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PLANE CURVES WITH MINIMAL DISCRIMINANT

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ABSTRACT. We give lower bounds for the degree of the discriminant with respect to y of squarefree polynomials $f \in \mathbb{K}[x,y]$ over an algebraically closed field of characteristic zero. Depending on the invariants involved in the lower bound, we give a geometrical characterization of those polynomials having minimal discriminant, and we give an explicit construction of all such polynomials in many cases. In particular, we show that irreducible monic polynomials with minimal discriminant coincide with coordinate polynomials. We obtain analogous partial results for the case of nonmonic or reducible polynomials by studying their $GL_2(\mathbb{K}[x])$-orbit and by establishing some combinatorial constraints on their Newton polygon. Our results suggest some natural extensions of the embedding line theorem of Abhyankar-Moh and of the Nagata-Coolidge problem to the case of unicuspidal curves of $\mathbb{P}^1 \times \mathbb{P}^1$.

1. Introduction.

Let $f \in \mathbb{K}[x,y]$ be a bivariate polynomial defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. We denote by d_x and d_y the respective partial degrees of f with respect to x and y. The *discriminant* $\Delta_y(f)$ of f with respect to y is the polynomial

$$\Delta_y(f) := \frac{(-1)^{d_y(d_y-1)/2}}{\mathrm{lc}_y(f)} \operatorname{Res}_y(f, \partial_y f) \in \mathbb{K}[x],$$

where $\partial_y f$ and $\mathrm{lc}_y(f)$, respectively, stand for the partial derivative and the leading coefficient of f with respect to y, and Res_y stands for the resultant with respect to y. In this note, we study polynomials with discriminants of low degrees. More precisely, we focus on the following problem:

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Problem 1.1. Give a lower bound for the degree of the discriminant in terms of some invariants attached to f and construct all polynomials whose discriminant reaches this lower bound.

Throughout the paper, we assume that f is *primitive* (with respect to y), that is, f has no factor in $\mathbb{K}[x]$. This hypothesis is not restrictive for our purpose, due to the well-known formula $\Delta_y(uf) = u^{2d_y-2}\Delta_y(f)$ when $u \in \mathbb{K}[x]$. We also assume that f is squarefree with respect to y in order to avoid zero discriminants.

1.1. The case of monic polynomials. We recall that a polynomial $f \in \mathbb{K}[x, y]$ is *monic* if its leading coefficient with respect to y is constant.

Theorem 1.2. *Let $f \in \mathbb{K}[x, y]$ be a primitive squarefree polynomial with r irreducible factors. Then*

$$\deg_x \Delta_y(f) \geq d_y - r.$$

If, moreover, f is monic, then the equality holds if and only if there exists a polynomial automorphism $\sigma = (\sigma_x, \sigma_y) \in \mathrm{Aut}(\mathbb{A}^2)$ and a degree r polynomial $g \in \mathbb{K}[y]$ such that $f = g \circ \sigma_y$.

Since the group $\mathrm{Aut}(\mathbb{A}^2)$ of automorphisms of $\mathbb{A}^2$ is generated by affine and elementary automorphisms, due to Jung's theorem **[11]**, Theorem 1.2 gives a solution of Problem 1.1 for monic polynomials in terms of d_y and r. Moreover, given f monic for which the equality holds, we can compute the automorphism σ recursively from the Newton polygon of any irreducible factor of f (remark after Proposition 4.12).

Theorem 1.2 implies, in particular, that, if f is monic and satisfies $\deg_x \Delta_y(f) = d_y - r$, then r divides d_y. Hence, either its discriminant is constant, or it satisfies the inequality

$$\deg_x \Delta_y(f) \geq \left\lceil \frac{d_y - 1}{2} \right\rceil.$$

It turns out that this fact still holds for nonmonic polynomials, and we have, moreover, a complete classification of polynomials for which equality holds, solving Problem 1.1 in terms of the degree d_y. The precise result requires some more notation and will be stated later in this introduction (Theorem 1.7).

Due to the multiplicative property of the discriminant,

$$\Delta_y(fg) = \pm\Delta_y(f)\Delta_y(g)\,\mathrm{Res}(f,g)^2$$

the inequality in Theorem 1.2 is equivalent to the fact that any *irreducible* polynomial satisfies the inequality

$$\deg_x \Delta_y(f) \geq d_y - 1.$$

A similar lower bound for irreducible polynomials appears in [**5,** Proposition 1], under the additional assumption that $\deg f = d_y$. The second part of Theorem 1.2 for $r = 1$ must be compared with [**10,** Theorem 4], where the authors showed that *if* d_y *coincides with the total degree of* f, then f is a coordinate of $\mathbb{C}^2$ if and only if f is a Jacobian polynomial such that $\deg_x \Delta_y(f) = d_y - 1$. Our result allows the replacement of the Jacobian hypothesis by irreducibility. Note, further, that being monic is a weaker condition than $\deg f = d_y$.

1.2. Bounds with respect to the genus. If, now, we take into account the genus g and the degree d_y of f, we can refine the lower bound $d_y - 1$ for irreducible polynomials:

Theorem 1.3. *Let* $f \in \mathbb{K}[x,y]$ *be a primitive irreducible polynomial. Then*

$$2g + d_y - 1 \leq \deg_x \Delta_y(f) \leq 2d_x(d_y - 1),$$

where g *stands for the geometric genus of the algebraic curve defined by* f*. Moreover, the equality*

$$\deg_x \Delta_y(f) = 2g + d_y - 1$$

holds if and only if the Zariski closure $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ *of the affine curve* $f = 0$ *is a genus* g *curve with a unique place supported on the line* $x = \infty$ *and smooth outside this place.*

Theorem 1.2 is mainly a consequence of Theorem 1.3, combined with the embedding line theorem of Abhyankar and Moh [**2**] that asserts that every embedding of the line in the affine plane $\mathbb{A}^2$ extends to a polynomial automorphism of the plane. In particular, the remarkable fact that a monic irreducible polynomial with minimal discriminant with respect to y is also monic with minimal discriminant with respect to x (Theorem 3.3) becomes apparent.

Remark 1.4. Our results are specific to fields of characteristic zero. For instance, if $\mathbb{K}$ has characteristic p, the polynomial $f(x,y) = y^p + y^k + x$ is irreducible and satisfies

$$\deg_x \Delta_y(f) = k - 1, \quad \text{for all } 1 \leq k < p.$$

Hence, there is no nontrivial lower bound for the degree of the discriminant if we do not take some care on the degree.

1.3. G-reduction of (nonmonic) minimal polynomials.

We say that $f \in \mathbb{K}[x,y]$ is *minimal* if it is irreducible and if its discriminant reaches the lower bound

$$\deg_x \Delta_y(f) = d_y - 1.$$

Theorem 1.2 characterizes monic minimal polynomials: they coincide with *coordinate* polynomials, that is, polynomials that form part of a Hilbert basis of the $\mathbb{K}$-algebra $\mathbb{K}[x,y]$. In the nonmonic case, the characterization of minimal polynomials is more complicated. Indeed, the second part of Theorem 1.2 is false in general since $\mathrm{Aut}(\mathbb{A}^2)$ does not preserve minimality of nonmonic polynomials. An idea is to introduce other group actions in order to reduce minimal polynomials to a "canonical form." Since the discriminant of f coincides with the discriminant of its homogenization F with respect to y, we may try to apply a reduction process to F. The multiplicative group $G := GL_2(\mathbb{K}[x])$ acts on the space $\mathbb{K}[x][Y]$ of homogeneous forms in $Y = (Y_0 : Y_1)$ with coefficients in $\mathbb{K}[x]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (F) = F(aY_0 + bY_1, cY_0 + dY_1). \tag{1.1}$$

The partial degree d_Y of F, the number r of irreducible factors and the degree of the discriminant are G-invariant (see Section 2). Group G is, thus, a good candidate for reducing nonmonic polynomials with small discriminant to a simpler form, in the same vein as in Theorem 1.2.

We say that $F, H \in \mathbb{K}[x][Y]$ are G-equivalent, denoted by $F \equiv H$, if there exists a $\sigma \in G$ such that $F = \sigma(H)$. The action (1.1) induces by dehomogenization a well-defined action on the set of *irreducible* polynomials in $\mathbb{K}[x,y]$ with $d_y > 1$, and, more generally, on the set of polynomials *with no linear factors* in y. In particular, we can talk about G-equivalence of (affine) minimal polynomials of degree $d_y > 1$.

The G-orbit of a monic minimal polynomial contains many nonmonic minimal polynomials, and it is natural to ask whether all nonmonic minimal polynomials arise in such a way. We prove that the answer is 'no,' in general, due to the following counterexample.

Theorem 1.5. *Let $\lambda \in \mathbb{K}^*$. The polynomial $f = x(x-y^2)^2 - 2\lambda y(x-y^2) + \lambda^2$ is minimal but is not G-equivalent to a monic polynomial.*

This result will follow as a corollary of the G-reduction Theorem 4.3, which shows, in particular, that, if the degree c of the leading coefficient of a minimal polynomial is not the smallest in the G-orbit, then d_y necessarily divides $d_x - c$. The proof is in the spirit of Wightwick's results **[18]** about orbits of $\mathrm{Aut}(\mathbb{C}^2)$. Although we can guess that this example is not unique, we were not able to find a single other such example despite a long computer search (see subsection B). Indeed, it turns out that being simultaneously minimal and G-reduced still imposes divisibility restrictions on the partial degrees. In particular, we can show that all minimal polynomials of prime degree d_y are G-equivalent to a monic polynomial, solving Problem 1.1 in that context. More precisely:

Theorem 1.6. *Let f be a minimal polynomial of prime degree d_y. Then, there exists a $g \in \mathbb{K}[y]$ of degree d_y such that*

$$f(x,y) \equiv g(y) + x.$$

In particular, f is G-equivalent to a monic polynomial, hence, to a coordinate polynomial.

Theorem 1.6 follows from the fact that minimality implies that either d_y divides $d_x - c$ or $d_x - c$ and d_y are not coprime except for some trivial cases (Theorem 4.14). The proof relies on a suitable toric embedding of the curve of f. It is natural to ask whether minimality implies the stronger fact that either d_y divides $d_x - c$ or $d_x - c$ divides d_y. This property holds for $c = 0$, a statement equivalent to the Abhyankar-Moh theorem [**1**]. In general, we need to study the singularity of smooth rational curves of $\mathbb{A}^1 \times \mathbb{P}^1$ with a unique place along $\infty \times \mathbb{P}^1$, generalizing the Abhyankar-Moh situation of smooth rational curves of $\mathbb{A}^2$ with a unique place at the infinity of $\mathbb{P}^2$.

1.4. Cremona equivalence of minimal polynomials. In a close context, we can pay attention to the Cremona reduction of minimal polynomials. Theorem 1.5 shows that it is hopeless to reduce a nonmonic minimal polynomial to a coordinate by successively applying $GL_2(\mathbb{K}[x])$ and $\mathrm{Aut}(\mathbb{A}^2)$. However, both groups can be considered as subgroups of the Cremona group $\mathrm{Bir}(\mathbb{A}^2)$ of birational transformations of the plane, and our results suggest asking whether all minimal polynomials define curves that are Cremona equivalent to a line. We will prove, for instance, that the nonmonic minimal polynomial in Theorem 1.5 satisfies this property (Proposition 4.11). This open problem can be seen as a generalization of the Coolidge-Nagata problem [**13**] to unicuspidal curves of $\mathbb{P}^1 \times \mathbb{P}^1$.

