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SURFACES OF CONSTANT MEAN CURVATURE IN MANIFOLDS OF CONSTANT CURVATURE

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0. Introduction

An immersed surface in a three-dimensional Euclidean space E^3 has constant (scalar) mean curvature if the length of the mean curvature vector H is constant. An arbitrary isometric immersion $M^n \hookrightarrow \overline{M}^{n+k}$ of Riemannian manifolds is said to have *constant mean curvature* if H is parallel in the normal bundle of the immersion (for definitions see § 1). This condition is stronger than the requirement $|H| =$ constant c. In the case of immersions of surfaces into manifolds of constant curvature we generalize many known facts and theorems about surfaces of constant (scalar) mean curvature in E^3. The main theorems of this paper were announced in Hoffman [7], and we refer the reader there for a more lengthy introduction and statement of results. What follows is a brief sketch of the principal results.

To a surface of constant mean curvature given in conformal coordinates we associate an analytic function φ constructed out of the second fundamental form in the mean curvature direction (Lemma 2.1). This was first done for surfaces in E^3 by Heinz Hopf [8]. Under certain additional assumptions, the same procedure works for other normal directions. These functions have direct geometrical meaning which is discussed in § 2. In particular they are used to prove Theorem 2.2(b): The only genus zero surfaces of constant mean curvature in E^4 or the standard 4-sphere S^4 are the standard 2-spheres.

Theorem 3.1 gives a local characterization of constant (nonzero) mean curvature immersions which have constant Gauss curvature; they are shown to be pieces of 2-spheres or products of 1-spheres, $S^1(r) \times S^1(\rho)$, $0 < r \leq \infty$, $0 < \rho < \infty$. Theorem 4.1 classifies complete surfaces of constant mean curvature in E^4 and S^4, whose Gauss curvature does not change sign; they must be minimal surfaces, 2-spheres or $S^1(r) \times S^1(\rho)$, $0 < r \leq \infty$, $0 < \rho < \infty$.

In § 5 we use the analytic functions of Lemma 2.1 to construct local examples of surfaces of constant mean curvature in 4-dimensional manifolds of constant curvature (Theorem 5.1). In these examples for the case of immersions into E^4 or S^4, the surfaces *do not* lie minimally in hyperspheres of E^4 or S^4

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(Corollary 5.2). Except for products of circles $S^1(r) \times S^1(\rho)$, $r \neq \rho$, these are the first known examples of such surfaces.

1. Preliminaries

Let $M^n \xrightarrow{i} \bar{M}^{n+k}$ be an isometric immersion of Riemannian manifolds of dimension n and $n+k$ respectively. If $\langle\ ,\ \rangle$ denotes the metric tensor on $T\bar{M}^{n+k}$, then that of TM^n is given by $i^*(\langle\ ,\ \rangle)$. We identify M^n with $i(M^n)$ and TM^n with $i_*(TM^n) \subset T\bar{M}^{n+k}$, deleting reference to i and its induced maps wherever possible. We consider $T\bar{M}^{n+k}$ restricted to the base space M^n. A vector field X on M^n, i.e., a member of $\Gamma(TM^n)$, the space of smooth sections of TM^n, is also a section of $T\bar{M}^{n+k}$. Let $[\ ]^T$ denote projection in $T\bar{M}^{n+k}$ onto TM. Then the normal bundle NM^n is the bundle whose fibre at p is $NM^n_p = \{X \in T\bar{M}^{n+k} \mid [X]^T = 0\}$. We let $[\ ]^N$ denote projection onto NM^n.

In the following, let $X, Y, Z \in \Gamma(TM^n)$. The Riemannian connection ∇ of M is related to the Riemannian connection $\bar{\nabla}$ of $\bar{M}$ (we suppress superscripts n and $n+k$ unless we wish to emphasize dimension) by

$$[\bar{\nabla}_X Y]^T = \nabla_X Y\ . \tag{1.1}$$

Definition. $B(X, Y) = [\bar{\nabla}_X Y]^N$. B is called the second fundamental form of the immersion and is a section of $\Gamma(TM \otimes TM, NM)$, the bundle of bilinear mappings form TM to NM. Let $N \in \Gamma(NM)$.

Definition. A is a section of $\Gamma(NM \otimes TM, TM)$ defined by

$$\langle A(N, X), Y\rangle = -\langle B(X, Y), N\rangle\ . \tag{1.2}$$

NM inherits a metric from $T\bar{M}$ and is a Riemannian vector bundle over M. Its Riemannian connection D is the connection defined on NM by

$$D_X N = [\bar{\nabla}_X N]^N\ , \qquad X \in \Gamma(TM),\ N \in \Gamma(NM)\ .$$

D is easily seen to be compatible with the metric of NM. Putting together the above decompositions, we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)\ , \qquad \bar{\nabla}_X N = D_X N + A(N, X)\ . \tag{1.3}$$

Given Riemanniann vector bundles E_i, $i = 1, \cdots, m+1$, with connections D^i the bundle $\mathscr{H}(\otimes_{i=1}^m E_i, E_{m+1}) \stackrel{\text{def}}{=} \mathscr{H}$ of fibre linear maps has a natural Riemannian structure ∇ defined as follows.

Definition. If B is a section of $\mathscr{H}$, and $X \in \Gamma(TM)$, then $\nabla_X B$ is the section of $\mathscr{H}$ given by

$$\nabla_X B(\ , \cdots,\ ) = D_X^{m+1}(B(\ ,\ )) - \sum_{i=1}^m B(\ , \cdots, D_X^i, \cdots,\ )\ . \tag{1.4}$$

The curvature associated with $\bar{\nabla}, \nabla$ and D are denoted $\bar{R}, R$ and $\tilde{R}$ respectively. For example $\tilde{R}$ is given by

$$\tag{1.5} \tilde{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X,Y]}N .$$

The curvatures are related to B and A by the Gauss and Codazzi-Mainardi equations

$$\tag{1.6} \begin{aligned} &\text{(i)} \quad [\bar{R}(X, Y)Z]^T = R(X, Y)Z + A(B(Y, Z), X) - A(B(X, Z), Y), \\ &\text{(ii)} \quad [\bar{R}(X, Y)N]^N = \tilde{R}(X, Y)N + B(A(N, Y), X) - B(A(N, X), Y), \\ &\text{(iii)} \quad [\bar{R}(X, Y)Z]^N = \nabla_X B(Y, Z) - \nabla_Y B(X, Z), \\ &\text{(iv)} \quad [\bar{R}(X, Y)N]^T = \nabla_X A(N, Y) - \nabla_Y A(N, X), \end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$, $N \in \Gamma(NM)$.

