{\mathbf{C}}$ is endowed with the Killing form, this action becomes skew symmetric.

$$\operatorname{ad}: \mathfrak{k}_{\mathbf{C}} \to \mathfrak{so}(\mathfrak{p}_{\mathbf{C}}).$$

Let

$$\tau^{\pm}: \mathfrak{k}_{\mathbf{C}} \to \operatorname{End}(S^{\pm})$$

be defined by $\tau^{\pm} = s^{\pm} \circ \operatorname{ad}$. For $w \in W$ let $V_{w\rho}$ be the irreducible $\mathfrak{k}_{\mathbf{C}}$-module with weight $w\rho$ and let

$$\tau_{w\rho}: \mathfrak{k}_{\mathbf{C}} \to \operatorname{End}(V_{w\rho})$$

be the corresponding representation. Then we have

$$S^{\pm} = \bigoplus_{\substack{w \in W \\ \det(w) = \pm 1}} V_{w\rho} \tag{5.7}$$

as $\mathfrak{k}_{\mathbf{C}}$-modules (see [34, Lemma 2.2]). Now, let $\mathfrak{p}_{\pm} = \oplus_{j=1}^n \mathbf{C}E_j^{\pm}$. Then $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. We consider $\Lambda^*\mathfrak{p}_-$. Since $E_{i_1}^- \wedge \cdots \wedge E_{i_p}^-$, $1 \leqslant i_1 < \cdots < i_p \leqslant n$, is a basis of $\Lambda^*\mathfrak{p}_-$, it follows from (1.1) that the weights of the $\mathfrak{k}_{\mathbf{C}}$-module $\Lambda^*\mathfrak{p}_-$ are given by $\{-\alpha_{i_1} - \cdots - \alpha_{i_p} | 1 \leqslant i_1 < \cdots < i_p \leqslant n\}$. Moreover, for each weight $-\alpha_{i_1} - \cdots - \alpha_{i_p}$ there exists a unique $w \in W$ such that $\rho - \alpha_{i_1} - \cdots - \alpha_{i_p} = w\rho$. Thus, the weights of $V_\rho \otimes \Lambda^*\mathfrak{p}_-$ are given by $\{w\rho | w \in W\}$ and (5.7) implies that

$$S^+ \oplus S^- = V_\rho \otimes \Lambda^*\mathfrak{p}_- \tag{5.8}$$

as $\mathfrak{k}_{\mathbf{C}}$-modules. In the same way one can show that

$$S^+ \oplus S^- = V_{-\rho} \otimes \Lambda^* \mathfrak{p}_+ \tag{5.9}$$

as $\mathfrak{k}_{\mathbf{C}}$-modules. Since $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, we get

$$\Lambda^* \mathfrak{p}_{\mathbf{C}} = (\Lambda^* \mathfrak{p}_+) \otimes (\Lambda^* \mathfrak{p}_-) = (S^+ \oplus S^-) \otimes (S^+ \oplus S^-).$$

On the other hand, it is known [4] that

$$[\Lambda^*_+ \mathfrak{p}_{\mathbf{C}}] - [\Lambda^*_- \mathfrak{p}_{\mathbf{C}}] = [S^+ \oplus S^-]([S^+] - [S^-])$$

in the representation ring $R(\mathfrak{k}_{\mathbf{C}})$. Therefore

$$\Lambda^*_\pm \mathfrak{p}_{\mathbf{C}} = (S^+ \oplus S^-) \otimes S^\pm \tag{5.10}$$

as $\mathfrak{k}_{\mathbf{C}}$-modules. For $w \in W$ we set

$$E_w^\pm = V_{w\rho} \otimes S^\pm, \tag{5.11}$$

where $V_{w\rho}$ is defined above. The representation $\tau_{w\rho} \otimes s^\pm \colon \mathfrak{k}_{\mathbf{C}} \to \operatorname{End}(E_w^\pm)$ lifts to a representation of K. Each $X \in \mathfrak{p}_{\mathbf{C}}$ defines a map $c(X)\colon S^\pm \to S^\mp$ which is the Clifford multiplication by X. Let $\{X_1, \cdots, X_{2n}\}$ be an orthonormal basis of $\mathfrak{p}$ and set

$$D_w^\pm = \sum_{i=1}^{2n} X_i \otimes \operatorname{Id}_{V_{w\rho}} \otimes c(X_i). \tag{5.12}$$

Then $D_w^\pm \in \mathfrak{U}(\mathfrak{g}_{\mathbf{C}}) \otimes \operatorname{Hom}(E_w^\pm, E_w^\mp)$ is K-invariant. Therefore, it defines a G-invariant first-order differential operator $\mathscr{D}_w^\pm$ from $(C^\infty(G) \otimes E_w^\pm)^K$ to $(C^\infty(G) \otimes E_w^\mp)^K$ by

$$\mathscr{D}_w^\pm = \sum_{i=1}^{2n} R(X_i) \otimes \operatorname{Id}_{V_{w\rho}} \otimes c(X_i). \tag{5.13}$$

$\mathscr{D}_w^\pm$ is the Dirac operator. Let $\tilde{\Delta}_w^\pm = \mathscr{D}_w^\mp \mathscr{D}_w^\pm$. We use Proposition 3.1 of [34] to compute $\tilde{\Delta}_w^\pm$. In our case we have $\rho = \rho_n$, $\rho_c = 0$ and $\|w\rho\|^2 = \|\rho\|^2$. Thus, we get

$$\tilde{\Delta}_w^\pm = -R(\Omega) \otimes \operatorname{Id}_{E_w^\pm}. \tag{5.14}$$

But $-R(\Omega) \otimes \operatorname{Id}_{E_w^\pm}$ is the restriction of $-R(\Omega) \otimes \operatorname{Id}_{\Lambda^*_\pm \mathfrak{p}}$ to the subspace $(C^\infty(G) \otimes E_w^\pm)^K$ and $\tilde{\Delta}^\pm = -R(\Omega) \otimes \operatorname{Id}_{\Lambda^*_\pm \mathfrak{p}}$ by Kuga's Lemma. $\tilde{\Delta}_w^\pm$, restricted to $(C_0^\infty(G) \otimes E_w^\pm)^K$, has a unique selfadjoint extension to an operator in $(L^2(G) \otimes E_w^\pm)^K$ [29]. We shall use the same notation $\tilde{\Delta}_w^\pm$ for this selfadjoint extension. Let $\exp(-t\tilde{\Delta}_w^\pm)$, $t \geqslant 0$, be the semigroup generated by $\tilde{\Delta}_w^\pm$. For each $t > 0$, $\exp(-t\tilde{\Delta}_w^\pm)$ is a G-invariant smoothing operator. Therefore, by §1, there exists a kernel function

$$h_{w,t}^\pm \colon G \to \operatorname{End}(E_w^\pm)$$

which is in $C^\infty \cap L^2$ and which satisfies (1.4) with respect to the representation

$$\tau_{w\rho} \otimes s^{\pm}: K \to \mathrm{GL}\big(E_w^{\pm}\big).$$

If π is a unitary representation of G, Ω the Casimir operator and $D_w^{\pm}$ the operator (5.12), let $\pi(\Omega)$ and $\pi(D_w^{\pm})$ be the operators defined by (1.3). One can generalize Proposition 3.1 of [34] to the case of any unitary representation π (see [6, 1.3.6]). If we apply this formula to our situation, then we get

$$\pi\big(D_w^{\mp}\big)\pi\big(D_w^{\pm}\big) = -\pi(\Omega) \otimes \mathrm{Id}_{E_w^{\pm}}, \tag{5.15}$$

because $\rho_c = 0$ and $\|w\rho\|^2 = \|\rho\|^2$. Now, let

$$\mathscr{D}_{w,d}^{\pm} = R_{\Gamma\backslash G}^{d}\big(D_w^{\pm}\big).$$

$\mathscr{D}_{w,d}^{\pm}$ is an operator from $(L_d^2(\Gamma \backslash G) \otimes E_w^{\pm})^K$ to $(L_d^2(\Gamma \backslash G) \otimes E_w^{\mp})^K$. Furthermore, let

$$\Delta_{w,d}^{\pm} = -R_{\Gamma\backslash G}^{d}(\Omega) \otimes \mathrm{Id}_{E_w^{\pm}}.$$

It follows from (5.15) that

$$\mathscr{D}_{w,d}^{\mp}\mathscr{D}_{w,d}^{\pm} = \Delta_{w,d}^{\pm}.$$

If we apply Proposition 2.1 of [6], then we get

$$\exp\big(-t\Delta_{w,d}^{\pm}\big) = R_{\Gamma\backslash G}^{d}\big(h_{w,t}^{\pm}\big). \tag{5.16}$$

On the other hand, we have $\Delta_d^{\pm} = -R_{\Gamma\backslash G}^{d}(\Omega) \otimes \mathrm{Id}_{\Lambda_{\pm}^{*}\mathfrak{p}}$, and, by (5.8) and (5.10),

$$\begin{aligned} L_d^2\Lambda_{\pm}^{*}(\Gamma \backslash \mathbf{H}^n) &= \big(L_d^2(\Gamma \backslash G) \otimes \Lambda_{\pm}^{*}\mathfrak{p}_{\mathbf{C}}\big)^K \\ &= \bigoplus_{w \in W} \big(L_d^2(\Gamma \backslash G) \otimes E_w^{\pm}\big)^K. \end{aligned}$$

Therefore, $\Delta_{w,d}^{\pm}$ is the restriction of $\Delta_d^{\pm}$ to the subspace $(L_d^2(\Gamma \backslash G) \otimes E_w^{\pm})^K$. Let P_w be the orthogonal projection of $L_d^2\Lambda_{\pm}^{*}(\Gamma \backslash \mathbf{H}^n)$ onto $(L_d^2(\Gamma \backslash G) \otimes E_w^{\pm})^K$. Then it follows from these remarks that

$$\exp\big(-t\Delta_d^{\pm}\big) = \sum_{w \in W} \exp\big(-t\Delta_{w,d}^{\pm}\big)P_w. \tag{5.17}$$

Let $p_w: \Lambda_{\pm}^{*}\mathfrak{p}_{\mathbf{C}} \to E_w^{\pm}$ be the orthogonal projection with respect to the identification of $\Lambda_{\pm}^{*}\mathfrak{p}_{\mathbf{C}}$ with $(S^{+} \oplus S^{-}) \otimes S^{\pm}$ by (5.10). It is clear that

$$h_t^{\pm}(g) = \sum_{w \in W} h_{w,t}^{\pm}(g) \circ p_w, \tag{5.18}$$

where $h_t^{\pm}$ is the kernel of $\exp(-t\tilde{\Delta}^{\pm})$. Then (5.16) and (5.17) imply that

$$\exp\big(-t\Delta_d^{\pm}\big) = R_{\Gamma\backslash G}^{d}\big(h_t^{\pm}\big). \tag{5.19}$$

Moreover, it follows from Proposition 2.4 of [6] that $h^{\pm}_{w,t} \in \mathscr{C}^p(G, \tau_{w\rho} \otimes s^{\pm})$ for all $p > 0$. Thus, by (5.18), $h^{\pm}_t \in \mathscr{C}^p(G, \sigma^{\pm})$ for all $p > 0$. If we use Proposition 4.6, then we can summarize our results by

Theorem 5.20. *Let $h^{\pm}_t$ be the kernel of the heat operator* $\exp(-t\tilde{\Delta}^{\pm})$, *acting on* $L^2\Lambda^*_{\pm}(\mathbf{H}^n)$. *Then $h^{\pm}_t \in \mathscr{C}^p(G, \sigma^{\pm})$ for each $p > 0$, where $\sigma^{\pm}$ is the representation* (5.5). *If $\Delta^{\pm}_d$ is the restriction of $\Delta^{\pm}$ to the subspace $L^2_d\Lambda^*_{\pm}(\Gamma \setminus \mathbf{H}^n)$, then*

$$\exp\left(-t\Delta^{\pm}_d\right) = R^d_{\Gamma\setminus G}\left(h^{\pm}_t\right),$$

and $R^d_{\Gamma\setminus G}(h^{\pm}_t)$ is a trace class operator.