1.5. A uniform lower bound for reducible polynomials. Our last result gives a uniform sharp lower bound for the degree of the discriminant of any squarefree (reducible) polynomial that depends only on d_y. Moreover, it establishes a complete classification of polynomials that reach this lower bound. We need to express this classification in homogeneous coordinates, and we let $F \in \mathbb{K}[x][Y]$ stand for the homogeneous form associated to f of degree $\deg_Y F = d_y$.

Theorem 1.7. *Let $f \in \mathbb{K}[x,y]$ be a primitive and squarefree polynomial. Then, f has constant discriminant if and only if $F \equiv H$ for some $H \in \mathbb{K}[Y]$. Otherwise, we have the inequality*

$$\deg_x \Delta_y(f) \geq \left\lceil \frac{d_y - 1}{2} \right\rceil,$$

and the equality holds if and only if one of the following conditions holds:

(i) *$d_y = 4$ and $F \equiv Y_0 Y_1 (Y_0^2 + (\mu x + \lambda) Y_0 Y_1 + Y_1^2)$, with $\mu, \lambda \in \mathbb{K}$, $\mu \neq 0$.*
(ii) *$d_y = 4$ and $F \equiv Y_1 (H(Y) + x Y_1^3)$, for some cubic form $H \in \mathbb{K}[Y]$.*
(iii) *d_y is odd and $F \equiv Y_1 H(Y_0^2 + x Y_1^2, Y_1^2)$ for some form $H \in \mathbb{K}[Y]$.*
(iv) *d_y is even and $F \equiv H(Y_0^2 + x Y_1^2, Y_1^2)$ for some form $H \in \mathbb{K}[Y]$.*

1.6. Organization of the paper. We prove Theorem 1.3 in Section 2. The proof is based upon the classical relations between the valuation of the discriminant and the Milnor numbers of the curve along the corresponding critical fiber. We prove Theorem 1.2 in Section 3, the main ingredients of the proof being Theorem 1.3 combined with the embedding line theorem of Abhyankar and Moh. In particular, we show that, for a monic polynomial, minimality with respect to y is equivalent to minimality with respect to x (Theorem 3.3). In Section 4, we focus on the $GL_2(\mathbb{K}[x])$-orbits of nonmonic minimal polynomials. We first characterize minimal polynomials that minimize the area of the Newton polygon in their orbit (subsection 4.1, Theorem 4.3). The counterexample of Theorem 1.5 follows as a corollary. Although this example is not G-equivalent to a coordinate, we show in subsection 4.2 that it defines a curve Cremona equivalent to a line, and we address the question of whether this property holds for all minimal polynomials. In a close context, we show in subsection 4.3 that the partial degrees of minimal polynomials obey to some strong divisibility constraints (Theorem 4.14). Theorem 1.6 follows as a corollary. At last, we prove Theorem 1.7 in Section 5. The paper concludes with two appendices on related problems. In Appendix A, we study the relations between small discriminants with respect to x and small discriminants with respect to y, extending Theorem 3.3 to the nonmonic case. In Appendix B, we give a parametric characterization of minimal polynomials and apply our result to the computer algebra challenge of computing nonmonic minimal polynomials.

2. Bounds for the degree of the discriminant. Proof of Theorem 1.3. The upper bound in Theorem 1.3 for the degree of the discriminant follows from classical results regarding the partial degrees of discriminants of homogeneous forms with indeterminate coefficients. The lower bound follows by studying the relations between the vanishing order of the discriminant at infinity and the singularities of the curve of f.

We recall that in all of the sequel, f is assumed to be primitive, hence with no factors in $\mathbb{K}[x]$. This assumption is not restrictive for our purpose, due to the well-known formula

$$\Delta_y(uf) = u^{2d_y-2}\Delta_y(f)$$

when $u \in \mathbb{K}[x]$.

2.1. Bihomogenization. Let $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ be two pairs of variables, and let $F \in \mathbb{K}[X, Y]$ be the bihomogenized polynomial of f

$$F := X_1^{d_x} Y_1^{d_y} f\left(\frac{X_0}{X_1}, \frac{Y_0}{Y_1}\right).$$

It is homogeneous of degree $d_X = d_x$ in X and of degree $d_Y = d_y$ in Y. We define the *discriminant* of F with respect to Y by

$$\Delta_Y(F) := \frac{(-1)^{d_Y(d_Y-1)/2}}{d_Y^{d_Y-2}} \operatorname{Res}_Y(\partial_{Y_0} F, \partial_{Y_1} F).$$

In the literature, several normalizations exist concerning the sign of this discriminant; however, these considerations have no impact here, where we are only interested in its degree. The present normalization satisfies the relation

$$\Delta_Y(F)(x, 1) = \Delta_y(f)(x).$$

The polynomial $\Delta_Y(F)$ is homogeneous of degree $2d_y - 2$ in the coefficients of F, which vanishes if and only if F is not squarefree with respect to Y. In our situation, it follows that $\Delta_Y(F)$ is a homogeneous polynomial in X of total degree

$$\deg_X \Delta_Y(F) = 2d_X(d_Y - 1) = 2d_x(d_y - 1).$$

We obtain the following relation

$$\deg_x \Delta_y(f) = \deg_X \Delta_Y(F) - \operatorname{ord}_\infty \Delta_Y(F),$$

where $\operatorname{ord}_\infty$ stands for the vanishing order at $\infty := (1 : 0) \in \mathbb{P}^1$. The upper bound in Theorem 1.3 follows. In order to get the lower bound, an upper bound is necessary for $\operatorname{ord}_\infty \Delta_Y(F)$. Let

$$C \subset \mathbb{P}^1 \times \mathbb{P}^1$$

be the curve $F = 0$. It coincides by construction with the Zariski closure of the affine curve $f = 0$ in the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$. For a point $\alpha \in \mathbb{P}^1$, we denote by

$$Z_\alpha := C \cap (X = \alpha)$$

the set theoretical intersection of C with the "vertical line" $X = \alpha$. It is zero-dimensional since, otherwise, F would have a linear factor in X, contradicting the primitivity assumption on f. Moreover, we have

$$\mathrm{Card}(Z_\alpha) \leq d_Y,$$

with strict inequality if and only if $\Delta_Y(F)(\alpha) = 0$, that is, if and only if $F(\alpha, Y)$ is not squarefree. In order to understand the order of vanishing of $\Delta_Y(F)$ at α, we need to introduce some classical local invariants of the curve C.

2.2. The ramification number. Let $p \in C$. A *branch* of C at p is an irreducible analytic component of the germ of curve (C, p).

Definition 2.1. The *ramification number* of C over $\alpha \in \mathbb{P}^1$ is defined as

$$r_\alpha := d_y - \sum_{p \in Z_\alpha} n_p,$$

where n_p stands for the number of branches of C at p.

In other words, the ramification number measures the defect to the expected number d_Y of branches of C along the vertical line $X = \alpha$. It is also equal to the sum $\sum(e_\beta - 1)$ over all places β of C over α, where e_β stands for the ramification index of β.

2.3. The delta invariant. Let B be a branch. The local ring $\mathcal{O}_B$ has finite index in its integral closure $\overline{\mathcal{O}}_B$. The quotient ring is a finite-dimensional vector space over $\mathbb{K}$, whose dimension

$$\delta(B) := \dim_{\mathbb{K}} \overline{\mathcal{O}}_B / \mathcal{O}_B$$

is called the *delta invariant* of B. More generally, we define the *delta invariant* of C at p as the nonnegative integer

$$\delta_p(C) := \sum_i \delta_p(B_i) + \sum_{i<j} (B_i \cdot B_j)_p,$$

where the B_i's run over the branches of C at p, and where $(B_i \cdot B_j)_p$ stands for the intersection multiplicity at p of the curves B_i and B_j. In some sense, the delta invariant $\delta_p(C)$ measures the complexity of the singularity of C at p. In particular, we have $\delta_p(C) = 0$ if and only if C is smooth at p.

Definition 2.2. The *delta invariant of C over $\alpha \in \mathbb{P}^1$* is

$$\delta_\alpha := \sum_{p \in Z_\alpha} \delta_p(C).$$

The integer δ_α thus measures the complexity of all singularities of C that lie over α.

2.4. PSL_2-invariance of the discriminant. The multiplicative group $GL_2(\mathbb{K}[x])$ of 2×2 invertible matrices with coefficients in $\mathbb{K}[x]$ acts naturally on the space $\mathbb{K}[x][Y]$ of homogeneous forms in $Y = (Y_0 : Y_1)$ with coefficients in $\mathbb{K}[x]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (F) = F(aY_0 + bY_1, cY_0 + dY_1). \tag{2.1}$$

This action preserves the degree in Y and, for $\tau \in GL_2(\mathbb{K}[x])$, we have

$$\Delta_Y(\tau(F)) = \det(\tau)^{d_Y(d_Y-1)} \Delta_Y(F), \tag{2.2}$$

so that the discriminant is $PSL_2(\mathbb{K}[x])$-invariant and the degree of the discriminant is $GL_2(\mathbb{K}[x])$-invariant. This action also preserves the irreducibility. It induces by dehomogenization a well-defined action on the set of *irreducible* polynomials in $\mathbb{K}[x, y]$ with $d_y > 1$, and, more generally, on the set of polynomials *with no linear factors* in y. The corresponding formula is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (f) = (cy + d)^{d_y} f\left(x, \frac{ay + b}{cy + d}\right). \tag{2.3}$$

We study in more detail the action of $GL_2(\mathbb{K}[x])$ in subsection 4.1.

2.5. Vanishing order of the discriminant. For $\alpha = (\alpha_0 : \alpha_1) \in \mathbb{P}^1$ and $H \in \mathbb{K}[X_0 : X_1]$ a homogeneous form, the vanishing order $\operatorname{ord}_\alpha H$ of H at α is the highest power of $\alpha_0 X_1 - \alpha_1 X_0$ that divides H. The vanishing order at $\alpha \neq \infty$ coincides with the usual valuation of the dehomogenization of H at $x - \alpha$. The vanishing order of the discriminant is related to the ramification degree and the delta invariant, due to the following key proposition:

Proposition 2.3. *Let $\alpha \in \mathbb{P}^1$, and let $F \in \mathbb{K}[X, Y]$ be a bihomogeneous form with no factors in $\mathbb{K}[X]$. We have the equality*

$$\operatorname{ord}_\alpha \Delta_Y(F) = r_\alpha + 2\delta_\alpha.$$

In particular, we have

$$\deg_x \Delta_y(f) = 2d_x(d_y - 1) - 2\delta_\infty - r_\infty.$$

Proof. Up to a change of coordinates of $\mathbb{P}^1$ and, without loss of generality, we can assume that $\alpha = (0 : 1)$, and we will simply write ord_0 for $\operatorname{ord}_{(0:1)}$. Note first that $\operatorname{ord}_0 \Delta_Y(F) = \operatorname{ord}_0 \Delta_y(f)$. Since $\mathbb{K}$ has infinite cardinality, there exists a $\beta \in \mathbb{K}$ such that $f(0, \beta) \neq 0$. For such a β, the leading coefficient with respect to y of the transformed polynomial $y^{d_y} f(x, \beta + 1/y)$ is a unit modulo x. Since by (2.2) the discriminant is invariant under $PSL_2(\mathbb{K})$, we can, thus, assume that the leading coefficient of f with respect to y is a unit modulo x, meaning that the point $(0, \infty)$ does not belong to C. In such a case, Hensel's lemma ensures that we have a unique factorization

$$f = u \prod_{p \in Z_0} f_p \in \mathbb{K}[[x]][y]$$

where $u \in \mathbb{K}[[x]]$ is a unit and where $f_p \in \mathbb{K}[[x]][y]$ is a monic polynomial giving the equation of the germ of curve (C, p). Note that f_p is not necessarily irreducible. By the well-known multiplicative relations between discriminants and resultants, we have

$$\Delta_y(f) = \pm u^{2d_y - 2} \prod_{p \in Z_0} \Delta_{f_p} \prod_{p \neq q} \operatorname{Res}_y(f_p, f_q)^2$$

where Res_y stands for the resultant with respect to y. The roots of $f_p(0, y)$ and $f_q(0, y)$ are distinct by assumption; thus, the resultant $\operatorname{Res}(f_p, f_q)$ is a unit in $\mathbb{K}[[x]]$. Hence,

$$\operatorname{ord}_0 \Delta_y(f) = \sum_{p \in Z_0} \operatorname{ord}_0 \Delta_y(f_p).$$

The polynomial f_p is monic with respect to $y - y_p$ and $f(0, y) = (y - y_p)^{d_p}$, where d_p stands for the degree in y of f_p. Such a polynomial is said to be distinguished with respect to $y - y_p$ (or a Weierstrass polynomial) and has the property that

$$\operatorname{ord}_0 \Delta_y(f_p) = (C \cdot C_y)_p,$$

where C_y stands for the polar curve $\partial_y f = 0$. Now, by Teissier's lemma [**16,** Chapter II, Proposition 1.2], we have

$$(C \cdot C_y)_p = \mu_p(C) + d_p - 1,$$

where μ_p stands for the *Milnor number* of C at p, that is,

$$\mu_p(C) := (C_x \cdot C_y)_p,$$

with C_x the polar curve $\partial_x f = 0$. The Milnor number and the delta invariant of a germ of curve are related by the Milnor-Jung formula [**17,** Theorem 6.5.9]

$$\mu_p(C) = 2\delta_p(C) - n_p(C) + 1,$$

where $n_p(C)$ stands for the number of branches of C at p. Finally, we obtain:

$$\operatorname{ord}_0 \Delta_y(f) = \sum_{p\in Z_0} (2\delta_p(C) - n_p(C) + d_p)$$

Proposition 2.3 then follows from equality $\sum_p d_p = d_y$. □

2.6. Adjunction formula. Let C be an irreducible algebraic curve on a smooth complete algebraic surface. We denote by $p_a(C) = \dim H^0(C, \Omega_C)$ the arithmetic genus of C and by $g(C) = \dim H^0(\widetilde{C}, \Omega_{\widetilde{C}})$ its geometric genus, where Ω_C and $\Omega_{\widetilde{C}}$ stand, respectively, for the canonical sheaves of C and of its normalization $\widetilde{C}$. The *adjunction formula* measures the difference between both integers, namely,

$$p_a(C) = g(C) + \sum_{p\in \mathrm{Sing}(C)} \delta_p(C), \tag{2.4}$$

see, for instance, [**3,** subsection 2.11]. This formula generalizes the famous Plücker formula that computes the geometric genus of a projective plane curve with ordinary singularities. We deduce the following bound for the valuation of the discriminant:

Proposition 2.4. *Let $\alpha \in \mathbb{P}^1$, and let $F \in \mathbb{K}[X, Y]$ be an irreducible bihomogeneous polynomial of partial degree $d_Y > 0$. Let g be the geometric genus of the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ defined by $F = 0$. We have the inequality*

$$\operatorname{ord}_\alpha \Delta_Y(F) \le (2d_X - 1)(d_Y - 1) - 2g.$$

Moreover, equality holds if and only if C has a unique place on the line $X = \alpha$ and is smooth outside this place.

Proof. Since C has at least one branch along the line $X = \alpha$, the ramification number r_α is bounded above by $d_y - 1$. Hence, Proposition 2.3 implies that

$$\operatorname{ord}_\alpha \Delta_Y(F) \leq 2 \sum_{p \in \operatorname{Sing}(C)} \delta_p(C) + d_y - 1.$$

It is well known that a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$, defined by a bihomogeneous polynomial of bidegree (d_x, d_y), has arithmetic genus

$$p_a(C) = (d_x - 1)(d_y - 1)$$

(see for instance **[8]**, subsection 4.4). The upper bound of Proposition 2.4 then follows from the adjunction formula (2.4). Equality holds in Proposition 2.4 if and only if both invariants δ_α and r_α are maximal once the genus is fixed. This is equivalent to the equalities

$$\delta_\alpha = \sum_{p \in \operatorname{Sing}(C)} \delta_p(C) \quad \text{and} \quad r_\alpha = d_y - 1.$$

The first equality is equivalent to $\delta_\beta = 0$ for all $\beta \neq \alpha$, meaning geometrically that C is smooth outside the line $X = \alpha$. The second equality is equivalent to the fact that C has a unique branch along this line. $\square$

Proof of Theorem 1.3. Theorem 1.3 follows by combining the equality

$$\deg_x \Delta_y(f) = \deg_X \Delta_Y(F) - \operatorname{ord}_\infty \Delta_Y(F)$$

with the inequality of Proposition 2.4. $\square$

Corollary 2.5. *Let $f \in \mathbb{K}[x, y]$ be an irreducible polynomial of partial degree $d_y > 0$. Then*

$$\deg_x \Delta_y(f) \geq d_y - 1,$$

and equality holds if and only if the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is rational, with a unique place over the line $x = \infty$, and smooth outside this place.

2.7. Almost minimal discriminants. Due to parity, we can also give a geometrical characterization of polynomials with an "almost minimal" discriminant, that is, for which equality $\deg_x \Delta(f) = d_y$ holds.

Corollary 2.6. *Let $f \in \mathbb{K}[x, y]$ be irreducible. Then, equality*

$$\deg_x \Delta_y(f) = d_y$$

holds if and only if the closed curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ defined by f is rational, with two places over the line $x = \infty$ and smooth outside these places.

Proof. From Proposition 2.3, we have $\deg_x \Delta_y(f) = d_y$ if and only if

$$d_y = 2d_x(d_y - 1) - 2\delta_\infty - r_\infty.$$

Since $\delta_\infty \leq (d_x - 1)(d_y - 1)$ by the adjunction formula, it follows that $r_\infty \geq d_y - 2$. But we have $r_\infty \leq d_y - 1$ and equality cannot hold for the reason of parity. Hence, the only solution is $r_\infty = d_y - 2$ and $\delta_\infty = (d_x - 1)(d_y - 1)$. This means exactly that C is rational with two places over the line $x = \infty$ and smooth outside of these two places. □

3. Classification of minimal monic polynomials. Proof of Theorem 1.2.

Definition 3.1. We say that $f \in \mathbb{K}[x, y]$ is *minimal* (with respect to y) if it is irreducible and satisfies the equality $\deg_x \Delta_y(f) = d_y - 1$.

Definition 3.2. We say that $f \in \mathbb{K}[x, y]$ is *monic* with respect to y (respectively, to x) if its leading coefficient with respect to y (respectively, to x) is constant. Caution must be used since in the literature this terminology often refers to polynomials with a leading coefficient equal to 1.

3.1. Characterization of monic minimal polynomial.

Theorem 3.3. *Let $f \in \mathbb{K}[x, y]$ be a nonconstant irreducible bivariate polynomial. The following assertions are equivalent*:

(a) $d_y = 0$, *or* $\deg_x \Delta_y(f) = d_y - 1$, *and* f *is monic with respect to* y.
(b) $d_x = 0$, *or* $\deg_y \Delta_x(f) = d_x - 1$, *and* f *is monic with respect to* x.
(c) *The affine curve* $f = 0$ *is smooth rational, and has a unique place at infinity of* $\mathbb{P}^2$.
(d) *There exists* $\sigma \in \mathrm{Aut}(\mathbb{A}^2)$ *such that* $f \circ \sigma = y$.

Due to Jung's theorem **[11]**, we have an explicit description of the group $\mathrm{Aut}(\mathbb{A}^2)$ of polynomial automorphisms of the plane, namely, it is generated by the transformations $(x, y) \to (y, x)$ and $(x, y) \to (x, \lambda y + p(x))$ with $\lambda \in \mathbb{K}^*$ and $p \in \mathbb{K}[x]$. Hence, Theorem 3.3 gives a complete and explicit description of all minimal monic polynomials. Note the remarkable fact that, for monic polynomials, minimality with respect to y is equivalent to minimality with respect to x. This symmetry can be extended to nonmonic polynomials by taking into account the number of roots of the leading coefficients, see Appendix A.

Proof.

$(a) \Rightarrow (b)$. If $d_x = 0$, the assertion is trivial. If $d_y = 0$, then, by the irreducibility assumption, we have $f = ax + b$ for some $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$ so that (b) also holds trivially. Suppose now that $d_y > 0$ and $d_x > 0$. By Theorem 1.3, the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ defined by f has a unique place p on the line $x = \infty$ and is smooth outside of this place. Since f must be monic with respect to y and $d_x > 0$, the curve C intersects the line $y = \infty$ at the unique point (∞, ∞). This forces equality $p = (\infty, \infty)$. Hence, C is rational with a unique place over the line $y = \infty$ and smooth outside of this line. Thus, f has a minimal discriminant with respect to x by Theorem 1.3. Since C has a unique place on the divisor at infinity

$$B := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{A}^2,$$

standard arguments (see Lemma 4.2) ensure that the Newton polygon of f has an edge that connects the points $(d_x, 0)$ and $(0, d_y)$. In particular, f is necessarily monic with respect to x.

$(b) \Rightarrow (a)$. Follows by the symmetric roles played by the variables x and y.

$(a) \Leftrightarrow (c)$. If $d_y = 0$ or $d_x = 0$, then the result is trivial. Suppose now that d_x and d_y are positive. We just saw that this is equivalent to the fact that C is rational, with (∞, ∞) as a unique place on the divisor at infinity $B := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{A}^2$ and smooth outside of this place. The result then follows from the fact that the number of places at the infinity of $\mathbb{P}^2$ is equal to the number of places on the boundary B of $\mathbb{P}^1 \times \mathbb{P}^1$.

$(c) \Leftrightarrow (d)$. This is an immediate consequence of the embedding line theorem **[2]** (also, see **[15]**). $\square$

3.2. The monic reducible case. Proof of Theorem 1.2.

Proposition 3.4. *Let $g, h \in \mathbb{K}[x, y]$ be two monic minimal polynomials. Then*

$$\mathrm{Res}_y(g, h) \in \mathbb{K}^* \iff h = \mu g + \lambda$$

for some nonzero constants $\mu, \lambda \in \mathbb{K}^*$.