Proposition 1.1. *Let $M \to \bar{M}$ be an isometric immersion. For fixed X, $Y \in \Gamma(TM)$, $\bar{R}(X, Y)$ leaves TM invariant $\Leftrightarrow \forall Z \in \Gamma(TM)$,*

$$\begin{aligned} &\nabla_X B(Y, Z) = \nabla_Y B(X, Z) \Leftrightarrow \forall N \in \Gamma(NM), \\ &\nabla_X A(N, Y) = \nabla_Y A(N, X) \Leftrightarrow \bar{R}(X, Y) \text{ leaves } NM \text{ invariant.} \end{aligned}$$

Proof. The first and third equivalences follow from (1.6) (iii) and (iv). The second equivalence follows from the fact that the adjoint of $\bar{R}(X, Y)$ is $-\bar{R}(X, Y)$. Hence the first and fourth statements are equivalent. More directly, the second equivalence follows from the easily verified equality $\langle \nabla_X B(Y, Z), N\rangle = \langle \nabla_X A(N, Y), Z\rangle$.

If $\bar{M}$ has constant sectional curvature c, then $\bar{R}(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$. In this case, the first and hence all the conditions of Proposition 1.1 are satisfied, and we may rewrite (1.6) as

$$\tag{1.7} \begin{aligned} &\text{i)} \quad c(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \\ &\qquad = R(X, Y)Z + A(B(Y, Z), X) - A(B(X, Z), Y), \\ &\text{ii)} \quad [\bar{R}(X, Y)N]^N = \tilde{R}(X, Y)N + B(A(N, Y), X) - B(A(N, X), Y), \\ &\text{iii)} \quad \nabla_X B(Y, Z) = \nabla_Y B(X, Z), \text{ or equivalently} \\ &\text{iv)} \quad \nabla_X A(N, Y) = \nabla_Y A(N, X). \end{aligned}$$

Let $F = \{e_1, \cdots, e_{n+k}\}$ be an orthonormal framing of $T\bar{M}$ defined in a neighborhood of $p \in M$. F is said to be *adapted* to M if $\{e_1, \cdots, e_n\}$ frames TM. Given coordinates $(u^i, \cdots, u^n)$ on M with coordinate vector fields $U_i = \partial/\partial u^i$, we shall also consider *adapted coordinate framings* of $T\bar{M}$ given by $\{U_1, \cdots, U_n\} \cup \{e_\alpha\}$, $n + 1 \le \alpha \le n + k$, where $\{e_\alpha\}$ is an orthonormal framing or NM. In this and what follows $1 \le i \le n$, $n + 1 \le \alpha \le n + k$.

Definition. For an adapted framing of $T\bar{M}$,

$$\lambda_{ij}^{\alpha} \stackrel{\text{def}}{=} \langle B(e_i, e_j), e_\alpha \rangle = -\langle A(e_\alpha, e_i), e_j \rangle . \tag{1.8}$$

Similarly, for an adapted coordinate framing,

$$L_{ij}^{\alpha} \stackrel{\text{def}}{=} \langle B(U_i, U_j), e_\alpha \rangle = -\langle A(e_\alpha, U_i), U_j \rangle . \tag{1.9}$$

For fixed α, the matrices (λ_{ij}^{α}) and (L_{ij}^{α}) are the *second fundamental forms in the e_α direction.* In the case where e_α is parallel in NM, i.e., $D_X e_\alpha = 0$, $\forall X \in \Gamma(TM)$, equations (1.7) take on decidedly classical appearance:

Proposition 1.2. *Let $M^2 \hookrightarrow \overline{M}^{n+k}(c)$ be an isometric immersion, where $\overline{M}^{n+k}(c)$ denotes an $(n + k)$-manifold of constant curvature c. If $\{U_1, \cdots, U_n\} \cup \{e_\alpha\}$ is a coordinate adapted framing such that one of the e_α, say e_{α_0}, is parallel, then*

$$(L_{ik}^{\alpha_0})_j - (L_{jk}^{\alpha_0})_i = \sum_{r=1}^{n} \Gamma_{jk}^{r} L_{ri}^{\alpha_0} - \Gamma_{ik}^{r} L_{rj}^{\alpha_0} , \tag{1.10}$$

and

$$\sum_{r,1} g^{jr} L_{r1}^{\alpha_0} L_{i1}^{\beta} - g^{ir} L_{r1}^{\alpha_0} \beta_{j1}^{\beta} = 0 \tag{1.11}$$

for all $i, j, k = 1, \cdots, n$, $\beta = n + 1, \cdots, n + k$, where Γ_{ij}^{k} are the Christoffel symbols, and $\nabla_{U_i} U_j = \sum \Gamma_{ij}^{k} U_k$.

Proof. For (1.10) use (1.7) (iv) with $X = U_i$, $Y = U_j$, $N = e_{\alpha_0}$. The fact that e_{α_0} is parallel implies

$$\nabla_{U_i} A(e_{\alpha_0}, U_j) = \nabla_{U_i}(A(e_{\alpha_0}, U_j)) - A(e_{\alpha_0}, \nabla_{U_i} U_j) .$$

Substitution of (1.9) will complete the proof. Equation (1.11) follows in a similar fashion from (1.7) (iii) using the fact that e_{α_0} parallel implies $\tilde{R}(U_i, U_j) e_{\alpha_0} = 0$.

Remark. (1.10) is a generalization of the classical Codazzi equation for surfaces.

Definition. For an isometric immersion $M^n \hookrightarrow \overline{M}^{n+k}$, the mean curvature vector field $H \stackrel{\text{def}}{=} \operatorname{Tr} B/n$. In terms of adapted or coordinate framings,

$$H = \frac{1}{n} \sum_{i,\alpha} \lambda_{ii}^{\alpha} e_\alpha = \frac{1}{n} \sum_{i,j,\alpha} g^{ij} L_{ij}^{\alpha} e_\alpha , \qquad n + 1 \leq \alpha \leq n + k .$$

Definition. $M \hookrightarrow \overline{M}$ is said to have *constant mean curvature* if H is parallel, i.e., if $D_X H = 0$, $\forall X \in \Gamma(TM)$. Since D is a Riemannian connection, we must have $X|H|^2 = 2\langle D_X H, H \rangle$. This equality (all but) proves the following *observations*:

1. H is parallel $\Rightarrow |H|$ is constant.
2. If $H \neq 0$, H is parallel $\Leftrightarrow |H|$ is constant and $H/|H|$ is parallel.