Corollary 5.21. *Let D: $\Lambda^*_+(\Gamma \setminus \mathbf{H}^n) \to \Lambda^*_-(\Gamma \setminus \mathbf{H}^n)$ be the signature operator. Then its L^2-index is given by*

$$\operatorname{Ind}_{L^2} D = \operatorname{Tr} R^d_{\Gamma\setminus G}\left(\operatorname{tr} h^+_t\right) - \operatorname{Tr} R^d_{\Gamma\setminus G}\left(\operatorname{tr} h^-_t\right).$$

Proof. Since $\exp(-t\Delta^{\pm}_d)$ are trace class operators, we can use (5.4). The corollary follows from Theorem 5.20 and Proposition 4.6. q.e.d.

Let $h_t = \operatorname{tr} h^+_t - \operatorname{tr} h^-_t$. Then, by Corollary 5.21, we have

$$\operatorname{Ind}_{L^2} D = \operatorname{Tr} R^d_{\Gamma\setminus G}(h_t). \tag{5.22}$$

We shall now use Selberg's trace formula to compute the right-hand side of (5.22). For this purpose we have to describe the function h_t explicitly. This problem can be reduced to the description of the heat kernel on the upper half-plane.

The representations (5.5) can be decomposed into one-dimensional representations:

$$\sigma^{\pm} = \bigoplus_{\chi \in \hat{K}} [\chi : \sigma^{\pm}]\chi,$$

where $[\chi : \sigma^{\pm}]$ denotes the multiplicity of the character χ in $\sigma^{\pm}$. For $\chi \in \hat{K}$ let

$$L^2(G, \chi) = \left\{ f \in L^2(G) \mid f(gk^{-1}) = \chi(k) f(g), k \in K \right\}.$$

Moreover, let $E_{\pm}(\chi) \subset \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}$ be the χ-isotypical subspace. Then

$$\left(L^2(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}\right)^K = \bigoplus_{\chi \in \hat{K}} \left(L^2(G, \chi) \otimes E_{\pm}(\chi)\right).$$

$L^2(G, \chi) \subset L^2(G)$ is invariant under $R(\Omega)$, because $\Omega \in \mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$. Let $\tilde{\Delta}_{\chi} = -R(\Omega)|_{L^2(G,\chi)}$ and let $P^{\pm}_{\chi}$ be the orthogonal projection of $(L^2(G) \otimes \Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}})^K$ onto the subspace $L^2(G, \chi) \otimes E_{\pm}(\chi)$. Then it is clear that

$$\exp(-t\tilde{\Delta}^{\pm}) = \sum_{\chi \in \hat{K}} \left(\exp\left(-t\tilde{\Delta}_{\chi}\right) \otimes \operatorname{Id}_{E_{\pm}(\chi)}\right) \cdot P^{\pm}_{\chi}. \tag{5.23}$$

$\exp(-t\tilde{\Delta}_{\chi})$ is a G-invariant smoothing operator. Let $h_t^{\chi} \in C^{\infty}(G)$ be its kernel and let $p_{\chi}^{\pm}$ be the orthogonal projection of $\Lambda^*_{\pm}\mathfrak{p}_{\mathbf{C}}$ onto $E_{\pm}(\chi)$. It follows from (5.23) that

$$h_t^{\pm} = \sum_{\chi} \left(h_t^{\chi} \otimes \mathrm{Id}_{E_{\pm}(\chi)} \right) \circ p_{\chi}^{\pm}.$$

Therefore

$$\operatorname{tr} h_t^{+} - \operatorname{tr} h_t^{-} = \sum_{\chi \in \hat{K}} ([\chi : \sigma^{+}] - [\chi : \sigma^{-}]) h_t^{\chi}. \tag{5.24}$$

On the other hand, we have

$$\operatorname{tr} \sigma^{+} - \operatorname{tr} \sigma^{-} = \sum_{\chi \in \hat{K}} ([\chi : \sigma^{+}] - [\chi : \sigma^{-}]) \chi. \tag{5.25}$$

Consider the half-spin representations $\tau^{\pm}: \mathfrak{k}_{\mathbf{C}} \to \mathrm{End}(S^{\pm})$. Then, by (5.10), we get $\sigma^{\pm} = (\tau^{+} \oplus \tau^{-}) \otimes \tau^{\pm}$. Thus $\operatorname{tr} \sigma^{+} - \operatorname{tr} \sigma^{-} = (\operatorname{tr} \tau^{+} + \operatorname{tr} \tau^{-})(\operatorname{tr} \tau^{+} - \operatorname{tr} \tau^{-})$ and, using (5.7), we get

$$\begin{aligned} \operatorname{tr} \sigma^{+} - \operatorname{tr} \sigma^{-} &= \prod_{\alpha \in \psi} (e^{\alpha/2} + e^{-\alpha/2}) \prod_{\alpha \in \psi} (e^{\alpha/2} - e^{-\alpha/2}) \\ &= \prod_{\alpha \in \psi} (e^{\alpha} - e^{-\alpha}) = \sum_{w \in W} \det(w) e^{2w\rho}. \end{aligned} \tag{5.26}$$

Let $\omega: \mathrm{SO}(2) \to \mathbf{C}^{\times}$ be defined by $\omega(k(\theta)) = e^{2i\theta}$. For each $w \in W$ we define the character $\chi_w \in \hat{K}$ by

$$\chi_w(k) = \prod_{j=1}^{n} \omega\left((k_j)^{w_j} \right), \tag{5.27}$$

where w_j is the jth component of w. Put $h_t^w = h_t^{\chi_w}$. If we combine (5.24)–(5.26), we get

$$\operatorname{tr} h_t^{+} - \operatorname{tr} h_t^{-} = \sum_{w \in W} \det(w) h_t^w. \tag{5.28}$$

This together with Corollary 5.21 gives

Proposition 5.29. *The L^2-index of the signature operator is given by*

$$\operatorname{Ind}_{L^2} D = \sum_{w \in W} \det(w) \operatorname{Tr} R^d_{\Gamma \backslash G}(h_t^w).$$

Our problem now is to determine h_t^w. Let $\chi \in \hat{K}$. Then $\chi = \otimes_{j=1}^{n} \chi_j$, where $\chi_j \in \hat{K}_0$. Thus

$$L^2(G, \chi) = \widehat{\bigotimes_{j=1}^{n}} L^2(G_0, \chi_j)$$

and $R(\Omega) = \Sigma_{j=1}^n R(\Omega_j)$, where Ω_j is the Casimir operator of the jth component of G. Therefore, it is sufficient to consider the case of $\mathrm{SL}(2,\mathbb{R})$. For $l \in \mathbb{Z}$ let σ_l: $\mathrm{SO}(2) \to \mathbb{C}^\times$ be defined by $\sigma_l(k(\theta)) = e^{2il\theta}$. $L^2(G_0, \sigma_l) \subset L^2(G_0)$ is invariant under $R(\Omega)$. Let Δ_l be the restriction of $-R(\Omega)$ to the subspace $L^2(G_0, \sigma_l)$. Δ_l is the Laplacian on the space of automorphic forms of weight l on the upper half-plane [16]. With respect to the coordinates (1.2), Δ_l is given by

$$\Delta_l = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2il\frac{\partial}{\partial x}.$$

Let $p_t^{(l)} \in C^\infty(G_0)$ be the kernel of the heat operator $\exp(-t\Delta_l)$, $t > 0$. For $w \in W$ let $I_w = \{i | w_i = \mathrm{Id}\}$, where $w = (w_1, \cdots, w_n)$. We shall write $p_t^\pm$ instead of $p_t^{(\pm 1)}$. From the considerations above it follows that

$$h_t^w(g) = \prod_{i \in I_w} p_t^+(g_i) \prod_{j \in \bar{I}_w} p_t^-(g_j). \tag{5.30}$$

We continue with the study of the kernel $p_t^{(l)}$. It is easy to relate Δ_l to a certain spinor Laplacian by using the same arguments as above. Let $(\mathfrak{h}_0)_\mathbb{C} = (\mathfrak{k}_0)_\mathbb{C}$ be the Cartan algebra of $(\mathfrak{g}_0)_\mathbb{C}$ and let α be the root which is given by $\alpha(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) = 2i$. Let $s_0^\pm$: $(\mathfrak{k}_0)_\mathbb{C} \to \mathrm{End}(S_0^\pm)$ be the representations induced by the half-spin representations. The weight of the $(\mathfrak{k}_0)_\mathbb{C}$-module $S_0^\pm$ is $\pm\alpha/2$. Let V_l be the $(\mathfrak{k}_0)_\mathbb{C}$-module with weight $(l - 1/2)\alpha$ and let $F_l^\pm = V_l \otimes S_0^\pm$. Then $L^2(G_0, \sigma_l) = (L^2(G) \otimes F_l^\pm)^K$. Let

$$D_l^\pm\colon \left(L^2(G_0) \otimes F_l^\pm\right)^K \to \left(L^2(G_0) \otimes F_l^\mp\right)^K$$

be the Dirac operator [34]. Then by Proposition 3.1 of [34]

$$\begin{aligned} D_l^- \circ D_l^+ &= -R(\Omega) \otimes \mathrm{Id}_{F_l^\pm} + 4|l(l-1)|\mathrm{Id} \\ &= \Delta_l + 4|l(l-1)|\mathrm{Id}. \end{aligned}$$

Thus $\exp(-t\Delta_l) = \exp(t4|l(l-1)|)\exp(-tD_l^- \circ D_l^+)$. If we apply Proposition 2.4 of [6], we get $p_t^{(l)} \in \mathscr{C}^p(G_0, \sigma_l)$ for all $p > 0$. Since $p_t^{(l)}$ is the kernel of the heat operator $\exp(-t\Delta_l)$ it has the following properties:

(i) $\partial p_t^{(l)}/\partial t = -R(\Omega)p_t^{(l)}$.

(ii) $p_t^{(l)}$, as $t \to 0$, converges to the Dirac delta measure at 1.

In addition, we have seen that

(iii) $p_t^{(l)} \in \mathscr{C}^p(G_0, \sigma_l)$ for all $p > 0$.

It is known that the characters of the discrete series and of the principal series are tempered distributions (§2). Since $p_t^{(l)} \in \mathscr{C}^p(G_0)$, $p > 0$, we can use the Plancherel formula for $\mathrm{SL}(2,\mathbb{R})$ [25, VIII, §4] to expand $p_t^{(l)}$ in terms of spherical functions. Note that $p_t^{(l)}(-g) = p_t^{(l)}(g)$. This follows from the fact that $p_t^{(l)}$ satisfies $p_t^{(l)}(gk) = \sigma_l(k)p_t^{(l)}(g)$, $k \in K$. Thus, the Plancherel expansion of $p_t^{(l)}$ is given by

$$p_t^{(l)}(g) = \sum_{m=2}^{\infty} \frac{m-1}{2\pi}\left(\operatorname{Tr}\left(\pi_m^+\left(p_t^{(l)}\right)\pi_m^+(g)^*\right) + \operatorname{Tr}\left(\pi_m^-\left(p_t^{(l)}\right)\pi_m^-(g)^*\right)\right)$$

$$+ \frac{1}{4\pi}\int_0^\infty \operatorname{Tr}\left(\pi_\lambda^+\left(p_t^{(l)}\right)\pi_\lambda^+(g)^*\right)\lambda \operatorname{th}\left(\frac{\pi\lambda}{2}\right)d\lambda. \tag{5.31}$$

We have to compute the traces occurring in (5.31). We start with $\operatorname{Tr}(\pi_\lambda^+(p_t^{(l)})\pi_\lambda^+(g)^*)$. Choose $v \in H_\lambda^+$ with $\|v\| = 1$ and $\pi_\lambda^+(k(\theta))v = e^{2il\theta}v$. Since $p_t^{(l)} \in \mathscr{C}^p(G_0, \sigma_l)$, it follows that

$$\operatorname{Tr}\left(\pi_\lambda^+\left(p_t^{(l)}\right)\pi_\lambda^+(g)^*\right) = \left\langle \pi_\lambda^+\left(p_t^{(l)}\right)v, \pi_\lambda^+(g)v\right\rangle.$$

Let $\Phi_\lambda(t, g) = \left\langle \pi_\lambda^+\left(p_t^{(l)}\right)v, \pi_\lambda^+(g)v\right\rangle$ and recall that the Casimir operator acts on H_λ^+ by $\pi_\lambda^+(\Omega) = -(1 + \lambda^2)\mathrm{Id}/4$ [17, I, §3]. Therefore, by (i), we get

$$\frac{\partial}{\partial t}\Phi_\lambda(t, g) = -\frac{1+\lambda^2}{4}\Phi_\lambda(t, g).$$

