JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 2, Summer 2018

FINITE COMMUTATIVE RINGS WHOSE UNITARY CAYLEY GRAPHS HAVE POSITIVE GENUS

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ABSTRACT. The unitary Cayley graph of a ring R is the simple graph whose vertices are the elements of R, and where two distinct vertices x and y are linked by an edge if and only if $x - y$ is a unit in R. The genus of a simple graph G is the smallest nonnegative integer g such that G can be embedded into an orientable surface $\mathbb{S}_g$. It is proven that, for a given positive integer g, there are at most finitely many finite commutative rings whose unitary Cayley graphs have genus g. We determine all finite commutative rings whose unitary Cayley graphs have genus 1, 2 and 3, respectively.

1. Introduction and preliminaries. This paper concerns the unitary Cayley graphs of finite commutative rings, focusing on the genera of unitary Cayley graphs. Let R be a ring with nonzero identity. We use $U(R)$ and $J(R)$ to denote the group of units of R and the Jacobson radical of R, respectively. The unitary Cayley graph of R, denoted by $\Gamma(R)$, is the simple graph whose vertices are the elements of R, and where two distinct vertices x and y are adjacent if and only if $x - y \in U(R)$.

The unitary Cayley graph of a ring was initially investigated for $\mathbb{Z}_n$ by Dejter and Giudici **[10]** and has been the topic of many publications (see, for example, **[2, 3, 5, 6, 10, 11, 13]**–**[19]**).

2010 AMS *Mathematics subject classification.* Primary 05C25, 13A99, Secondary 05C75, 13M05.

Keywords and phrases. Unitary Cayley graph, complete graph, complete bipartite graph, genus, finite commutative ring.

This work was supported by a Discovery grant from the NSERC of Canada. The first author was also supported by the National Natural Science Foundation of China, grant Nos. 11661013, 11661014, 11461010, the Guangxi Natural Sciences Foundation, grant Nos. 2016GXNSFCA380014, 2015GXNSFAA139009, 2016GXSFDA380017, the Guangxi Science Research and Technology Development Project, grant No. 1599005-2-13, and the Scientific Research Foundation of Guangxi Educational Committee, grant No. KY2016YB280.

Received by the editors on January 19, 2015, and in revised form on January 8, 2016.

DOI:10.1216/JCA-2018-10-2-275

We recall some necessary notation in graph theory. Let $G=(V,E)$ be a simple graph. For $v \in V$, the degree of v, denoted by $\deg(v)$, is the number of edges of G incident with v. For an integer $k>0$, the graph G is called *k-regular* if the degree of each vertex of G is equal to k. We use $K_{m,n}$ and K_n to denote the complete bipartite graph with partitions of size m and n, and the complete graph of n vertices, respectively. An *orientable surface* $\mathbb{S}_g$ is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. A graph G that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g-1$, is called a *graph* of genus g. A *planar graph* is a graph with genus zero, and a *toroidal graph* is a graph with genus one. We write $\gamma(G)$ for the genus of the graph G. Clearly, if H is a subgraph of a graph G, then $\gamma(H) \le \gamma(G)$. Determining the genus of a graph is one of the fundamental problems in topological graph theory. In [**23**], Thomassen proved that the graph genus problem is indeed NP-complete.

The genera of graphs associated with rings is an active research subject. For instance, the planarity of zero divisor graphs was studied in [**1, 4, 21**]. The rings with toroidal zero divisor graphs were classified in Wang [**24**] and Wickham [**26, 27**]. Genus two zero divisor graphs of local rings were investigated by Bloomfield and Wickham [**7**]. Recently, Maimani et al. [**20**] determined all isomorphism classes of finite rings whose total graphs have genus at most one, and Tamizh Chelvam and Asir [**22**] characterized all isomorphism classes of finite rings whose total graphs have genus two. In [**2,** Theorem 8.2], all finite commutative rings having planar unitary Cayley graphs are completely classified. The goal of this paper is to classify all finite commutative rings whose unitary Cayley graphs have genus 1, 2 and 3, respectively. It is also proven that, for a given positive integer g, there are at most finitely many finite commutative rings whose unitary Cayley graphs have genus g.

As is standard, $\mathbb{Z}_n$ and $\mathbb{F}_q$ will denote the ring of integers $\mod n$ and the field with q elements, respectively. The cardinal of a set A is denoted $|A|$. Throughout, graphs are finite simple graphs.

2. Unitary Cayley graphs with genus g. In this section, we prove that, for any integer $g>0$, there are only finitely many finite commutative rings R with $\gamma(\Gamma(R))=g$. Some lemmas are required.

Lemma 2.1 ([**25,** Theorems 6.37, 6.38])**.** *Let $m \geq 2$, $n \geq 3$ and $p \geq 2$ be integers. Then,*

$$\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil,$$
$$\gamma(K_{m,p}) = \left\lceil \frac{1}{4}(m-2)(p-2) \right\rceil,$$

where $\lceil x \rceil$ is the least integer that is greater than or equal to x.