3. If codimension $k = 1$, H is parallel $\Leftrightarrow |H|$ is constant.

We remark briefly that constant mean curvature may be expressed in terms of Cartan forms as follows. If $\{e_i\} \cup \{e_\alpha\}$ is an adapted framing with $e_{n+1} = H/|H|$ and $\{\omega^i\}$ are the dual 1-forms on M, then $de_i = \omega_i^k e_k + \omega_i^\alpha e_\alpha$ where $\{\omega_i^k\}$, $1 \leq i, k \leq n$ are the connection forms and $\omega_i^\alpha = \lambda_{ik}^\alpha \omega^k$, $1 \leq i \leq n$, $n + 1 \leq \alpha \leq n + k$. Similarly, $de_\alpha = -\omega_\alpha^k e_k + \omega_\alpha^\beta e_\beta$ where $\{\omega_\alpha^\beta\}$, $n + 1 \leq \alpha, \beta \leq n + k$, are the torsion forms of the immersion. By observation 2 above, H is parallel $\Leftrightarrow \sum_{i=\ell}^n \lambda_{ii}^{n+\ell}$ is constant and $\omega_{n+\ell}^\beta = 0$, $n + 1 \leq \beta \leq n + k$.

For closed hypersurfaces M^n in E^{n+1}, constant mean curvature is equivalent to requiring the n-dimensional "area" of M^n to be stationary with respect to variations which leave fixed the $(n + 1)$-volume of the part of E^{n+1} enclosed by M^n. This condition for hypersurfaces can also be stated in a local manner (see Hopf [8, p. 83]). For immersions with arbitrary codimension in E^{n+k}, Ruh and Vilms [15] have shown that constant mean curvature is equivalent to the requirement that the Gauss map into $G(n, n + k)$ be harmonic in the sense of Eells and Sampson [4].

2. Surfaces with constant mean curvature

We shall now consider a surface M^2 isometrically immersed in $M^{2+k}(c)$, a $(2 + k)$-manifold with constant sectional cuavature c. Without loss of generality we many assume that the immersion is given locally in conformal coordinates (u^1, u^2), so that $ds^2 = E[(du^1)^2 + (du^2)^2]$, (i.e., $\langle U_i, U_j\rangle = E\delta_{ij}$). Let $z = u^1 + iu^2$. To the coordinate framing $\{U_1, U_2\}$ there is a naturally associated adapted framing $\{e_i = U^i/\sqrt{E}\}$. For a unit normal section $e_\alpha \in \Gamma(NM)_1$.

Definition. $$\varphi_\alpha(z) \overset{\text{def}}{=} (L_{11}^\alpha - L_{22}^\alpha) - iL_{12}^\alpha$$

(or equivalently $\varphi_\alpha = E(\frac{1}{2}(\lambda_{11}^\alpha - \lambda_{22}^\alpha) - i\lambda_{12}^\alpha)$ since by (1.9), $L_{ij}^\alpha = \langle B(U_i, U_j), e_\alpha\rangle = E^{-1}\langle B(e_i, e_j), e_\alpha\rangle$).

Lemma 2.1. *Let $M^2 \hookrightarrow \overline{M}^{2+k}(c)$ be an isometric immersion given locally in conformal coordinates (u^1, u^2) with conformal parameter E. Let e_α be a unit section of NM which is parallel.*

(a) *If $E^{-1}(L_{11}^\alpha + L_{22}^\alpha) = (\lambda_{11}^\alpha + \lambda_{22}^\alpha)$ is constant, then φ^α is an analytic function of z. In particular,* i) *if $H \neq 0$ is parallel and $e_3 = H/|H|$, then φ_3 is analytic;* ii) *if e_α satisfies $\langle e_\alpha, H\rangle = 0$, then φ_α is analytic.*

(b) *If $e_\beta \in \Gamma(NM)$ is any other unit section and $\langle e_\beta, e_\alpha\rangle = 0$, then $\varphi_\alpha \equiv 0$ or $\varphi_\beta = f\varphi_\alpha$ where f is a smooth function of z with possible isolated poles.*

(c) *If e_α and e_β are parallel unit sections of NM both of which satisfy the hypothesis of* (a), *then one is a* (*real*) *constant multiple of the other.*

Proof. (a) In conformal coordinates $g^{ij} = \delta_{ij}/E$. Moreover the Christoffel symbols are given by

$$(2.1) \quad \Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \tfrac{1}{2}E_1/E\,, \qquad \Gamma_{22}^2 = \Gamma_{21}^2 = -\Gamma_{11}^2 = \tfrac{1}{2}E_2/E\,.$$

Equation (1.10) becomes (for $i = j = 1$, $k = 2$ and $i = k = 1$, $j = 2$, resp.)

$$(2.2)\qquad \begin{aligned} (L^\alpha_{11})_2 - (L^\alpha_{12})_1 &= \tfrac{1}{2}E_2(L^\alpha_{11} + L^\alpha_{22}) \,, \\ (L^\alpha_{12})_1 - (L^\alpha_{22})_2 &= \tfrac{1}{2}E_1(L^\alpha_{11} + L^\alpha_{22}) \,. \end{aligned}$$

Since $(L^\alpha_{11} + L^\alpha_{22})/E$ is constant by assumption, (2.2) can be rewritten as

$$(2.3)\qquad \begin{aligned} \{\tfrac{1}{2}(L^\alpha_{11} - L^\alpha_{22})\}_2 - (L^\alpha_{12})_2 &= 0 \,, \\ \{\tfrac{1}{2}(L^\alpha_{11} - L^\alpha_{22})\}_1 + (L^\alpha_{12})_2 &= 0 \,. \end{aligned}$$

These are the Cauchy-Riemann equations for φ_α. Statement (i) follows from *observation* 2 of § 1 and the fact that $|H| = \frac{1}{2}E^{-1}(L^3_{11} + L^3_{22})$. Statement (ii) is true since $\lambda^\alpha_{11} + \lambda^\alpha_{22} = (\operatorname{tr} \lambda^\alpha_{ij}) = \langle \operatorname{tr} B, e_\alpha \rangle = \langle 2H, e_\alpha \rangle = 0$.

(b) In conformal coordinates, equation (1.11) is

$$(2.4)\qquad \left(\sum_{k=1}^{2} L^\alpha_{k2}L^\beta_{k1} - L^\alpha_{k1}L^\beta_{k2} \right) \Big/ E = 0 \,.$$

If $\varphi_\alpha \not\equiv 0$, then $\varphi_\beta/\varphi_\alpha = \varphi_\beta\bar{\varphi}_\alpha/|\varphi_\alpha|^2$. But

$$\begin{aligned} \operatorname{Im}(\varphi_\beta\bar{\varphi}_\alpha) &= (\operatorname{Re}\bar{\varphi}_\alpha)(\operatorname{Im}\varphi_\beta) - (\operatorname{Im}\bar{\varphi}_\alpha)(\operatorname{Re}\varphi_\beta) \\ &= \sum L^\alpha_{k1}L^\beta_{k2} - L^\alpha_{k2}L^\beta_{k1} = 0 \qquad \text{by (2.4)} \,. \end{aligned}$$

Therefore $f = \varphi_\beta\bar{\varphi}_\alpha/|\varphi_\alpha|^2$ is real and smooth and has only isolated poles since φ_α is analytic.

