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A GEOMETRIC CHARACTERIZATION OF POINTS OF TYPE m ON REAL SUBMANIFOLDS OF C^n

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1. Introduction

Let D be a domain in C^n with smooth boundary bD. bD is said to be pseudoconvex (respectively strongly pseudoconvex) if the Levi form is non-negative (respectively positive definite) on the complex tangent space at all points of bD.

Pseudoconvexity of bD is a necessary and sufficient condition for D to be a domain of holomorphy [4]. However, if one makes the assumption of strong pseudoconvexity, more precise results are possible than mere existence statements, e.g., solutions of $\bar\partial$ within the class of bounded functions, boundary regularity of solutions of $\bar\partial$ (see [1] and the references there). The existence of holomorphic support functions and peak functions plays an important role in analysis on strongly pseudoconvex domains.

Pseudoconvexity alone is not a sufficient condition for local regularity of $\bar\partial$ at the boundary (for global regularity see [6]). A counterexample appears in [8] in which bD contains a complex submanifold. Nor does pseudoconvexity guarantee the existence of peak functions (see [9] for an interesting counterexample). Thus conditions between pseudoconvexity and strong pseudoconvexity are of interest [5], [7].

In [5], J. J. Kohn introduced the notion of points of type m (m is a positive integer or $+\infty$) on the boundary of a domain D in C^2. A point at which the Levi form does not vanish is of type 1. If bD contains a complex submanifold, then all points on this submanifold are of infinite type [5]. Pseudoconvexity together with finite type yields a subelliptic estimate for (0, 1) forms which implies local regularity at the boundary for the canonical solution of $\bar\partial$, [5]. P. Greiner [3] showed that these assumptions are necessary for this estimate. Kohn also introduced the notion of strict type m which is sufficient to guarantee the existence of local peak functions [5].

Kohn's definition of points of type m is in terms of properties of commutators of tangential holomorphic vector fields. In [11] Naruki studies real submanifolds of C^n of arbitrary codimension. A similar condition involving commutators of tangential holomorphic vector fields appears. Using this con-

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dition together with total indefiniteness of the Levi form, Naruki obtains a subelliptic estimate for $\bar{\partial}_b$ on functions.

Our main result is a geometric characterization of points of type m on a hypersurface M in C^n ('type' is defined in § 2):

Theorem 2.4. *A point $P \in M$ is of type $m < \infty$ if and only if there is a complex submanifold of codimension one tangent to M at P to order m but no codimension one complex submanifold tangent to a higher order. A point $P \in M$ is of infinite type if and only if there are complex submanifolds of codimension one tangent to M at P to arbitrarily high order. (There may or may not be a complex submanifold tangent to infinite order.)*

The proof of this theorem is contained in § 2. It would be of interest to relate a 'type' condition to the maximum degree of tangency of a complex submanifold of dimension one. This is the idea behind § 3, but our results are incomplete. However, some interesting examples are given. In § 4 we generalize Theorem 2.4 to the case of generic submanifolds of arbitrary codimension. However the commutator condition is not the same as Naruki's [11].

We are indebted to Peter Greiner for numerous helpful discussions concerning this work.

1. Basic definitions

1.1. Let M be a real C^∞ submanifold of an open subset U in C^n, and let P be a point of M. The complexified tangent space to C^n at P, denoted by $CT(C^n, P)$ splits naturally into a direct sum of two subspaces $T^{1,0}(C^n, P) \oplus T^{0,1}(C^n, P)$ the holomorphic and anti-holomorphic parts. The injection of M into C^n induces an injection of the complexified tangent space to M at P, $CT(M, P)$ into $CT(C^n, P)$ and we consider $CT(M, P)$ as a subset of $CT(C^n, P)$.

1.2. Definition. The holomorphic tangent space to M at P is defined to be the intersection $CT(M, P) \cap T^{1,0}(C^n, P)$ and is denoted by $T^{1,0}(M, P)$.

Suppose that $M = \{z \in U \mid r_1 = r_2 = \cdots = r_k = 0\}$, where the r_i are real-valued C^∞ functions such that $dr_1 \wedge \cdots \wedge dr_k \neq 0$ at all points of M. Then we may identify $T^{1,0}(M, P)$ with all $w \in C^n$ satisfying

$$(1.2.1) \qquad \sum_{j=1}^{n} \frac{\partial r_i}{\partial z_j} w_j = 0 \qquad \text{for } i = 1, \cdots, k .$$

We note that $\dim_C T^{1,0}(M, P)$ satisfies [12] the inequalities

$$\max(0, n-k) \leq \dim_C T^{1,0}(M, P) \leq n - \left[\frac{k+1}{2}\right] .$$

If M is a real hypersurface then $\dim_C T^{1,0}(M, P) = n - 1$.