Lemma 2.2 ([**25,** Corollary 6.19])**.** *The genus of a graph is the sum of the genera of its components.*

Lemma 2.3 ([**27,** Proposition 2.1])**.** *Let G be a graph with n (≥ 3) vertices. Let $\delta(G)$ be the minimal degree of G. Then*

$$\delta(G) \leq 6 + \frac{12(\gamma(G)-1)}{n}.$$

Lemma 2.4 ([**2,** Proposition 2.2])**.** *Let R be a ring with $|U(R)| = k < \infty$. Then, $\Gamma(R)$ is k-regular.*

Lemma 2.5. *Let R be a finite commutative ring with $\gamma(\Gamma(R)) = g > 0$. Then, either $|R| \leq 12(g-1)$ or $|U(R)| \leq 6$.*

Proof. We can assume $|U(R)| > 6$. From Lemma 2.3,

$$\delta(\Gamma(R)) \leq 6 + \frac{12(g-1)}{|R|}.$$

By Lemma 2.4, $\delta(\Gamma(R)) = |U(R)| > 6$. Thus,

$$\frac{12(g-1)}{|R|} = \left(6 + \frac{12(g-1)}{|R|}\right) - 6 \geq \delta(\Gamma(R)) - 6 \geq 1,$$

giving $12(g-1) \geq |R|$. $\square$

If R is a finite commutative ring containing n zero divisors with $n > 1$, then $|R| \leq n^2$ (see [**12,** Theorem 1]). For a finite local ring R with maximal ideal $\mathfrak{m}$, there exists a prime p such that $|R/\mathfrak{m}| = p^t$ for some integers $t \geq 1$, and hence, $|R| = p^n$ for some integers $n \geq t$.

Lemma 2.6 ([**8,** page 687])**.** *Let R be a commutative local ring. Then*:

(i) $|R| = 4$ *if and only if* $R \in \{\mathbb{F}_4, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)\}$.
(ii) $|R| = 8$ *if and only if* $R \in \{\mathbb{F}_8, \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_4[x]/(2x, x^2),\ \mathbb{Z}_4[x]/(2x, x^2-2)\}$.
(iii) $|R| = 9$ *if and only if* $R \in \{\mathbb{F}_9, \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2)\}$.

Lemma 2.7 ([**9,** Proposition 3.4])**.** *Let R be a finite commutative local ring. Then*:

(i) $|U(R)| \neq 5$.
(ii) $|U(R)| = 2$ *if and only if* $R \in \{\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)\}$.
(iii) $|U(R)| = 3$ *if and only if* $R = \mathbb{F}_4$.
(iv) $|U(R)| = 4$ *if and only if* $R \in \{\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_4[x]/(2x, x^2-2),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_4[x]/(2x, x^2)\}$.
(v) $|U(R)| = 6$ *if and only if* $R \in \{\mathbb{Z}_7, \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2)\}$.

Let R be an Artinian ring. If $R/J(R)$ has no summands isomorphic to $\mathbb{Z}_2$, then $\Gamma(R)$ is connected by [**18,** Lemma 4.1]. If $R/J(R)$ has s (≥ 1) summands isomorphic to $\mathbb{Z}_2$, then $\Gamma(R)$ contains 2^{s-1} connected components (see [**18,** Theorem 1.2]). Note that, for a finite commutative ring R, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times R)$ is two copies of $\Gamma(\mathbb{Z}_2 \times R)$. Indeed, suppose that $a - b$ is an edge in $\Gamma(\mathbb{Z}_2 \times R)$. Then, $(0, a) - (1, b)$ and $(1, a) - (0, b)$ are two disjoint edges in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times R)$. This, together with Lemma 2.2, gives

Lemma 2.8. *Let R be a finite commutative ring and s a positive integer. Then, we have* $\gamma(\Gamma((\mathbb{Z}_2)^s \times R)) = 2^{s-1}\gamma(\Gamma(\mathbb{Z}_2 \times R))$.

Lemma 2.9. *Let R be a finite* (*commutative*) *ring. If $|U(R)| \leq 3$, then $\Gamma(R)$ is planar.*

Proof. If $|U(R)| \leq 2$, then the maximal degree of $\Gamma(R)$ is at most two; thus, $\Gamma(R)$ must be planar.

Suppose that $|U(R)| = 3$. Then, $2 = 0$ in R, and $\Gamma(R)$ is 3-regular by Lemma 2.4. Write $U(R) = \{u_1, u_2, u_3\}$. For a given $r \in R$, r is adjacent to $u_i + r$ $(i = 1, 2, 3)$. Two situations now arise.

Case 1. $u_1 + r$ is adjacent to $u_2 + r$. Then, $(u_1 + r) - (u_2 + r) = u_3$. Thus, $u_1 + r$ is adjacent to $u_3 + r$. In fact, $u_1 + r$ is adjacent to

$u_2 + r \Leftrightarrow u_1 + r$ is adjacent to $u_3 + r \Leftrightarrow u_2 + r$ is adjacent to $u_3 + r$. Hence, in this case, $r,\ u_1 + r,\ u_2 + r,\ u_3 + r$ form a complete graph K_4, which is 3-regular.