(c) If $\varphi_\alpha \equiv \varphi_\beta \equiv 0$, there is nothing to prove. Without loss of generality, assume $\varphi_\alpha \not\equiv 0$. Then by (b), $\varphi_\beta/\varphi_\alpha$ is real with possible poles. But it is meromorphic since φ_β and φ_α are both analytic. Hence $\varphi_\beta/\varphi_\alpha$ is a (real) constant.

Before using Lemma 2.1 to prove a generalization of Hopf's theorem on closed surfaces of constant mean curvature in E^3, we make some remarks about the functions φ_α. In this and what follows we assume that $H \neq 0$ and set $e_{n+1} = H/|H|$.

Definition. $M^n \hookrightarrow \bar{M}^{n+k}$ is *pseudo-umbilical* at p if $(\lambda^{n+1}_{ij}) = \lambda\delta_{ij}$ at p. $M^n \hookrightarrow \bar{M}^{n+k}$ is *totally umbilical* if $M^n \hookrightarrow \bar{M}^{n+k}$ is pseudo-umbilical and $\lambda^\alpha_{ij} = 0$, $\alpha > n + 1$. A point where φ_α is real is a point where (L^α_{ij}) and (λ^α_{ij}) are diagonalized. A zero of φ_α is a point where the eigenvalues are equal. If $e_3 = H/|H|$, then zeros of φ_3 are precisely the pseudo-umbilic points of the immersion. Lemma 2.1 (a) (i) implies that an immersion with constant mean curvature is either everywhere pseudo-umbilic or has isolated pseudo-umbilic points. Part (b) implies that, away from pseudo-umbilic points, one can simultaneously diagonalize the second fundamental forms in every direction of a normal framing $\{e_3, \cdots, e_{2+k}\}$. Part (c) says that, under the added assumption that e_α is parallel, (λ^α_{ij}) is completely determined by (λ^3_{ij}).

Theorem 2.2. (a) *A closed oriented surface M^2 of genus 0 immersed in*

$\bar{M}^{2+k}(c)$, $c \geq 0$, *with constant nonzero mean curvature is pseudo-umbilical and lies minimally in a hypersphere of radius* $(|H|^2 + c)^{-\frac{1}{2}}$.

(b) *If* $k = 2$, *then* M^2 *is a small* 2-*sphere of radius* $(|H|^2 + c)^{-\frac{1}{2}}$.

Proof. (a) In a neighborhood of each $p \in M^2$ we consider the immersion to be given conformally. Let $e_3 = H/|H|$. By Lemma 2.1 (a), φ_3 is analytic. Since φ_3 transforms quadratically, the differential Φ_3 which in local coordinates is given by $\varphi_3 dz^2$ is well defined.

M^2 is a Riemann surface via the local conformal structure and the definition of z. Since M^2 is of genus 0, $\Phi_3 \equiv 0$. Hence in local coordinates, $\varphi_3 \equiv 0$. By the remarks preceding this theorem, this means the immersion is pseudo-umbilical at each point. It is then straightforward to show that M^2 lies minimally in some hypersphere of radius $(|H|^2 + c)^{-\frac{1}{2}}$. In fact, an immersion $M^n \hookrightarrow \bar{M}^{n+2}(c)$, $c \geq 0$, with constant nonzero mean curvature is pseudo-umbilical $\Leftrightarrow M^n$ lies minimally in some hypersphere of $\bar{M}^{n+2}(c)$. (To make sense of this in case $c > 0$ we take as model for $M^{n+k}(c)$, $S^{n+k}(1/\sqrt{c})$ considered as a hypersurface in E^{n+k+1}. Hyperspheres are then intersections of S^{n+k} with affine $(n + k)$-planes in E^{n+k+1}.) This result is proved in [6]. The Euclidean case of the theorem for surfaces occurs in Chen [2] and Ruh [14].

(b) In a neighborhood of each $p \in M^2$, let e_4 be a smooth unit section of NM^2 such that $\langle e_3, e_4 \rangle = 0$. Since $|e_4| = 1$, $D_x e_4 = \omega(X)e$ for some 1-form ω. But

$$0 = X\langle e_3, e_4 \rangle = \langle D_x e_4, e_3 \rangle + \langle D_x e_4, e_3 \rangle + \langle D_x e_3, e_4 \rangle = \omega(X) .$$

The second equality follows from the fact that $D_x e_3 = 0$. Hence $D_x e_4 = 0$, i.e., e_4 is parallel. By Lemma 2.1 (a), φ_4 is analytic. Repeating the argument of (a) of this proof shows $\varphi_4 \equiv 0$. Since $\lambda_{11}^3 + \lambda_{22}^4 = 0$ (see proof of Lemma 2.1 (a) (ii)) we must have $\lambda_{ij}^4 = 0$, $1 \leq i, j \leq 2$. Hence the immersion is totally umbilic. It is well known that totally umbilic manifolds are pieces of spheres. In our case we need only observe that by (a) of this theorem, M^2 lies in a 3-sphere of radius $(|H|^2 + c)^{-\frac{1}{2}}$ in such a way that e_4 is its unit normal in that sphere. Because $\lambda_{ij}^4 = 0$, it is totally geodesic and must then be an equatorial 2-sphere of this 3-sphere.

Remarks. 1. Theorem 2.3 (b) is a natural case of a more general result about surfaces in $\bar{M}^{2+k}(c)$ with constant nonzero mean curvature and normal bundles which admit framings $\{e_3 = H/|H|, e_4, \cdots, e_{2+k}\}$ such that each of the e_α is parallel. The proof of Theorem 2.2 (b) shows that such a surface of genus 0 must be a standard 2-sphere. (See also Proposition (3.3).)

2. Minimal surfaces (more generally manifolds) in Euclidean spheres give examples of surfaces (submanifolds) with constant mean curvature in Euclidean space. The examples of Lawson [12] of compact minimal surfaces in S^3 of every genus are also examples of surfaces of constant nonzero mean curvature in E^4. Thus there are compact surfaces of constant mean curvature of every genus in E^4.

3. It is important to know that minimal surfaces in hyperspheres are *not* the only examples of surfaces of constant mean curvature. In § 5 we prove the existence of a large class of surfaces in E^4 and S^4 which have constant mean curvature but do not lie minimally in hyperspheres.

