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THE CONJECTURES ON CONFORMAL TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS

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Introduction

Let (M, g) be a Riemannian n-manifold with Riemannian metric g. *Throughout this paper manifolds under consideration are always assumed to be connected and smooth.*

For a smooth function ρ on M, a Riemannian metric $e^{2\rho}g$ is said to be *conformally related*, or *conformal*, to g. Let h be a smooth map of (M, g) into another Riemannian manifold (M', g'). If the Riemannian metric h^*g' induced on M by h is conformal to g, then h is called a *conformal map* of (M, g) into (M', g'). It is well-known that h is conformal if and only if it preserves the angle between any two tangent vectors. h remains conformal under any conformal changes of Riemannian metrics on M and M' as well. If h is a conformal diffeomorphism of (M, g) onto (M', g'), then it is called briefly a *conformorphism* of (M, g) onto (M', g'), and (M, g) is said to be *conformorphic* to (M', g') Via h. If furthermore $(M, g) = (M', g')$, then h is called a *conformal transformation* or a *conformorphism* of (M, g).

It is known that the group $C(M, g)$ of all conformorphisms of (M, g) is a Lie group with respect to the compact-open topology. Let $C_0(M, g)$ denote the connected component of the identity of $C(M, g)$. If g and $\bar{g}$ are conformal to each other, then $C(M, g) = C(M, \bar{g})$. The group $I(M, g)$ of all isometries of (M, g) is a closed subgroup of $C(M, g)$. A subgroup G of $C(M, g)$ is said to be *essential* if G is not contained in $I(M, e^{2\rho}g)$ for any smooth function ρ on M, and is said to be *inessential* otherwise.

In this paper, unless otherwise stated, *we always assume* $\dim M > 2$, although some of our propositions are valid even for $\dim M = 2$.

There have been two conjectures:

Conjecture I. *Let (M, g) be a compact Riemannian n-manifold. Then $C_0(M, g)$ is essential if and only if (M, g) is conformorphic to a Euclidean n-sphere S^n.*

Conjecture II. *Let (M, g) be a compact Riemannian n-manifold with constant scalar curvature k. Then $C_0(M, g)$ is essential if and only if k is positive and (M, g) is isometric to a Euclidean n-sphere $S^n(k)$ of radius* $1/\sqrt{k}$.

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In each of the conjectures, "if" part is obvious.

Conjecture I has been proved under some additional conditions, for example, in the following cases:

(a) $C_0(M, g)$ contains a one-parameter subgroup generated by a gradient vector field (Ishihara-Tashiro [11], Tashiro [33]).

(b) $C_0(M, g)$ is transitive on M (Nagano [21], Ba [2]).

(c) $C_0(M, g)$ contains a one-parameter subgroup generated by a vector field with singular points at each of which its divergence does not vanish (Avez [1], Obata [26]).

(d) (M, g) is conformally flat and has a finite fundamental group (Obata [27]).

(e) (M, g) is analytic and has a finite fundamental group (Ledger-Obata [16]).

Recently Lelong-Ferrand [17] has proved Conjecture I by using a technique involving quasi-conformal transformations. In the present paper Conjecture I will be proved along with the proof in [16]; indeed, the assumptions of analyticity and finite fundamental group will be removed. Our method is different from Lelong-Ferrand's in idea.

As for Conjecture II there have been also many results under some additional conditions, for example, in the following cases:

(f) (M, g) is an Einstein space (Yano-Nagano [31]).

(g) (M, g) is a Riemannian manifold with parallel Ricci tensor (Tanaka [31], Nagano [22]).

(h) (M, g) is homogeneous (Goldberg-Kobayashi [6], [7]).

(i) The magnitude of the Ricci tensor or the curvature tensor is constant (Lichnerowicz [19], Barbance [3], Hsiung [9]).

(j) $C_0(M, g)$ contains a one-parameter subgroup generated by a gradient vector field (Ishihara-Tashiro [11], Obata [24], [25], Lichnerowicz [19], Tashiro [33], Yano-Obata [36], Bishop-Goldberg [4], Tanno-Weber [32]).