Proof. We have $\mathrm{Res}_y(g, h) \in \mathbb{K}^*$ if and only if the curves $C_1, C_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ respectively defined by g and h do not intersect in the open set $\mathbb{A}^1 \times \mathbb{P}^1$. Let $\sigma \in \mathrm{Aut}(\mathbb{A}^2)$, and let $\widetilde{g} = g \circ \sigma$ and $\widetilde{h} = h \circ \sigma$. Assume that $\deg_y \widetilde{g} > 0$. Since g is assumed to be monic minimal, so is $\widetilde{g}$ by Theorem 3.3. It follows that the respective curves $\widetilde{C}_1$ and $\widetilde{C}_2$ of $\widetilde{g}$ and $\widetilde{h}$ do not intersect in $\mathbb{A}^1 \times \{\infty\}$. Since C_1 and C_2 do not intersect in $\mathbb{A}^2$, by assumption, the curves $\widetilde{C}_1$ and $\widetilde{C}_2$ cannot intersect in $\mathbb{A}^2$ since σ is an automorphism of the plane. Hence, $\widetilde{C}_1$ and $\widetilde{C}_2$ do not intersect in $\mathbb{A}^1 \times \mathbb{P}^1$, that is,

$$\mathrm{Res}_y(\widetilde{g}, \widetilde{h}) \in \mathbb{K}^*. \tag{3.1}$$

By Theorem 3.3, there exists a $\sigma \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\widetilde{g} = y$. Combined with (3.1), this implies that $\widetilde{h}(x, 0) \in \mathbb{K}^*$. Since $\widetilde{h}$ is a coordinate polynomial by Theorem 3.3, it is irreducible, as well as $\widetilde{h}(x, y) - \widetilde{h}(x, 0)$, forcing the equality $\deg_y \widetilde{h} = 1$. Since h is monic, so is $\widetilde{h}$, and the condition $\widetilde{h}(x, 0) \in \mathbb{K}^*$ implies that

$$\widetilde{h} = \mu y + \lambda = \mu \widetilde{g} + \lambda$$

for some constant $\mu, \lambda \in \mathbb{K}^*$. The result follows by applying σ^{-1}. $\square$

Proof of Theorem 1.2. Let f be a monic squarefree polynomial with r irreducible factors $f_1, \ldots, f_r$ of respective degrees $d_1, \ldots, d_r$. Corollary 2.5 combined with the multiplicative properties of the discriminant gives the inequality

$$\begin{aligned}\deg_y(\Delta_y(f)) &= \sum_{i=1}^{r} \deg_y(\Delta_y(f_i)) + \sum_{i\neq j}^{r} \deg_y(\mathrm{Res}_y(f_i, f_j)) \\ &\geq \sum_{i=1}^{r}(d_i - 1) \geq d_y - r.\end{aligned}$$

Moreover, equality holds if and only if all factors f_i are minimal and satisfy $\mathrm{Res}_y(f_i, f_j) \in \mathbb{K}^*$ for all $i \neq j$. If f is monic, all of its factors are also monic. We conclude, due to Theorem 3.3 and Proposition 3.4, that there exists an automorphism $\sigma \in \mathrm{Aut}(\mathbb{K}^2)$ such that $f \circ \sigma$ is a degree r univariate polynomial. Note that r automatically divides d_y. □

4. $GL_2(\mathbb{K}[x])$-orbits of minimal polynomials. We saw that monic minimal polynomials are particularly easy to describe and construct since they coincide with coordinate polynomials. What can be said for nonmonic minimal polynomials? Due to relation (2.2), a simple way to produce nonmonic minimal polynomials is to let $G := GL_2(\mathbb{K}[x])$ act on a monic minimal polynomial. It is natural to ask whether all nonmonic minimal polynomials arise in such a way. We prove here that the answer is 'no,' a counterexample being given by

$$f = x(x - y^2)^2 - 2\lambda y(x - y^2) + \lambda^2$$

(Theorem 1.5 of the introduction). However, we will show that, if we assume that d_y is prime, then the answer is 'yes' (Theorem 1.6). Both results will follow from divisibility constraints on the partial degrees of a minimal polynomial (Theorems 4.3 and 4.14).

Definition 4.1. Let $f, g \in \mathbb{K}[x, y]$ be two irreducible polynomials with partial degrees $\deg_y f > 1$ and $\deg_y g > 1$. We say that f and g are G-equivalent, denoted by $f \equiv g$, if there exists a $\sigma \in G$ such that $f = \sigma(g)$, the action of σ being defined in (2.1).

4.1. G-reduction of minimal polynomials. Proof of Theorem 1.5. In this subsection, we focus on the G-reduction of minimal polynomials: what is the 'simplest' form of a polynomial in the G-orbit of one which is minimal?

4.1.1. Newton polygon. We define the *generic* Newton polygon of $f \in \mathbb{K}[x, y]$ as the convex hull

$$P(f) := \operatorname{Conv}\Big((0,0) \cup (0, d_y) \cup \operatorname{Supp}(f)\Big),$$

where $\operatorname{Supp}(f)$ stands for the support of f, i.e., the set of exponents that appear in its monomial expansion. It is well known that the edges of the generic polygon that do not pass through the origin give information about the singularities of f at infinity. In our context, we have the following lemma:

Lemma 4.2. *Suppose that f is minimal. Then*

$$P(f) := \operatorname{Conv}\Big((0,0), (0, d_y), (b, d_y), (a, 0)\Big)$$

for some integers a, b.

Proof. Since f has a unique place along $x = \infty$, the claim follows from the Newton-Puiseux factorization theorem applied along the line $x = \infty$. See, for instance, [**4,** Chapter 6]. $\square$

The integers $a = a(f)$ and $b = b(f)$ of Lemma 4.2 coincide with the respective degrees in x of the constant and leading coefficients of f with respect to y. Due to Lemma 4.2, we have the relation

$$d_x = \max(a, b)$$

for any minimal polynomial f, and we define the integer $c = c(f)$ as

$$c := \min(a, b).$$

We say that f is in *normal position* if $b \leq a$, that is, if $(d_x, c) = (a, b)$.

4.1.2. Reduced minimal polynomials. We can now state our main result about G-reduction of minimal polynomials. Given $n : \mathbb{K}[x, y] \to \mathbb{Q}^+$ and $f \in \mathbb{K}[x, y]$, we define

$$n_{\min}(f) := \inf\{n(g),\ g \equiv f\}.$$

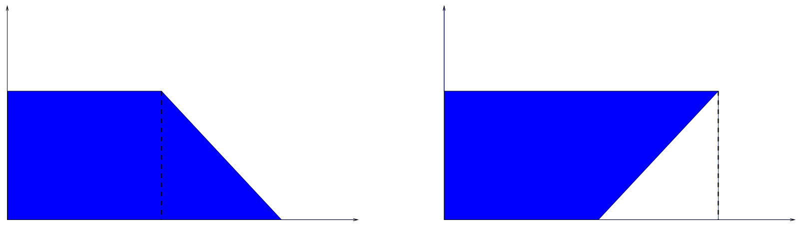

FIGURE 1. The generic Newton polygons of a minimal polynomial in normal and non normal position.

Theorem 4.3. *Let f be a minimal polynomial with parameters (d_y, d_x, c). Denote by V the Euclidean area of $P(f)$. Suppose that $d_y \geq 2$. Then, the following assertions are equivalent*:

(i) $V = V_{\min}$;
(ii) $d_x = d_{x,\min}$ *and* $c = c_{\min}$;
(iii) d_y *does not divide* $d_x - c$.

Definition 4.4. We say that f is *reduced* if it is minimal and satisfies one of the equivalent conditions of Theorem 4.3.

The remaining part of this subsection is dedicated to the proof of Theorem 4.3.

4.1.3. The characteristic polynomial. It turns out that Newton-Puiseux theorem gives strong information about the edge polynomial of f attached to the right hand side of $P(f)$, namely, we have:

Lemma 4.5. *Suppose that f is minimal with parameters (d_y, a, b). Then*

$$f(x,y) = x^b(\alpha y^p + \beta x^q)^n + \sum_{\substack{j \leq pn \\ pi+qj<pqn+np}} c_{ij} x^i y^j, \tag{4.1}$$

where $p \in \mathbb{N}^*$ and $q \in \mathbb{Z}$ are coprime integers such that

$$pn = d_y, \qquad qn = a - b, \tag{4.2}$$

where $\alpha, \beta \in \mathbb{K}^*$. We call the polynomial $f_\infty := (\alpha y^p + \beta x^q)^n$ the characteristic polynomial of f at $x = \infty$.

Proof. By Corollary 2.5, the Zariski closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the curve defined by f has a unique place along the line $x = \infty$. Thus, it follows once again from the Newton-Puiseux theorem applied along the line $x = \infty$ that the edge polynomial attached to the right hand edge of $P(f)$ is of the form $x^b g(x, y)$, where g is the power of an irreducible quasi-homogeneous polynomial [**4,** Chapter 6]. □

Corollary 4.6. *If $V = V_{\min}$, then d_y does not divide $d_x - c$.*

Proof. By Lemma 4.7 below, the parameters (d_x, d_y, c, V) are invariant under the inversion τ, while the parameters (a, b) are permuted. Hence, without loss of generality, we can suppose that f is in normal position, that is, $(d_x, c) = (a, b)$. By (4.2), we get that $q \geq 0$, and that d_y divides $d_x - c$ if and only if $p = 1$. In such a case, the polynomial

$$g(x, y) := f(x, y - \beta/\alpha x^q)$$

satisfies $b(g) = b(f)$ and $a(g) < a(f)$. Since g is equivalent to f, it is also minimal of partial degree d_y, and we deduce from Lemma 4.2 that

$$V(g) = \frac{d_y(a(g) + b(g))}{2} < V(f) = \frac{d_y(a(f) + b(f))}{2}.$$

The corollary follows. □

4.1.4. Basic transformations. We first study the behavior of the parameters d_x and c under the inversion and the polynomial De Jonquières transformations. We define the inversion $\tau \in G$ by

$$\tau(f) := y^{d_y} f(x, 1/y).$$

We have the following, obvious lemma:

Lemma 4.7. *Let $f \in \mathbb{K}[x, y]$ not be divisible by y. The parameters d_y, d_x, c are invariant by τ, and parameters a and b are permuted.*

Proof. It is straightforward to verify that $d_y(g) < d_y(f)$ if and only if $f(x, y) = y^k h(x, y)$ with $k > 0$, which is excluded by hypothesis. The remaining part of the lemma is straightforward. □

Let $U \subset G$ stand for the polynomial De Jonquières subgroup of G, that is, the subgroup of transformations σ of type

$$\sigma(f) : (x, y) \longmapsto f(x, \lambda y + h(x)),$$

where $\lambda \in \mathbb{K}^*$ and $h \in \mathbb{K}[x]$. We then define $\deg(\sigma) := \deg(h)$, with the convention $\deg(0) = 0$. If σ is a homothety, that is, if $h = 0$, then the Newton polygon and all of the parameters of f and $\sigma(f)$ obviously coincide. Otherwise, we obtain:

Lemma 4.8. *Let f be a minimal polynomial of degree $d_x > 0$, and let $\sigma \in U$ not be a homothety. Let $g = \sigma(f)$. Then*:

(i) *If f is in normal position and d_y does not divide $d_x - c$, then*

$$d_x(g) = \max(c(f) + d_y(f) \deg \sigma, d_x(f)) \quad \textit{and} \quad c(g) = c(f).$$

(ii) *If f is not in normal position, then*

$$d_x(g) = d_x(f) + d_y(f) \deg \sigma \quad \textit{and} \quad c(g) = d_x(f).$$

In both cases, g is in normal position.