3. Surfaces with constant mean curvature and constant Gauss curvature

In this section we classify immersions $M^2 \hookrightarrow \bar{M}^4(c)$, $c \geq 0$, which have constant nonzero mean curvature H and constant Gauss curvature K. For $c > 0$ we take as a model for $\bar{M}^4(c)$ the hypersurface

$$S^4(1/\sqrt{c}) = \{X \in E^5 \,|\, |X|^2 = 1/c\} \subset E^5 .$$

By a *standard product immersion of* $S^1(\rho) \times S^1(r)$ *in* E^4 we mean the product of of two Euclidean plane circles (of radii ρ and r respectively). ρ may take on the value $+\infty$, so this includes right circular cylinders. By a *standard product immersion in* $\bar{M}^4(c)$ we mean an immersion $M^2 \hookrightarrow \bar{M}^4(c) \simeq S^4(1/\sqrt{c}) \subset E^5$ which lies in some affine 4-plane $\Pi \subset E^5$ and as such is a standard product immersion in the Euclidean sense. In particular, standard product immersions into 4-spheres lie in great 3-spheres if Π passes through the origin, and in small 3-spheres otherwise.

Theorem 3.1. *Let* $M^2 \hookrightarrow \bar{M}^4(c)$ *be an isometric immersion with constant nonzero mean curvature and constant Gauss curvature* K. *Then* $K \equiv 0$ *or* $K \equiv |H|^2 + c$. *If* $c \geq 0$, *then* M^2 *is a piece of a product of circles* ($K \equiv 0$) *or a piece of a 2-sphere* ($K \equiv |H|^2 + c$).

Proof. The theorem follows from Propositions 3.3 and 3.4.

Lemma 3.2. *Let* $M^2 \hookrightarrow \bar{M}^{2+k}(c)$ *be a conformal immersion with conformal parameter* E. *Let* $K' = K - c$ *be the relative curvature of the immersion, and* $\{e_3, \cdots, e_{2+k}\}$ *an orthonormal framing of* NM^2. *Then*

$$E^2(|H|^2 - K') = \sum_{\alpha=3}^{2+k} |\varphi_\alpha|^2 \overset{\text{def}}{=} \eta . \tag{3.1}$$

If $|H|^2 - K' \neq 0$, *then*

$$K = -\frac{\Delta \log [\eta/(|H|^2 - K')]}{4\eta(|H|^2 - K')^{\frac{1}{2}}} . \tag{3.2}$$

Proof. (3.1) follows from the Gauss equation (1.7) (i) and the definition of K:

$$K = E^{-2}\langle \bar{R}(U_1, U_2)U_2, U_1\rangle .$$

Equation (3.2) follows from (3.1) and the intrinsic equation for K:

$$K = -\tfrac{1}{2}E^{-1}\Delta \log E \ .$$

Proposition 3.3. *Let $M^2 \hookrightarrow \bar{M}^{2+k}(c)$ be a conformal immersion with constant nonzero mean curvature and K = constant. Suppose further that $\{e_3 = H/|H|, e_4, \cdots, e_{2+k}\}$ is an orthonormal framing of NM^2 such that each e_α is parallel. Then either $K \equiv |H|^2 + c$ or $K \equiv 0$. If $K \equiv |H|^2 + c$ and $c \geq 0$, then M^2 is immersed as a piece of a standard 2-sphere.*

In particular, if $M^2 \hookrightarrow \bar{M}^4(c)$, $c \geq 0$, has constant nonzero mean curvature and K = constant, then either $K = |H|^2 + c$ and M^2 is a piece of a sphere or $K \equiv 0$.

Proof. If $K \not\equiv |H|^2 + c$, then $|H|^2 - K' \not\equiv 0$ and by (3.1) $\eta = \sum |\varphi_\alpha|^2 \neq 0$. Thus at least one φ_α, say φ_{α_0}, is nonzero. By Lemma 2.1 (a) and (c), $\varphi_\alpha = k_\alpha \varphi_{\alpha_0}$ where k_α is a real constant. Hence $\eta = (\sum_\alpha k_\alpha^2)|\varphi_{\alpha_0}|^2$. Therefore $\log \eta$ is harmonic since φ_{α_0} is analytic. But (3.2) implies $K \equiv 0$. If $K \equiv |H|^2 + c$, then by (3.2) $\eta \equiv 0$ and consequently each $\varphi_\alpha \equiv 0$. Therefore the immersion is totally umbilic and hence a piece of a 2-sphere. q.e.d.

The special case $M^2 \hookrightarrow \bar{M}^4(c)$, $c \geq 0$, follows from the above and the proof of Theorem 2.2 (b) where we have shown that if e_4 is a unit normal section such that $\langle e_3, e_4 \rangle = 0$, then e_4 is parallel.

Proposition 3.4. *Let $M^2 \hookrightarrow M^4(c)$, $c \geq 0$, have constant nonzero mean curvature, and assume $K \equiv 0$. Then M^2 is a standard product immersion of $S^1(r) \times S^1(\rho)$, where $|H|^2 = \rho^{-2} + r^{-2}$.*

Proof. Since $K \equiv 0$, M^2 is isometric to the plane, and we may choose conformal coordinates locally on M^2 with $E \equiv 1$. As usual, let $\{e_3 = H/|H|, e_4\}$ be a normal framing.

Case A: $\varphi_3 \equiv 0$. Then the immersion is pseudo-umbilical, and by the proof of Theorem 2.3 (a), M^2 lies minimally in some 3-sphere of radius $1/|H|$. By a result of Lawson [12] a minimal surface in $S^3(r)$ with $K \equiv 0$ must be a piece of the Clifford torus $S^1(\sqrt{r}/2) \times S^1(\sqrt{r}/2)$ in $S^3(r)$. Hence the immersion is a standard product immersion. One can also obtain this result by a method similar to

Case B: $\varphi_4 = k\varphi_3$. By (3.1), $|H|^2 - K' = |H|^2 + c = (1 + k^2)|\varphi_3|^2$, and $|\varphi_3|$ is constant. Therefore φ_3 is constant, and after a possible rotation of coordinates φ_3 may be assumed to be real. If $\varphi_3 \overset{\text{def}}{=} \gamma$, then

$$(3.3) \qquad (\lambda_{ij}^3) = (L_{ij}^3) = \begin{pmatrix} |H| + \gamma & 0 \\ 0 & |H| - \gamma \end{pmatrix}, \qquad \gamma^2 = \frac{|H|^2 + c}{1 + k^2} ,$$

since $\lambda_{11}^3 + \lambda_{22}^3 = 2|H|$. Since $\lambda_{11}^4 + \lambda_{22}^4 = 0$,

$$(3.4) \qquad (\lambda_{ij}^4) = \begin{pmatrix} k\gamma & 0 \\ 0 & -k\gamma \end{pmatrix} .$$

Now assume $c = 0$ and therefore $M^2 \hookrightarrow E^4$.