Moreover, by (ii),

$$\lim_{t\to+0} \pi_\lambda^+\left(p_t^{(l)}\right)v = \lim_{t\to+0}\int_{G_0} p_t^{(l)}(g)\pi_\lambda^+(g)v\,dg = v.$$

This implies

$$\begin{aligned}\operatorname{Tr}\left(\pi_\lambda^+\left(p_t^{(l)}\right)\pi_\lambda^+(g)^*\right) &= \exp\left(-t\frac{1+\lambda^2}{4}\right)\left\langle v, \pi_\lambda^+(g)v\right\rangle \\ &= \exp\left(-t\frac{1+\lambda^2}{4}\right)\overline{\Phi_{\lambda,2l}^+}(g).\end{aligned} \tag{5.32}$$

In the same way one can determine $\operatorname{Tr}(\pi_m^\pm(p_t^{(l)})\pi_m^\pm(g)^*)$. The Casimir operator acts on the discrete series representation $\pi_m^\pm$ by $\pi_m^\pm(\Omega) = m(m-2)\mathrm{Id}/4$, (see §2). Let $H^{(\pm m)}$ be the space of the representation $\pi_m^\pm$ and let $H_p^{(\pm m)} = \{v \in H^{(\pm m)} | \pi_m^\pm(k(\theta))v = e^{ip\theta}v\}$. Then

$$H^{(m)} = \widehat{\bigoplus_{\substack{p\geqslant m\\ p\equiv m\,(2)}}} H_p^{(m)} \quad \text{and} \quad H^{(-m)} = \widehat{\bigoplus_{\substack{p\leqslant -m\\ p\equiv m\,(2)}}} H_p^{(-m)}.$$

Assume that $H_{2l}^{(m)} = 0$ ($H_{2l}^{(-m)} = 0$). Since $p_t^{(l)} \in \mathscr{C}^p(G_0, \sigma_l)$, it follows that $\pi_m^+(p_t^{(l)}) = 0$ ($\pi_m^-(p_t^{(l)}) = 0$). Thus, if $l > 0$ ($l < 0$), only those discrete series representations π_m^+ (π_m^-) can make a nontrivial contribution to the Plancherel expansion of $p_t^{(l)}$, for which $m = 2k$ and $1 \leqslant k \leqslant |l|$. Now let $m = 2k > 0$ be such that $H_{2l}^{(\pm m)} \neq 0$. Choose $v \in H_{2l}^{(\pm m)}$ with $\|v\| = 1$. Then

$$\operatorname{Tr}\left(\pi_m^\pm\left(p_t^{(l)}\right)\pi_m^\pm(g)^*\right) = \left\langle \pi_m^\pm\left(p_t^{(l)}\right)v, \pi_m^\pm(g)v\right\rangle.$$

If we use the properties (i) and (ii) of $p_t^{(l)}$ in the same way as above, we get

(5.33) $$\operatorname{Tr}\left(\pi_{2k}^{\pm}\left(p_t^{(l)}\right)\pi_{2k}^{\pm}(g)^*\right) = e^{k(k-1)t}\overline{\Phi}_{2k,2l}^{\pm}(g),$$

where $\Phi_{2k,2l}^{\pm}$ are the spherical trace functions introduced in §2. Thus, we have proved

Lemma 5.34. *The Plancherel expansion of the kernel $p_t^{(l)}$ of the heat operator* $\exp(-t\Delta_l)$ *is given by*

$$p_t^{(l)}(g) = \sum_{1 \leqslant k \leqslant |l|} \frac{2k-1}{2\pi} e^{k(k-1)t}\left(\overline{\Phi}_{2k,2l}^{+}(g) + \overline{\Phi}_{2k,2l}^{-}(g)\right)$$
$$+ \frac{1}{4\pi}\int_0^\infty \exp\left(-t\frac{1+\lambda^2}{4}\right)\overline{\Phi}_{\lambda,2l}^{+}(g)\lambda\operatorname{th}\left(\frac{\pi\lambda}{2}\right)d\lambda.$$

Moreover, the discrete and the principal series characters have the following values at $p_t^{(l)}$:

$$\Theta_\lambda^{+}\left(p_t^{(l)}\right) = \exp\left(-t\frac{1+\lambda^2}{4}\right), \qquad \Theta_\lambda^{-}\left(p_t^{(l)}\right) = 0,$$

$$\Theta_m^{+}\left(p_t^{(l)}\right) = \begin{cases}\exp(k(k-l)t) & \text{if } l > 0,\ m = 2k,\ 1 \leqslant k \leqslant l,\\ 0 & \text{otherwise},\end{cases}$$

$$\Theta_m^{-}\left(p_t^{(l)}\right) = \begin{cases}\exp(k(k-l)t) & \text{if } l < 0,\ m = 2k,\ 1 \leqslant k \leqslant -l,\\ 0 & \text{otherwise}.\end{cases}$$

Remark. The values of the characters of the discrete and principal series representations at $p_t^{(l)}$ are independent of the choice of the invariant measure on G_0. If the Haar measure dg is multiplied by $C > 0$ then the heat kernel with respect to the measure $C dg$ is $p_t^{(l)}/C$.

We can now evaluate the contribution given by each term in the trace formula to the L^2-index of the signature operator. We shall use the expression for the index which is given by Proposition 5.29 together with formula (5.30) for h_t^w. For the description of the various terms occurring in the trace formula we refer to §4.

(i) *The central contribution.* By Proposition 5.29, the central contribution to $\operatorname{Ind}_{L^2} D$ is given by

$$|Z_\Gamma|\operatorname{Vol}(\Gamma\setminus G)\sum_{w\in W}\det(w)h_t^w(1).$$

It follows from Lemma 5.34 that

$$p_t^{\pm}(1) = \frac{1}{2\pi} + \frac{1}{4\pi}\int_0^\infty \exp\left(-t\frac{1+\lambda^2}{4}\right)\lambda\operatorname{th}\left(\frac{\pi\lambda}{2}\right)d\lambda.$$

Thus, by (5.30), $h_t^w(1)$ is independent of $w \in W$. Since $W = \{\pm 1\}^n$, we get

$$\sum_{w \in W} \det(w) = \sum_{q=0}^{n} (-1)^q \binom{n}{q} = 0. \tag{5.35}$$

Hence, the central contribution to the L^2-index of D is zero.

(ii) *The elliptic contribution.* The elliptic contribution is

$$(E) \sum_{\{\gamma\}_\Gamma} \operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma) \sum_{w \in W} \det(w) \int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg, \tag{5.36}$$

where the sum runs over all elliptic conjugacy classes of Γ. There are two cases depending on whether $-1 \in \Gamma$ or $-1 \notin \Gamma$. First, we assume that $-1 \in \Gamma$. Let $\gamma \in \Gamma$ be elliptic and let γ_0 be a generator of Γ_γ. γ_0 is primitive elliptic. Since $-1 \in \Gamma$, the order of γ_0 is even. Namely, assume that $\gamma \in \Gamma$ is elliptic of order m, m odd. Then $(-\gamma)^{m+1} = \gamma$ and $-\gamma \in \Gamma$ is of order $2m$. Therefore, each elliptic element in Γ has even order. Thus, $\Gamma_\gamma \cong \mathbb{Z}/2l\mathbb{Z}$. We consider the contribution of the elliptic conjugacy classes $\{\gamma_0^q\}$, $\{-\gamma_0^q\}$, $1 \leqslant q < l$, to (5.36). γ_0 is conjugate in G to an element $k \in K$ with

$$k_j = \begin{pmatrix} \cos\frac{\pi}{l} r_j & \sin\frac{\pi}{l} r_j \\ -\sin\frac{\pi}{l} r_j & \cos\frac{\pi}{l} r_j \end{pmatrix}, \qquad (2l, r_j) = 1; j = 1, \cdots, n. \tag{5.37}$$

If we use (4.10) and Lemma 5.34, we get

$$\begin{aligned} \int_{K_0 \backslash G_0} p_t^{\pm}\left(g^{-1}(k_j)^q g\right) dg &= \pm \frac{i}{4\pi \sin(\pi r_j q/l)} \exp\left(\pm i \frac{\pi}{l} r_j q\right) \\ &\quad + \frac{1}{4\pi \sin(\pi r_j q/l)} \int_{-\infty}^{\infty} \exp\left(-t \frac{1+\lambda^2}{4}\right) \frac{\exp(-2\pi r_j q \lambda/l)}{1 + e^{-2\pi\lambda}} d\lambda. \end{aligned} \tag{5.38}$$

Let $b_j(t) = -1/4\pi +$ 2nd summand of (5.38), $j = 1, \cdots, n$. For $w \in W$ let $I_w = \{j | w_j = \mathrm{Id}\}$. Then it follows from (4.9), (5.30) and (5.38) that

$$\begin{aligned} \int_{G_\gamma \backslash G} h_t^w(g^{-1}(\gamma_0)^q g)\, dg &= \prod_{\iota \in I_w} \left(\frac{i}{4\pi} \cot\left(\frac{\pi}{l} r_\iota q\right) + b_\iota(t)\right) \\ &\quad \cdot \prod_{j \in \bar{I}_w} \left(-\frac{i}{4\pi} \cot\left(\frac{\pi}{l} r_j q\right) + b_j(t)\right). \end{aligned}$$

Since $p_t^{\pm}(-g) = p_t^{\pm}(g)$, we get the same result for $-(\gamma_0)^q$. Moreover, $\operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma) = (2\pi)^n/2l$. Further, note that $\det(w) = (-1)^{|\bar{I}_w|}$. Thus, if we sum over $w \in W$, then we get the following contribution of the elliptic conjugacy classes $\{(\gamma_0)^q\}$, $\{-(\gamma_0)^q\}$, $1 \leqslant q < l$, to the index:

$$(5.39)\quad 2\operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma)2^n \frac{i^n}{(4\pi)^n} \sum_{q=1}^{l-1} \prod_{j=1}^{n} \cot\left(\frac{\pi}{l} r_j q\right) = \frac{i^n}{l} \sum_{q=1}^{l-1} \prod_{j=1}^{n} \cot\left(\frac{\pi}{l} r_j q\right).$$

We turn now to the second case, where $Z_\Gamma = \{1\}$. In this case each elliptic element $\gamma \in \Gamma$ has odd order. Indeed, if $\gamma \in \Gamma$ is elliptic of order $2p$, then $\gamma^p \neq 1$ and $(\gamma^p)^2 = 1$. Since Γ is irreducible, it follows that $\gamma^p = -1$. Thus $-1 \in \Gamma$, which contradicts our assumption. Let $\gamma \in \Gamma$ be elliptic of order l, l odd, and let γ_0 be a generator of Γ_γ. γ_0 is conjugate in G to $k \in K$ with

$$(5.40)\qquad k_j = \begin{pmatrix} \cos\frac{2\pi}{l} r_j & \sin\frac{2\pi}{l} r_j \\ -\sin\frac{2\pi}{l} r_j & \cos\frac{2\pi}{l} r_j \end{pmatrix}, \qquad (l, r_j) = 1.$$

If we use the same arguments as in the first case, we get the following contribution of the elliptic conjugacy classes $\{(\gamma_0)^q\}$, $1 \leqslant q < l$:

$$(5.41)\qquad \frac{i^n}{l} \sum_{q=1}^{l-1} \prod_{j=1}^{n} \cot\left(\frac{2\pi}{l} r_j q\right).$$

If $1 \leqslant q < l$, then $2q$ also runs over all nonzero residue classes mod l. Therefore, (5.41) is equal to

$$(5.42)\qquad \frac{i^n}{l} \sum_{q=1}^{l-1} \prod_{j=1}^{n} \cot\left(\frac{\pi}{l} r_j q\right).$$