Proof. We write $\sigma(f) = f(x, \lambda y + \mu x^k + r(x))$, with $\lambda, \mu \in \mathbb{K}^*$, $k = \deg \sigma \geq 0$ and $\deg r < k$, and we write $f = \sum c_{ij} x^i y^j$. We have

$$g(x, 0) = f(x, \mu x^k + r(x)) = \sum_{i+kj=M} c_{ij} \mu^j x^{i+kj} + R(x), \tag{4.3}$$

where

$$M := \max_{(i,j) \in \mathrm{Supp}(f)} (i + kj) \quad \text{and} \quad \deg R < M.$$

Since the line $i + kj = 0$ has negative slope $-1/k$ (vertical if $k = 0$), Lemma 4.2 forces M to be reached at one of the two vertices $(a(f), 0)$ or $(b(f), d_y)$ of N_f, forcing the equality

$$M = \max(a(f), b(f) + kd_y).$$

Suppose that f is not in normal position. Then, $a(f) < b(f)$, and $M = b(f) + kd_y$ is reached at the unique point $(b(f), d_y)$ of N_f. Thus, there is a unique monomial in (4.3) with maximal degree. This forces the equality

$$a(g) := \deg_x(g(x, 0)) = b(f) + kd_y.$$

On the other hand, it is clear that, for any f, we have

$$\mathrm{lc}_y(g) = \lambda^{d_y} \mathrm{lc}_y(f).$$

In particular, $b(g) = b(f) = a(g) - kd_y \leq a(g)$, forcing equality $(a(g), b(g)) = (d_x(g), c(g))$. Since f is not in normal position, we have $(a(f), b(f)) = (c(f), d_x(f))$. Claim (ii) follows.

Suppose now that f is in normal position and that d_y does not divide $d_x - c$. In particular, we have $a(f) \neq b(f) + kd_y$, so that, once again, M is reached at a unique point of N_f, forcing equality

$$a(g) := \deg_x(g(x,0)) = \max(b(f) + kd_y, a(f)).$$

Since $b(g) = b(f) \leq a(g)$, we have $(a(g), b(g)) = (d_x(g), c(g))$. Since f is in normal position, we have $(a(f), b(f)) = (d_x(f), c(f))$. Claim (i) follows. $\square$

4.1.5. Decomposition of $GL_2(\mathbb{K}[x])$. Let $V := GL_2(\mathbb{K}) \subset G$. It is well known that $GL_2(\mathbb{K}[x])$ is the amalgamated free product of the subgroups U and V along their intersections **[14]**: every element of $GL_2(\mathbb{K}[x])$ can be uniquely written as a finite alternating product of elements of U and V, with no elements in $U \cap V$, except possibly the first or last factor. On the other hand, it is a classical fact that V is generated by translations

$$y \longrightarrow y + \lambda,$$

homotheties

$$y \longrightarrow \lambda y,$$

$$\lambda \in \mathbb{K}^*,$$

and the inversion τ. Since translations and homotheties lie in $U \cap V$, it follows that any transformation $\sigma \in G$ can be decomposed as an alternate product

$$\sigma = \sigma_n \tau \sigma_{n-1} \tau \cdots \sigma_2 \tau \sigma_1, \tag{4.4}$$

with $\sigma_i \in U$ for all i. We can assume, moreover, that $\sigma_i \notin U \cap V$, except possibly for $i = 1$ or $i = n$, that is,

$$\deg \sigma_i > 0 \quad \text{for all } i = 2, \ldots, n-1.$$

Now, let $\sigma \in G$ have decomposition (4.4), and let $f \in \mathbb{K}[x,y]$. We introduce the notation

$$f_1 = \sigma_1(f) \quad \text{and} \quad f_i = (\sigma_i \tau)(f_{i-1}),\ i = 2, \ldots, n,$$

and we write, for short, $d_i = d_x(f_i)$ and $c_i = c(f_i)$. The following proposition must be compared to **[18]**, where the author considers the behavior of the total degree of a bivariate polynomial under the action of $\mathrm{Aut}(\mathbb{A}^2)$.

Proposition 4.9. *Let f be a minimal polynomial in normal position such that d_y does not divide $d_x - c$, and let $\sigma \in G$. With the notation introduced before, we have*

$$d_x \leq d_1 < d_2 < \cdots < d_{n-1} \leq d_n$$

and

$$c = c_1 < c_2 < \cdots < c_{n-1} \leq c_n.$$

Moreover, d_y does not divide $d_n - c_n$ if and only if $(d_n, c_n) = (d_x, c)$.

Proof. By Lemma 4.8, the proposition is true if $n = 1$. We have $a(f) > b(f)$ by assumption, so that $a(f_1) > b(f_1)$ by Lemma 4.8. From Lemma 4.7, it follows that $\tau(f_1)$ is not in normal position. If σ_2 is a homothety, then $n = 2$, and $f_2 = \sigma(\tau(f_1))$ has the same parameters as f_1, proving the proposition in that case. If σ_2 is not a homothety, then $d_2 > d_1$, $c_2 > c_1$, and d_y divides $d_2 - c_2$ by Lemma 4.8. Hence, Proposition 4.9 follows for $n = 2$. Moreover, we have

$$n > 2 \Longrightarrow \deg \sigma_2 > 0 \Longrightarrow a(f_2) > b(f_2),$$

the second implication again using Lemma 4.8. Thus, $n > 2$ implies, moreover, that $\tau(f_2)$ is not in normal position. The proposition then follows by induction. □

Proof of Theorem 4.3. We have (i) ⇒ (iii) by Corollary 4.6, while the implication (iii) ⇒ (ii) is an immediate consequence of Proposition 4.9. The remaining implication (ii) ⇒ (i) follows from equality $V = d_y(c + d_x)/2$ that holds for minimal polynomials due to Lemma 4.2. □

As mentioned earlier in this section, Theorem 1.5 is an easy corollary of Theorem 4.3.

Proof of Theorem 1.5. A direct computation shows that the polynomial $f = x(x-y^2)^2 - 2\lambda y(x-y^2) + \lambda^2$ is minimal with parameters $(d_y, d_x, c) = (4, 3, 1)$ for all $\lambda \in \mathbb{K}^*$ (for $\lambda = 0$, the polynomial f is reducible). Since d_y does not divide $d_x - c$, f is reduced by Theorem 4.3. Hence, $c = c_{\min} = 1 \neq 0$, and f is not equivalent to a monic polynomial by Theorem 4.3. □

4.2. Cremona equivalence of minimal polynomials. Theorem 1.5 shows that we cannot hope that a nonmonic minimal polynomial can be transformed to a coordinate via a composition of an element of $GL_2(\mathbb{K}[x])$ with an element of $\mathrm{Aut}(\mathbb{A}^2)$. However, both groups act on the curve of f as subgroups of the Cremona group $\mathrm{Bir}(\mathbb{A}^2)$ of birational transformations of the plane, and both Theorem 3.3 and $GL_2(\mathbb{K}[x])$-invariance of the degree of the discriminant lead us to ask the following natural question:

Question 4.10. *Do minimal polynomials define curves Cremona equivalent to lines?*

Theorem 3.3 gives a positive answer in the case of monic polynomials, and more generally for all members of their $GL_2(\mathbb{K}[x])$-orbits. This is also the case for the nonmonic minimal polynomial of Theorem 1.5, as will be shown in the next proposition. Note that being Cremona equivalent to a line does not imply minimality. In a close context, it has recently been proved in **[13]** that any rational cuspidal curve of $\mathbb{P}^2$ is Cremona equivalent to a line, solving a famous problem of Coolidge and Nagata. In the present context, minimal polynomials define rational unicuspidal curves of $\mathbb{P}^1 \times \mathbb{P}^1$ (Corollary 2.5), and we may ask whether the result of Koras-Palka extends to this case. These kinds of problems are closely related to the geometry of the minimal embedded resolution.

Proposition 4.11. *The curve defined by the polynomial* $f = x(x-y^2)^2 - 2\lambda y(x-y^2) + \lambda^2$ *is Cremona equivalent to a line.*

Proof. Since the polynomial f is minimal with parameters $(d_x, d_y, c) = (3, 4, 1)$, it is easy to see that it defines a unicuspidal curve of $\mathbb{P}^2$. Hence, the claim follows from [**13**]. It should be noticed that we can 'read' the underlying birational transformation on the Newton polygon of f. We have $f_1(x,y) := f(x+y^2,y) = x^3 + (xy-\lambda)^2$, and

$$f_2(x,y) := f_1(x, y/x+\lambda) = x^3 + y^2$$

defines a curve which is clearly Cremona equivalent to $f = 0$. Let $C \subset \mathbb{P}^2$ be the projective plane curve defined by the homogenization $F(X,Y,Z) = X^3 + Y^2 Z$ of f_2. Consider the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ (X:Y:Z) &\longmapsto (XY^2 : Y^3 : X^3 + Y^2 Z). \end{aligned}$$

The restriction of σ to the chart $Y=1$ coincides with the affine map $(x,z) \to (x, x^3+z)$, which is clearly invertible. Hence, $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$ is a Cremona transformation that satisfies $\sigma^{-1}(Y=0) = C$. □

4.3. Divisibility constraints for minimal reduced polynomials. Proof of Theorem 1.6. Due to Theorem 1.2, monic minimal polynomials coincide with coordinate polynomials. In particular, it follows from [**1**] that they obey the crucial property:

Proposition 4.12 (Abhyankar-Moh's theorem reformulated)**.** *Let f be a monic minimal polynomial. Then, d_x divides d_y or d_y divides d_x.*

Proposition 4.12 is another reformulation of the embedding line theorem of Abhyankar and Moh [**1**]. Indeed, this property allows reduction of the degree of f with translations $x \mapsto x - \alpha y^k$ or $y \mapsto y - \alpha x^k$. Since these translations preserve the property of being simultaneously monic and minimal, we can reach $f = y$. In the nonmonic case, a similar reduction process requires a positive answer to the following question:

Question 4.13. *If f is minimal, is it true that $d_x - c$ divides d_y or d_y divides $d_x - c$?*

Here, the parameter c is that defined in the previous subsection. This property holds for all polynomials in the G-orbit of a monic minimal polynomial by Propositions 4.12 and 4.9. It also holds for the minimal reduced polynomial f of Theorem 1.5 ($d_x - c = 2$ divides $d_y = 4$), and may be seen as a key point in the explicit construction of the birational map of Proposition 4.11. Although Questions 4.10 and 4.13 are closely related, translations on x do not preserve the minimality of a nonmonic minimal polynomial, and it is not clear that a positive answer to Question 4.13 leads to a positive answer to Question 4.10. Anyway, it would be an important property for reducing minimal polynomials to a "nice canonical form." We prove a partial result here that shows that, if f is minimal and d_y does not divide $d_x - c$, then d_y and $d_x - c$ are not coprime as soon as $d_x > 1$.