Case 2. $u_i + r$ is not adjacent to $u_j + r$ whenever $i \neq j$. Let the neighbors of $u_1 + r$ be $r,\ a,\ b$. We may assume $u_1 + r - a = u_2$ and $u_1 + r - b = u_3$. Then, $u_2 + r - a = u_1$ and $u_3 + r - b = u_1$. This means that a is adjacent to $u_2 + r$ and b is adjacent to $u_3 + r$. Let c be the third neighbor of $u_2 + r$. It can easily be verified that $c \notin \{u_1 + r,\ u_2 + r,\ u_3 + r,\ a, b, r\}$. Moreover, it must be that $u_2 + r - c = u_3$; thus, $u_3 + r - c = u_2$. This means that c is also a neighbor of $u_3 + r$. Now, consider the vertex a. Let x be the third neighbor of a. Then, it must be that $x - a = u_3$. Since $x - b = u_3 + a - b = u_1 + r - b + a - b = u_1 + r - a = u_2$, x is adjacent to b. Since $x - c = (u_3 + a) - (u_3 + r - u_2) = u_2 + r - a = u_1$, x is adjacent to c. It can also be verified that $x \notin \{a,\ b,\ u_1 + r,\ u_2 + r,\ u_3 + r,\ r,\ c\}$. Thus, the vertices $r,\ u_1 - r,\ u_2 - r,\ u_3 - r,\ a,\ b,\ c$ and x form a cube (see Figure 1), which is 3-regular.

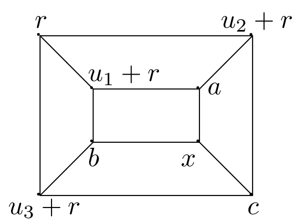

FIGURE 1.

Note that $\Gamma(R)$ cannot contain both K_4 and a cube as subgraphs. In fact, if $r,\ u_1 + r,\ u_2 + r,\ u_3 + r,\ a,\ b,\ c,\ x$ form a cube (as shown in Figure 1), then $u_1 - u_2 = (u_1 + r) - (u_2 + r)$ is not a unit; however, if $s,\ u_1 + s,\ u_2 + s,\ u_3 + s$ form a complete graph K_4, then $u_1 - u_2 = (u_1 + s) - (u_2 + s) = u_3$ is a unit. Hence, as $\Gamma(R)$ is 3-regular, either $\Gamma(R)$ is a disjoint union of copies of a cube, or $\Gamma(R)$ is a disjoint union of copies of K_4. Since a cube and K_4 are planar graphs, $\Gamma(R)$ is planar. $\square$

We are ready to prove our first result.

Theorem 2.10. *For a given positive integer g, there are at most finitely many finite commutative rings whose unitary Cayley graphs have genus g.*

Proof. Let R be a finite commutative ring with $\gamma(\Gamma(R)) = g$. It suffices to show that $|R|$ is bounded above by a constant depending only upon g.

If R is a field, then $\Gamma(R)$ is a complete graph $K_{|R|}$. Thus,

$$g = \gamma(\Gamma(R)) = \gamma(K_{|R|}) = \left\lceil \frac{(|R|-3)(|R|-4)}{12} \right\rceil$$

by Lemma 2.1. This yields $(|R|-3)(|R|-4) \leq 12g$, or $|R| \leq (7+\sqrt{49+48(g-1)})/2$, as desired.

If R is a local ring which is not a field, then $\mathfrak{m} = Z(R)$ is the maximal ideal of R, and $|R| \leq |Z(R)|^2$ by **[12,** Theorem 1]. Note that every element in $\mathfrak{m}$ is adjacent to each element in $1+\mathfrak{m} := \{1+a|a \in \mathfrak{m}\}$. Hence, $K_{|\mathfrak{m}|,|\mathfrak{m}|}$ is a subgraph of $\Gamma(R)$. Thus, we have

$$g = \gamma(\Gamma(R)) \geq \gamma(K_{|\mathfrak{m}|,|\mathfrak{m}|}) = \left\lceil \frac{(|\mathfrak{m}|-2)^2}{4} \right\rceil$$

by Lemma 2.1. This implies that $(|\mathfrak{m}|-2)^2 \leq 4g$ or $|\mathfrak{m}| \leq 2\sqrt{g}+2$. Therefore, $|R| \leq (2\sqrt{g}+2)^2$, as desired.

Now, suppose that R is not a local ring. We may assume that $R = (\mathbb{Z}_2)^s \times R_1 \times \cdots \times R_t$, where $s \geq 0$ and each R_i is a local ring with at least three elements. Since $g > 0$, we have $s \leq 1 + \log_2 g$ by Lemma 2.8. If $|R| \leq 12(g-1)$, we are done. Otherwise, by Lemmas 2.5 and 2.9, we have $4 \leq |U(R)| \leq 6$. As $|U(R)| = |U(R_1)| \times \cdots \times |U(R_t)|$, there are the following possibilities:

Case 1. $|U(R)| = 4$. Then, either $t = 1$ and $|U(R_1)| = 4$, or $t = 2$ and $|U(R_1)| = |U(R_2)| = 2$. From Lemma 2.7, the former gives $R \cong (\mathbb{Z}_2)^s \times R_1$, where $s \geq 1$ and

$$R_1 \in \left\{ \mathbb{Z}_5,\ \mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{(x^3)},\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)} \right\},$$

and the latter shows $R \cong (\mathbb{Z}_2)^s \times R_1 \times R_2$ where $s \geq 0$ and $R_1,\ R_2 \in \{\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2)\}$.