(5.39) and (5.42) are precisely the cotangent sums associated in [24] to the quotient singularities of $\Gamma \backslash \mathbf{H}^n$ via the equivariant signature theorem of Atiyah-Bott-Singer. More precisely, let $\tilde{\Gamma} = \Gamma/Z_\Gamma$ and recall that $Z_\Gamma \subset \{\pm 1\}$. $\tilde{\Gamma}$ acts effectively on $\mathbf{H}^n$. Let $z \in \mathbf{H}^n$ be a fixed point of $\gamma \in \Gamma$, $\gamma \neq 1$. Then γ is elliptic and $\tilde{\Gamma}_z = \Gamma_\gamma/Z_\Gamma$. $\tilde{\Gamma}_z$ is a cyclic group of order l. We choose around z a sufficiently small geodesic ball B_z. B_z is invariant under $\tilde{\Gamma}_z$. Let $\zeta = e^{2\pi i/l}$ and let γ_0 be a generator of $\tilde{\Gamma}_z$. There exist integers $(r_1, \cdots, r_n)$, which are prime to l, such that the action of $\tilde{\Gamma}_z$ on B_z is given by

$$(\gamma_0)^q(u_1, \cdots, u_n) = (\zeta^{r_1 q} u_1, \cdots, \zeta^{r_n q} u_n),$$

where $(u_1, \cdots, u_n)$ are geodesic coordinates at z. The integers $(r_1, \cdots, r_n)$ are determined by either (5.37) or (5.40). The cotangent sum associated with the quotient singularity of $\Gamma \backslash \mathbf{H}^n$, represented by $z \in \mathbf{H}^n$, is given by

$$(5.43)\qquad \delta(z) = \frac{i^n}{l} \sum_{q=1}^{l-1} \prod_{j=1}^{n} \cot\left(\frac{\pi}{l} r_j q\right)$$

(see [24, p. 225]). If $-1 \in \Gamma$, then Γ_γ is of order $2l$ and $\tilde{\Gamma}_z$ has order l. Thus, (5.43) coincides with the contribution (5.39) of the elliptic conjugacy classes $\{(\gamma_0)^q\}$, $\{-(\gamma_0)^q\}$, $1 \leqslant q < l$, to the L^2-index of D. If $-1 \notin \Gamma$, then $Z_\Gamma = \{1\}$ and $\tilde{\Gamma}_z = \Gamma_\gamma$. In this case, (5.43) coincides with the contribution (5.41) of the elliptic conjugacy classes $\{(\gamma_0)^q\}$, $1 \leqslant q < l$, l the order of $\Gamma_\gamma = \tilde{\Gamma}_z$. Let $z_1, \cdots, z_r \in \mathbf{H}^n$ be a complete system of Γ-inequivalent elliptic fixed points of Γ. Then, the contribution of the elliptic conjugacy classes of Γ to the L^2-index of D is given by

$$(5.44) \qquad \sum_{j=1}^{r} \delta(z_j),$$

where $\delta(z_j)$ is the cotangent sum (5.43) associated with z_j.

(iii) *The type* I *hyperbolic and the mixed contribution.* This contribution is given by

$$(HM) \sum_{\{\gamma\}_\Gamma} \operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma) \sum_{w \in W} \det(w) \int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg,$$

where the sum runs over all type I hyperbolic and all mixed conjugacy classes. First, consider the case where $\gamma \in \Gamma$ is hyperbolic of type I. By Lemma 5.34, we have $\Theta_\lambda^+(p_t^\pm) = \exp(-t(1 + \lambda^2)/4)$. Therefore, if we use (5.30) and (4.11), we obtain

$$(5.45) \qquad \int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg = \prod_{j=1}^{n} \frac{\exp\left(-t/4 - \left(\log N(\gamma_j)\right)^2/t\right)}{2\sqrt{4\pi t}\, \operatorname{sh}\left(N(\gamma_j)\right)},$$

where γ_j is the jth component of γ and γ_j is conjugate in G_0 to the diagonal matrix with entries $N(\gamma_j)$ and $N(\gamma_j)^{-1}$. In particular, (5.45) is independent of $w \in W$. Therefore, (5.35) implies that the contribution of the type I hyperbolic conjugacy classes to the L^2-index of D is zero. Now let $\gamma \in \Gamma$ be mixed. Each component of γ is either elliptic or hyperbolic, and there is at least one component, say γ_j, which is hyperbolic. The orbit integral $\int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg$ splits into a product of orbit integrals on G_0. Each of these integrals can be calculated as in the case where γ is elliptic or hyperbolic of type I. The orbit integral, which corresponds to the hyperbolic component γ_j is equal to

$$\frac{\exp\left(-t/4 - \left(\log N(\gamma_j)\right)^2/t\right)}{2\sqrt{4\pi t}\, \operatorname{sh}\left(N(\gamma_j)\right)}$$

for all $w \in W$. This follows as above from (5.30), (4.11) and the fact $\Theta_\lambda^+(p_t^\pm) = \exp(-t(1 + \lambda^2)/4)$. Let $w, w' \in W$ be such that $w_i = w_i'$, if $i \neq j$, and $w_j = -w_j'$. Since the orbit integral which corresponds to γ_j is independent of

$w \in W$, we get

$$\int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg = \int_{G_\gamma \backslash G} h_t^{w'}(g^{-1}\gamma g)\, dg.$$

But $\det(w) = -\det(w')$. This shows that

$$\sum_{w \in W} \det(w) \int_{G_\gamma \backslash G} h_t^w(g^{-1}\gamma g)\, dg = 0$$

for any mixed element $\gamma \in \Gamma$. Thus, the contribution of the mixed conjugacy classes to the L^2-index of D is zero too.

(iv) *The type* II *hyperbolic contribution*. For each j, $j = 1, \cdots, n$, we define σ_j: $W \to \{\pm 1\}$ by $\sigma_j(w) = +1$ if $w_j = \mathrm{Id}$, and $\sigma_j(w) = -1$ if $w_j = -\mathrm{Id}$. It follows from (4.14), (5.30) and Lemma 5.34, that the contribution of the type II hyperbolic conjugacy classes to $\operatorname{Tr} R^d_{\Gamma\backslash G}(h_t^w)$ is given by

$$\begin{aligned}
&\frac{1}{2} \operatorname{Vol}(\Gamma_M \backslash M) \frac{\exp(-(n-1)t/4)}{(\pi t)^{(n-1)/2}} \Bigg\{ \sum_{\substack{v \in V_1 \\ v \neq \pm 1}} C_v \frac{e^{-t/4}}{\sqrt{\pi t}} \prod_{j=1}^{n} e^{-(\log|v^{(j)}|)^2/t} \\
&\qquad + \sum_{\substack{v \in V_1 \\ v \neq \pm 1}} |N(v - v^{-1})| \sum_{j=1}^{n} I\left(p_g^{(\sigma_j(w))}\right)(v^{(j)}) \prod_{k=1}^{n}{}^{(j)} e^{-(\log|v^{(k)}|)^2/t} \Bigg\},
\end{aligned} \tag{5.46}$$

where

$$I(p_t^\pm)(\lambda) = \int_{U_0} p_t^\pm(u^{-1}a_\lambda u)\,\alpha_0(H(w_0 u))\, du, \tag{5.47}$$

and the notation is the same as in §4, (v). The first sum in (5.46) is independent of $w \in W$. In order to determine the dependence of the second sum in (5.46) on $w \in W$, we have to investigate the integral (5.47). For this purpose we prove the following

Lemma 5.48. *The heat kernels $p_t^\pm$ satisfy*

$$p_t^+(u^{-1}au) = p_t^-(uau^{-1})$$

for $u \in U_0$, $a \in A_0$.

Proof. According to Lemma 5.34 we have

$$p_t^+(g) = \frac{1}{2\pi}\overline{\Phi}_{2,2}^+(g) + \frac{1}{4\pi}\int_0^\infty \exp\left(-t\frac{1+\lambda^2}{4}\right)\overline{\Phi}_{\lambda,2}^+(g)\lambda \operatorname{th}\left(\frac{\pi\lambda}{2}\right) d\lambda,$$

$$p_t^-(g) = \frac{1}{2\pi}\overline{\Phi}_{2,-2}^-(g) + \frac{1}{4\pi}\int_0^\infty \exp\left(-t\frac{1+\lambda^2}{4}\right)\overline{\Phi}_{\lambda,-2}^+(g)\lambda \operatorname{th}\left(\frac{\pi\lambda}{2}\right) d\lambda.$$

Therefore, it is sufficient to prove that $\Phi^{+}_{2,2}(u^{-1}au) = \Phi^{-}_{2,-2}(uau^{-1})$ and $\Phi^{+}_{\lambda,2}(u^{-1}au) = \Phi^{-}_{\lambda,-2}(uau^{-1})$ for $u \in U_0$, $a \in A_0$. If we use (2.4) and the fact that $\varphi_{\pi,n} = \varphi_{\pi,-n}$, then an easy computation gives the desired result. q.e.d

Let $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Then $\alpha_0(H(w_0 u)) = -\log(1 + x^2)$. Thus

$$\alpha_0\big(H(w_0 u)\big) = \alpha_0\big(H\big(w_0 u^{-1}\big)\big), \qquad u \in U_0. \tag{5.49}$$

If we change variables in (5.47) by $u \mapsto u^{-1}$ and use (5.49) and Lemma 5.48, then we get $I(p_t^{+})(\lambda) = I(p_t^{-})(\lambda)$. This shows that (5.46) is independent of $w \in W$. Thus, by Proposition 5.29 and (5.35), the contribution of the type II hyperbolic conjugacy classes to the L^2-index of D is zero.

(v) *The parabolic contribution.* If we use (4.22), (4.28), (5.30) and Lemma 5.34, then we get the following expression for the parabolic contribution to the index:

$$\begin{aligned}
&\operatorname{Vol}(\Gamma \cap U \backslash U) \sum_{w \in W} \det(w) \sum_{\varepsilon} a_0(\varepsilon) \int_M \int_A h_t^w\big(m^{-1}a^{-1}u(\varepsilon)am\big) e^{-2\ln a}\, da\, dm \\
&\qquad + a_{-1} \operatorname{Vol}(\Gamma \cap U \backslash U) \frac{\exp(-(n-1)t/4)}{(4\pi t)^{(n-1)/2}} \\
&\qquad \cdot \sum_{w \in W} \det(w) \sum_{j=1}^{n} \int_{-\infty}^{\infty} \ln|x|\, p_t^{\sigma_j(w)}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx,
\end{aligned} \tag{5.50}$$

where $\sigma_j(w) = 1$ if $w_j = \mathrm{Id}$, and $\sigma_j(w) = -1$ if $w_j = -\mathrm{Id}$. Recall that $\varepsilon \in \{\pm 1\}^n$. The integral in the first sum can be computed by using (4.23) and (4.26). It follows from (4.26) and Lemma 5.34 that

$$\int_{U_0 \backslash G_0} p_t^{\pm}\big(g^{-1}u(\varepsilon)g\big)\, dg = \pm \frac{i}{2\pi}\varepsilon_j + \frac{e^{-t/4}}{4\sqrt{\pi t}}. \tag{5.51}$$

Put $b(t) = e^{-t/4}/4\sqrt{\pi t}$ and let $I_w = \{\, j | w_j = \mathrm{Id}\}$ for $w \in W$. If we use (4.23), (5.30) and (5.51) then we get

$$\begin{aligned}
&\sum_{w \in W} \det(w) \int_M \int_A h_t^w\big(m^{-1}a^{-1}u(\varepsilon)am\big) e^{-2\ln a}\, da\, dm \\
&\qquad = \sum_{w \in W} (-1)^{|\bar{I}_w|} \prod_{l \in I_w} \left(\frac{i}{2\pi}\varepsilon_l + b(t)\right) \prod_{j \in \bar{I}_w} \left(-\frac{i}{2\pi}\varepsilon_j + b(t)\right) \\
&\qquad = \frac{i^n}{\pi^n} N(\varepsilon),
\end{aligned}$$

where $N(\varepsilon) = \varepsilon_1 \cdots \varepsilon_n$. Hence, the first sum in (5.50) is equal to

$$\frac{i^n}{\pi^n} \operatorname{Vol}(\Gamma \cap U \backslash U) \sum_{\varepsilon} N(\varepsilon) a_0(\varepsilon). \tag{5.52}$$

We introduce the following L-series:

$$(5.53)\qquad L(\mathsf{M},\mathsf{V},s)=\sum_{\mu\in(\mathsf{M}-0)/\mathsf{V}}\frac{\operatorname{sign}N(\mu)}{|N(\mu)|^2},\qquad \operatorname{Re}(s)>1,$$

where $\mathsf{M}\subset F$ and $\mathsf{V}\subset U_{\mathsf{M}}^+$ are given by (3.6). If $\zeta_\varepsilon(\mathsf{M},\mathsf{V},s)$ is the zeta function defined by (4.18), then we have

$$(5.54)\qquad L(\mathsf{M},\mathsf{V},s)=\sum_{\varepsilon}N(\varepsilon)\zeta_\varepsilon(\mathsf{M},\mathsf{V},s).$$

