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ON THE MEAN CURVATURE FUNCTION FOR COMPACT SURFACES

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Dedicated to Professor Buchin Su on his 80th birthday

It is a classical fact that any surface in $\mathbf{R}^3$ is determined up to congruences by its first and second fundamental forms. We shall prove in this article that *compact* surfaces are essentially determined by the first fundamental form and only the trace of the second, that is, by the metric and the mean curvature function. The only possible exception to this phenomenon occurs in the case of constant mean curvature. Of course, it is a long-standing conjecture of Hopf that the only such (compact) surfaces are the round spheres.

An explicit statement of our main result is as follows. Denote by $M^3(c)$ the complete simply-connected 3-manifold of constant sectional curvature c.

Theorem. *Let Σ be a compact oriented surface equipped with a riemannian metric, and let H: $\Sigma \to \mathbf{R}$ be a smooth function. If H is not constant, then there exist at most two geometrically distinct isometric immersions of Σ into $M^3(c)$ with mean curvature H.*

Remarks. 1. Two immersions are said to be *geometrically distinct* if they do not differ by an isometry of $M^3(c)$, i.e., by a congruence.

2. The theorem above can be immediately applied to nonorientable surfaces. Here the function H: $\Sigma 7 \to \mathbf{R}$ must be replaced by a function $\tilde{H}$: $\tilde{\Sigma} \to \mathbf{R}$ on the 2-sheeted orientable covering surface π: $\tilde{\Sigma} \to \Sigma$ with the property that $H(\alpha(p)) = -H(p)$ where α: $\tilde{\Sigma} \xrightarrow{\approx} \tilde{\Sigma}$ is the deck transformation of the covering π.

3. The result above represents a generalization to genus greater than one, of a theorem proved in the doctoral dissertation of the second author [5]. The first author insists on stating that the hard part of the proof and the principal ideas originated there.

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4. In the case that Σ is homeomorphic to the sphere S^2, the theorem can be strengthened. In that case there exists at most one isometric immersion with a given mean curvature function. This is proved in [5] and follows also from the arguments below.

Proof of the theorem. The given metric determines a conformal structure on Σ, and we shall always work in the corresponding local conformal, or "isothermal", coordinates. With respect to such a local coordinate $z = x_1 + ix_2$, the metric can be written as

$$ds^2 = \lambda^2|dz|^2.$$

Suppose now that $F: \Sigma \to M^3(c)$ is an isometric immersion with unit normal vector field ν. Let

$$b_{ij} = -\left\langle \nabla_{\frac{\partial}{\partial x_i}} \nu, \frac{\partial}{\partial x_j} \right\rangle$$

for $1 \leqslant i, j \leqslant 2$, denote the components of the second fundamental form of this immersion. Of fundamental importance to this study is the associated quadratic differential

$$Q \equiv \{b_{11} - b_{22} - 2ib_{12}\} dz^2 \equiv f\, dz^2, \tag{1}$$

which is well-defined globally on Σ. On the metric induced naturally on the bundle $T^{1,0} \otimes T^{1,0}$ we have that

$$\|Q\|^2 = H^2 - 4(K - c), \tag{2}$$

where $H = \lambda^{-2}(b_{11} + b_{22})$ is the mean curvature of the immersion, and $K = \lambda^{-2}(b_{11}b_{22} - b_{12}^2) + c$ is the Gaussian curvature of the surface. In terms of the principal curvatures k_1 and k_2 of Σ we see that

$$\|Q\|^2 = (k_1 - k_2)^2,$$

and so Q vanishes precisely at the umbilic points of the immersion.

We begin by recalling the following well-known consequence of the Mainardi-Codazzi equations (cf. [5]).

Lemma 5. *The quadratic form Q is holomorphic if and only if the immersion F has constant mean curvature.*

We now suppose that we are given three isometric immersions $F_k: \Sigma \to \mathbf{R}^3$, $k = 1, 2, 3$, with the same mean curvature function H. We let

$$Q_k = f_k(z)\, dz^2, \quad k = 1, 2, 3$$

be the corresponding associated quadratic differentials on Σ. The following principal results are proved in [5].

Proposition 6. *Each of the differences $Q_{ij} \equiv Q_i - Q_j$ for $1 \leqslant i, j \leqslant 3$, is a holomorphic quadratic differential form on Σ.*

Theorem 7. *If the three immersions F_k, $k = 1, 2, 3$, are mutually noncongruent, then*

$$\Delta^0 \log(f_k) = \left| \frac{\partial f_k}{\partial \bar{z}} \right|^2 \tag{3}$$

for each k, where $\Delta^0 = 4(\partial/\partial z)(\partial/\partial \bar{z})$ is the standard laplacian in the local coordinate z.

From this point on we shall assume that F_1, F_2 and F_3 are mutually noncongruent. However, we shall really only use the fact that (3) holds for the two immersions F_1 and F_2.