Case 2. $|U(R)| = 5$. Thus, $t = 1$ and $|U(R_1)| = 5$. However, this is impossible by Lemma 2.7.

Case 3. $|U(R)| = 6$. Then, either $t = 1$ and $|U(R_1)| = 6$, or $t = 2$, $|U(R_1)| = 2$ and $|U(R_2)| = 3$. By Lemma 2.7, the former shows $R \cong (\mathbb{Z}_2)^s \times R_1$, where $s \geq 1$ and $R_1 \in \{\mathbb{Z}_7,\ \mathbb{Z}_9,\ \mathbb{Z}_3[x]/(x^2)\}$, and the latter yields $R \cong (\mathbb{Z}_2)^s \times R_1 \times \mathbb{F}_4$, where $s \geq 0$ and $R_1 \in \{\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2)\}$.

In conclusion, in each case, we always have $|R_1 \times \cdots \times R_t| \leq 16$. It then follows that $|R| \leq 2^s \cdot 16 \leq 2^{(1+\log_2 g)} \cdot 16 = 32g$, as desired. $\square$

3. The unitary Cayley graphs of genus at most three. In this section, we determine the finite commutative rings whose unitary Cayley graphs have genus at most three. This is a preparation for the classification of the finite commutative rings whose unitary Cayley graphs have genus 1, 2 or 3, respectively.

Lemma 3.1 ([**25,** Corollaries 6.14, 6.15]). *Suppose a simple graph G is connected with $p \geq 3$ vertices and q edges. Then, $\gamma(G) \geq q/6 - p/2 + 1$. Furthermore, if G has no triangles, then $\gamma(G) \geq q/4 - p/2 + 1$.*

Lemma 3.2. *Let R be a finite ring. If $\gamma(\Gamma(R)) \leq 3$, then $|U(R)| \leq 8$.*

Proof. Let $|R| = n$ and $|U(R)| = k$. Then, $\Gamma(R)$ is k-regular with n vertices. Thus, $\Gamma(R)$ has $(kn)/2$ edges. From Lemma 3.1, $\gamma(\Gamma(R)) \geq kn/12 - n/2 + 1$. If $k \geq 9$, then $\gamma(\Gamma(R)) \geq 4$, a contradiction. Thus, $k \leq 8$. $\square$

Lemma 3.3. *Let R be a finite commutative local ring. If $\gamma(\Gamma(R)) \leq 3$, then $|R| \leq 13$.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. Since every element in $\mathfrak{m}$ is adjacent to each element in $1 + \mathfrak{m} = \{1 + a \mid a \in \mathfrak{m}\}$, $\Gamma(R)$ contains a subgraph $K_{|\mathfrak{m}|,|\mathfrak{m}|}$. If $|\mathfrak{m}| \geq 6$, then $\gamma(\Gamma(R)) \geq 4$ by Lemma 2.1, a contradiction. Therefore, $|\mathfrak{m}| \leq 5$. Thus, $|R| = |U(R)| + |\mathfrak{m}| \leq 8 + 5 = 13$ by Lemma 3.2. $\square$

Lemma 3.4. *Let S and T be finite commutative local rings that are not fields.*

(i) *If* $R = \mathbb{Z}_2 \times S$ *with* $|S| = 8$, *then* $\gamma(\Gamma(R)) = 2$.
(ii) *If* $R = \mathbb{Z}_3 \times S$ *with* $|S| = 4$, *then* $\gamma(\Gamma(R)) = 1$.
(iii) *If* $R = S \times T$ *with* $|S| = |T| = 4$, *then* $\gamma(\Gamma(R)) = 2$.

Proof.

(i) It is clear that $|U(S)| = |J(S)| = 4$. Thus, $|U(R)| = |J(R)| = 4$, and hence, $\Gamma(R)$ is 4-regular. Since each element in $J(R)$ is adjacent to every element in $U(R)$, we deduce that $\Gamma(R)$ comprises two copies of $K_{4,4}$. Hence, $\gamma(\Gamma(R)) = 2$ by Lemmas 2.1 and 2.2.

(ii) We have $S \in \{\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2)\}$. Note that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))$ have the same graph structure. Clearly, $\Gamma(R)$ is 4-regular and contains no triangles. Thus, $\gamma(\Gamma(R)) \geq 1$ by Lemma 3.1. On the other hand, we can embed $\Gamma(R)$ into $\mathbb{S}_1$, as shown in Figure 2. Hence, $\gamma(\Gamma(R)) = 1$.