We can now appeal to Lemma 4.20, which tells us that $L(\mathsf{M},\mathsf{V},s)$ has an analytic continuation to the entire complex plane with at most one simple pole at $s=1$. Moreover, the residue $a_{-1}(\varepsilon)$ of the pole $s=1$ of $\zeta_\varepsilon(\mathsf{M},\mathsf{V},s)$ is independent of ε. Hence, the residue of $L(\mathsf{M},\mathsf{V},s)$ at $s=1$ is equal to

$$a_{-1}\sum_{\varepsilon}N(\varepsilon).$$

But

$$(5.55)\qquad \sum_{\varepsilon}N(\varepsilon)=\sum_{q=0}^{n}(-1)^q\binom{n}{q}=0.$$

This implies

Lemma 5.56. *The L-series* $L(\mathsf{M},\mathsf{V},s)$ *defined by* (5.53) *has an analytic continuation to an entire function in the complex plane. If* $a_0(\varepsilon)$ *is the constant term of the Laurent expansion of* $\zeta_\varepsilon(\mathsf{M},\mathsf{V},s)$ *at* $s=1$, *then*

$$L(\mathsf{M},\mathsf{V},1)=\sum_{\varepsilon}N(\varepsilon)a_0(\varepsilon).$$

Let $(\beta_1,\cdots,\beta_n)$ be a basis of M and set

$$d(\mathsf{M})=\left|\det\left(\beta_i^{(j)}\right)\right|,$$

where $x\in F\mapsto x^{(j)}\in\mathbb{R}$ is the jth embedding of F in $\mathbb{R}$. Then $\operatorname{Vol}(\Gamma\cap U\backslash U)=d(\mathsf{M})$. This gives the following expression for (5.52):

$$(5.57)\qquad \frac{i^n}{\pi^n}d(\mathsf{M})L(\mathsf{M},\mathsf{V},1).$$

It remains to investigate the second sum occurring in (5.50). By arguments similar to those given in the proof of Lemma 5.48, one can show that $p_t^+(u)=p_t^-(u^{-1})$, $u\in U_0$. Therefore

$$\int_{-\infty}^{\infty}\ln|x|\,p_t^+\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)dx=\int_{-\infty}^{\infty}\ln|x|\,p_t^-\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)dx.$$

This together with (5.35) implies that the second sum in (5.50) is zero and the parabolic contribution to the L^2-index of D is precisely (5.57).

(vi) *The intertwining and the residual contribution.* Recall that $h_t^w \in \mathscr{C}^p(G, \chi_w)$, where $\chi_w \in \hat{K}$ is given by (5.27). Moreover, $\Theta_{\omega,s}$ is the character of $\mathrm{Ind}_P^G(\chi_{\omega,s})$, where $\chi_{\omega,s}(uam) = \tau_\omega(m)e^{s\ln a}$ and τ_ω is defined by (3.16). Thus by (5.30) and Lemma 5.34, we get

$$\begin{aligned}\Theta_{\omega,i\lambda}(h_t^w) &= \prod_{j=1}^n \Theta^+_{\lambda+\pi(B\omega)_j}\left(p_t^{\sigma_j(w)}\right)\\ &= \prod_{j=1}^n \exp\left(-\frac{1+(\lambda+\pi(B\omega)_j)^2}{4}\right).\end{aligned} \tag{5.58}$$

By (4.29) and Proposition 5.29, the contribution of the intertwining term to the L^2-index of the signature operator is

$$\frac{1}{4\pi}\sum_{\omega\in\mathbf{Z}^{n-1}}\sum_{w\in W}\det(w)\int_{\mathrm{Re}(s)=0}\Theta_{\omega,s}(h_t^w)\frac{d}{ds}C_\omega(\chi_w,s)C_\omega(\chi_w,-s)\,|ds|. \tag{5.59}$$

$C_\omega(\chi_w, s)$ was computed in §3. Let $I_w = \{i \mid w_i = \mathrm{Id}\}$ and $J_w = \bar{I}_w$. Then $\chi_w = \sigma_{I_w,J_w}$, where σ_{I_w,J_w} is given by (3.20). It follows from (3.21) and (3.23) that $C_\omega(\chi_w, s)$ is independent of $w \in W$. Thus, by (5.58), the integral occurring in (5.59) is independent of $w \in W$ and (5.35) implies that the intertwining contribution (5.59) is zero. The same argument, applied to the residual term, shows that the contribution of the residual term to the L^2-index of D is zero too.

This completes our computation of the L^2-index of the signature operator. The calculations above have been carried out under the assumption that $\Gamma\setminus\mathbf{H}^n$ has a single cusp. But it is clear that everything works equally well in the general case. Every cusp gives a contribution like (5.57). This is the only difference to our assumption. We shall now summarize our results. Recall that to every parabolic fixed point of Γ there corresponds a lattice $\mathsf{M} \subset F$ and a subgroup $\mathsf{V} \subset U_{\mathsf{M}}^+$ of finite index [24], [38]. The strict equivalence class of M and the group V are uniquely determined by the parabolic orbit. Let $L(\mathsf{M}, \mathsf{V}, s)$ be the corresponding L-series (5.53). We have proved the following

Theorem 5.60. *Let $F/\mathbf{Q}$ be a totally real number field of degree n and let $\Gamma \subset \mathrm{SL}(2, F)$ be an arithmetic subgroup. Let z_j $(1 \leqslant j \leqslant r)$ be a complete system of Γ-inequivalent elliptic fixed points of Γ and let x_ι $(1 \leqslant \iota \leqslant p)$ be a complete system of Γ-inequivalent parabolic fixed points of Γ. With each z_j we associate the cotangent sum (5.43) which we denote by $\delta(z_j)$. For each x_ι let $\mathsf{M}_\iota \subset F$ and $\mathsf{V}_\iota \subset U_{\mathsf{M}_\iota}^+$ be the lattice and the group of units which correspond to x_ι. Then*

$$\mathrm{Ind}_{L^2} D = \sum_{j=1}^r \delta(z_j) + \frac{i^n}{\pi^n}\sum_{\iota=1}^p d(\mathsf{M}_\iota)L(\mathsf{M}_\iota, \mathsf{V}_\iota, 1).$$

It is well known that for a closed Riemannian manifold X, the index of the signature operator on X coincides with the signature $\operatorname{Sign}(X)$. In our case it turns out that the L^2-index of the signature operator is indeed the signature of the rational homology manifold $\Gamma \setminus \mathbf{H}^n$. Recall formula (5.3). We use the results of Harder [18], [19] to relate $\mathscr{H}^*_{(2),\pm}(\Gamma \setminus \mathbf{H}^n)$ to the usual cohomology. For this purpose we have to pass to a torsion free subgroup $\Gamma_1 \subset \Gamma$. According to Selberg [36] there exists a torsion free normal subgroup $\Gamma_1 \subset \Gamma$ of finite index. It follows from reduction theory that $\Gamma_1 \setminus \mathbf{H}^n$ has the homotopy type of a compact manifold with boundary [8].

Let

$$\mathscr{H}^*_{\mathrm{cus}}(\Gamma \setminus \mathbf{H}^n) \subset \mathscr{H}^*_{(2)}(\Gamma \setminus \mathbf{H}^n)$$

be the space of harmonic cusp forms. If we identify $L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$ with $L^2(\Gamma \setminus G, \Lambda^* \mathrm{Ad}^*_{\mathfrak{p}})$, then it follows from the results concerning the spectral decomposition of $L^2(\Gamma \setminus G)$ that the orthogonal complement of $\mathscr{H}^*_{\mathrm{cus}}(\Gamma \setminus \mathbf{H}^n)$ in $\mathscr{H}^*_{(2)}(\Gamma \setminus \mathbf{H}^n)$ is generated by harmonic residues of Eisenstein series. We denote this space by $\mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n)$. Thus

$$\mathscr{H}^*_{(2)}(\Gamma \setminus \mathbf{H}^n) = \mathscr{H}^*_{\mathrm{cus}}(\Gamma \setminus \mathbf{H}^n) \oplus \mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n). \tag{5.61}$$

In our case, $\mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n)$ has an explicit description. Let ω_i be the volume form $(dz_i \wedge d\bar{z}_i)/y_i^2$ on the ith factor of $\mathbf{H}^n$ and equal to one on the others. For any subset $I \subset \{1,\cdots,n\}$ we put

$$\omega_I = \bigwedge_{i \in I} \omega_i. \tag{5.62}$$

Each ω_I is a G-invariant differential form on $\mathbf{H}^n$. Thus, it defines a differential form in $\Lambda^*(\Gamma \setminus \mathbf{H}^n)$, which is easily seen to be harmonic and square-integrable. Moreover, ω_I is orthogonal to $\mathscr{H}^*_{\mathrm{cus}}(\Gamma \setminus \mathbf{H}^n)$. Hence $\omega_I \in \mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n)$ by (5.61).

Lemma 5.63. *The set* $\{\omega_I | \subset \{1,\cdots,n\}\}$ *is a basis of* $\mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n)$.

Proof. It is clear that the forms ω_I, $I \subset \{1,\cdots,n\}$, are linearly independent. We show that they generate $\mathscr{H}^*_{\mathrm{res}}(\Gamma \setminus \mathbf{H}^n)$. According to (3.18) and Lemma 3.14, the only possible harmonic residues of Eisenstein series can arise from the pole $s = 1$ of the Eisenstein series with $\omega = 0$. For $I, J \subset \{1,\cdots,n\}$ let $E_0(\sigma_{I,J}, s, z)$ be the Eisenstein series associated to $\omega = 0$ and $\sigma_{I,J}$, where $\sigma_{I,J}$ is given by (3.20). The intertwining operator $C_0(\sigma_{I,J}, s)$ is given by (3.23). The poles of $E_0(\sigma_{I,J}, z, s)$ coincide with the poles of $C_0(\sigma_{I,J}, s)$ and $C_0(\sigma_{I,J}, s)$ has at most a simple pole at $s = 1$ [20, IV]. Assume that $I \neq J$ and that $C_0(\sigma_{I,J}, s)$ has a pole at $s = 1$. Then $I \cup J - (I \cap J) \neq \varnothing$. Thus, by (3.21), $\Gamma_{0,I,J}(s)$ has a zero at $s = 1$. It follows from (3.23) that

$$\sum_{\Gamma \cap P \backslash \Gamma / \Gamma \cap U} |N(c)|^{-(s+1)}$$

has a pole of order $\geqslant 2$ at $s = 1$. On the other hand, using (3.21), we see that $\Gamma_{0,I,I}(1) \neq 0$. Thus, by (3.23), $C_0(\sigma_{I,J}, s)$ has a pole of order $\geqslant 2$ at $s = 1$, which is impossible. Therefore, the only possible Eisenstein series with a pole at $s = 1$ are the $E_0(\sigma_{I,I}, s, z)$. But $E_0(\sigma_{I,I}, s, z) = E(s, z)\omega_I$, where

$$E(s, z) = \sum_{\Gamma \cap P \backslash \Gamma} \prod_{j=1}^{n} \operatorname{Im}\left(\gamma^{(j)}(z_j)\right)^{(s+1)/2}.$$

$E(s, z)$ has a simple pole at $s = 1$. This shows that the set $\{\omega_I | I \subset \{1, \cdots, n\}\}$ generates $\mathscr{H}^*_{\text{res}}(\Gamma \setminus \mathbf{H}^n)$. q.e.d.