1.3. Definition. A holomorphic vector field on U is a C^∞ vector field F

whose value at each point $q \in U$ satisfies

$$F(q) \in T^{1,0}(C^n, q) \ .$$

Such a vector field may be written in the form $\sum_{i=1}^n a_i(\partial/\partial z_i)$ with a_i a complex-valued C^∞ function on U.

1.4. Definition. A vector field F is tangential to M if $F(q) \in CT(M, q)$ for all $q \in M$.

1.5. Definition. A holomorphic vector field tangential to M is a vector field F such that $F(q) \in T^{1,0}(M, q)$ for all $q \in M$ and $F(q) \in T^{1,0}(C^n, q)$ for all $q \in U$.

If F is written in the form $\sum_{i=1}^n a_i(\partial/\partial z_i) + \sum_{i=1}^n b_i(\partial/\partial \bar{z}_i)$, then it is tangential if and only if

$$\sum_{i=1}^n a_i \frac{\partial r_s}{\partial z_i} + \sum_{i=1}^n b_i \frac{\partial r_s}{\partial \bar{z}_i} = 0 \qquad \text{on } M$$

for $s = 1, \cdots, k$. That is, $F(r_s) = 0$ on M for $s = 1, \cdots, k$.

1.6. Definition. For F a vector field we define its conjugate $\bar{F}$ via the equation

$$\bar{F}(u) = \overline{F(\bar{u})} \qquad \text{for all } u \in C^\infty(U) \ .$$

If $F = \sum a_i(\partial/\partial z_i) + \sum b_i(\partial/\partial \bar{z}_i)$, then

$$\bar{F} = \sum \bar{a}_i \frac{\partial}{\partial \bar{z}_i} + \sum \bar{b}_i \frac{\partial}{\partial z_i} \ .$$

Note that F is tangential if and only if $\bar{F}$ is.

1.7. Definition. For each integer $\mu \geq 0$ we define $\mathscr{L}_\mu$ to be the module, over $C^\infty(U)$, of vector fields generated by the holomorphic tangential vector fields, their conjugates and commutators of order $\leq \mu$ of such vector fields.

Thus $\mathscr{L}_0$ is the module of vector fields spanned by the tangential holomorphic vector fields and their conjugates. $\mathscr{L}_\mu$ is spanned by elements of the form $[F, G]$ with $F \in \mathscr{L}_{\mu-1}$ and $G \in \mathscr{L}_0$.

$\mathscr{L}_\mu$ is closed under conjugation and consists solely of tangential vector fields. Note that $\mathscr{L}_\mu \subset \mathscr{L}_{\mu+1}$, and setting $\mathscr{L} = \bigcup_{\mu=0}^\infty \mathscr{L}_\mu$ we note that $\mathscr{L}$ is a Lie algebra [5, p. 526].

2. The geometric characterization for hypersurfaces

Let M be a real C^∞ hypersurface in an open subset $U \subset C^n$. Let $M = \{z \in U \mid r(z) = 0\}$ where r is a real-valued C^∞ function such that $dr \neq 0$ on M.

2.1. Definition [5, p. 525]. A point $P \in M$ is of type m if $\langle \partial r(P), F(P) \rangle = 0$ for all $F \in \mathscr{L}_{m-1}$ while $\langle \partial r(P), F(P) \rangle \neq 0$ for some $F \in \mathscr{L}_m$. Here $\langle \ , \ \rangle$

denotes contraction between a cotangent vector and a tangent vector.

Note that m is an integer ≥ 1 or $+\infty$. We will use the notation $t(P) = m$.

2.2. Remarks. 1. The function $t(P)$ is upper-semicontinuous on M.

2. If the Levi form is nonzero at P then $t(P) = 1$, [5].

Let X be an $(n-1)$-dimensional complex submanifold of a neighborhood of P which is tangent to M at P.

2.3. Definition. X is tangent to M at P to order s if the restriction $r|_X$ of r to X vanishes to order $s+1$ at P.

For s an integer ≥ 1 we will use the notation $a(P) = s$ if there exists a complex $(n-1)$-dimensional submanifold tangent to M at P to order s but none tangent to order $s+1$. We will write $a(P) = +\infty$ if either

1. there is a complex $(n-1)$-dimensional submanifold tangent to M at P to order $+\infty$, or

2. for every integer N no matter how large, there is a complex $(n-1)$-dimensional submanifold of some neighborhood of P tangent to M at P to order N (see § 2.14). Thus $a(P)$ is an integer ≥ 1 or $+\infty$.