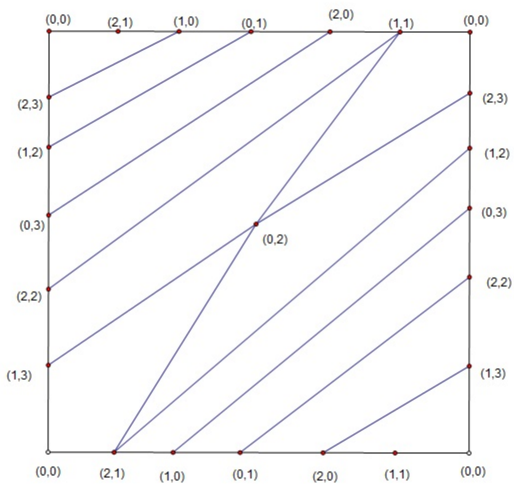

FIGURE 2. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$.

(iii) It is clear that $|U(S)| = |U(T)| = 2$ and $|J(S)| = |J(T)| = 2$. Thus, $|U(R)| = |J(R)| = 4$, and hence, $\Gamma(R)$ is 4-regular. Since each element in $J(R)$ is adjacent to every element in $U(R)$, we deduce that $\Gamma(R)$ comprises two copies of $K_{4,4}$. Therefore, $\gamma(\Gamma(R)) = 2$ by Lemmas 2.1 and 2.2. $\square$

Lemma 3.5. *Let S be a finite commutative local ring that is not a field.*

(i) *If $R = \mathbb{Z}_2 \times \mathbb{Z}_7$, then $\gamma(\Gamma(R)) \geq 5$.*
(ii) *If $R = \mathbb{Z}_2 \times S$ with $|S| = 9$, then $\gamma(\Gamma(R)) \geq 6$.*
(iii) *If $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$, then $\gamma(\Gamma(R)) \geq 7$.*
(iv) *If $R = \mathbb{F}_4 \times S$ with $|S| = 4$, then $\gamma(\Gamma(R)) \geq 5$.*

Proof.

(i) Since $|U(R)| = 6$, $\Gamma(R)$ is 6-regular. Note that $\Gamma(R)$ is a bipartite graph; thus, it contains no triangles. As $\Gamma(R)$ contains 14 vertices and 42 edges, $\gamma(\Gamma(R)) \geq 5$, by Lemma 3.1.

(ii) It is clear that $|U(R)| = |U(S)| = 6$; thus, $\Gamma(R)$ is 6-regular. Since $\Gamma(R)$ has 54 edges and 18 vertices and contains no triangles, $\gamma(\Gamma(R)) \geq 6$, by Lemma 3.1.

(iii) Since $|U(R)| = 6$, $\Gamma(R)$ is 6-regular. As $\Gamma(R)$ has 72 edges and 24 vertices and contains no triangles, $\gamma(\Gamma(R)) \geq 7$, by Lemma 3.1.

(iv) We have $|U(S)| = 2$; thus, $|U(R)| = 6$, and hence, $\Gamma(R)$ is 6-regular. Note that $\Gamma(R)$ contains no triangles. Indeed, if $(a_1, b_1) - (a_2, b_2) - (a_3, b_3) - (a_1, b_1)$ is a triangle in $\Gamma(R)$, then $b_1 - b_2$, $b_2 - b_3$ and $b_3 - b_1$ comprise three units in $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Thus, $0 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_1)$ is a sum of three units in $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. This is impossible. Since $\Gamma(R)$ has 48 edges and 16 vertices, $\gamma(\Gamma(R)) \geq 5$, by Lemma 3.1. □

Proposition 3.6. *Let R be a finite commutative ring. If $1 \leq \gamma(\Gamma(R)) \leq 3$, then R is isomorphic to one of the following rings*:

(i) $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{F}_8$, $\mathbb{F}_9$;
(ii) $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$;
(iii) $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$;
(iv) $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$.

Proof. Suppose that R is a field. In view of the proof of Theorem 2.10, the assumption that $\gamma(\Gamma(R)) \leq 3$ implies $|R| \leq 9$. Moreover, $R \notin \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4\}$ by Lemma 2.9; thus, $R \in \{\mathbb{Z}_5, \mathbb{Z}_7, \mathbb{F}_8, \mathbb{F}_9\}$.

Suppose that R is a local ring that is not a field. The assumption that $\gamma(\Gamma(R)) \leq 3$ implies $|R| \leq 13$ by Lemma 3.3. Note that, in this case, $|R| = p^n$ for some prime p and some $n > 1$. We deduce $|R| \in \{4, 8, 9\}$. However, $|R| \neq 4$ by Lemma 2.9; thus, $|R| \in \{8, 9\}$. Hence, by Lemma 2.6,

$$R \in \left\{\mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{(x^3)},\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)},\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{(x^2)}\right\}.$$

Suppose that R is not a local ring. Since $\gamma(\Gamma(R)) \leq 3$, in view of the proof of Theorem 2.10, we either have $|R| \leq 24$, or $R \cong (\mathbb{Z}_2)^s \times R_1 \times \cdots \times R_t$ with $0 \leq s \leq 2$ and $|U(R_1 \times \cdots \times R_t)| = 4$ or 6.