It should be remarked that in some of the above results, compactness is replaced by a weaker condition of completeness, and an essential conformal vector field is replaced by a nonisometric conformal vector field. In most of the above cases the additional conditions are made to reduce the problem to the case of a conformal vector field which is a gradient of some function; this situation is typical in an Einstein space. A gradient conformal vector field is essential in our terminology.

Once Conjecture I is proved, the manifold under consideration in Conjecture II may be thought of as an n-sphere with a Riemannian metric which is conformal to the standard one and has constant scalar curvature. Thus Conjecture II follows from a result on a conformal change of metrics, namely, in the above case such a conformal metric is of constant (sectional) curvature (Proposition 6.1).

This paper is divided into two parts, one for Conjecture I and the other for Conjecture II.

§ 1 contains some preliminary facts about the conformal structure on a Riemannian manifold. In particular, by following the theory of G-structures the topology of the group $C(M, g)$ of all conformorphisms will be given. Proofs are omitted mostly because they are just translations of known results in the general theory.

§ 2 contains several propositions on essential groups of conformorphisms, each of which will be used later. Whenever the references are known, the proofs are omitted.

In § 3, known results on conformorphisms of a Euclidean sphere will be quoted mainly from [26], and an improvement of a theorem in [27] will be given and proved.

In § 4, after showing that if $C_0(M, g)$ is essential then (M, g) is conformally flat, we shall prove Conjecture I by using the same idea as that given in [16].

In Part II, after preparing general formulas for conformal changes of metric in § 5, a special case for a Euclidean sphere will be considered in § 6, which seems to be a clue for the solution of Conjecture II.

In Part I and consequently in Part II as well, the following theorem of Kuiper [14] will be of essential use, in particular, in the proof of Proposition 3.4.

Theorem K. *A conformally flat simply connected Riemannian n-manifold is conformorphic to an open submanifold of a Euclidean n-sphere.*

An outline of this paper has been announced in [28].

PART I

1. Preliminary remarks

Let (M, g), or simply M, be a Riemannian n-manifold with Riemannian metric g. From the general theory of G-structures of finite type (Kobayashi [12], [13], Sternberg [30]) it follows that the group $C(M, g)$ of all conformorphisms of (M, g) is a Lie group with respect to the compact-open topology, since a conformal structure is indeed of finite type for $n > 2$. We shall give some necessary facts in our terminology. A *conformal frame* at a point p of M is, by definition, a triple $b = (\lambda, b_0, \mu)$, where λ is a positive number, b_0 an orthonormal frame with respect to the Riemannian structure on M, and μ a non-zero tangent vector at p (Cartan [5]). The set P of all the conformal frames of M is equivalent to a sub-bundle of the bundle $P^2(M)$ of 2-jects of M (Ogiue [29]). The bundle P is called the *conformal frame bundle* of M, which is known to be completely parallelizable, i.e., to enjoy an $\{e\}$-structure. A conformorphism of (M, g) is identified with a fibre preserving diffeomorphism of $P^2(M)$ leaving P invariant, i.e., with an automorphism of the $\{e\}$-structure of P by a prolongation. By a theorem of Kobayashi (Kobayashi [12], [13]; Sternberg [30]) on automorphisms of a manifold with complete parallelism, we can state the following propositions.

Proposition 1.1. *$C(M, g)$ acts on the conformal frame bundle P without fixed points.*

Proposition 1.2. *Let $\{f_k\}$ be a sequence of elements of $C(M, g)$ such that $f_k(b) \to f(b)$ for some conformal frame b on M and some $f \in C(M, g)$. Then $f_k \to f$ in the topology of $C(M, g)$.*

Proposition 1.3. *Let $\{f_k\}$ be a sequence of elements of $C(M, g)$ such that $f_k(b) \to b'$ for some conformal frames b and b' on M. Then there exists an $f \in C(M, g)$ such that $f(b) = b'$.*

As an easy consequence of these propositions the following is obtained.