2.4. Theorem. $t(P) = a(P)$.

For $M \subset C^2$ this result is implicit in the article of Kohn [5]. In fact our proof is quite similar to his proof.

The proof of Theorem 2.4 will be carried out in Lemmas 2.6 to 2.12. We will show $t(P) \geq a(P)$ (Lemma 2.11) and $t(P) \leq a(P)$ (Lemma 2.12). Lemma 2.11 depends only on Lemma 2.9 and the preceding lemmas. Lemma 2.10 is needed for Lemma 2.12.

2.5. First we suppose that we have local coordinates $z_1, \cdots, z_{n-1}, w$ centered at P so that r has the form

$$r = 2\,\mathrm{Re}\,(w) + \phi\,, \tag{2.5.1}$$

where ϕ vanishes to order ≥ 2 at P.

Thus

$$r_w(P) = r_{\bar{w}}(P) = 1\,, \tag{2.5.2}$$

while

$$r_{z_i}(P) = r_{\bar{z}_i}(P) = 0 \qquad \text{for } i = 1, \cdots, n-1\,. \tag{2.5.3}$$

If F is a vector field written in the form

$$F = \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \bar{z}_i} + c\frac{\partial}{\partial w} + d\frac{\partial}{\partial \bar{w}}\,,$$

then $\langle \partial r(P), F(P)\rangle = c(P)$. Thus $t(P) = m$ precisely when $c(P) \neq 0$ for some $F \in \mathscr{L}_m$ but $c(P) = 0$ for all $F \in \mathscr{L}_{m-1}$. Also note that if F is tangential, then $c(P) + d(P) = 0$. The vector fields

$$L_i = r_w \frac{\partial}{\partial z_i} - r_{z_i} \frac{\partial}{\partial w} \qquad \text{for } i = 1, \cdots, n-1 \tag{2.5.4}$$

are tangential.

2.6. Lemma. *$\mathscr{L}_\mu$ is generated modulo vector fields vanishing on M as a C^∞ module by the commutators of order $\leq \mu$ of the $2n-2$ vector fields $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}$.*

Proof. Let F be a holomorphic tangential vector field:

$$F = \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial z_i} + c \frac{\partial}{\partial w} .$$

Then $\sum_{i=1}^{n-1} a_i r_{z_i} + c r_w = 0$ on M while $r_w \neq 0$ on a neighborhood of P (assumed to be U). Thus

$$F - \sum_{i=1}^{n-1} \frac{a_i}{r_w} L_i$$

is a vector field which vanishes on M. That is, $\mathscr{L}_0$ is spanned by $L_1, \cdots, L_{n-1}$, $\bar{L}_1, \cdots, \bar{L}_{n-1}$ and vector fields of the form rH where H is any vector field. It follows by induction on μ that $\mathscr{L}_\mu$ is spanned by the commutators of $L_1, \cdots$, $L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}$ of order $\leq \mu$ and vector fields of the form rH, [5, p. 526].

2.7. Lemma. *Let F be a vector field written in the form*

$$F = \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \bar{z}_i} + c \frac{\partial}{\partial w} + d \frac{\partial}{\partial \bar{w}} .$$

Then the coefficient of $\partial/\partial w$ in $[L_\alpha, F]$ is

$$r_w \frac{\partial c}{\partial z_\alpha} - r_{z_\alpha} \frac{\partial c}{\partial w} + \sum_{i=1}^{n-1} a_i r_{z_i z_\alpha} + \sum_{i=1}^{n-1} b_i r_{z_\alpha \bar{z}_i} + c r_{z_\alpha w} + d r_{z_\alpha \bar{w}} . \tag{2.7.1}$$

The coefficient of $\partial/\partial z_j$ in $[L_\alpha, F]$ is

$$r_w \frac{\partial a_j}{\partial z_\alpha} - r_{z_\alpha} \frac{\partial a_j}{\partial w} - \delta_{j\alpha} \left[\sum_{i=1}^{n-1} a_i r_{w z_i} + \sum_{i=1}^{n-1} b_i r_{w \bar{z}_i} + c r_{ww} + d r_{w\bar{w}} \right] . \tag{2.7.2}$$

Of course there are similar formulas for the coefficients of $\partial/\partial \bar{w}$ and $\partial/\partial \bar{z}_j$ and for the coefficients in $[\bar{L}_\alpha, F]$.

Proof. Direct computation.

We will use the notation $z = (z_1, \cdots, z_{n-1})$.