(i) If $k = 0$, then $\lambda^4_{ij} = 0$. If $X \in \Gamma(TM^2)$, then

$$\begin{aligned} \bar{\nabla}_X e_4 &= A(e_4, X) + D_x e_4 = D_x e_4 \qquad \text{by (1.8)} \\ &= 0 \qquad \text{since } e_4 \text{ is parallel .} \end{aligned}$$

Therefore e_4 is a constant vector in E^4. Let $p \in M^2$. If $\Pi_3 = \{X \in E^4 \mid \langle x - p, e_4 \rangle = 0\}$, then it follows that M^2 lies in Π_3. As a surface in this 3-dimensional Euclidean space, its unit normal is e_3. Furthermore (3.1) determines γ as $\pm |H|$. Therefore by (3.3),

$$(\lambda^3_{ij}) = \begin{pmatrix} 2|H| & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.5}$$

and since $E \equiv 1$ we have the familiar second fundamental form of a right circular cylinder. By the uniqueness theorem for hypersurfaces, the immersion must be a right circular cylinder $S^1(1/|H|) \times S^1(\infty)$.

(ii) If $k \neq 0$, the equations

$$\begin{aligned} a(|H| + \gamma) + bk\gamma &= 0 , \\ -ak\gamma + b(|H| - \gamma) &= 0 , \qquad a^2 + b^2 = 1,\ a > 0 \end{aligned}$$

can be solved uniquely for a and b since $|H|^2 - \gamma^2 + k^2\gamma^2 = K$ by (3.1) and $K = 0$ by assumption. Let $(\tilde{e}_3, \tilde{e}_4)$ be a new framing defined by

$$\tilde{e}_3 = ae_3 - be_4 , \qquad \tilde{e}_4 = be_3 + ae_4 . \tag{3.6}$$

Both $\tilde{e}_3$ and $\tilde{e}_4$ are parallel since e_3 and e_4 are, and their second fundamental forms are given by

$$\begin{aligned} (\tilde{\lambda}^3_{ij}) &= \begin{pmatrix} 0 & 0 \\ 0 & a(|H| - \gamma)k\gamma \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{W}_3 \end{pmatrix}, \\ (\tilde{\lambda}^4_{ij}) &= \begin{pmatrix} b(|H| + \gamma)k\gamma & 0 \\ 0 & 0 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \tilde{W}_4 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.7}$$

It is now a straightforward matter to verify that the immersion is in fact a product of circles. Toward that end we first notice that $U_2 \wedge \tilde{e}_3$ is a constant plane in E^4 since

$$\frac{\partial}{\partial u_i}(U_2 \wedge \tilde{e}_3) = \tilde{W}_3 \tilde{e}_3 \wedge \tilde{e}_3 + U_2 \wedge (-\tilde{W}_3 U_2) = 0 .$$

Similarly $U_1 \wedge \tilde{e}_4$ is a constant plane. These 2 planes are orthogonal. Futhermore for fixed u^2 (resp. u^1), the immersion is a circle of radius $1/\tilde{W}_4$ (resp. $1/\tilde{W}_3$) in the plane $U_1 \wedge \tilde{e}_4$ (resp. $U_2 \wedge \tilde{e}_3$). This clearly gives the immersion as a product of circles.

All that remains to complete the proof of this proposition is to study (i) and (ii) for the case $c > 0$. We do that by reducing the case to the Euclidean case $c = 0$.

Lemma 3.5. *Let $M^2 \hookrightarrow S^4(r)$ be an immersion with nonzero constant mean curvature and $K \equiv 0$. Then M^2 lies in some small 3-sphere of $S^4(r)$, which is $M^2 \hookrightarrow S^4(r) \cap \Pi^4$, where Π^4 is an affine 4-plane.*

Proof. Equations (3.3) and (3.4) give (λ^3_{ij}) and (λ^4_{ij}) as constant, diagonalized matrices of a specific form. Let e_5 be a unit normal vector field to $S^4(r) \hookrightarrow E^5$. Restricted to $M^2 \hookrightarrow S^4(r) \hookrightarrow E^5$, e_5 is still a unit normal vector field and $\lambda^5_{ij} = \delta_{ij}/r$, $i, j = 1, 2$. We can find real constants a, b, c such that

$$a(\lambda^3_{ij}) + b(\lambda^5_{ij}) + c(\lambda^5_{ij}) = 0 \; , \qquad a^2 + b^2 + c^2 = 1 \; . \tag{3.8}$$

Let $\tilde{e} = ae_3 + be_4 + ce_5$. $\tilde{e}$ is parallel since e_3, e_4, e_5 are parallel and a, b and c are constants. Equation (3.8) says that the second fundamental form in the $\tilde{e}$ direction is identically zero. This implies that $\tilde{e}$ is a constant vector in E^5. To wit,

$$\tilde{\nabla}_{U_i}\tilde{e} = -A(\tilde{e}, U_i) + D_{U_i}\tilde{e} = 0 \; ,$$

since $\tilde{\lambda}_{ij} = 0$ and $\tilde{e}$ is parallel. For $p \in M^2$, let $\Pi^4 = \{X \in E^5 \,|\, \langle X - p, \tilde{e}\rangle = 0\}$. Clearly $M^2 \subset \Pi^4$.

Remark. Proposition 3.4 also follows from Erbacher [5, Theorem 1].

4. Complete surfaces with constant mean curvature

In this section we prove a generalization of a theorem due to Klotz and and Osserman [11] which states that a complete surface in E^3 with constant mean curvature and Gauss curvature which does not change sign is a minimal surface, a sphere or a right circular cylinder.

Theorem 4.1. *A complete immersed surface $M^2 \hookrightarrow \overline{M}^4(c)$ with constant mean curvature and Gauss curvature K which does not change sign must be minimal ($H \equiv 0$), a sphere of radius $(|H|^2 + c)^{-\frac{1}{2}}$ or a product of circles $S^1(r) \times S^1(p)$, $0 < r \leq \infty$, $0 < p < \infty$, with the standard product immersion.*

Proof. By observation 1 of § 1, $|H|$ is constant, so either $H \equiv 0$ or H has no zeros. Henceforth we assume $H \neq 0$ and choose a normal framing $\{e_3 = H/|H|, e_4\}$. In local conformal coordinates $ds^2 = E[(du^1)^2 + (du^2)^2]$, the functions φ_3 and φ_4 of Lemma 2.1 are analytic functions of $z = u^1 + iu^2$. Covering M^2 by local conformal charts induces a Riemann surface structure of M^2.