Proposition 1.4. *Let M' be an open submanifold of M, which is invariant under the action of $C(M, g)$. Then $C(M, g)$ acts on (M', g') effectively as a closed subgroup of (M', g'), where g' is the restriction of g to M'.*

To close this section, for later use we give a condition for a group of isometries to be compact.

Proposition 1.5 (*Ledger-Obata* [16]). *A closed subgroup G of $I(M, g)$ is compact if and only if there exists a point $p \in M$ such that the orbit $G(p)$ through p is compact.*

2. Essential groups of conformorphisms

A subgroup G of $C(M, g)$ is said to be *essential* if G is not contained in $I(M, e^{2\rho}g)$ for any smooth function ρ.

Proposition 2.1 (*Ishihara* [10]). *An essential group of conformorphisms is not compact.*

Since the group of isometries of any compact Riemannian manifold is compact, the converse is true for a compact Riemannian manifold. More precisely, we can state in the following way.

Proposition 2.2. *Let (M, g) be a compact Riemannian manifold. Then a closed subgroup G of $C(M, g)$ is essential if and only if it is not compact.*

By a theorem of Montgromery and Zippin [20], any noncompact Lie group of positive dimensions contains a closed one-parameter subgroup isomorphic to the additive group of reals. Therefore by Proposition 2.2 we obtain

Proposition 2.3. *If $C_0(M, g)$ is essential on a compact Riemannian manifold M, then it contains a closed essential one-parameter subgroup.*

The conformal curvature tensor W of type $(1, 3)$ on (M, g) is a conformal invariant and therefore is invariant under the action of $C(M, g)$. It is well-known that if W vanishes identically and $\dim M > 3$, then (M, g) is conformally flat.

If $\dim M = 3$, then W automatically vanishes and it is known that there is a tensor field $\tilde{W}$ of type $(0, 3)$ constructed from the Riemannian structure g such that (M, g) is conformally flat if and only if $\tilde{W}$ vanishes identically. $\tilde{W}$ is a conformal invariant and is invariant by the action of $C(M, g)$ as well in case $\dim M = 3$ (see, for example, Yano [34]).

Proposition 2.4 (*Hlavatý* [8], *Nagano* [21]). *If (M, g) has non-vanishing conformal curvature tensor, then $C(M, g)$ is inessential.*

In fact, $\|W\| g$ is invariant under the action of $C(M, g)$, and so is $\|\tilde{W}\|^{2/3} g$ for $\dim M = 3$, where $\|W\|$ denotes the magnitude of W with respect to g.

Let G be a one-parameter group of conformorphisms, and X the vector field defined by G, which is called a *conformal vector field*. X is obviously invariant under the action of G itself. A fixed point of G is a *zero*, or a *singular point*, of X.

Proposition 2.5 (*Avez* [1], *Obata* [26]). *An essential one-parameter group of conformorphisms always has a fixed point.*

In fact, if G has no fixed point, then X never vanishes. Since X is invariant under the action of G, so is $\bar{g} = g/\|X\|^2$, which is conformal to g. Thus G is a subgroup of $I(M, \bar{g})$.

Proposition 2.6. *Let (M, g) be a Riemannian manifold, $(\tilde{M}, \tilde{g})$ a Riemannian covering manifold, and $\pi\colon \tilde{M} \to M$ the projection with $\pi^*g = \tilde{g}$.*

(i) Then *$C_0(M, g)$ acts on $\tilde{M}$ as a closed subgroup of $C_0(\tilde{M}, \tilde{g})$.*

(ii) *If a closed one-parameter subgroup G is essential on M, then so is it on $\tilde{M}$.*

Proof. (i) By the covering homotopy theorem any one-parameter subgroup of $C_0(M, g)$ acts on $\tilde{M}$. Thus we have a map $\alpha\colon C_0(M, g) \to C_0(\tilde{M}, \tilde{g})$ with $\pi \circ \alpha(f) = f \circ \pi$ for all $f \in C_0(M, g)$. By Proposition 1.1, α is injective. By Propositions 1.2 and 1.3 it is easy to see that a sequence $\{f_k\}$ converges in $C_0(M, g)$ if and only if $\{\alpha(f_k)\}$ converges in $C_0(\tilde{M}, \tilde{g})$. Thus α is continuous and closed.