Theorem 4.14. *Let f be a minimal polynomial of degree $d_y \geq 1$. If f is nonreduced, then d_y divides $d_x - c$. If f is reduced, we have*:

(i) *If $d_x = 0$, then $c = 0$ and $d_y = 1$.*
(ii) *If $d_x = 1$, then $c = 0$ and $d_y > 1$.*
(iii) *If $d_x > 1$ and $c = 0$, then d_x divides d_y.*
(iv) *If $d_x > 1$ and $c > 0$, then $2 \leq \gcd(d_x - c, d_y) \leq d_y/2$.*

Proof. If f is nonreduced, then d_y divides $d_x - c$ by Theorem 4.3. Assume that f is reduced. If $d_x = 0$, then $c = 0$ is obvious and $d_y = 1$ since, otherwise, f would not be irreducible. If $d_x = 1$, then $c \leq 1$. Since f is reduced, we must have $c = 0$ and $d_y > 1$ by Theorem 4.3 since, otherwise, d_y would divide $d_x - c$. Suppose now that $d_x > 1$. If $c = 0$, then we can suppose that f is monic up to application of the inversion $y \to 1/y$. The claim thus follows from Proposition 4.12 combined with the fact that d_y cannot divide d_x since f is assumed to be reduced (Theorem 4.3). Now, suppose that $d_x > 1$ and $c > 0$. Then, $\gcd(d_x - c, d_y) \leq d_y/2$ by Theorem 4.3.

It remains to show that d_y and $d_x - c$ are not coprime. We will use the theory of toric varieties. Since they appear only in this proof, we do not give a detailed account on toric varieties, but instead, we refer the reader to the books of Fulton [**8**] and Cox, Little and Schenck [**6**]. Let $P := P(f)$ be the generic Newton polygon of a bivariate polynomial f such that $f(x, 0)$ is non constant. Consider the map

$$\begin{aligned} \phi_P : \mathbb{A}^2 &\longrightarrow \mathbb{P}^N \\ t &\longmapsto [t^{m_0} : \cdots : t^{m_N}], \end{aligned}$$

where $P \cap \mathbb{Z}^2 = \{m_0, \dots, m_N\}$. Since $(0,0),(1,0),(0,1) \in P$ by hypothesis, the map ϕ_P is an embedding. The *projective toric surface* $X = X_P$ associated to P is, by definition, the Zariski closure of $\phi_P(\mathbb{A}^2)$. Set theoretically, we have that

$$X = \mathbb{A}^2 \sqcup (D_1 \cup \cdots \cup D_r),$$

where the divisors $D_i \simeq \mathbb{P}^1$ are in one-to-one correspondence with the edges $\Lambda_1, \dots, \Lambda_r$ of P that do not pass through the origin. Moreover, if we let $C \subset X$ be the Zariski closure of the image of the affine curve $f = 0$, then C properly intersects the divisors D_i with total intersection degree

$$\deg(C \cdot D_i) = \mathrm{Card}(\Lambda_i \cap \mathbb{Z}^2) - 1.$$

Let us return to our context of f minimal. We can assume that $P(f)$ is in normal position so that $P(f)$ satisfies the previous conditions. Moreover, since $c > 0$, P has exactly four edges, by Lemma 4.2. Let $E \subset X$ be the divisor corresponding to the right hand edge Λ of P. Since P has two horizontal edges and one vertical edge passing throw the origin (see Figure 1), the normal fan of P refines the fan of $\mathbb{A}^1 \times \mathbb{P}^1$ (see [**8,** subsection 1.4]), and it follows that

$$X \setminus E = \mathbb{A}^1 \times \mathbb{P}^1.$$

In particular, we have, by the minimality of f, that C is smooth in $X \setminus E$. On the other hand, we have that

$$\deg(C \cdot E) = \mathrm{Card}(\Lambda \cap \mathbb{Z}^2) - 1 = \gcd(d_x - c, d_y).$$

Suppose that $\gcd(d_x - c, d_y) = 1$. Then, $\deg(C \cdot E) = 1$ forces C to intersect E transversally at a unique point. In particular, it is smooth along E, hence, smooth in X by what was previously stated. The arithmetic genus formula for curves in toric surface [**12**], combined with the fact that $C \subset X$ is smooth and rational, leads to the equality

$$0 = g(C) = p_a(C) = \mathrm{Card}(\mathrm{Int}(P) \cap \mathbb{Z}^2),$$

where $\mathrm{Int}(P)$ stands for the interior of P. However, this contradicts the fact that P is the convex hull of $(0,0),(0,d_y),(c,d_y),(d_x,0)$ with $d_x > 1$, $c > 0$ and $d_y \geq 2$. $\square$

5. A uniform lower bound for reducible polynomials. We now focus on the non monic reducible case, and we prove Theorem 1.7

of the introduction: all polynomials $f \in \mathbb{K}[x,y]$ with non constant discriminant satisfy

$$\deg_x \Delta_y(f) \geq \left\lceil \frac{d_y - 1}{2} \right\rceil,$$

and we have a complete classification of polynomials for which equality holds. The proof requires some preliminary lemmas. In order to study the discriminant of reducible polynomials, it is more convenient to consider homogeneous polynomials in $Y = (Y_0 : Y_1)$. Homogeneity in x is unnecessary. We, thus, consider polynomials $F \in \mathbb{K}[x][Y]$.

Lemma 5.1. *Let $F \in \mathbb{K}[x][Y]$ be a squarefree polynomial of degree $\deg_Y F = d \geq 0$ with no factor in $\mathbb{K}[x]$. Assume that F has only linear factors. Then, exactly one of the following occurs*:

(i) $\deg_x \Delta_Y F = 0$, *and F is G-equivalent to some polynomial of $\mathbb{K}[Y]$.*
(ii) $d = 2$, *and* $\deg_x \Delta_Y F \geq 2 > d/2$.
(iii) $d \geq 3$, *and* $\deg_x \Delta_Y F \geq 2(d-2) > d/2$.

Proof. Cases $d = 0$ and $d = 1$ are trivially in case (i). We now assume that $d \geq 2$. We have $F = \prod_{i=1}^d F_i$ with $F_i = a_i Y_0 + b_i Y_1$, for some a_i and b_i in $\mathbb{K}[x]$. For all nonempty subsets $I \subset \{1, \ldots, d\}$, we write

$$F_I = \prod_{i \in I} F_i.$$

If I has only one element, then, clearly, $\Delta_Y F_I \in \mathbb{K}$. Among all subsets I such that $\Delta_Y F_I \in \mathbb{K}$, we consider one with a maximal number of elements and write m for its cardinality. We have $1 \leq m \leq d$.

Consider first the case $m = 1$. For all $i \neq j$, we have $\deg_x \mathrm{Res}_Y(F_i, F_j) \geq 1$. This implies that $\deg_x \Delta_Y F \geq d(d-1)$. This proves case (ii) if $d = 2$ and case (iii) if $d > 2$.

Consider now the case $2 \leq m$. We can assume that $I = \{1, 2, \ldots, m\}$. We then have $\mathrm{Res}(F_1, F_2) \in \mathbb{K}$. The matrix

$$\sigma = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$$

is, therefore, an element of $GL_2(\mathbb{K}[x])$. Via the action of σ, F_1 and F_2 are transformed into Y_0 and Y_1. Without loss of generality, we assume that $F_1 = Y_0$ and $F_2 = Y_1$. For all $3 \le i \le m$, we have $\mathrm{Res}_Y(F_i, Y_0) \in \mathbb{K}$; hence, $b_i \in \mathbb{K}$. Similarly, we have $\mathrm{Res}_Y(F_i, Y_1) \in \mathbb{K}$; hence, $a_i \in \mathbb{K}$. This proves that $F_i \in \mathbb{K}[Y]$ for all $i \in I$. If $m = d$, then we have proved that F is equivalent to a polynomial in $\mathbb{K}[Y]$; hence, we are in case (i).

It remains to consider the case $2 \le m < d$. This case is possible only if $d \ge 3$. As before, we can assume that $I = \{1, 2, \ldots, m\}$ and $F_i \in \mathbb{K}[Y]$ for all $i \in I$. For an integer $j \notin I$, there exists at most one value of $i \in I$ such that $\mathrm{Res}_Y(F_i, F_j) \in \mathbb{K}$. Otherwise, using a similar argument as before, we would have $F_j \in \mathbb{K}[Y]$ and $\mathrm{Res}_Y(F_i, F_j) \in \mathbb{K}$ for all $i \in I$, contradicting the maximality of I. Since each F_j for $j \notin I$ has at least $m-1$ nonconstant resultants with F_i for $i \in I$, this proves that $\deg_x \Delta_Y F \ge 2(m-1)(d-m)$. It is an exercise to verify the inequalities $2(m-1)(d-m) \ge 2(d-2) > d/2$. □

Lemma 5.2. *Let $F \in \mathbb{K}[x][Y]$ be an irreducible polynomial of degree $d \ge 2$. Assume that F is minimal. Consider an integer n and polynomials $F_i = a_iY_0 + b_iY_1$, for $1 \le i \le n$ and $a_i, b_i \in \mathbb{K}$, that are pairwise coprime. If $\mathrm{Res}_Y(F_i, F) \in \mathbb{K}$ for all $1 \le i \le n$, then $n \le 1$.*

Proof. It is sufficient to prove that the case $n = 2$ is impossible. Suppose that two such polynomials exist. Using the action of $GL_2(\mathbb{K})$, we can assume that $F_1 = Y_0$ and $F_2 = Y_1$. We write $r_1 = \mathrm{Res}_Y(Y_1, F) \in \mathbb{K}$ and $r_0 = \mathrm{Res}_Y(Y_0, F) \in \mathbb{K}$. Since F is irreducible of degree $d \ge 2$, it cannot be divisible by Y_0; hence, $r_0 \ne 0$. Without loss of generality, we can assume that $r_0 = 1$. The relation $\mathrm{Res}_Y(Y_0, F) = 1$, therefore, implies that $F(Y_0, Y_1)$ is monic in Y_1. By Theorem 3.3, $F(1, y)$ is equivalent to y up to an automorphism of $\mathbb{A}^2$. This implies that $F(1, y)$ is irreducible of degree $d \ge 2$, as well as $F(1, y) - r_1$. However, this last polynomial is, by construction, divisible by y. We obtain a contradiction. □

Lemma 5.3. *Let $F \in \mathbb{K}[x][Y]$ be a squarefree polynomial of degree $\deg_Y F = d \ge 2$ with no factor in $\mathbb{K}[x]$. Assume that $F = PQ$, where P is irreducible of degree $\deg_Y P \ge 2$, and Q has only linear factors.*

Then

$$\deg_x \Delta_Y F \geq \left\lceil \frac{d-1}{2} \right\rceil .$$

Furthermore, equality holds if and only if F is G-equivalent to one of the following exceptional polynomials:

(*Case* $d = 2$). $Y_0^2 + (x+a)Y_1^2$, ($a \in \mathbb{K}$);

(*Case* $d = 3$). $Y_1(Y_0^2 + (x+a)Y_1^2)$, ($a \in \mathbb{K}$);

(*Case* $d = 4$). $Y_1(Y_0^3 + aY_0Y_1^2 + (x+b)Y_1^3)$, ($a, b \in \mathbb{K}$);

(*Case* $d = 4$). $Y_0Y_1(Y_0^2 + (ax+b)Y_0Y_1 + Y_1^2)$, ($a \in \mathbb{K}^*$ *and* $b \in \mathbb{K}$).

Proof. We write $F = PQ$. In order to shorten some expressions, we write $d_P = \deg_Y P$ and $d_P = \deg_Y Q$. We have $d = d_P + d_Q$. By Theorem 1.3, we already have $\deg_x \Delta_Y P \geq d_P - 1$. The proof splits into different cases according to which case corresponds to the polynomial F in Lemma 5.1.