2.8. Lemma. *Suppose $F \in \mathscr{L}_\mu - \mathscr{L}_{\mu-1}$ is formed by commutators of $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}$. Then the coefficients a_i, b_i, c, d are sums of terms of the form $\pm D^1(r) \cdots D^{\mu+1}(r)$, where each D^i is differentiation to order d_i, and the integers d_i satisfy*

1. $d_1 + \cdots + d_{\mu+1} = 2\mu + 1$,
2. $1 \leq d_i \leq \mu + 1$.

In addition each such term in a_j or b_j involves differentiation a total of μ times with respect to z and $\mu + 1$ times with respect to w. Each term in c, d involves differentiation a total of $\mu + 1$ times with respect to z and μ times with respect to w.

Proof. The proof is by induction on μ and an examination of formulas (2.7.1) and (2.7.2). The statement about the a_j and b_j coefficients is needed only for the inductive proof of the statement about the c and d coefficients.

2.9. Lemma. *Suppose $F \in \mathscr{L}_\mu - \mathscr{L}_{\mu-1}$ is formed by commutators of $L_1, \cdots, L_{n-1}$ and $\bar{L}_1, \cdots, \bar{L}_{n-1}$. Then each term in the c and d coefficients contains a factor of the form $D(r)$ where D is differentiation in $z, \bar{z}$ only (i.e., no w) of order $\leq \mu + 1$.*

Proof. By Lemma 2.8 each term contains $\mu + 1$ factors, and the total order of differentiation in w is just μ.

2.10. Lemma. *Let $D = (\partial/\partial z)^\sigma(\partial/\partial\bar{z})^\tau$ where σ, τ are multi-indices and $|\sigma| \geq 1$, $|\tau| \geq 1$ and $|\sigma| + |\tau| = \mu + 1$ (thus $\mu \geq 1$). Then there exists $F \in \mathscr{L}_\mu$ whose c coefficient has the following properties:*

1. *There is one term $r_w^{|\sigma|-1} r_{\bar{w}}^{|\tau|} D(r)$.*
2. *All other terms $D^1(r) \cdots D^{\mu+1}(r)$ have the property that some D^i is a differentiation in $z, \bar{z}$ (i.e., no w) of order $\leq \mu$.*

Proof. The proof is by induction on μ. When $\mu = 1$ we have $D = \partial^2/\partial z_i \partial\bar{z}_j$. The c coefficient of $[L_i, \bar{L}_j]$ is $r_{\bar{w}} r_{z_i\bar{z}_j} - r_{\bar{z}_j} r_{z_i\bar{w}}$ which satisfies (1) and (2).

For the inductive step we have either $|\sigma| > 1$ or $|\tau| > 1$ say $|\sigma| > 1$. We write $D = (\partial/\partial z_\alpha)(\partial/\partial z)^{\sigma'}(\partial/\partial\bar{z})^\tau$ where $|\sigma'| = |\sigma| - 1$. By the induction hypothesis we can find $F \in \mathscr{L}_{\mu-1}$ with properties (1) and (2) for $(\partial/\partial z)^{\sigma'}(\partial/\partial\bar{z})^\tau$. An examination of formula (2.7.1) shows that $[L_\alpha, F]$ satisfies (1) and (2) for D. In fact, the form $r_w^{|\sigma|-1} r_{\bar{w}}^{|\tau|} D(r)$ comes from $r_w(\partial c/\partial z_\alpha)$.

2.11. Lemma. $t(P) \geq a(P)$.

Proof. Let X be an $(n - 1)$-dimensional complex manifold tangent to M at P to order s $(1 \leq s < +\infty)$. We may assume the coordinate w (of formula (2.5.1)) chosen so that $X = \{(z, w) \in U \mid w = 0\}$.

Now, $r(z, 0)$ vanishes at P to order $s + 1$. Consequently, $D(r)$ vanishes at P if D involves differentiation of order $\leq s$ with respect to $z, \bar{z}$ (i.e., no w differentiation). Thus Lemma 2.9 shows that the c coefficient of any $F \in \mathscr{L}_{s-1}$ vanishes at P and hence $t(P) \geq s$. Thus $t(P) \geq a(P)$.

2.12. Lemma. $t(P) \leq a(P)$.

Proof. Suppose that $t(P) \geq m$ where m is an integer ≥ 1. We may assume that the coordinate w (of formula (2.5.1)) is chosen so that $D(r)(P) = 0$ where D is any pure differentiation with respect to z or $\bar{z}$ (i.e., no mixture of derivatives with respect to z and $\bar{z}$) of order $\leq m + 1$. We will show that $w = 0$ is tangent to M at P to order $\geq m$.