(ii) Let G be a closed one-parameter subgroup of $C(M, g)$, which is essential on M. To show that G is essential on $\tilde{M}$, assume the contrary, where we write simply G instead of $\alpha(G)$. Let $\tilde{h}$ be a Riemannian metric on $\tilde{M}$ conformal to $\tilde{g}$ such that $G \subset I_0(\tilde{M}, \tilde{h})$. By Proposition 2.5, G has a fixed point p on M, and thus any point $\tilde{p}$ of $\tilde{M}$ covering p is a fixed point of G on $\tilde{M}$. Therefore G is contained in the isotropy subgroup of $I(\tilde{M}, \tilde{h})$ at $\tilde{p}$, which is compact. Since G is closed, it is compact, contrary to our assumption that G is non-compact. Hence the proposition is proved.

Next, we consider a sufficient condition for a one-parameter group G of conformorphisms to be essential. Let X be the conformal vector field defined by G, and assume that G has fixed points. It is known that the values of the divergence ϕ of X at the fixed points are unchanged by any conformal change of metric, even though ϕ itself changes as a scalar function. So, if G is inessential, then the divergence must vanish at each of the fixed points of G, because any Killing vector field has vanishing divergence. Hence we have

Proposition 2.7 (*Obata* [26]). *If a conformal vector field has non-vanishing divergence at one of its singular points, then it is essential.*

However, it should be remarked that on S^n there exists an essential conformal vector field with vanishing divergence at each of its singular points.

The following has been proved.

Proposition 2.8 (*Avez* [1], *Obata* [26]). *If a Riemannian manifold M admits a one-parameter group of conformorphisms with fixed points at each of which the divergence of the corresponding vector field does not vanish, then M is conformorphic to a Euclidean n-sphere S^n or a once-punctured n-sphere $S^n - \{p_\infty\}$.*

3. Conformorphisms of S^n

As a model of Riemannian manifold admitting an essential group of conformorphisms, we consider a Euclidean n-sphere S^n with standard metric and list some known facts for later use.

Proposition 3.1 (*Ledger-Obata* [16]). *A local one-parameter group of local conformorphisms of S^n can be extended uniquely to a global one-parameter group of global conformorphisms.*

This is based on a fact that S^n is analytic and simply connected.

A classification of essential one-parameter groups of conformorphisms is made by the following.

Proposition 3.2 (*Obata* [26]). *Let $G = \{f_t\}$ be an essential one-parameter group of conformorphisms of S^n, and X the vector field defined by G. Then G has one of the following properties.*

(i) *G has exactly one fixed point p_0 at which the divergence of X vanishes, and the orbit $G(p)$, for any $p \in S^n$, satisfies*

$$\lim_{t \to \pm\infty} f_t(p) = p_0 .$$

(ii) *G has exactly two fixed points p_0 and p_∞ at each of which the divergence of X does not vanish and the orbit $G(p)$, for $p \in S^n - \{p_0, p_\infty\}$, connects p_0 and p_∞.*

Proposition 3.3. *Let M be an open submanifold of S^n, which is invariant by a one-parameter group G of conformorphisms of S^n. If G is essential on M, then M is either S^n itself or $S^n - \{p_\infty\}$.*

Proof. Since G is essential on M, so is it on S^n. Then Proposition 3.2 implies that G has at most two fixed points on S^n. On the other hand, G has at least one fixed point on M by Proposition 2.5. Thus by Proposition 3.2, M is either S^n or $S^n - \{p_\infty\}$.

Proposition 3.4. *Let (M, g) be a conformally flat Riemannian n-manifold.*

It there is a closed essential one-parameter subgroup G of $C_0(M, g)$, then M is conformorphic to either S^n or $S^n - \{p_\infty\}$.