Case 1. $|R| \leq 24$. We write $R = (\mathbb{Z}_2)^s \times R_1 \times \cdots \times R_t$, where each R_i is a local ring with at least three elements. If $s = 0$, then $t = 2$. It follows that R is one of the following rings:

$$\begin{aligned}
&\mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\\
&\mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_7,\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_4 \times \mathbb{F}_4,\\
&\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4,\\
&\mathbb{F}_4 \times \mathbb{F}_4,\ \mathbb{Z}_4 \times \mathbb{Z}_5.
\end{aligned}$$

Since $|U(R)| \leq 8$ by Lemma 3.2, we have $R \neq \mathbb{Z}_3 \times \mathbb{Z}_7$ and $R \neq \mathbb{F}_4 \times \mathbb{F}_4$. If $R = \mathbb{Z}_3 \times \mathbb{Z}_5$, then $\Gamma(R)$ is a graph with 15 vertices and 60 edges; thus, $\gamma(\Gamma(R)) \geq 4$, by Lemma 3.1. If $R = \mathbb{Z}_4 \times \mathbb{Z}_5$, then $\Gamma(R)$ is a graph with 20 vertices and 80 edges; thus, $\gamma(\Gamma(R)) \geq 5$, by Lemma 3.1. If $R = \mathbb{Z}_4 \times \mathbb{F}_4$ or $R = \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$, then $\gamma(\Gamma(R)) \geq 5$ by Lemma 3.5 (iv). Hence, we obtain

$$R \in \left\{\mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}\right\}.$$

If $s = 1$, then $t = 1$ or 2. It follows that R is one of the following rings:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_{11}$;
(ii) $\mathbb{Z}_2 \times \mathbb{F}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$;

(iii) $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2\times\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_2\times\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2\times\mathbb{Z}_4[x]/(2x,x^2)$;
(iv) $\mathbb{Z}_2\times\mathbb{Z}_7$;
(v) $\mathbb{Z}_2\times\mathbb{Z}_5$;
(vi) $\mathbb{Z}_2\times\mathbb{F}_4$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2\times\mathbb{Z}_3$;
(vii) $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$;
(viii) $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{F}_4$.

From Lemma 3.2, the ring $\mathbb{Z}_2\times\mathbb{Z}_{11}$ should be ruled out. By Lemma 3.5, the rings $\mathbb{Z}_2\times\mathbb{Z}_9$, $\mathbb{Z}_2\times\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_2\times\mathbb{Z}_7$ and $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{F}_4$ should be ruled out. Note that $\Gamma(\mathbb{Z}_2\times\mathbb{F}_9)$ is an 8-regular graph. Thus, $\gamma(\Gamma(\mathbb{Z}_2\times\mathbb{F}_9))\geq 4$, by Lemma 3.1.

If $s=2$, then $t=1$. Hence, $|R_1|\leq 6$. It follows that R is one of the following rings:

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5,\qquad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{F}_4,$$
$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4,\qquad \mathbb{Z}_2\times\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{(x^2)}.$$

Applying Lemma 2.9, we have $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$ in this case.

If $s=3$, then $t=1$. Thus, $|R_1|\leq 3$. It follows that $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$. However, in this case, $\Gamma(R)$ is planar by Lemma 2.9.

Case 2. $R\cong(\mathbb{Z}_2)^s\times R_1\times\cdots\times R_t$ with $0\leq s\leq 2$ and $|U(R_1\times\cdots\times R_t)|=4$ or 6. If $s=0$, then $t=2$. Thus, $|U(R_1)|=|U(R_2)|=2$ or $|U(R_1)|=2$ and $|U(R_2)|=3$. From Lemma 2.7, R is one of the following rings:

$$\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_3\times\mathbb{Z}_4,\ \mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)},$$
$$\mathbb{Z}_4\times\mathbb{Z}_4,\ \mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^2)}\times\frac{\mathbb{Z}_2[x]}{(x^2)},$$
$$\mathbb{Z}_3\times\mathbb{F}_4,\ \mathbb{Z}_4\times\mathbb{F}_4,\ \frac{\mathbb{Z}_2[x]}{(x^2)}\times\mathbb{F}_4.$$

As shown in Case 1, the rings $\mathbb{Z}_4\times\mathbb{F}_4$ and $\mathbb{Z}_2[x]/(x^2)\times\mathbb{F}_4$ should be ruled out. If $s=1$ or 2, then R is one of the following rings:

(a) $\mathbb{Z}_2\times\mathbb{Z}_5$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$;
(b) $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2\times\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_2\times\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2\times\mathbb{Z}_4[x]/(2x,x^2)$;

(c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2)$;
(d) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$;
(e) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$;
(f) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$;
(g) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$;
(h) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$;
(i) $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$;
(j) $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$;
(k) $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$,
(l) $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$;
(m) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$;
(n) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{F}_4$;
(o) $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$.

From Lemmas 2.8 and 3.4 (i), (ii), the rings appearing in (c) and (f) should be ruled out. By Lemmas 2.8 and 3.4 (iii), the rings appearing in (g)–(i) should be ruled out. From Lemma 3.5, the rings appearing in (j)–(o) should be ruled out. Therefore, R is one of the rings appearing in (a), (b), (d) or (e). This completes the proof. $\square$

4. The classification. Akhtar, et al. **[2]** gave a list of finite commutative rings whose unitary Cayley graphs are planar (see **[2,** Theorem 8.2]). In view of their proof, the list should include the rings $\mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_2[x]/(x^2) \times B$, where B is a finite Boolean ring. We restate the result as follows.