Now, let $\Gamma_1 \subset \Gamma$ be a torsion free normal subgroup of finite index and let $H^*_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ be the image of the cohomology with compact supports in the usual cohomology. $H^*_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ has the following description. Let $A^*(\mathbf{H}^n)$ be the space of G-invariant differential forms on $\mathbf{H}^n$. The forms ω_I, $I \subset \{1, \cdots, n\}$, defined by (5.62), form a basis of $A(\mathbf{H}^n)$. Since each ω_I is closed, we get a homomorphism

$$A^*(\mathbf{H}^n) \to H^*(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}). \tag{5.64}$$

Let $H^*_A(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ be the image of $A^*(\mathbf{H}^n)$ in $H^*(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$. If $p > 0$, then $H^p_A(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}) \subset H^p_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ and the kernel of the homomorphism (5.64) is equal to $A^{2n}(\mathbf{H}^n)$ [19, Proposition 2.3]. Moreover, the canonical homomorphism

$$\mathscr{H}^*_{\text{cus}}(\Gamma_1 \setminus \mathbf{H}^n) \to H^*_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$$

is injective and, if $p > 0$,

$$H^p_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}) = \mathscr{H}^p_{\text{cus}}(\Gamma_1 \setminus \mathbf{H}^n) \oplus H^p_A(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$$

(see [19]). If we combine these results with (5.61) and Lemma 5.63, we see that we have proved the following

Propostion 5.65. *The canonical map*

$$\mathscr{H}^p_{(2)}(\Gamma_1 \setminus \mathbf{H}^n) \to H^p(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$$

is injective for $p < 2n$. *If* $0 < p < 2n$, *then the image of this map is* $H^p_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$.

The involution τ acts on $H^n_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ and we denote by $H^n_{!,\pm}(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ the ± 1-eigenspaces. Then we get

Corollary 5.66. $\mathscr{H}^n_{(2),\pm}(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}) = H^n_{!,\pm}(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$.

Let $N = \Gamma_1 \setminus \Gamma$. N acts on $\Gamma_1 \setminus \mathbf{H}^n$ by isometries and $\Gamma \setminus \mathbf{H}^n = N \setminus (\Gamma_1 \setminus \mathbf{H}^n)$. By Borel [7, Chapter III] one has $H^n_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}) = H^n_!(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})^N$. Moreover, $\mathscr{H}^n_{(2)}(\Gamma \setminus \mathbf{H}^n) = \mathscr{H}^n_{(2)}(\Gamma_1 \setminus \mathbf{H}^n)^N$. Note that

$H_!^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})^N$ is the image of the projection

$$P\colon H_!^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}) \to H_!^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C}),$$

defined by

$$P = \frac{1}{|N|} \sum_{g \in N} g.$$

Since the cup product is preserved under P, P takes $H_{!,\pm}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})$ to $H_{!\pm}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{C})^N$. Thus, we get

$$\begin{aligned}
\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n) &= \dim H_{!,+}^n(\Gamma \setminus \mathbf{H}^n; \mathbf{R}) - \dim H_{!,-}^n(\Gamma \setminus \mathbf{H}^n; \mathbf{R}) \\
&= \dim P\big(H_{!,+}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{R})\big) - \dim P\big(H_{!,-}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{R})\big) \\
&= \frac{1}{|N|} \sum_{g \in N} \big\{\operatorname{tr}\big(g | H_{!,+}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{R})\big) - \operatorname{tr}\big(g | H_{!,-}^n(\Gamma_1 \setminus \mathbf{H}^n; \mathbf{R})\big)\big\} \\
&= \frac{1}{|N|} \sum_{g \in N} \big\{\operatorname{tr}\big(g | \mathscr{H}_{(2),+}^n(\Gamma_1 \setminus \mathbf{H}^n)\big) - \operatorname{tr}\big(g | \mathscr{H}_{(2),-}^n(\Gamma_1 \setminus \mathbf{H}^n)\big)\big\} \\
&= \dim \mathscr{H}_{(2),+}^n(\Gamma \setminus \mathbf{H}^n) - \dim \mathscr{H}_{(2),-}^n(\Gamma \setminus \mathbf{H}^n).
\end{aligned}$$

This proves

Proposition 5.67. *The signature of* $\Gamma \setminus \mathbf{H}^n$ *is given by*

$$\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n) = \dim \mathscr{H}_{(2),+}^n(\Gamma \setminus \mathbf{H}^n) - \dim \mathscr{H}_{(2),-}^n(\Gamma \setminus \mathbf{H}^n).$$

Corollary 5.68. *Let the notations be the same as in Theorem* 5.60. *Then*

$$\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n) = \sum_{j=1}^{r} \delta(z_j) + \frac{i^n}{\pi^n} \sum_{\iota=1}^{p} d(\mathsf{M}_\iota) L(\mathsf{M}_\iota, \mathsf{V}_\iota, 1).$$

Proof. The corollary follows from Theorem 5.60, (5.3) and Proposition 5.67.

Now, we compare our result with Hirzeburch's formula for the signature of $\Gamma \setminus \mathbf{H}^n$ [24, p. 228]. We recall the definition of the signature defect associated to a cusp of $\Gamma \setminus \mathbf{H}^n$ [24, §3]. We assume that $n = 2k$. Let $x \in (P_1(\mathbf{R}))^n$ be a parabolic fixed point of Γ. There exists $\rho \in \mathrm{SL}(2, F)$ such that $\rho x = \infty$. Let Γ_x be the stabilizer of x. $\rho \Gamma_x \rho^{-1}$ acts on

$$W(d) = \left\{ z \in \mathbf{H}^n \,\middle|\, \prod_{j=1}^{n} \operatorname{Im}(z_j) \geqslant d \right\}, \qquad d > 0.$$

Let $\tilde{W} = (\rho \Gamma_x \rho^{-1}) \setminus W(d)$ and $Y = \partial \tilde{W}$. There exists a natural framing of the stable tangent bundle $TY \oplus \mathbf{R}$, which is induced from a framing of $TW(d)$. Therefore, Y bounds a $4k$-dimensional compact oriented manifold X. Since

$TY \oplus \mathbb{R}$ is framed, we can push down the tangent bundle TX to a SO-bundle over X/Y. Let $\tilde{p}_j \in H^{4j}(X/Y; \mathbb{Z})$ be its Pontrjagin classes. The signature defect of the cusp x is defined as

$$\delta(x) = L_k(\tilde{p}_1, \cdots, \tilde{p}_k)[X, Y] - \operatorname{Sign}(X), \tag{5.69}$$

where L_k is the Hirzebruch polynomial. In [24, p. 228] Hirzebruch proved the following formula for the signature of $\Gamma \setminus \mathbf{H}^n$:

$$\operatorname{Sign}(\Gamma \setminus \mathbf{H}^n) = \sum_{j=1}^{r} \delta(z_j) + \sum_{\iota=1}^{p} \delta(x_\iota), \tag{5.70}$$

where $z_1, \cdots, z_r$ is a complete system of Γ-inequivalent elliptic fixed points of Γ and $x_1, \cdots, x_p$ is a complete system of Γ-inequivalent parabolic fixed points of Γ. $\delta(x_\iota)$ is the signature defect (5.69), associated to x_ι, and $\delta(z_j)$ is the cotangent sum (5.43), associated with the elliptic fixed point z_j. If we compare (5.70) with Corollary 5.68, we get our main result:

Theorem 5.71. *Let $F/\mathbf{Q}$ be a totally real number field of degree $n = 2k$ and let $\Gamma \subset \mathrm{SL}(2, F)$ be an arithmetic subgroup. Let x_ι, $1 \leqslant \iota \leqslant p$, be a complete system of Γ-inequivalent parabolic fixed points of Γ and let $\delta(x_\iota)$ be the signature defect* (5.69) *of x_ι. Moreover, let $(\mathsf{M}_\iota, \mathsf{V}_\iota)$, $\mathsf{M}_\iota \subset F$ a lattice and $\mathsf{V}_\iota \subset U^+_{\mathsf{M}_\iota}$, be associated with x_ι and let $L(\mathsf{M}_\iota, \mathsf{V}_\iota, s)$ be the L-series defined by* (5.53). *Then*

$$\sum_{\iota=1}^{p} \delta(x_\iota) = \frac{i^n}{\pi^n} \sum_{\iota=1}^{p} d(\mathsf{M}_\iota) L(\mathsf{M}_\iota, \mathsf{V}_\iota, 1).$$

If $\Gamma \setminus \mathbf{H}^n$ has a single cusp x, then it follows that

$$\delta(x) = \frac{i^n}{\pi^n} d(\mathsf{M}) L(\mathsf{M}, \mathsf{V}, 1).$$

This is part of Hirzebruch's conjecture for groups Γ with a single parabolic orbit.

We turn now to the Dolbeault operator. Its L^2-index can be computed by the same method. We shall not carry out all the details, because most of the arguments are similar to those used in the case of the signature operator. It turns out that the L^2-index of the Dolbeault operator is related to the dimension of the space $\mathscr{H}^{p,q}_{\mathrm{cus}}(\Gamma \setminus \mathbf{H}^n)$ of harmonic cusp forms of bidegree (p, q). In this way we get a formula for the dimension of the space of harmonic cusp forms of a given type. This generalizes the results of Matsushima and Shimura [27] who treated the case when $\Gamma \subset G$ is cocompact and torsion free, and it answers a question raised in [19, §3].

Let $\Lambda^{p,q} = \Lambda^{p,q}(\Gamma \setminus \mathbf{H}^n)$ be the space of Γ-invariant C^∞-differential forms of bidegree (p, q) on $\mathbf{H}^n$. We consider the $\bar{\partial}$-complex:

$$0 \to \Lambda^{p,0} \xrightarrow{\bar{\partial}_0} \Lambda^{p,1} \xrightarrow{\bar{\partial}_1} \cdots \xrightarrow{\bar{\partial}_{n-1}} \Lambda^{p,n} \to 0$$

and the corresponding elliptic operator

$$\bar{\partial} + \bar{\partial}^*: \sum_q \Lambda^{p,2q} \to \sum_q \Lambda^{p,2q+1}.$$

This is the Dolbeault operator D_p. Let $\Lambda_0^{p,q} = \Lambda_0^{p,q}(\Gamma \setminus \mathbf{H}^n)$ be the subspace of $\Lambda^{p,q}$ consisting of forms with compact support mod Γ, and let $L^2\Lambda^{p,q}$ be the closure of $\Lambda_0^{p,q}$ in $L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$. Let $\overline{D}_p$ be the closure in L^2 of D_p, restricted to $\Sigma_q \Lambda_0^{p,2q}$. The L^2-indx of D_p is by definition

$$\operatorname{Ind}_{L^2} D_p = \dim \ker \overline{D}_p - \dim \operatorname{coker} \overline{D}_p.$$

Let $\mathscr{H}_{(2)}^{p,q}(\Gamma \setminus \mathbf{H}^n) \subset \mathscr{H}_{(2)}^*(\Gamma \setminus \mathbf{H}^n)$ be the subspace of square integrable harmonic forms of bidegree (p, q). By arguments similar to those which we used in the case of the signature operator, one can show that

$$\operatorname{Ind}_{L^2} D_p = \sum_{q=0}^{n} (-1)^q \dim \mathscr{H}_{(2)}^{p,q}(\Gamma \setminus \mathbf{H}^n).$$

From (5.61) we obtain the decomposition

(5.72) $$\mathscr{H}_{(2)}^{p,q}(\Gamma \setminus \mathbf{H}^n) = \mathscr{H}_{\text{cus}}^{p,q}(\Gamma \setminus \mathbf{H}^n) \oplus \mathscr{H}_{\text{res}}^{p,q}(\Gamma \setminus \mathbf{H}^n).$$

Let $I \subset \{1, \cdots, n\}$ be such that $|I| = q$. The form ω_I, defined by (5.62), is of bidegree (q, q). Thus, if we appeal to Lemma 5.63, we get

$$\dim \mathscr{H}_{\text{res}}^{p,q}(\Gamma \setminus \mathbf{H}^n) = \begin{cases} 0, & p \neq q, \\ \binom{n}{p}, & p = q \end{cases}.$$

This implies

(5.73) $$\operatorname{Ind}_{L^2} D_p = \sum_{q=0}^{n} (-1)^q \dim \mathscr{H}_{\text{cus}}^{p,q}(\Gamma \setminus \mathbf{H}^n) + (-1)^p \binom{n}{p}.$$

One can extend the method of Matsushima and Shimura [27, §3, 4] to prove the following vanishing theorem:

Theorem 5.74. *If $p + q \neq n$, then $\mathscr{H}_{\text{cus}}^{p,q}(\Gamma \setminus \mathbf{H}^n) = 0$.*