Case (0). If $d_Q = 0$. We have

$$\deg_x \Delta_Y F = \deg_x \Delta_Y P \geq d - 1 \geq \left\lceil \frac{d-1}{2} \right\rceil .$$

Equality holds if and only if $d = 2$ and P is minimal. From Theorem 1.6, P is G-equivalent to a polynomial of the form $Y_0^2 + (x+c)Y_1^2$, with $c \in \mathbb{K}$.

Case (i). If $d_Q > 0$ and $\deg_x \Delta_Y Q = 0$. From Lemma 5.1, we can assume that $Q \in \mathbb{K}[Y]$.

Subcase (i.1). If $d_Q \leq d_H - 2$. Here, we simply have

$$\deg_x \Delta_Y F \geq d_P - 1 \geq \frac{d_P + d_Q}{2} .$$

In this case, the anticipated inequality is proven. We then observe that equality implies that P is minimal, $d_Q = d_P - 2$, and $\mathrm{Res}_Y(P, Q) \in \mathbb{K}$. From Lemma 5.2, this is possible only if $d_Q = 1$ and $d_P = 3$. By Theorem 1.6, we deduce that P is G-equivalent to a polynomial of the form $Y_0^3 + aY_0Y_1^2 + (x+b)Y_1^3$. In this case, Q can only be Y_1.

Subcase (i.2). If $d_Q = d_Q - 1$, we have

$$\deg_x \Delta_Y F \geq d_P - 1 = \frac{d-1}{2} .$$

This proves the inequality. The equality holds if and only if P is minimal and $\mathrm{Res}_Y(P,Q) \in \mathbb{K}$. From Lemma 5.2, this is possible only if $d_Q = 1$ and $d_P = 2$. By Theorem 1.6, we deduce that P is G-equivalent to a polynomial of the form $Y_0^2 + (x+a)Y_1^2$. In this case, Q can only be Y_1.

Subcase (i.3). If $d_Q = d_P$. In this case, we have $d = 2d_P$. If P is minimal, then, by Lemma 5.2, $\deg_x \mathrm{Res}_Y(P,Q) \geq d_P - 1$; hence, $\deg_x \Delta_Y F \geq d_P - 1 + 2(d_P - 1)$. This is always larger than $d/2$. If P is not minimal, we have $\deg_x \Delta_Y P \geq d_P$, whence the inequalities

$$\deg_x \Delta_Y F \geq d_P = \frac{d}{2}.$$

This proves the inequality. We see here that equality holds only if $\deg_x \Delta_Y P = d_P$ and $\mathrm{Res}_Y(P,Q) \in \mathbb{K}$. Let

$$Q = \prod_{i=1}^{d_P} Q_i$$

be the factorization of Q into linear factors in $\mathbb{K}[Y]$, and let Q_0 be another linear polynomial in $\mathbb{K}[Y]$, coprime to Q. We define $R_0 = \mathrm{Res}_Y(P,Q_0) \in \mathbb{K}[x]$. Using interpolation at the Q_i's, we see that P can be written as $P = \lambda Q R_0 + b$, with $\lambda \in \mathbb{K}^*$ and $b \in \mathbb{K}[Y]$. We clearly have

$$\deg_x R_0 = \deg_x P = \deg_x F.$$

We denote by $r_0 \in \mathbb{K}^*$ the leading coefficient of R_0. $\Delta_Y P$ is a homogeneous polynomial of degree $2(d_P - 1)$ in terms of the coefficients of P, hence, of degree at most $D = 2(d_P - 1)\deg_x P$ in x. The coefficient in x^D in its expansion is equal to $\mathrm{Disc}_Y(\lambda Q r_0)$, which is not zero since Q is squarefree. This proves that

$$\deg_x \Delta_Y P = 2(d_P - 1)\deg_x P.$$

Since this is also equal to d_P, the only possibility is $\deg_x P = 1$ and $d_P = 2$. Using the action of $GL_2(\mathbb{K})$, we can, therefore, assume that $Q = Y_0 Y_1$. Under all of these conditions, P is of the form

$$P = Y_0^2 + (ax + b)Y_0 Y_1 + Y_1^2,$$

for some $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$.

Subcase (i.4). If $d_Q \geq d_P + 1$. It is impossible for P to have constant resultants with strictly more than d_P linear polynomials in $\mathbb{K}[Y]$, since otherwise, by interpolation, it would have coefficients in $\mathbb{K}$, contradicting its irreducibility. This proves that $\deg_x \mathrm{Res}_Y(P,Q) \geq d_Q - d_P$. We then have the inequalities

$$\deg_x \Delta_Y F \geq d_P - 1 + 2(d_Q - d_P) \geq d_Q \geq \frac{d+1}{2}.$$

This proves the previously mentioned inequality and, in this case, an equality is impossible.

Cases (ii) and (iii). In the remaining cases, we have $d_Q \geq 2$ and $\deg_x \Delta_Y Q > d_Q/2$. This gives

$$\deg_x \Delta_Y F > d_P - 1 + \frac{d_Q}{2} \geq \frac{d_P}{2} + \frac{d_Q}{2} = \frac{d}{2},$$

whence the conclusion. □

Lemma 5.4. *Let $q = y^2 + ay + b$ be a polynomial in $\mathbb{K}[x][y]$, with a and b in $\mathbb{K}[x]$. Assume that $\deg_x a^2 - 4b$ is odd.*

For a polynomial $p \in \mathbb{K}[x][y]$, we have $\mathrm{Res}_y(p,q) \in \mathbb{K}$ if and only if $p = \alpha q + \beta$ for some $\alpha \in \mathbb{K}[x][y]$ and $\beta \in \mathbb{K}$.

Proof. Let $p = \alpha q + uy + v$ be the Euclidean division of p by q, with u and v in $\mathbb{K}[x]$. We have

$$\mathrm{Res}_y(p,q) = \mathrm{Res}_y(uy+v,q) = (v - au/2)^2 - \frac{a^2-4b}{4}u^2.$$

By assumption, this is an element of $\mathbb{K}$. Since $\deg_x a^2 - 4b$ is odd, inspection by degrees shows that this is possible only if $u = 0$ and $v - au/2 \in \mathbb{K}$. This gives the conclusion. □

We are now ready to prove Theorem 1.7, which we reformulate in a more convenient form for the proof.

Theorem 5.5. *Let $F \in \mathbb{K}[x][Y]$ be a squarefree polynomial of degree $\deg_Y F = d \geq 0$ with no factor in $\mathbb{K}[x]$. Then, exactly one of the following occurs*:

(i) $\deg_x \Delta_Y F = 0$, *and* F *is* G*-equivalent to some polynomial of* $\mathbb{K}[Y]$.
(ii) $d \geq 2$, *and* $\deg_x \Delta_Y F \geq \lceil (d-1)/2 \rceil$.

Furthermore, if $d \geq 2$*, equality* $\deg_x \Delta_Y F = \lceil (d-1)/2 \rceil$ *occurs if and only if* F *is* G*-equivalent to one of the following polynomials*:

- (Case d odd). $Y_1 \prod_{i=1}^n (Y_0^2 + (x+a_i)Y_1^2)$ $(a_i \in \mathbb{K})$;
- (Case d even). $\prod_{i=1}^n (Y_0^2 + (x+a_i)Y_1^2)$ $(a_i \in \mathbb{K})$;
- (Case $d = 4$). $Y_1(Y_0^3 + aY_0Y_1^2 + (x+b)Y_1^3)$ $(a, b \in \mathbb{K})$;
- (Case $d = 4$). $Y_0Y_1(Y_0^2 + (ax+b)Y_0Y_1 + Y_1^2)$ $(a \in \mathbb{K}^*$ *and* $b \in \mathbb{K})$.

Proof. Write $F = PQ$, where Q has only linear factors and P has no linear factor. Let

$$P = \prod_{i=1}^n P_i$$

be the decomposition of P into irreducible factors in $\mathbb{K}[x][Y]$. If $n = 0$, then the result is given by Lemma 5.1. Assume now that $n \geq 1$. The polynomial $F_1 = P_1 Q$ satisfies Lemma 5.3, hence,

$$\deg_x \Delta_Y F_1 \geq \frac{\deg Q + \deg P_1 - 1}{2}.$$

For $i \geq 2$, the polynomials P_i satisfy Theorem 1.3, thus,

$$\deg_x \Delta_Y P_i \geq \deg P_i - 1 \geq \frac{\deg P_i}{2}.$$

Putting these inequalities together gives

$$\deg_x \Delta_Y F \geq \frac{\deg Q + \deg P_1 - 1}{2} + \sum_{i \geq 2} \frac{\deg P_i}{2} = \frac{d-1}{2} \tag{5.1}$$

Consider now the question of equality. The easiest case is when d is odd. In this situation, all inequalities in (5.1) are equalities. This implies that $\deg P_i = 2$ for all $i \geq 2$, and $\deg F_1$ is odd with $\deg_x \Delta_Y F_1 = (\deg F_1 - 1)/2$. From Lemma 5.3, we can, therefore, assume that $F_1 = Y_1(Y_0^2 + (x+a_1)Y_1^2)$. The P_i's have a constant resultant with Y_1 and $Y_0^2 + (x+a_1)Y_1^2$. Using Lemma 5.4, we deduce that they are of the form

$$P_i = b_i(Y_0^2 + (x+a_i)Y_1^2)$$

with $a_i, b_i \in \mathbb{K}$. The constant $\prod b_i$ can be removed using G-equivalence. This gives the conclusion for d odd.

If d is even, Lemma 5.3 shows that F cannot have more than two linear factors. We have, therefore, three cases to consider:

• If F has no linear factor, then, by Lemma 5.3, we can assume that $P_1 = Y_0^2 + (x + a_1)Y_1^2$ for some $a_1 \in \mathbb{K}$. The proof in this case is very similar to the previous case and is left to the reader.

• If F has one linear factor, then, by Lemma 5.3, it is sufficient to consider the case $F_1 = Y_1 P_1$ with

$$P_1 = Y_0^3 + aY_0Y_1^2 + (x + b)Y_1^3$$

for some $a, b \in \mathbb{K}$. The other factors P_i must be quadratic and minimal, and also have constant resultant with F_1. The resultant with Y_1 shows that the P_i's are monic in Y_0. If $n \geq 2$, the resultant of P_2 and $P_1 = Y_0^3 + aY_0Y_1^2 + (x+b)Y_1^3$ is constant, and Lemma 5.4 imposes that $P_1 = Y_0P_2 + \beta Y_1^3$ with $\beta \in \mathbb{K}$. This is incompatible with $\beta = x + b$; hence, we must deduce that $n = 1$ and $F = F_1$.

• If F has two linear factors, then, by Lemma 5.3, it is sufficient to consider the case $F_1 = Y_0Y_1P_1$ with $P_1 = Y_0^2 + (ax + b)Y_0Y_1 + Y_1^2$ for some $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$. The other factors P_i must be quadratic and minimal, and also have constant resultant with F_1. In particular, if $n \geq 2$, $\mathrm{Res}_Y(Y_0Y_1, P_2) \in \mathbb{K}$ imposes that

$$P_2 = a_2Y_0^2 + b_2Y_0Y_1 + c_2Y_1^2$$

with a_2 and c_2 in $\mathbb{K}$. However, this is incompatible with $\deg_x \Delta_Y P_2 = 1$; hence, we must deduce that $n = 1$ and $F = F_1$. $\square$

APPENDICES

A. Small Δ_y versus small Δ_x. The equivalence (a) $\Leftrightarrow$ (b) of Theorem 3.3 asserts that a monic polynomial is minimal with respect to y if and only it is monic and minimal with respect to x. We prove here a generalization of this statement to the case of nonmonic polynomials.