The c coefficient of any $F \in \mathscr{L}_{m-1}$ vanishes at P. By Lemma 2.10 we may conclude that $(\partial/\partial z)^\sigma(\partial/\partial\bar{z})^\tau r(P) = 0$ for σ, τ any multi-indices satisfying $|\sigma| \geq 1$, $|\tau| \geq 1$, $|\sigma| + |\tau| \leq m$. (We proceed by induction on $|\sigma| + |\tau|$ using the fact that $r_w(P) = r_{\bar{w}}(P) = 1$. Both statements in Lemma 2.10 are needed.) That is, $r(z, 0)$ vanishes at P to order $\geq m + 1$. q.e.d.

Lemmas 2.11 and 2.12 complete the proof of Theorem 2.4.

2.13. Corollary. *Let M be real analytic and $P \in M$ a point of type $+\infty$. Then M contains a complex $(n-1)$-dimensional submanifold of a neighborhood of P.*

Proof. Using the assumption that r is real analytic we may assume the coordinate w chosen so that $D(r)(P) = 0$ where D is pure differentiation with respect to z or $\bar{z}$ of any order. Then the reasoning in the proof of Lemma 2.12 shows that $\{(z, w) \mid w = 0\}$ is contained in M.

2.14. Counterexamples. The conclusion of Corollary 2.13 need not hold if M is only C^∞. We give two examples:

1. Consider $r = 2\operatorname{Re} w + \exp(-(|z|^2 + (\operatorname{Im} w)^2)^{-1})$ and $M = \{z, w \in \boldsymbol{C}^2 \mid r = 0\}$. Then $(0, 0)$ is a point of type ∞. However, M is strongly pseudoconvex (type 1) in a deleted neighborhood of $(0, 0)$ and cannot contain a complex submanifold.

2. Consider the formal power series

$$\operatorname{Re}\left(w - \sum_{n=2}^{\infty} n!\, z^n\right).$$

By a theorem of E. Borel [10, p. 28] there exists a C^∞ function r in $\boldsymbol{C}^2$ having this series as its formal Taylor series at $(0, 0)$. Let $M = \{z, w \in \boldsymbol{C}^2 \mid r(z, w) = 0\}$. The complex submanifold $w = \sum_{n=2}^{m} n!\, z^n$ is tangent to M to order m at $(0, 0)$. However, there is no complex submanifold tangent to M at $(0, 0)$ to infinite order.

3. The case of a single vector field

As before, M is a real C^∞ hypersurface in an open subset of $\boldsymbol{C}^n$, and P denotes a point of M.

Let L be a tangential holomorphic vector field to M. We let $\mathscr{L}_\mu(L)$ denote the C^∞ module of vector fields spanned by $L, \bar{L}$ and their commutators of order $\leq \mu$.

3.1. Definition. We say L is of type m at P if there exists $F \in \mathscr{L}_m(L)$ such that $\langle \partial r(P), F(P)\rangle \neq 0$ while for all $F \in \mathscr{L}_{m-1}(L)$ we have

$$\langle \partial r(P), F(P)\rangle = 0 .$$

We shall use the notation $t(L, P) = m$. If $\langle \partial r(P), F(P)\rangle = 0$ for all $F \in \mathscr{L}_\mu(L)$ and all integers $\mu \geq 1$ we will write $t(L, P) = +\infty$.

3.2. Proposition. *Suppose there is a 1-dimensional complex submanifold*

X of a neighborhood of P, tangent to M at P to order s. Then there exists a tangential holomorphic vector field L such that $L(P)$ is tangent to X at P and $t(L, P) \geq s$.

Proof. Choose coordinates $z_1, \cdots, z_{n-1}, w$ centered at P so that

1. $X = \{(z, w) \mid w = z_1 = \cdots = z_{n-2} = 0\}$,
2. $r = 2\operatorname{Re}(w) + \phi$ where ϕ vanishes to order ≥ 2 at P.

Consider the tangential holomorphic vector field $L_{n-1} = r_w \dfrac{\partial}{\partial z_{n-1}} - r_{z_{n-1}} \dfrac{\partial}{\partial w}$. We shall show that L_{n-1} is of type $\geq s$ at p.

Now $r|_X$ has a zero of order $s + 1$ at P. Thus the description of the commutators of L_{n-1} and $\bar{L}_{n-1}$ contained in Lemmas 2.8 and 2.9 is sufficient to prove the proposition.

3.3. Remarks. 1. If in these coordinates we have $D(r)(0, 0) \neq 0$ for some impure differentiation D in $z_{n-1}, \bar{z}_{n-1}$ of order $s + 1$, then L_{n-1} has type precisely s at P.

2. We do not know if there is a converse to Proposition 3.2. The condition that all nonzero holomorphic vector fields be of finite type is conjectured by Kohn [7] to be necessary and sufficient for the $\bar{\partial}$-Neumann problem to be subelliptic at a boundary point of a pseudoconvex domain.