Since F_1 and F_2 are isometric and have the same mean curvature function, we see from (2) that $\|Q_1\| \equiv \|Q_2\|$. Hence we may write

$$Q_2 = e^{i\theta} Q_1, \tag{4}$$

where θ is well defined (modulo 2π) outside the zeros of $\|Q_k\|^2 = H^2 - 4(K - c)$. We now consider the holomorphic quadratic form

$$q \equiv Q_1 - Q_2 = (1 - e^{i\theta}) f_1 \, dz^2. \tag{5}$$

Clearly, the zeros of Q_k (the umbilic points) are contained in the zeros of q. In particular, the zeros of Q_k, which we shall denote by $Z = \{p_j\}_{j=1}^n$, are *isolated.*

We now consider the quotient

$$\Psi \equiv \frac{q}{Q_1} = 1 - e^{i\theta},$$

which is well defined on $\Sigma \setminus Z$. Since q is holomorphic, we have from (3) that

$$\Delta \log \Psi = \Delta \log|\Psi| + i\Delta \arg \Psi \leqslant 0, \tag{6}$$

where $\Delta = \lambda^{-2}\Delta^0$ is the Laplace-Beltrami operator on Σ. Equation (6) can be rewritten by saying that

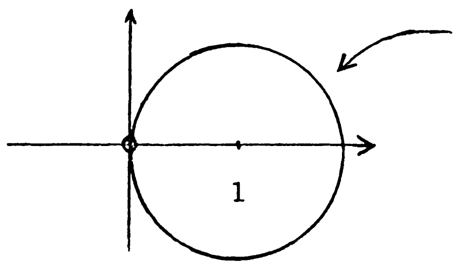
$$\Delta \log|\Psi| \leqslant 0, \quad \Delta \arg \Psi = 0 \tag{7}$$

on $\Sigma \setminus Z$.

We now observe that since Ψ is not zero in the connected set $\Sigma \setminus Z$, the function θ cannot be zero (modulo 2π) in this set. Hence we can choose a continuous branch θ: $\Sigma \setminus Z \to (0, 2\pi) \subset \mathbf{R}$. It follows that there exists a continuous branch

$$\arg(\Psi(z)) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{8}$$

for $z \in \Sigma \setminus Z$. In particular it follows from (7) and (8) that



$$|\Psi(z) - 1| \equiv 1.$$

arg Ψ is a bounded harmonic function on $\Sigma \setminus Z$, where Z consists of a finite set of points. By a classical theorem on removable singularities, arg Ψ extends to a smooth harmonic function on all of Σ, and hence arg Ψ is constant. It follows immediately that Ψ is constant. Consequently, Q_1 is holomorphic, and so by Lemma 5, the mean curvature function H is constant. This completes the proof.

Final comments

1. It should be pointed out that the main result of this paper is definitely global in nature; that is, there exist compact surfaces in $\mathbf{R}^3$ having small neighborhoods which can be continuously deformed through noncongruent isometric embeddings with the same mean curvature function.

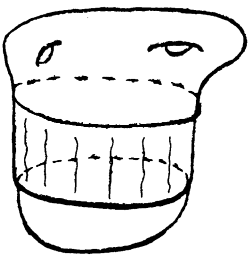
2. The local question of isometric immersions with the same mean curvature function into $\mathbf{R}^3$ has been studied in [3], [4] and [2]. In these works it is proved that if a nontrivial family of such immersions does not exist, then there are at most two noncongruent ones. (It follows from [5] that this result is valid also for immersions into $M^3(c)$, any c.) A superficial reading of these papers can indicate that in the absence of a nontrivial family, the immersion must be unique. However, in none of these papers do the arguments actually prove this.

3. Complete, simply-connected surfaces of constant mean curvature in $M^3(c)$ always admit 1-parameter families of isometric deformations through noncongruent surfaces with the same constant mean curvature (see [1]).

4. It remains an open question whether there can exist two geometrically distinct isometric immersions $\Sigma \hookrightarrow M^3(c)$ with the same mean curvature function for a compact surface Σ of genus > 0.

5. For any compact surface Σ, there do exist families of noncongruent (and *nonisometric*) immersions into $\mathbf{R}^3$ with the same mean curvature function. Such families can be constructed as follows. Let γ be a closed curve in the

plane $\mathbf{R}^2 \subset \mathbf{R}^3$, and consider the cylinder $\gamma \times [0, t]$ of height t over γ. Cap off (smoothly) the bottom of the cylinder with a disk and the top of the cylinder with a surface of desired topological type. These "caps" should be the same, i.e., congruent, for all time t. The mean curvature of the annulus at a point $(x, s) \in \gamma \times [0, t]$ is just $\kappa(x) \equiv$ the curvature of the planar curve γ at x. It is easy to reparameterize these surfaces by a single surface Σ in such a manner that the resulting family of immersions ψ_t: $\Sigma \to \mathbf{R}^3$ has mean curvature function independent of t. (Stretch the parameter along the generators of the cylinder.)



$\updownarrow$ stretch

Of course, many such cylinders could be added, giving k-fold deformations $\psi_{t_1, \dots, t_k}$: $\Sigma \to \mathbf{R}^3$ with the same H.

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