Case 1: $K \leq 0$. In this case $K' = K - c < 0$. By (3.1) we have in each local chart

$$\eta = |\varphi_3|^2 + |\varphi_4|^2 = E^2(|H|^2 - K') \geq 0 \; . \tag{4.1}$$

By Lemma 2.1 (c), either $\varphi_3 = \varphi_4 = 0$ or one is a constant multiple of the other. Therefore $\log E^2(|H|^2 - K') = \log \eta$ is harmonic. Let $d\tilde{s}^2 = \sqrt{\eta}\,[(du^1)^2 + (du^2)^2]$. The Gauss curvature $\tilde{K}$ of this new metric is given by

$$\tilde{K} = -\tfrac{1}{2}\eta^{-\frac{1}{2}}\Delta \log \sqrt{\eta} = 0\,, \tag{4.2}$$

since $\log \eta$ is harmonic. Therefore $d\tilde{s}^2$ is a flat metric conformally equivalent to ds^2. By a standard argument the simply connected covering surface of M^2 is conformally equivalent to the plane. On M^2 the function $\log(\sqrt{\eta}/E)$ is a globally defined function, and is bounded below by $\log|H| > -\infty$ due to (4.1). Moreover, it is superharmonic since

$$\begin{aligned} \Delta \log(\sqrt{\eta}/E) &= \Delta \log \sqrt{\eta} - \Delta \log \sqrt{E} \\ &= -\Delta \log \sqrt{E} \qquad \text{by (4.2)} \\ &= EK \leq 0 \qquad \text{since } K \leq 0\,. \end{aligned}$$

Lifting $\log(\sqrt{\eta}/E)$ to the simply connected covering space of M^2 we have a superharmonic function bounded below on a surface which is conformally equivalent to the plane (parabolic). Therefore $\log(\sqrt{\eta}/E)$ is constant. This implies $\eta = |H|^2 - K'$ and hence K are also constants. Using conformal equivalence with the plane again, K must be identically zero. By Theorem 3.1, M^2 must be immersed as the standard product of circles $S^1(r) \times S^1(p)$. This completes the proof if $K \leq 0$.

Remark. By Proposition 3.4 and its proof, is the case where $\varphi_3 \equiv 0$ M^2 is a product of circles with $r = p$ (minimal Clifford torus in a hypersphere), while in the case where $\varphi_3 \not\equiv 0$, $\varphi_4 \equiv 0$, M^2 is a right circular cylinder ($r = \infty$). If neither φ_3 nor φ_4 are identically zero, M^2 is $S^1(r) \times S^1(p)$ with $r \neq p$ and $r \neq \infty$.

Case 2: $K \geq 0$. By a theorem of Huber [9], a complete surface with $K \geq 0$ is either compact or parabolic. Suppose M^2 is compact. If $K \equiv 0$ we are done by Proposition 3.4. If not, M^2 must be of genus 0 by Gauss-Bonnet, and is a sphere of radius $(|H|^2 + c)^{-\frac{1}{2}}$ by Theorem 2.3 (b). Suppose M^2 is parabolic. We claim that M^2 must then be flat ($K \equiv 0$). To see this, observe that $\eta = E^2(|H|^2 - K')$ is not identically zero; for otherwise $K \equiv |H|^2 + c > 0$, and then M^2 would carry a complete metric of constant positive curvature, an impossibility since M^2 is parabolic. As in Case 1, $\log \eta$ is harmonic. Therefore

$$\begin{aligned} 0 = \Delta \log \eta &= 2[\Delta \log E + \Delta \log(\sqrt{\eta}/E)] \\ &= 2[-2KE + \Delta \log(\sqrt{\eta}/E)] \\ &\leq 2(\Delta \log(\sqrt{\eta}/E) \qquad \text{since } K \geq 0\,. \end{aligned}$$

Thus $\log(\sqrt{\eta}/E)$ is subharmonic, and is further bounded above since $\log(\sqrt{\eta}/E) = \log(|H|^2 - K^1)^{\frac{1}{2}} \geq \log(|H|^2 + c)^{\frac{1}{2}}$. Therefore $\sqrt{\eta}/E$ is constant since M^2 is parabolic. By the definition of η, K is also constant. As is Case 1,

$K \equiv 0$ since M^2 is parabolic. Hence Proposition 3.4 completes the proof.

Remark. The special case of pseudo-umbilical immersion in E^4 with constant mean curvature has been treated by Itoh [10].

5. A local existence theorem for surfaces of constant mean curvature in $\bar{M}^4(c)$

In the previous sections the only examples of surfaces with constant mean curvature have been products of circles or minimal surfaces in hyperspheres. The following theorem shows that there are indeed a good many more examples.

Theorem 5.1. *Let $\varphi \equiv 0$ be an analytic function of $z = u^1 + iu^2$ defined in a neighborhood of the origin in the (u^1, u^2) plane. Let h and α be real constants with $h > 0$. Then there exist a neighborhood $\mathscr{U}_0$ of the origin, a conformal metric $E(u^1, u^2)$ defined on $\mathscr{U}_0$ and an isometric immersion $(\mathscr{U}_0, E) \hookrightarrow \bar{M}^4(c)$ with the following properties:*

The immersion has constant mean curvature. The mean curvature vector field H has length $|H| = h$. If $\{e_3 = H/h, e_4\}$ is an orthonormal framing of $N\mathscr{U}_0$, then $\varphi_3 = \varphi$ and $\varphi_4 = \alpha\varphi$.

Proof. Suppose such a metric E and such an immersion existed. By (3.1) we must have

$$E^2[(h^2 + c) - K] = (1 + \alpha^2)|\varphi|^2 , \tag{5.1}$$

where $K = -\frac{1}{2}E^{-1}\Delta \log E$. The existence of a positive E satisfying (5.1) is equivalent to the existence of a positive E which is a solution of

$$\Delta \log E = 2\{(l + \alpha^2)|\varphi|^2 E^{-1} - (h^2 + c)E\} . \tag{5.2}$$

It is therefore a necessary condition (for the existence of an immersion as stated in the theorem) that a solution of (5.2) exist.

Claim. There exists a solution of (5.2) defined in a neighborhood of the origin.