Proof. Take the universal Riemannian covering manifold $(\tilde{M}, \tilde{g})$ of (M, g). Then $(\tilde{M}, \tilde{g})$ is a simply connected conformally flat Riemannian manifold, and is therefore conformorphic to an open submanifold N of S^n by Theorem K (Kuiper [14]). Since G is closed in $C_0(M, g)$ and essential on M, so is it on $\tilde{M}$ by Proposition 2.6. By the conformorphism between $\tilde{M}$ and N, G acts on N, and by Proposition 3.1 the action is extended to S^n. Then by Proposition 3.3, N is S^n itself or $S^n - \{p_\infty\}$. Thus $(\tilde{M}, \tilde{g})$ is conformorphic to either S^n or $S^n - \{p_\infty\}$.

The fixed points of G on $\tilde{M}$ are exactly the points of $\tilde{M}$ covering the fixed points of G on M. Since G has at most two fixed points on $\tilde{M}$, $\tilde{M}$ is M itself or a double covering of M. We are going to show that M itself is simply connected.

If $\tilde{M}$ is a double covering of M, then G must have two fixed points on $\tilde{M}$, both of which cover a single fixed point of G on M. Then by Proposition 3.2 the corresponding vector field $\tilde{X}$ on $\tilde{M}$ has nonvanishing divergence at each of these fixed points on M, and so does the corresponding vector field X on M. Thus by Proposition 2.8, M itself is conformorphic to S^n or $S^n - \{p_\infty\}$, each of which is simply connected, a contradiction.

Proposition 3.5. *Let (M, g) be a compact conformally flat Riemanniann n-manifold. If $C_0(M, g)$ is essential, then M is conformorphic to S^n.*

Proof. Since M is compact, and $C_0(M, g)$ is essential, by Proposition 2.3 there is a closed essential one-parameter subgroup. Then by Proposition 3.4, M is conformorphic to S^n or $S^n - \{p_\infty\}$. Since M is compact, it is conformorphic to S^n.

4. Conjecture I

Theorem I. *Conjecture I is a true.*

On account of Proposition 3.5 we have only to show that (M, g) under consideration is conformally flat. Thus the following Proposition 4.1 together with Proposition 3.5 gives the proof of Theorem 1.

Proposition 4.1. *Let (M, g) be a compact Riemannian manifold with the essential group $C_0(M, g)$ of conformorphisms. Then (M, g) is conformally flat.*

Proof. Assume that (M, g) is not conformally flat, and let $N = \{p \in M: W_p \neq 0\}$. In case $\dim M = 3$, $N = \{p \in M: \tilde{W}_p \neq 0\}$. Then N is an open subset of M, and any connected component N_0 of N is an open submanifold of M. Since W, as well as $\tilde{W}$ for $\dim M = 3$, is invariant under the action of $C(M, g)$, it follows that N is fixed under this action. Hence N_0 is fixed under the action of $C_0(M, g)$. Let g_0 be the restriction of g to N_0. Then by Proposition 1.4, $C_0(M, g)$ acts on N_0 effectively as a closed subgroup of $C_0(N_0, g_0)$, which is identical with the group $I_0(N_0, \bar{g}_0)$ of isometrics for some $\bar{g}_0$ conformal to g_0 by

Proposition 2.4. Since $C_0(M, g)$ is essential, by Proposition 2.3 it contains a closed essential one-parameter subgroup G. Then G is closed in $I_0(N_0, \bar{g}_0)$, and hence the orbit $G(p)$, for $p \in N_0$, is a closed submanifold of N_0. Since G is closed in $I_0(N_0, \bar{g}_0)$ and noncompact, it follows from Proposition 1.5 that $G(p)$ is noncompact for any $p \in N_0$ and is diffeomorphic to G itself by the natural projection $G \to G(p)$.

Let X be the conformal vector field on (M, g) defined by G. Then X is nowhere zero in N_0, since $G(p)$ is diffeomorphic to G.

Now on M we put

$$F(p) = \|X \otimes X \otimes W\|_p$$
$$(F(p) = \|X \otimes X \otimes X \otimes \tilde{W}\| \text{ if } \dim M = 3) ,$$

where we write $\|T\|$ for the magnitude of a tensor T with respect to g.