3.4. The type of a vector field is not determined solely by its value at P. Consider $M \subset C^3$ defined as the zero set of

$$r = 2\operatorname{Re}(w) + |z_1|^2 - |z_2|^4 , \qquad P = (0, 0, 0) .$$

Here L_1 is of type 1, and L_2 is of type 3. (L_1 and L_2 are defined by (2.5.4).)

Note however that M contains the complex submanifold

$$X = \{(w, z_1, z_2) \mid w = 0 \text{ and } z_1 = z_2^2\} .$$

Now $L = 2z_2L_1 + L_2$ is a tangential holomorphic vector field which restricts to a holomorphic vector field on X. Thus it is of type $+\infty$. Of course $L(P) = L_2(P)$.

3.5. It is possible to have a point $P \in M$ such that all nonzero holomorphic tangential vector fields are of finite type at P but there are points arbitrarily close to P where these are nonzero holomorphic tangential vector fields not of finite type. We will give one such example with M pseudoconvex.

Let M be given as the zero set of $r = 2\operatorname{Re}(w) + |z_1^2 - z_2^3|^2$ and $P = (0, 0, 0)$. Since r is plurisubharmonic M is pseudoconvex (when considered as the boundary of $r < 0$).

We will first show that every tangential holomorphic vector field L such that $L(P) \neq 0$ is of finite type at P (in fact of type ≤ 5).

Note that L_1 is of type 3 at P, and L_2 is of type 5 at P. Any tangential holomorphic vector field L can be written $L = \phi_1L_1 + \phi_2L_2$ where ϕ_1 and ϕ_2 are C^∞ functions. If $\phi_1(P) \neq 0$, it is easily seen that $t(L, P) = t(L_1, P) = 3$. If $\phi_1(P) = 0$, and $L(P) \neq 0$, then $\phi_2(P) \neq 0$. Therefore we may assume that

$$L = \phi L_1 + L_2 \quad \text{with} \quad \phi(0) = 0 .$$

Expressing the commutator $[[[[[L, \bar{L}], L], \bar{L}], L], \bar{L}]$ as a linear combination of commutators of $L_1, \bar{L}_1, L_2$ and $\bar{L}_2$, each commutator S has the property that it occurs with a coefficient having a factor ϕ or else $\langle \partial r(P), S(P) \rangle = 0$ except for the commutator $[[[[[L_2, \bar{L}_2], L_2], \bar{L}_2], L_2], \bar{L}_2]$. Thus $t(L, P) \leq 5$ (in fact $t(L, P) = 5$).

Now M contains the complex analytic set

$$X = \{w, z_1, z_2 \,|\, w = 0,\ z_1^2 = z_2^3\} .$$

X has a singular point at P, but at all other points it is nonsingular. Thus for any point $q \in X - P$ there is a nonzero tangential holomorphic vector field which is not of finite type.

4. Generic submanifolds of higher codimension

Let M be a real C^∞ submanifold of dimension $2n - k$ $(k < n)$ of an open subset U of C^n. Let $r_1, \cdots, r_k$ be real-valued C^∞ functions such that $M = \{z \in U \,|\, r_1 = \cdots = r_k = 0\}$ and $dr_1 \wedge \cdots \wedge dr_k \neq 0$ on M.

4.1. Definition [12]. M is generic if $\partial r_1 \wedge \cdots \wedge \partial r_k \neq 0$ on M.

This condition is equivalent to $\dim_C T^{1,0}(M, q) = n - k$ for all $q \in M$. (Hence it is independent of the functions $r_1, \cdots, r_k$.) This is, of course, the minimum possible dimension for the holomorphic tangent space.

4.2. Definition. A point $P \in M$ is of type m (m an integer ≥ 1 or $+\infty$) if there exists $F \in \mathscr{L}_m$ such that $F(P) \notin T^{1,0}(M, P) \oplus T^{0,1}(M, P)$ while $\mathscr{L}_{m-1}$ contains no such F.

We use the notation $t(P) = m$.

The requirement that $F(P) \notin T^{1,0}(M, P) \oplus T^{0,1}(M, P)$ is equivalent to the following: if $r_1, \cdots, r_k$ are defining functions for M, then $\langle \partial r_i(P), F(P) \rangle \neq 0$ for some i.

4.3. Remark. This is not the most interesting type condition. Naruki's estimate [11] depends on there being an integer m such that $\{F(P) \,|\, F \in \mathscr{L}_m\} = CT(M, P)$. The point P is then termed $(m + 1)$-regular by Naruki.

Let X be an $(n - k)$-dimensional complex submanifold of a neighborhood U of P which is tangent to M at P.