Proof. We shall use the notations introduced in §3. Each form $\Phi \in \Lambda^*(\Gamma \setminus \mathbf{H}^n)$ can be decomposed as

$$\Phi = \sum_{I,J} f_{I,J} \frac{dz^I}{y_I} \wedge \frac{d\bar{z}^J}{y_J},$$

where I, J run over ordered subsets of $\{1, \cdots, n\}$ and $f_{I,J}$ satisfies

$$f_{I,J}(\gamma z) = j_{I,J}(\gamma, z) f_{I,J}(z)$$

for all $\gamma \in \Gamma$. Let

$$\Phi_{I,J} = f_{I,J} \frac{dz^I}{y_I} \wedge \frac{d\bar{z}^J}{y_J}.$$

If $\sigma_{I,J}$ is the character defined by (3.20), then $\Phi_{I,J}$ can be considered as an element of $C^\infty(\Gamma \setminus G, \sigma_{I,J})$. This space is invariant under $R(\Omega)$. Therefore, by Kuga's Lemma, it follows that Φ is harmonic iff each $\Phi_{I,J}$ is harmonic. Thus, it is sufficient to consider harmonic forms of type (I, J). Let $\Phi \in \mathscr{H}^{p,q}_{\text{cus}}(\Gamma \setminus \mathbf{H}^n)$ be a harmonic form of type (I, J), i.e.

$$\Phi = f\frac{dz^I}{y_I} \wedge \frac{d\bar{z}^J}{y_J},$$

with $|I| = p$, $|J| = q$. On a complete Riemannian manifold, a L^2-form ω is harmonic iff $d\omega = 0$ and $\delta\omega = 0$ [12]. If Γ has elements of finite order, we can choose a torsion free normal subgroup $\Gamma_1 \subset \Gamma$ of finite index [36]. Thus $\Gamma_1 \setminus \mathbf{H}^n \to \Gamma \setminus \mathbf{H}^n$ is a finite covering of $\Gamma \setminus \mathbf{H}^n$ by a complete Riemannian manifold. This shows that $d\Phi = 0$ and $\delta\Phi = 0$. Now, let $p + q \neq n$. First, we assume that $p + q < n$. Then $|I \cup J| < n$ and there exists j, $1 \leqslant j \leqslant n$, such that $j \notin I \cup J$. Since $d\Phi = 0$, we get

$$\frac{\partial f}{\partial z_j} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}_j} = 0.$$

Hence, f does not depend on the variable z_j. From the definition of the automorphy factor $j_{I,J}(\gamma, z)$ it follows that $j_{I,J}(\gamma, z)$ does not depend on the variable z_j and the component γ_j of γ. Let $G' = (\mathrm{SL}(2, \mathbb{R}))^{n-1}$ and let π: $G \to G'$ be the projection defined by $(g_1, \cdots, g_n) \mapsto (g_1, \cdots, \hat{g}_j, \cdots, g_n)$. Let $\Gamma' = \pi(\Gamma)$. Then f can be identified with a function $\tilde{f} \in C^\infty(\mathbf{H}^{n-1})$, which satisfies $\tilde{f}(\gamma' z') = j_{I,J}(\gamma', z')\tilde{f}(z')$ for $z' \in \mathbf{H}^{n-1}$ and $\gamma' \in \Gamma'$. Since $\Gamma \subset G$ is an irreducible lattice, $\Gamma' = \pi(\Gamma)$ is dense in G' [35, Corollary 5.21]. Therefore, $\tilde{f}$ satisfies

$$\tilde{f}(g'z') = j_{I,J}(g', z')\tilde{f}(z') \tag{5.75}$$

for all $g' \in G'$, $z' \in \mathbf{H}^{n-1}$. Let $z_0 \in \mathbf{H}^{n-1}$ be the point $(i, \cdots, i)$. If $\tilde{f}(z_0) = 0$, then (5.75) implies $\tilde{f} = 0$. Assume $\tilde{f}(z_0) \neq 0$. It follows from (5.75) that $j_{I,J}(k', z_0) = 1$ for all $k' \in K' = (\mathrm{SO}(2))^{n-1}$. Suppose that $I \neq J$. Let $\iota \in I$, $\iota \notin J$ and let $k' \in K'$ be such that $k'_l = 1$ for $l \neq \iota$ and

$$k_\iota = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then we get $j_{I,J}(k', z_0) = e^{-2i\theta}$. Thus $I = J$ and therefore $\Phi = f\omega_I$, where ω_I is the form (5.62). Since ω_I is G-invariant, we have $f \in C^\infty(\Gamma \setminus \mathbf{H}^n)$. By our assumption, Φ is a harmonic cusp form. In particular, $\Phi \in L^2\Lambda^*(\Gamma \setminus \mathbf{H}^n)$ and this implies $f \in L^2(\Gamma \setminus \mathbf{H}^n)$. Moreover, $\Delta\phi = (\Delta f)\omega_I$. Hence $\Delta f = 0$. Therefore, f is a constant C. But $C\omega_I$ is a cusp form iff $C = 0$. Thus $\mathscr{H}^{p,q}_{\text{cus}}(\Gamma \setminus \mathbf{H}^n) = 0$ if $p + q < n$. Now, note that the Hodge $*$-operator is an isomorphism of

$\mathcal{H}_{(2)}^{p,q}(\Gamma \setminus \mathbf{H}^n)$ onto $\mathcal{H}_{(2)}^{n-p,n-q}(\Gamma \setminus \mathbf{H}^n)$ which carries cusp forms into cusp forms. Thus $\mathcal{H}_{\mathrm{cus}}^{n-p,n-q}(\Gamma \setminus \mathbf{H}^n) \cong \mathcal{H}_{\mathrm{cus}}^{p,q}(\Gamma \setminus \mathbf{H}^n)$ and this proves the theorem.

From (5.73) and Theorem 5.74 we get

$$\dim \mathcal{H}_{\mathrm{cus}}^{p,n-p}(\Gamma \setminus \mathbf{H}^n) = (-1)^{n-p} \operatorname{Ind}_{L^2} D_p + (-1)^{n+1}\binom{n}{p}. \tag{5.76}$$

Let $\Delta^{p,q} = \bar{\partial}_p^* \bar{\partial}_p + \bar{\partial}_{p-1} \bar{\partial}_{p-1}^*$ be the Laplacian on $\Lambda^{p,q}$. As above one can show that $\Delta^{p,q}$, restricted to $\Lambda_0^{p,q}$, has a unique selfadjoint extension to an operator in $L^2\Lambda^{p,q}$. We denote this extension again by $\Delta^{p,q}$. Let $L_d^2\Lambda^{p,q} \subset L^2\Lambda^{p,q}$ be the subspace spanned by the eigenforms of $\Delta^{p,q}$ and let $\Delta_d^{p,q}$ be the restriction of $\Delta^{p,q}$ to $L_d^2\Lambda^{p,q}$. Let $\exp(-t\Delta_d^{p,q})$ be the semigroup generated by $\Delta_d^{p,q}$. By arguments similar to those used in the case of the signature operator one can show that $\exp(-t\Delta_d^{p,q})$ is a trace class operator for each $t > 0$ and

$$\operatorname{Ind}_{L^2} D_p = \sum_{q=0}^{n} (-1)^q \operatorname{Tr}\left(\exp(-t\Delta_d^{p,q})\right). \tag{5.77}$$

Let $\Lambda^{p,q}\mathfrak{p}_{\mathbf{C}} \subset \Lambda^*\mathfrak{p}_{\mathbf{C}}$ be the subspace spanned by the vectors $v_{I,J}$, defined by (3.19), with $|I| = p$, $|J| = q$, and let

$$\sigma^{p,q}\colon K \to \mathrm{GL}(\Lambda^{p,q}\mathfrak{p}_{\mathbf{C}})$$

be the corresponding representation. Consider the Laplacian $\tilde{\Delta}^{p,q}$ on $L^2\Lambda^{p,q}(\mathbf{H}^n)$. $\exp(-t\tilde{\Delta}^{p,q})$ is a G-invariant smoothing operator. Therefore, it has a kernel

$$h_t^{p,q}\colon G \to \operatorname{End}(\Lambda^{p,q}\mathfrak{p}_{\mathbf{C}}),$$

which is in $C^\infty \cap L^2$ and satisfies (1.4) with respect to the representation $\sigma^{p,q}$. As above it turns out that $h_t^{p,q} \in \mathscr{C}^r(G, \sigma^{p,q})$ for each $r > 0$ and

$$\exp(-t\Delta_d^{p,q}) = R_{\Gamma\setminus G}^d(h_t^{p,q}). \tag{5.78}$$

Let

$$e_t^p = \sum_{q=0}^{n} (-1)^q \operatorname{tr} h_t^{p,q}.$$

Then Proposition 4.6 and (5.77) imply

$$\operatorname{Ind}_{L^2} D_p = \operatorname{Tr} R_{\Gamma\setminus G}^d(e_t^p). \tag{5.79}$$

The representation $\sigma^{p,q}$ splits into characters

$$\sigma^{p,q} = \bigoplus_{\tau \in \hat{K}} [\tau : \sigma^{p,q}]\tau,$$

$[\tau: \sigma^{p,q}]$ being the multiplicity of τ in $\sigma^{p,q}$. For $\tau \in \hat{K}$ let $L^2(G, \tau) = \{ f \in L^2(G) | f(gk^{-1}) = \tau(k)f(g), k \in K \}$ and let $\Delta_\tau = -R(\Omega)|_{L^2(G,\tau)}$. If h_t^τ is the kernel of $\exp(-t\Delta_\tau)$, then we get as in (5.24)

$$e_t^p = \sum_{\tau \in \hat{K}} \sum_{q=0}^{n} (-1)^q [\tau : \sigma^{p,q}] h_t^\tau.$$

Thus

$$\operatorname{Ind}_{L^2} D_p = \sum_{\tau \in \hat{K}} \sum_{q=0}^{n} (-1)^q [\tau : \sigma^{p,q}] \operatorname{Tr} R^d_{\Gamma \backslash G}(h_t^\tau). \tag{5.80}$$

Each of the characters τ occurring in $\sigma^{p,q}$ is of the form $\sigma_{I,J}$, where $\sigma_{I,J}$ is defined by (3.20). Suppose that $I \cap J = \varnothing$ and let $L = \{1, \cdots, n\} - (I \cup J)$. By using the same considerations which led to (5.30), we get

$$h_t^\tau(g) = \prod_{i \in I_\tau} p_t^+(g_i) \prod_{j \in J_\tau} p_t^-(g_j) \prod_{l \in L_\tau} p_t^0(g_l), \tag{5.81}$$

where $\tau = \sigma_{I_\tau, J_\tau}$, $p_t^\pm = p_t^{(\pm 1)}$ and $p_t^0 = p_t^{(0)}$ in the notation of Lemma 5.34. Now, one can use Selberg's trace formula to compute $\operatorname{Ind}_{L^2} D_p$ as above. It follows from (4.11), Lemma 5.34 and (5.81) that the contribution of the type I hyperbolic conjugacy classes to $\operatorname{Tr} R^d_{\Gamma \backslash G}(h_t^\tau)$ is independent of τ. Since

$$\sum_{q=0}^{n} (-1)^q \dim \Lambda^{p,q} \mathfrak{p}_{\mathbf{C}} = \binom{n}{p} \sum_{q=0}^{n} (-1)^q \binom{n}{q} = 0,$$

the type I hyperbolic contribution to the L^2-index of D_p is zero. By a more subtle argument one can see that the mixed and the type II hyperbolic contribution to the L^2-index of D_p is zero too. The contribution of the remaining conjugacy classes can be easily determined by passing to the limit as $t \to \infty$, because the left-hand side of (5.80) is independent of t. The intertwining and the residual terms approach zero if $t \to \infty$. To compute the central contribution, we use Lemma 5.34. The central term of $\operatorname{Tr} R^d_{\Gamma \backslash G}(h_t^\tau)$ is

$$|Z_\Gamma| \operatorname{Vol}(\Gamma \backslash G) h_t^\tau(1).$$

It follows from Lemma 5.34 that $p_t^\pm(1) = 1/2\pi + a(t)$ and $p_t^0(1) = a(t)$ with $\lim_{t \to \infty} a(t) = 0$. Thus, if we pass to the limit as $t \to \infty$, then the central term of $\operatorname{Tr} R^d_{\Gamma \backslash G}(h_t^\tau)$ tends to zero, except when $\tau = \sigma_{I,J}$ with $I \cap J = \varnothing$, $|I| + |J| = n$ and $|I| = p$. In the latter case we get in the limit $|Z_\Gamma|(2\pi)^{-n} \operatorname{Vol}(\Gamma \backslash G)$. Thus, by (5.80), the central contribution to the L^2-index of D_p is

$$(-1)^{n-p} \binom{n}{p} |Z_\Gamma| \frac{\operatorname{Vol}(\Gamma \backslash G)}{(2\pi)^n}.$$