For $f \in \mathbb{K}[x, y]$, a nonconstant bivariate polynomial, we let $\mathrm{lc}_y(f)$ (respectively, $\mathrm{lc}_x(f)$) stand for the leading coefficient of f seen as a polynomial in y (respectively, in x). We denote by n_x (respectively, n_y) the number of *distinct roots* of lc_y (respectively, of lc_x). We have the inequalities

$$n_x \leq \deg_x \mathrm{lc}_y(f) \quad \text{and} \quad n_y \leq \deg_y \mathrm{lc}_x(f),$$

and we say that f is *nondegenerate* if both equalities hold, that is, if both leading coefficients of f are squarefree. We write, for short, $f(\infty,\infty) = 0$ if the bihomogenization F of f vanishes at the point $X_1 = Y_1 = 0$, that is, if f has no monomial of bidegree (d_x, d_y).

Proposition A.1. *Let $f \in \mathbb{K}[x,y]$ be a nondegenerate, irreducible, bivariate polynomial such that $f(\infty,\infty) = 0$. The following assertions are equivalent*:

(a) $\deg_x \Delta_y(f) = d_y + n_y - 1$;
(b) $\deg_y \Delta_x(f) = d_x + n_x - 1$;
(c) *The Zariski closure*

$$C \subset \mathbb{P}^1 \times \mathbb{P}^1$$

of the affine curve $f = 0$ is rational, unicuspidal and smooth outside of (∞,∞).

Moreover, the equivalence (c) $\Leftrightarrow$ (a) $\cap$ (b) *still holds for degenerate polynomials.*

Proof. First, we prove (c) $\Leftrightarrow$ (a) $\cap$ (b). Hence, f is allowed to be degenerate.

• (c) $\Rightarrow$ (a) $\cap$ (b). By Proposition 2.3, we have the equality

$$\deg_x \Delta_y(f) = 2d_x(d_y - 1) - 2\delta_\infty - r_\infty,$$

where δ_∞ and r_∞ stand, respectively, for the delta invariant and the ramification index of f over $x = \infty$. Since C is assumed to be rational with a unique possible singularity at (∞,∞), the adjunction formula leads to the equality

$$\delta_\infty = p_a(C) = (d_x - 1)(d_y - 1).$$

Moreover, the curve is assumed everywhere to be locally irreducible. Hence, the number of places of C over $x = \infty$ coincides with the number of intersection points of C with $x = \infty$, that is, $n_x + 1$. It follows that

$$r_\infty = d_y - (n_x + 1).$$

Equality (a) then follows from Proposition 2.3. The implication (c) $\Rightarrow$ (b) follows from (c) $\Rightarrow$ (a) by symmetry.

• (a) $\cap$ (b) $\Rightarrow$ (c). Let us assume that (a) holds. From Proposition 2.3, we have:

$$2\delta_\infty = 2d_x(d_y - 1) - (d_y + n_x - 1) - r_\infty. \tag{A.1}$$

By assumption, the curve C of f has at least $n_y + 1$ places over $x = \infty$ so that

$$r_\infty \leq d_y - n_y - 1.$$

Combined with (A.1), we obtain the inequality

$$2\delta_\infty \geq 2(d_x - 1)(d_y - 1).$$

On the other hand, the genus being nonnegative, the adjunction formula leads to the inequality

$$\delta_\infty = p_a(C) - g \leq p_a(C) = (d_x - 1)(d_y - 1).$$

This forces $\delta_\infty = p_a(C)$. Hence, $g = 0$, and the singularities of C are located along the line $x = \infty$. This also forces $r_\infty = d_y - n_y - 1$ so that the curve C has exactly $n_y + 1$ places over $x = \infty$ and, hence, is locally irreducible along the line $x = \infty$. If, moreover, (b) holds, we obtain by symmetry that C has all its singularities located on the line $y = \infty$ and that C has exactly $n_x + 1$ places over $y = \infty$. Hence, (a) $\cap$ (b) forces C to be rational, with a unique possible singularity at (∞, ∞), this singularity being irreducible.

To finish the proof, we need to show that implication (a) $\Rightarrow$ (c) holds when f is nondegenerate. We just proved that (a) implies that C is rational with all of its singularities irreducible and located on the line $x = \infty$. The nondegenerate assumption ensures that C is transversal to the line $x = \infty$ (hence, smooth), except possibly at (∞, ∞). Hence, (c) holds. $\square$

Corollary A.2. *Let $f \in \mathbb{K}[x,y]$ be an irreducible bivariate polynomial such that $f(\infty,\infty)=0$. Then*

$$\deg_x \Delta_y(f) = d_y - 1 \Longrightarrow \begin{cases} \deg_y \Delta_x(f) = d_x + n_x - 1, \\ n_y = 0, \end{cases}$$

and the converse holds for nondegenerate polynomials. In particular, polynomials vanishing at (∞,∞) and minimal with respect to y are monic with respect to x.

Proof. If f is minimal, its curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is rational unicuspidal with a unique place on $x=\infty$ by Theorem 1.3. This place must be (∞,∞), by assumption. This forces $n_y=0$. The equality $\deg_y \Delta_x(f) = d_x + n_x - 1$ follows from Proposition A.1. If f is nondegenerate, the converse holds, again by Proposition A.1. □

B. Parametrization of minimal polynomials. Let

$$f = \sum \alpha_{ij} x^i y^j \in \mathbb{K}[x,y]$$

be a polynomial with parameters (d_x, d_y, c) and with indeterminate coefficients

$$\alpha = (\alpha_{ij})_{(i,j)\in P(f)\cap\mathbb{Z}^2}.$$

The discriminant of f is a polynomial in (x,α) of degree $2d_x(d_y-1)$ in x. Thus, in order to find which specializations of α which lead to a minimal polynomial, $\Delta_y(f)$ must be computed and then a system of

$$2d_x(d_y-1) - (d_y-1) \in \mathcal{O}(d_x d_y)$$

polynomial equations in α with

$$\mathrm{Card}(P(f)\cap\mathbb{Z}^2) \in \mathcal{O}(d_x d_y)$$

unknowns must be solved. This polynomial system very quickly turns out to be too complicated to be solved on a computer, even for the reasonable size of d_x and d_y. Moreover, an irreducibility test for each solution must be performed. However, we know that minimal polynomials define a rational curve, a strong information that was not used in the previous basic strategy. In particular, the curve admits a rational parametrization, that is to say, there exist two rational functions $u, v \in \mathbb{K}(s)$ such that the equality

$$f(u(s), v(s)) = 0$$

holds in $\mathbb{K}(s)$. The next result summarizes the relations between minimality and parametrization.

Proposition B.1. *An irreducible polynomial $f \in \mathbb{K}[x, y]$ is minimal if and only if there exist two rational functions $u, v \in \mathbb{K}(s)$ such that*:

(i) $f(u, v) = 0$ *in* $\mathbb{K}(s)$ (*rationality*);
(ii) $\mathbb{K}(s) = \mathbb{K}(u, v)$ (*proper parametrization*);
(iii) $u \in \mathbb{K}[s]$ (*unique place along* $x = \infty$);
(iv) $\mathbb{K}[s] = \mathbb{K}[u, v] \cap \mathbb{K}[u, v^{-1}]$ (*smoothness in* $\mathbb{A}^1 \times \mathbb{P}^1$).

Moreover, given such a pair u, v, we have the equality

$$d_y = \deg_s u, \qquad a(f) = \deg_s v_1 \quad \textit{and} \quad b(f) = \deg_s v_2$$

where $v_1, v_2 \in \mathbb{K}[t]$ are coprime polynomials such that $v = v_1/v_2$.

Proof. We know from Theorem 1.3 that f is minimal if and only if the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is rational, with a unique place along $x = \infty$ and smooth outside this line. Rationality is equivalent to the existence of a proper parametrization, that is, the existence of rational functions $u, v \in \mathbb{K}(s)$ such that items (i) and (ii) hold. The rational map

$$(u, v) : \mathbb{K} \dashrightarrow \mathbb{K}^2$$

extends to a morphism

$$\rho : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

whose image is C. Moreover, the parametrization being proper, this morphism establishes a one-to-one correspondence between $\mathbb{P}^1$ and the places of C. The fact that C has a unique place along the line $x = \infty$ is equivalent to the fact that u has a unique pole on $\mathbb{P}^1$. Up to a Mobius transformation on $\mathbb{P}^1$, it is no less to assume that this pole is $s = \infty$, meaning precisely that $u \in \mathbb{K}[s]$. The restriction of C to $\mathbb{A}^1 \times \mathbb{P}^1$ is smooth if and only if so are its restrictions to the two affine charts

$$U := \mathbb{A}^1 \times \{y \neq \infty\} \simeq \mathbb{A}^2$$

and

$$V := \mathbb{A}^1 \times \{y \neq 0\} \simeq \mathbb{A}^2.$$

However, this is also equivalent to the fact that the coordinate rings

$$\frac{\mathbb{K}[x,y]}{(f(x,y))} \simeq \mathbb{K}[u,v] \quad \text{and} \quad \frac{\mathbb{K}[x,y]}{(y^{d_y} f(x,1/y))} \simeq \mathbb{K}[u,v^{-1}]$$

of the affine curves $C_{|U}$ and $C_{|V}$ are integrally closed in their field of fractions $\mathbb{K}(u,v) = \mathbb{K}(s)$. Since $u \in \mathbb{K}[s]$, we deduce that s is integrally closed over $\mathbb{K}[u,v]$ and over $\mathbb{K}[u,v^{-1}]$, whence the inclusion

$$\mathbb{K}[s] \subset \mathbb{K}[u,v] \cap \mathbb{K}[u,v^{-1}].$$

The reverse inclusion always holds by a Gauss lemma argument, and we obtain item (iv). Conversely, if item (iv) holds, then

$$\mathbb{K}[u,v] \cap \mathbb{K}[u,v^{-1}] = \mathbb{K}[s]$$

is integrally closed so that the curve is smooth in $\mathbb{A}^1 \times \mathbb{P}^1$. The formulas for $\deg_y f$, $a(f)$ and $b(f)$ follow, for instance, from **[7]**, where the authors compute the Newton polygon of a parametrized curve. $\square$

Due to Proposition B.1, computing all minimal polynomials of given parameters (d_x, d_y, c) is equivalent to computing the discriminant of the implicit equation of the parametrization (u,v) with indeterminate coefficients that satisfies items (i), (ii), (iii) and solving a system of

$$(2d_x - 1)(d_y - 1) \in \mathcal{O}(d_x d_y)$$

polynomial equations with

$$d_y + d_x + c \in \mathcal{O}(d_x + d_y)$$

unknowns. When compared to the previous approach, this drastically reduces the number of unknowns, and irreducibility tests are avoided. This is the approach which allowed us to find the crucial example of Theorem 1.5 by computer. It must be noted, however, that the degree of the polynomial system then increases. Finally, we mention that item (iv) (hence, minimality) can also be directly verified by requiring that the so-called D-resultant of the pair (u,v) is constant (see **[9]**), a computational problem of an a priori equivalent complexity.

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