J. DIFFERENTIAL GEOMETRY
2 (1968) 391-392

FOUR BRIEF EXAMPLES CONCERNING POLYNOMIALS ON CERTAIN BANACH SPACES

ROBERT A. BONIC

Let E denote one of the spaces l^p $(1 \leq p < \infty)$ or c_0, and $\{e_1, e_2, \cdots\}$ be the standard basis in E. An element x in E will be written as $x = \Sigma_n x_n e_n$. The following examples are perhaps justified by the fact that their proofs are shorter than their statements.

Example A. *Suppose E is real,* and

$$\phi(t) = 3t^2 - 2t^3 \ (t \text{ real}), \qquad \phi_n(t) = \phi(\alpha_n t)/2^{n-1},$$

where $\alpha_n = 2^{n/4}$. Then the mapping $A(x) = \Sigma_n \phi_n(x_n)$ is a continuous real-valued polynomial of degree 3, *and the image of the critical points contains* $[0, 2]$.

Proof. Any x of the form $x = \Sigma_n \varepsilon_n \alpha_n^{-1} e_n$, where ε_n is 0 or 1, is a critical point of A, and $A(x) = \Sigma_n \varepsilon_n / 2^{n-1}$.

Example A is based on examples of Kupka [2] and Bonic [1], and the remark "bien sur" of Douady [Baton Rouge, April 1967].

Example B. *Suppose E is complex, and*

$$\begin{aligned}
&\psi(z) = az^2 + bz^3 + cz^4 + dz^5\,,\\
&\psi_n(z) = \psi(\beta_n z)/2^{n-1}\,,\\
&\quad 4a = 4i + 4, \qquad 4b = -5i + 5,\\
&\quad 4c = -2i - 2, \qquad 4d = 3i - 3, \qquad \beta_n = 2^{n/6},
\end{aligned}$$

where z is complex. Then the mapping

$$B(x) = \Sigma_n \psi_n(x_{2n-1}) + \Sigma_n \psi_n(x_{2n})$$

is a continuous complex-valued polynomial of degree 5, *and the image of the critical points contains* $[0, 2] \times [0, 2]$.

Proof. Any x of the form

$$x = \Sigma_n \varepsilon_n \beta_n^{-1} e_{2n-1} - \Sigma_n \delta_n \beta_n^{-1} e_{2n}\,,$$

Communicated by R. S. Palais, March 18, 1968. Research supported in part by NSF Grant GP-7026.

where ε_n and δ_n are 0 or 1, is a critical point of B, and $B(x) = \Sigma_n \varepsilon_n / 2^{n-1} + i \Sigma_n \delta_n / 2^{n-1}$.

Example C. *Suppose E is real or complex. Then the mapping $C(x) = \Sigma_n x_n^2 e_n$ is a continuous, but not completely continuous, polynomial of degree 2 from E into E, and each derivative of the polynomial is completely continuous.*

Proof. C is clearly continuous but cannot be completely continuous since $Ce_n = e_n$ for all n. Since $DC(x)h = \Sigma_n 2x_n h_n$ and $x_n \to 0$, we have that $DC(x)$ is completely continuous.

Example C answers a question asked the author by A. Tromba who pointed out that it solves negatively a problem posed in Vainberg [4, p. 51].

Example D. *Suppose E is real c_0. Then the mapping $D(x) = \Sigma_n (x_n + x_n^3) e_n$ is a continuous polynomial of degree three from c_0 into c_0, and is a proper map (the inverse image of a compact set is compact). Moreover, D has the form $D = I + D_0$, where D_0 is not completely continuous, but each derivative of D_0 is completely continuous.*

Proof. Letting ϕ denote the inverse of the mapping $\phi(t) = t + t^3$ we have that $D^{-1}(x) = \Sigma_n \phi(x_n) e_n$ is a continuous map of c_0 into c_0 and hence that D is proper. The facts about D_0 follow exactly as in Example C.

Example D is of some interest in degree theory since the usual Leray-Schauder degree is given for maps of the form $I + F$, where F is a completely continuous mapping. In [3] Tromba develops a degree theory for smooth proper maps of the form $I + G$ only assuming that each $DG(x)$ is completely continuous. Therefore this example gives an instance, where the latter but not the former degree is defined.

Bibliography

[1] R. A. Bonic, *A note on Sard's theorem in Banach spaces,* Proc. Amer. Math. Soc. **17** (1966) 1218.

[2] I. Kupka, *Counterexample to the Morse-Sard theorem in the case of infinite dimensions,* Proc. Amer. Math. Soc. **16** (1965) 954-957.

[3] A. J. Tromba, *Degree theory on Banach manifolds,* Princeton University Thesis, 1968.

[4] M. M. Vainberg, *Variational methods for the study of nonlinear operators,* Holden-Day, San Francisco, 1964.

Northeastern University