If we use on G_0 the measure $e^{-2 \ln a}\, du\, da\, dk$ with dk normalized by $\operatorname{Vol}(K_0) = 1$, then, under the isomorphism $G_0/K_0 \cong \mathbf{H}$, this measure corresponds to $(dx\, dy/y^2)/2$ with respect to the coordinates (1.1). Thus $\operatorname{Vol}(\Gamma \backslash G) = (|Z_\Gamma|)^{-1} 2^{-n} \operatorname{Vol}(\Gamma \backslash \mathbf{H}^n)$ and the central contribution is

$$(-1)^{n-p} \binom{n}{p} \frac{\operatorname{Vol}(\Gamma \backslash \mathbf{H}^n)}{(4\pi)^n}.$$

In the same way one can determine the elliptic and the parabolic contribution. The parabolic contribution to $\operatorname{Tr} R^{d}_{\Gamma\backslash G}(h^{\tau}_{t})$ is given by (4.22) and (4.28), where $f = h^{\tau}_{t}$. First we consider (4.28). It follows from Lemma 5.34, (2.4) and (2.5) that

$$\int_{-\infty}^{\infty} (\ln|x|)\, p_t^{(l)}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx, \qquad l = 0, \pm 1,$$

is bounded as $t \to \infty$. By Lemma 5.34 we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_{\lambda}^{+}\left(p_t^{(l)}\right) d\lambda = (\pi t)^{-1/2} e^{-t/4}.$$

Thus, if $n > 1$ and if we put $f = h^{\tau}_{t}$, then it turns out that (4.28) tends to zero if $t \to \infty$. Moreover, by (4.26) and Lemma 5.34, we get

$$\int_{U_0\backslash G_0} p_t^{\pm}\left(g^{-1}u(\varepsilon_j)g\right) dg = \pm \frac{i}{2\pi}\varepsilon_j + b(t),$$

$$\int_{U_0\backslash G_0} p_t^{0}\left(g^{-1}u(\varepsilon_j)g\right) dg = b(t),$$

with $\lim_{t\to\infty} b(t) = 0$. Let $n > 1$. Then it follows from (4.22), (4.23) and (5.81) that the parabolic term of $\operatorname{Tr} R^{d}_{\Gamma\backslash G}(h^{\tau}_{t})$ tends to zero as $t \to \infty$, except when $\tau = \sigma_{I,J}$ with $I \cap J = \varnothing$ and $|I| + |J| = n$, $|I| = p$. In the latter case the limit is equal to

$$(-1)^{n-p}\frac{i^n}{(2\pi)^n}\sum_{\varepsilon} a_0(\varepsilon)N(\varepsilon),$$

$N(\varepsilon) = \varepsilon_1 \cdots \varepsilon_n$. Now, we can proceed as in the case of the signature operator. Let $x_1, \cdots, x_h$ be a complete system of Γ-inequivalent parabolic fixed points of Γ, and let $(\mathsf{M}_\iota, \mathsf{V}_\iota)$ be associated with x_ι as above. Then the parabolic contribution to the L^2-index of D_p is given by

$$\binom{n}{p}\frac{i^n}{(2\pi)^n}\sum_{\iota=1}^{h} d(\mathsf{M}_\iota)L(\mathsf{M}_\iota, \mathsf{V}_\iota, 1).$$

The computation of the elliptic contribution is similar. One has to use (4.9) and (4.10). Together with (5.76) this gives the following

Theorem 5.82. *Let $F/\mathbf{Q}$ be a totally real number field of degree $n > 1$ and let $\Gamma \subset \mathrm{SL}(2, F)$ be an arithmetic subgroup. Let z_j, $1 \leqslant j \leqslant r$, and x_ι, $1 \leqslant \iota \leqslant h$, be complete systems of Γ-inequivalent elliptic and parabolic fixed points of Γ respectively. Each $\gamma \in \Gamma_{z_j}$ is conjugate to some $k \in K$ and we denote by $\theta_l(\gamma)$ the angle of the component k_l of k. Further, let $Z_p \subset \{\pm 1\}^n$ be the subset of those ε which have precisely p components equal to one. Let $\mathsf{M}_\iota \subset F$, $\mathsf{V}_\iota \subset U^{+}_{\mathsf{M}_\iota}$, be* *associated with x_ι as above. Then*

$$\dim \mathscr{H}^{p,n-p}_{\mathrm{cus}}(\Gamma\backslash \mathbf{H}^n) = \binom{n}{p}\frac{\operatorname{Vol}(\Gamma\backslash\mathbf{H}^n)}{(4\pi)^n} + (-1)^{n+1}\binom{n}{p}$$
$$+(-1)^{n-p}\frac{i^n}{2^n}\sum_{j=1}^{r}\sum_{\substack{\gamma\in\Gamma_{z_j}\\ \gamma\neq 1}}\sum_{\varepsilon\in Z_p}\frac{1}{|\Gamma_{z_j}|}\prod_{l=1}^{n}\frac{\exp(i\varepsilon_l\theta_l(\gamma))}{\sin\theta_l(\gamma)}$$
$$+(-1)^{n-p}\binom{n}{p}\frac{i^n}{(2\pi)^n}\sum_{\iota=1}^{h} d(\mathsf{M}_\iota)L(\mathsf{M}_\iota,\mathsf{V}_\iota,1).$$

Note, that the parabolic contribution vanishes if n is odd. This theorem together with Corollary 5.68 gives

Corollary 5.83.

$$\operatorname{Sign}(\Gamma\backslash\mathbf{H}^n) = \sum_{p,q=0}^{n}(-1)^q\dim\mathscr{H}^{p,q}_{\mathrm{cus}}(\Gamma\backslash\mathbf{H}^n).$$

This is the analogue of the signature formula for compact Kähler manifolds [23, §15.8].

6. The Hirzebruch conjecture

In this section we discuss briefly how one can prove Hirzebruch's conjecture in general. As explained in the introduction, we will not carry out the details since this is part of a future publication which treats spectral theory of the Laplacian on Riemannian manifolds which are locally symmetric near infinity.

We recall Hirzebruch's conjecture [24, p. 230]. Let $F/\mathbb{Q}$ be a totally real number field of degree n. Let $\mathsf{M}\subset F$ be a lattice $\mathsf{V}\subset U^+_{\mathsf{M}}$ a subgroup of finite index in the group U^+_{M} of totally positive units which transform M into itself. Suppose that G is a group of matrices $\left(\begin{smallmatrix}\varepsilon & \mu\\ 0 & 1\end{smallmatrix}\right)$ (with $\varepsilon\in\mathsf{V}$, $\mu\in F$, and $\mu\in\mathsf{M}$ if $\varepsilon=1$) such that the sequence

$$0\to\mathsf{M}\to\mathsf{G}\to\mathsf{V}\to 1 \tag{6.1}$$

is exact. The group G acts freely and properly discontinuously on $\mathbf{H}^n$. $\overline{\mathsf{G}\backslash\mathbf{H}^n} = \mathsf{G}\backslash\mathbf{H}^n\cup\{\infty\}$ is a normal complex space with an isolated singularity. We call this singular point a cusp of type (M,V). With the cusp ∞ one can associate its signature defect $\delta(\mathsf{G})$, which is defined in the same way as the signature defect (5.69). On the other hand, we have the L-series $L(\mathsf{M},\mathsf{V},s)$ associated with (M,V) via (5.53). Hirzebruch conjectured that

$$\delta(\mathsf{G}) = \frac{i^n}{\pi^n}d(\mathsf{M})L(\mathsf{M},\mathsf{V},1)$$

for every extension (6.1). In particular, $\delta(\mathsf{G})$ depends only on (M,V) and not on the extension (6.1). There are two problems which prevent us from proving Hirzebruch's conjecture by the methods of the previous paragraphs:

(i) The group G may not occur as $\rho\Gamma_x\rho^{-1}$, where Γ_x is the stabilizer of a cusp x of some irreducible discrete subgroup $\Gamma \subset G$ with finite covolume and $\rho \in G$ is such that $\rho x = \infty$.

(ii) $\Gamma \setminus \mathbf{H}^n$ can have several cusps.

To overcome these difficulties, we replace $\Gamma \setminus \mathbf{H}^n$ by a manifold which consists of a single cusp, chopped off near infinity and glued together with a compact Riemannian manifold, which has the same boundary. More precisely, consider an extension (6.1). For $d > 0$ let $W(d) = \{z \in \mathbf{H}^n | \prod_{j=1}^n \operatorname{Im}(z_j) \geqslant d\}$ and let $Y(d) = \mathsf{G} \setminus W(d)$. The stable tangent bundle of the boundary $\partial Y(d)$ has a canonical parallelization. Therefore, there exists a compact oriented manifold N with boundary $\partial Y(d)$. N and $Y(d)$ can be glued together along their common boundary. Let X be the resulting manifold. We choose a smooth Riemannian metric on X which coincides with the given metric on $Y(d)$. We call X a manifold with a cusp of type (M,V). X has a decomposition $X = X_0 \cup X_1$, where X_0 is a compact Riemannian manifold with boundary and X_1 is isometric to $Y(d)$ for some $d > 0$. The point is that one can extend all results, concerning the spectral resolution of the Laplacian on the locally symmetric space $\Gamma \setminus \mathbf{H}^n$, to Riemannian manifolds with a cusp of type (M,V). This program has been carried out by the author for manifolds which are natural generalizations of the $\mathbb{R}$-rank one locally symmetric spaces [30], [31]. In this case the cusps are Riemannian warped products. This means that each cusp is isometric to a product $\mathbb{R}^+ \times X$, where X is a closed Riemannian manifold with metric tensor g and the metric ds^2 on the product is given by $ds^2 = dy^2 + e^{-2y}g$. In principle, the same methods can be applied in our situation. The hard work is to do analysis on the cusp. But in our case this reduces to harmonic analysis on $\mathsf{G} \setminus \mathbf{H}^n$. Selberg's trace formula, which we used in the locally symmetric case, has to be replaced by the asymptotic expansion of the heat kernel. Let

$$D = d + \delta : \Lambda^*_+(X) \to \Lambda^*_-(X)$$

be the signature operator. Then, using these methods, one can compute the L^2-index of D. It is given by

$$\operatorname{Ind}_{L^2} D = \int_X L(p) + \frac{i^n}{\pi^n} d(\mathsf{M}) L(\mathsf{M},\mathsf{V},1), \tag{6.2}$$

where $L(p)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of X. To prove that $\operatorname{Ind}_{L^2}(D)$ is equal to $\operatorname{Sign}(X)$, we have to extend the results of Harder [18], [19] on cohomology of $\Gamma \setminus \mathbf{H}^n$ to manifolds with a cusp of type

(M, V). Harder uses the theory of Eisenstein series. In our case we have a corresponding theory of Eisenstein forms, which satisfy the same properties as the Eisenstein series in the locally symmetric case. In particular, they satisfy the same system of functional equations. Using the Eisenstein forms one can extend the results of Harder to our situation and in this way we get

$$\text{Sign}(X) = \text{Ind}_{L^2} D. \tag{6.3}$$

Finally, we prove a formula which is similar to Hirzebruch's formula (5.70). For $d > 0$ let $X_d = X - Y(d)$. We orient $\partial Y(d)$ by the orientation induced from the canonical orientation of $Y(d)$. Let $\tilde{p}_j \in H^{4j}(X_d/\partial X_d; \mathbb{Z})$ be the Pontrjagin classes of the SO-bundle over $X_d/\partial X_d$ obtained by pushing down the stable tangent bundle of X_d. Suppose that $\dim X = 4k$. It follows from the definition of the signature defect $\delta(\mathsf{G})$ that

$$\text{Sign}(X) = L_k(\tilde{p}_1, \cdots, \tilde{p}_k)[X_d, \partial X_d] + \delta(\mathsf{G}).$$

If we apply the arguments used by Hirzebruch in the proof of formula (20), §3, in [24] to the Pontrjagin forms, then we get

$$\text{Sign}(X) = \int_X L(p) + \delta(\mathsf{G}).$$

This result combined with (6.2) and (6.3) gives a proof of Hirzebruch's conjecture.

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AKADEMIE DER WISSENSCHAFTEN DER DDR, BERLIN
GERMAN DEMOCRATIC REPUBLIC