4.4. Definition. X is tangent to M at P to order s (s an integer ≥ 1 or $+\infty$) if $s = \inf\{t \,|\, \text{there exists a real valued } C^\infty \text{ function } r \text{ on } U \text{ such that } r|_M = 0,\ dr \neq 0 \text{ on } M \text{ and } r|_X \text{ vanishes at } P \text{ to order} \geq t + 1\}$.

Thus s is the least order of tangency of X with a hypersurface containing M.

Note that the roles of X and M cannot be interchanged in this definition, for $\dim_R X < \dim_R M$. Also whenever $r_1, \cdots, r_k$ are functions such that $M = \{z \,|\, r_1 = \cdots = r_k = 0\}$ and $dr_1 \wedge \cdots \wedge dr_k \neq 0$ on M, there is an index i for which $r_i|_X$ vanishes at P to order $s + 1$.

We set $a(P) = \sup\{s \mid$ there exists an $(n-k)$-dimensional complex submanifold tangent to M at P to order $s\}$. Thus $a(P)$ is an integer ≥ 1 or $+\infty$.

4.5. Theorem. $a(P) = t(P)$.

Proof. The proof is analogous to that of Theorem 2.4. Since M is generic, given defining functions $r_1, \cdots, r_k$ for M we can choose local coordinates $z_1, \cdots, z_{n-k}, w_1, \cdots, w_k$ at P such that

$$r_i = 2\,\mathrm{Re}\,(w_i) + \phi_i\,, \qquad i = 1, \cdots, k\,, \tag{4.5.1}$$

where ϕ_i vanishes to order ≥ 2 at P. Thus

$$\frac{\partial r_i}{\partial w_j}(P) = \frac{\partial r_i}{\partial \bar{w}_j}(P) = \delta_{ij}\,, \qquad i, j = 1, \cdots, k\,, \tag{4.5.2}$$

$$\frac{\partial r_i}{\partial z_j}(P) = \frac{\partial r_i}{\partial \bar{z}_j}(P) = 0\,, \qquad i = 1, \cdots, k\,,\ j = 1, \cdots, n-k\,. \tag{4.5.3}$$

Consider the vector fields

$$L_i = E\frac{\partial}{\partial z_i} + \sum_{j=1}^{k} E_j^i \frac{\partial}{\partial w_j}\,, \qquad i = 1, \cdots, n-k\,, \tag{4.5.4}$$

where $E, E_1^i, \cdots, E_k^i$ are the cofactors of the elements in the first row of the $(k+1)\times(k+1)$ matrix

$$\begin{pmatrix} e & e_1 & \cdots & e_k \\ \dfrac{\partial r_1}{\partial z_i} & \dfrac{\partial r_1}{\partial w_1} & \cdots & \dfrac{\partial r_1}{\partial w_k} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial r_k}{\partial z_i} & \dfrac{\partial r_k}{\partial w_1} & \cdots & \dfrac{\partial r_k}{\partial w_k} \end{pmatrix}. \tag{4.5.5}$$

Note that $L_i(r_s) = 0$ for $i = 1, \cdots, n-k$, $s = 1, \cdots, k$ since $E(\partial r_s/\partial z_i) + \sum_{j=1}^k E_j^i(\partial r_s/\partial w_j)$ is equal to the expansion of the determinant of (4.5.5) when $e = \partial r_s/\partial z_i$ and $e_j = \partial r_s/\partial w_j$. Of course, in that case the matrix has two identical rows.

Now the relations (4.5.2) and (4.5.3) imply that $E(P) = 1$ while $E_j^i(P) = 0$ for $i = 1, \cdots, n-k$, and $j = 1, \cdots, k$.

The following lemmas are proved in a manner similar to the corresponding lemmas in § 2. Details are omitted for the most part.

4.6. Lemma. *$\mathscr{L}_\mu$ is generated modulo vector fields vanishing on M as a C^∞ module by the commutators of order $\leq \mu$ of the $2n-2k$ vector fields $L_1, \cdots, L_{n-k}, \bar{L}_1, \cdots, \bar{L}_{n-k}$.*

Any vector field F can be written in the form

$$F = \sum_{i=1}^{n-k} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n-k} b_i \frac{\partial}{\partial \bar{z}_i} + \sum_{j=1}^{k} c_j \frac{\partial}{\partial w_j} + \sum_{j=1}^{k} d_j \frac{\partial}{\partial \bar{w}_j} .$$

If F is tangential, then by our choice of coordinates $c_j(0) + d_j(0) = 0$.