Since X and W (or $\tilde{W}$ if $\dim M = 3$) are invariant under the action of G, so is $X \otimes X \otimes W$ (or $X \otimes X \otimes X \otimes \tilde{W}$ if $\dim M = 3$). As F is of type (3,3), its magnitude is invariant by G as well. Thus F is a nonzero constant on $G(p)$, $p \in N_0$. Now take $q \in \mathrm{Cl}\, G(p)$, $p \in N_0$, the closure of $G(p)$ in M. Then, by the continuity of F, F is a nonzero constant on $\mathrm{Cl}\, G(p)$ so that $F(q) \neq 0$. Thus $W_q \neq 0$ (or $\tilde{W}_q \neq 0$ if $\dim M = 3$) and $q \in N$. Since $G(p)$ is closed in N_0, we have $q \in G(p)$. Thus $G(p)$ is closed in the compact manifold M and hence is a compact submanifold of M and N_0 as well. Since G is a closed subgroup of $I_0(N_0, \bar{g}_0)$, it follows from Proposition 1.5 that G is compact and so we have a contradiction. Thus W (and $\tilde{W}$ if $\dim M = 3$) must vanish identically and M must be conformally flat.

Remark. A Euclidean n-sphere has the essential group of conformorphisms, and so the "if" part of the conjecture is obvious.

PART II

5. General formulas for conformal changes of metric

Let M be a Riemannian n-manifold. With respect to a local coordinate system we use g_{ij}, $\left\{ {h \atop ji} \right\}$, ∇_i, $K_{kji}{}^h$, $K_{ji} = K_{hji}{}^h$, $K = K_{ji}g^{ji}$, and $k = K/[n(n-1)]$, to denote, respectively, the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$, the curvature tensor, the Ricci tensor, the contracted curvature scalar and the scalar curvature of M. Put

$$G_{ji} = K_{ji} - Kg_{ji}/n , \tag{5.1}$$

which measures the deviation of M from being an Einstein space.

Consider a conformal change of metric

$$g^*_{ji} = e^{2\rho} g_{ji} . \tag{5.2}$$

When Ω is a quantity formed with g, we denote by Ω^* the corresponding quantity formed with g^*. For later convenience, we put

$$u = e^{-\rho} , \qquad u_i = \nabla_i u . \tag{5.3}$$

Then the following formulas are known (Yano-Obata [37]):

$$K^* = u^2 K + 2(n-1)u\Delta u - n(n-1)u_i u^i , \tag{5.4}$$

$$G^*_{ji} = G_{ji} + (n-2)P_{ji} , \tag{5.5}$$

where

$$\Delta u = g^{ji}\nabla_j u_i , \tag{5.6}$$

$$P_{ji} = u^{-1}(\nabla_j u_i - \Delta u g_{ji}/n) , \qquad P_j{}^i = P_{jk}g^{ki} . \tag{5.7}$$

From (5.7) we obtain

$$P_{ji}P^{ji} = u^{-2}[\nabla^j u^i \nabla_j u_i - (\Delta u)^2/n] . \tag{5.8}$$

6. Conformal changes of metrics on S^n

On a Riemannian manifold one can consider a conformal change corresponding to an arbitrarily given function ρ in (5.2), However, if there is given a curvature condition for the changed metric, then in general the existence of a conformal change satisfying the condition is not known. We are going to prove the following proposition, which is a clue for the solution of Conjecture II.

Proposition 6.1. *Let* (S^n, g) *be a Euclidean n-sphere of radius* 1, *and* g^* *another Riemannian metric on* S^n *conformal to* g. *Then* g^* *is of constant scalar curvature* 1 *if and only if it is of constant* (*sectional*) *curvature* 1.