We proceed with the proof of the theorem modulo the claim. (A proof of the claim follows at the end.) Let E be a solution of (7.3) in a neighborhood $\mathscr{U}_0$ of the origin. Consider $\mathscr{U}_0$ with the conformal metric E. Let $N = \mathscr{U}_0 \times R^2$. We consider N as the total space of a vector bundle over $(\mathscr{U}_0, E)$. With the usual inner product on R^2, N is a Riemannian vector bundle endowed with the usual connection on R^2 which we denote by D. Let $\{e_3, e_4\}$ be an orthonormal parallel framing of N. Such a parallel framing clearly exists since D is the usual flat connection on R^2. Let B be a section of $\mathscr{H}(T(\mathscr{U}_0, E) \otimes T(\mathscr{U}_0, E), N)$ defined as follows. If $U_i = \partial/\partial u_i$ are the coordinate vector fields on $\mathscr{U}_0$,

$$\begin{aligned} B(U_1, U_1) &= (h + \operatorname{Re}\varphi)e_3 + (\alpha \operatorname{Re}\varphi)e_4 , \\ B(U_2, U_2) &= (h - \operatorname{Re}\varphi)e_3 - (\alpha \operatorname{Re}\varphi)e_4 , \\ B(U_1, U_2) &= (-\operatorname{Im}\varphi)e_3 - (\alpha \operatorname{Im}\varphi)e_4 . \end{aligned} \tag{5.3}$$

By a theorem of Szczarba [18] there exists an immersion of $(\mathscr{U}_0, E)$ into $\bar{M}^4(c)$ with N as normal bundle and B as second fundamental form if and only if E and B satisfy the Gauss and Codazzi equations (1.7). (These equations are clearly necessary. Their sufficiency in the codimension-one case is the classical theorem on existence and rigidity of hypersurfaces.) The Gauss equation (1.7) (i) reduces this case to

$$cE^2 = KE^2 + E^2[h^2 + (1 + \alpha^2)|\varphi|^2] \,, \tag{5.4}$$

which is an immediate consequence of (5.1). The Codazzi equation (1.7) (iii) reduces to the Cauchy-Riemann equations for φ once one uses the fact that h is constant and e_3 and e_4 are parallel. The second Gauss equation (1.7) (ii) is

$$\tilde{R}(U_i, U_j)e_\alpha = B(U_j, A(e_\alpha, U_i)) - B(U_i, A(e_\alpha, U_j)) \,.$$

The left-hand side is always zero since e_α is parallel. The right-hand side is seen to be (after a calculation exactly like that in Lemma 2.1 (b)) equal to $\operatorname{Im}(\alpha|\varphi|^2) = 0$.

By the aformentioned theorem of Szczarba, there exists an immersion $(\mathscr{U}_0, E) \hookrightarrow \bar{M}^4(c)$ with N as normal bundle and B as second fundamental form. We remark that it is also unique up to isometries of $\bar{M}^4(c)$. Expressing this immersion in terms of the coordinates (u^1, u^2) yields the conformal metric E. From the definition of B in (5.3) it is immediate that $H = he_3$ and $\varphi_3 = \varphi$, $\varphi_4 = \alpha\varphi$.

Proof of claim. Let $\beta = h^2 + c$, $\eta = (1 + \alpha^2)|\varphi|^2$ and $f = \log E$. By assumption, β is a real constant and η is real analytic. We may write (5.2) as

$$\partial^2 f/\partial u^2 = -\partial^2 f/\partial u^2 + 2(\eta e^{-f} - \beta e^f) \,. \tag{5.5}$$

If we consider (5.5) with the initial values

$$f(0, u^2) = 0 \,, \qquad \partial f/\partial u^1(0, u^2) = 0 \,, \tag{5.6}$$

we may assert the existence (and uniqueness) of an analytic solution to this initial-value problem by the Cauchy-Kovalewski theorem [3, p. 39]. Then $E = e^f$ will be a solution to (5.2).

Corollary 5.2. *Let $(\mathscr{U}_0, E) \hookrightarrow \bar{M}^4(c)$, $c \geq 0$, be an immersion corresponding to a specified $\varphi \not\equiv 0$, α and $h > 0$ as in Theorem 5.1. Then each of the following holds:*

(i) *The image of $\mathscr{U}_0$ does not lie in any hypersphere of $\bar{M}^4(c)$ as a minimal surface.*

(ii) *The immersion is a piece of a standard product of circles $\Leftrightarrow |\phi/E|$ is constant. In particular, if φ has zeros, $(\mathscr{U}_0, E)$ is not immersed as a product of circles.*

(iii) $\alpha = 0 \Leftrightarrow \mathscr{U}_0$ *lies in a 3-dimensional hyperplane or hypersphere as a surface of constant mean curvature.*

Proof. (i) $\mathscr{U}_0$ lies minimally in a hypersphere $\Leftrightarrow$ the immersion is pseudo-umbilical $\Leftrightarrow \varphi_3 = \varphi \equiv 0$. This last condition is prohibited by hypothesis.

(ii) By equation (5.1), K is constant $\Leftrightarrow |\varphi/E|$ is constant. The immersion must be a piece of standard 2-sphere or a product of circles by Theorem 3.1, cannot be a piece of a standard 2-sphere by (i) of this corollary, and so is a product of circles. In particular if φ has a zero, then $|\phi/E|$ cannot be constant.

(iii) $\alpha = 0 \Leftrightarrow B(U_i, U_j)$ is always a multiple of $e_3 \Leftrightarrow$ the immersion lies in a 3-dimensional hyperplane (in the case $c = 0$) or hypersphere ($c > 0$). The first equivalence follows from the definition of B in (5.3), and the second from the following lemma by taking as model for $\bar{M}^4(c)$ a hyperplane or hypersphere in E^5.

Lemma 5.3. *Suppose* $M^n \xrightarrow{X} E^{n+k}$ *has an r-dimensional distribution* $\mathscr{D}$ *in* NM^n *such that* (a) *the range of* B *is in* $\mathscr{D}$ *and* (b) *if* V *is a smooth section of* $\mathscr{D}$, *then* $D_X V \in \mathscr{D}$ *for all* $W \in \Gamma(TM^n)$. *Then* M^n *lies in an* $(n + r)$*-plane* $\tilde{\Pi} \subset E^{n+k}$.

Proof. Choose $V_i, \cdots, V_r$, differentiable vector fields which span $\mathscr{D}$. For coordinates $(u^i, \cdots, u^n)$, let $X_i = \partial x/\partial u_i$ be coordinate vector fields, and set $W = X_1 \wedge \cdots \wedge X_n \wedge V_1 \wedge \cdots \wedge V_r$. Then conditions (a) and (b) imply

$$\partial W/\partial u^k = f_k W\ , \qquad k = 1, \cdots, n\ .$$

For real-valued functions f_k. This says that $(n + r)$-vector W spans a constant $(n + r)$-plane Π. Let $p \in M^n$. Clearly the affine $(n + r)$-plane $\tilde{\Pi} = \Pi + p$ contains M^n.

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