4.7. Lemma. *Suppose $F \in \mathscr{L}_\mu - \mathscr{L}_{\mu-1}$ and is formed from commutators of $L_1, \cdots, L_{n-k}, \bar{L}_1, \cdots, \bar{L}_{n-k}$. Then the coefficients a_i, b_i, c_j, d_j of F are sums of terms of the form*

$$\pm D^1(r) \cdots D^{\mu+1}(r) ,$$

where each $D^l(r)$, $l = 1, \cdots, \mu + 1$ is the determinant of a $k \times k$ matrix whose entries are partial derivatives of $r_1 \cdots r_k$ with respect to $z_1, \cdots, z_{n-k}, w_1, \cdots, w_k$ with the following properties:

1. *The ith row contains derivatives only of r_i.*
2. *The differentiation operator is the same for all entries in a given column.*
3. *The order d of the differentiation in a given column satisfies $1 \leq d \leq \mu + 1$.*
4. *The total order of differentiation in each term is $(\mu + 1)k + \mu$.*
5. *Each term in c_j or d_j involves $\mu + 1$ derivatives with respect to $z, \bar{z}$ and $(\mu + 1)k - 1$ derivatives with respect to $w, \bar{w}$.*

4.8. Lemma. *Suppose $F \in \mathscr{L}_\mu - \mathscr{L}_{\mu-1}$ and is formed from commutators of $L_1, \cdots, L_{n-k}, \bar{L}_1, \cdots, \bar{L}_{n-k}$. Then among the columns of the determinants in each term $\pm D^1(r) \cdots D^{\mu+1}(r)$ of the c_j and d_j coefficients, there is one in which the differentiation is in $z, \bar{z}$ only (and of order $\leq + 1$).*

Proof. According to Lemma 4.7 there are $(\mu + 1)k$ columns altogether, and the order of differentiation in w is $(\mu + 1)k - 1$.

4.9. Lemma. *Let $D = (\partial/\partial z)^\sigma(\partial/\partial \bar{z})^\tau$ where σ and τ are multi-indices and $|\sigma| \geq 1$, $|\tau| \geq 1$. Let $\mu + 1 = |\sigma| + |\tau|$. Let T be the $k \times k$ determinant*

$$T = \det \begin{bmatrix} Dr_1 & \dfrac{\partial r_1}{\partial w_1} \cdots \dfrac{\partial r_1}{\partial w_{j-1}} & \dfrac{\partial r_1}{\partial w_{j+1}} \cdots \dfrac{\partial r_1}{\partial w_k} \\ \vdots & & \\ Dr_k & \dfrac{\partial r_k}{\partial w_1} \cdots \dfrac{\partial r_k}{\partial w_{j-1}} & \dfrac{\partial r_k}{\partial w_{j+1}} \cdots \dfrac{\partial r_k}{\partial w_k} \end{bmatrix} .$$

(Note that $T(0) = \pm Dr_j(0)$.) Then there exists $F \in \mathscr{L}_\mu$ whose c_j coefficient has the following properties:

1. There is one term $E^{|\sigma|-1}\bar{E}^{|\tau|}T$.
2. For each of the remaining terms, one determinant contains a column in which the differentiation is in $z, \bar{z}$ only and of order $\leq \mu$.

Proof. By induction using the analog of formula (2.7.1). (Cf. Lemma 2.10.)

4.10. Lemma. $t(P) \geq a(P)$.

Proof. Let X be an $(n - k)$-dimensional complex submanifold tangent to

M at P to order s ($1 \leq s < \infty$). We may choose coordinates at P so that $X = \{(z, w) \mid w = 0\}$. Then $r_i(z, 0)$ vanishes to order $\geq s + 1$, $i = 1, \cdots, k$. Lemma 4.8 shows that the c_j and d_j coefficients of any $F \in \mathscr{L}_{s-1}$ vanish at P for $j = 1, \cdots, k$. Hence $t(P) \geq s$.

4.11. Lemma. $t(P) \leq a(P)$.

Proof. Suppose that $t(P) \geq m$ where m is an integer ≥ 1. We may assume that the coordinate w_j is chosen so that $D(r_j)(P) = 0$, $j = 1, \cdots, k$ where D is any pure differentiation with respect to z or $\bar{z}$ of order $\leq m + 1$. Lemma 4.9 shows that for any impure differentiation D in $z, \bar{z}$ of order $\leq m$, $Dr_j(0) = 0$, $j = 1, \cdots, k$. That is, $w = 0$ is tangent to $r_j = 0$ to order $\geq m$, $j = 1, \cdots, k$. We conclude that $w = 0$ is tangent to M to order $\geq m$. Thus $a(P) \geq m$.

Lemmas 4.10 and 4.11 complete the proof of Theorem 4.5.

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