Theorem 4.1 ([**2**]). *Let R be a finite commutative ring. Then, $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings*:

$$\mathbb{Z}_3,\ \mathbb{F}_4,\ B,\ \mathbb{Z}_3 \times B,\ \mathbb{F}_4 \times B,\ \mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_4 \times B,\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times B,$$

where B is a finite Boolean ring.

The goal is to classify the finite commutative rings whose unitary Cayley graphs have genus 1, 2, 3, respectively. In order to do so, we merely need to determine the genera of the unitary Cayley graphs of the rings appearing in Proposition 3.6.

Lemma 4.2. *The following statements hold*:

(i) $\gamma(\Gamma(\mathbb{Z}_5)) = \gamma(\Gamma(\mathbb{Z}_7)) = 1$.
(ii) $\gamma(\Gamma(\mathbb{F}_8)) = 2$.
(iii) $\gamma(\Gamma(\mathbb{F}_9)) = 3$.

Proof. Note that the unitary Cayley graph of a field is a complete graph. The claims follow by Lemma 2.1. □

A graph G is said to be complete 3-partite if the set of vertices of G can be partitioned into three disjoint sets V_1, V_2 and V_3 such that no two vertices within any V_i are adjacent, but, for $i \neq j$, every $a \in V_i$ is adjacent to every $b \in V_j$. The complete 3-partite complete graph with partitions $|V_i| = k_i$, $i = 1, 2, 3$, is denoted by K_{k_1,k_2,k_3}.

Lemma 4.3 ([**25,** Theorem 6.39]). $\gamma(K_{mn,n,n}) = [(mn-2)(n-1)]/2$. *In particular,* $\gamma(K_{3,3,3}) = 1$.

Lemma 4.4. *Let R be a finite commutative local ring that is not a field. If $|R| = 8$ or 9, then $\gamma(\Gamma(R)) = 1$.*

Proof. If $|R| = 8$, then $|U(R)| = |J(R)| = 4$. Thus, $\Gamma(R)$ is the complete bipartite graph $K_{4,4}$. Hence, $\gamma(\Gamma(R)) = 1$, by Lemma 2.1. If $|R| = 9$, then $\Gamma(R)$ is a complete 3-partite graph $K_{3,3,3}$. Hence, $\gamma(\Gamma(R)) = 1$, by Lemma 4.3. □

Lemma 4.5. *The following statements hold*:

(i) $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$.
(ii) $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 1$.
(iii) $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)) = 1$.

Proof.

(i) From Theorem 4.1, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is not planar. Thus, $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) \geq 1$. Figure 3 shows that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ can be embedded into $\mathbb{S}_1$. Hence, $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$.

(ii) By Theorem 4.1, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$ is not planar. However, we can embed $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$ into $\mathbb{S}_1$ as shown in Figure 4. Hence, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 1$.

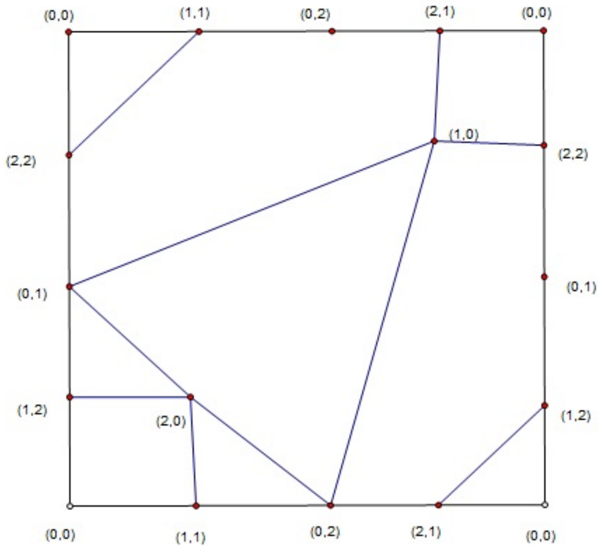

FIGURE 3. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

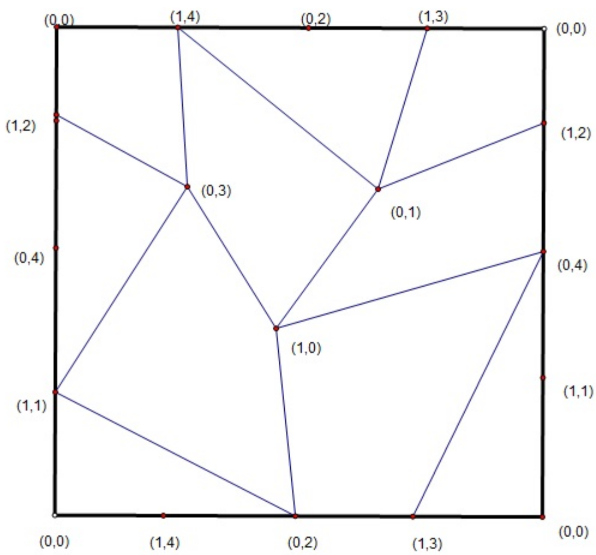

FIGURE 4. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$.