Proof. We have $G_{ji} = 0$ and $K = n(n-1)$ on (S^n, g), and $K^* = n(n-1)$ on (S^n, g^*). Therefore from (5.4) and (5.5) it follows that

$$\Delta u = \frac{n}{2}u^{-1}(1 - u^2 + u_i u^i) , \tag{6.1}$$

$$G^*_{ji} = (n-2)P_{ji} . \tag{6.2}$$

By using (6.1) and (6.2) we shall show that P_{ji} and therefore G^*_{ji} vanish identically. To do this, consider a nonnegative quantity

$$A = u^{3-n}P_{ji}P^{ji} = u^{1-n}[\nabla_j u_i \nabla^j u^i - (\Delta u)^2/n] ,$$

and a vector field

$$v^i = u^{2-n}u^j P_j{}^i = u^{1-n}[u^j \nabla_j u^i - (\Delta u)u^i/n] .$$

A straightforward computation then gives

$$\nabla_i v^i = A + B ,$$

where

$$B = (1-n)u^{-n}u^j u^i \nabla_j u_i + \frac{n-1}{n}u^{-n}[uu^j \nabla_j(\Delta u) + u_i u^i \Delta u] + u^{1-n}u^j(\nabla_i \nabla_j u^i - \nabla_j \nabla_i u^i) . \tag{6.3}$$

Substituting (6.1) in the second term and applying the Ricci formula to the last term on the right hand side of (6.3), we get

$$\begin{aligned} &\text{the second term} = (n-1)u^{-n}(u^j u^i \nabla_j u_i - uu_i u^i) , \\ &\text{the last term} \quad = (n-1)u^{1-n}u_i u^i . \end{aligned}$$

Thus $B=0$, and therefore $\nabla_i v^i = A \geq 0$. By the well-known Bochner's lemma we obtain that $A = 0$, so that $P_{ji} = 0$ and therefore

$$G^*_{ji} = 0 , \tag{6.4}$$

which implies that (S^n, g^*) is an Einstein space. Since g^* is conformal to g, (S^n, g^*) is conformally flat. It is known that a conformally flat Einstein space is always a space of constant (sectional) curvature. Hence g^* has constant (sectional) curvature 1.

Remark 1. A little more general argument, similar to this proof, may be seen in Yano-Obata [37, Proposition 3.3].

Remark 2, In the proof of Proposition 6.1, one can see that $G_{ji} = 0$ implies $G^*_{ji} = (n-2)P_{ji} = 0$ under the condition $k = k^* = 1$. Since $P_{ji} = 0$ implies that the manifold under consideration is isometric to a unit sphere (Lichnerowicz [19], Yano-Obata [36]), we obtain

Proposition 6.2. *Let (M, g) be a compact Einstein space with scalar curvature* 1. *If there is a Riemannian metric g^* ($\neq g$) on M such that g^* is conformal to g and g^* has a constant scalar curvature* 1, *then (M, g) and (M, g^*) are isometric to a unit n-sphere.*

Remark 3. (S^n, g^*) can be obtained by a conformorphism of (S^n, g). The proof of this will be given in a forthcoming paper.

7. Cojecture II

Theorem II. *Conjecture II is true.*

Proof. It is known (Kurita [15], Lichnerowicz [18], Obata [23]) that if a compact Riemannian manifold with constant scalar curvature admits a nonisometric conformorphism, then the constant scalar curvature is positive. Therefore without loss of generality we may assume that the Riemannian manifold (M, g) under consideration has constant scalar curvature 1. Since $C_0(M, g)$ is essential, it follows from Theorem I that there exists a conformorphism f of (M, g) onto (S^n, g_0) where g_0 is a standard metric on a unit sphere S^n. Thus $(f^{-1})^*g = g^*$ is a Riemannian metric on S^n and $f\colon (M, g) \to (S^n, g^*)$ is an isometry. Since g^* is conformal to g_0 and g^* has scalar curvature 1, it follows from Proposition 6.1 that (S^n, g^*) is of constant (sectional) curvature. Thus (M, g) is isometric to a Euclidean n-sphere (S^n, g^*).

Remark. It is not difficult to show that the one-parameter subgroup of $C_0(M, g)$ generated by the gradient of a certain function on M is a closed essential subgroup of $C_0(M, g)$.

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