(iii) By Theorem 4.1, $\Gamma(\mathbb{Z}_3\times\mathbb{F}_4)$ is not planar. Write $\mathbb{F}_4 = \{0,1,a,b\}$. Then, we can embed $\Gamma(\mathbb{Z}_3\times\mathbb{F}_4)$ into $\mathbb{S}_1$ as shown in Figure 5. Therefore, $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)) = 1$. □

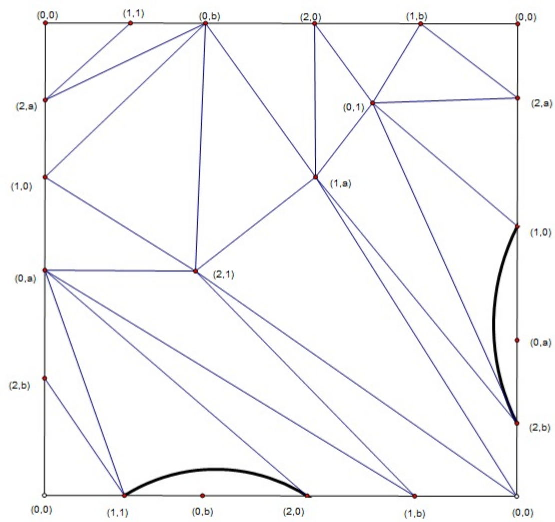

FIGURE 5. $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$.

We now prove the main result of this paper.

Theorem 4.6. *Let R be a finite commutative ring. Then*:

(i) $\gamma(\Gamma(R)) = 1$ *if and only if R is isomorphic to one of the following rings*:

$$\mathbb{Z}_5,\ \mathbb{Z}_7,\ \mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{(x^3)},\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)},$$
$$\mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{(x^2)},\ \mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_3\times\mathbb{Z}_4,\ \mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_3\times\mathbb{F}_4,$$
$$\mathbb{Z}_2\times\mathbb{Z}_5,\ \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3.$$

(ii) $\gamma(\Gamma(R)) = 2$ *if and only if R is isomorphic to one of the following rings*:

$$\mathbb{F}_8,\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)},$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5,\ \mathbb{Z}_2 \times \mathbb{Z}_8,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^3)},\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x, x^2-2)},$$

$$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x,y)^2},\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x, x^2)},$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}.$$

(iii) $\gamma(\Gamma(R)) = 3$ *if and only if* $R \cong \mathbb{F}_9$.

Proof. It suffices to classify the rings appearing in Proposition 3.6.

From Lemmas 3.4, 4.2, 4.4 and 4.5, the classification is settled as claimed, except for the rings

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \qquad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4,$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \qquad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

and

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

However, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5)) = 2\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 2$ by Lemma 4.5 (ii) and Lemma 2.8. Note that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4)$ comprises two copies of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$. Thus, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4)) = 2$, by Lemma 3.4 (ii). Similarly, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))) = 2$.

Since $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ has 36 edges, 18 vertices and no triangles, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) \geq 1$, by Lemma 3.1. However, we have $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_6)$, which can be embedded into $\mathbb{S}_1$, as shown in Figure 6. Therefore, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$.

Thus, by Lemma 2.8, we have $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$. This completes the proof. $\square$

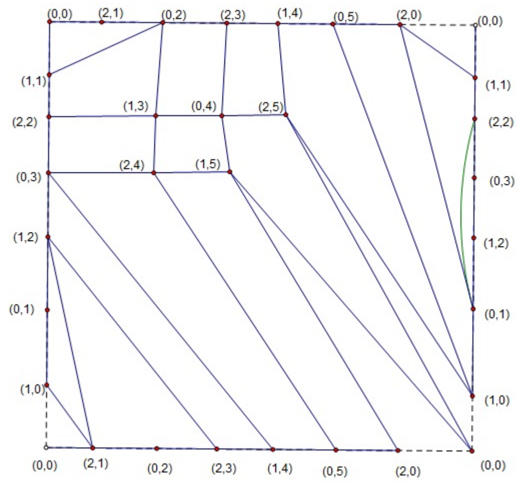

FIGURE 6. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_6)$.

We conclude this paper with a remark. The unitary Cayley graph of a ring R was generalized in [**14**], as follows: let S be a non-empty subset of $U(R)$ such that $s^{-1} \in S$ for all $s \in S$. The graph $\Gamma(R, S)$ is a simple graph with vertex set R and where two distinct vertices x and y are adjacent if $x + sy \in U(R)$ for some $s \in S$. When $S = \{-1\}$, $\Gamma(R, S) = \Gamma(R)$ is the unitary Cayley graph.

Remark 4.7. When finalizing the paper, we noticed the online paper [**3**], where the authors characterized the commutative Artinian rings R with $\Gamma(R, S)$ toroidal, i.e., having genus one (see [**3,** Theorem 4.2]). As its application, they presented a complete list of commutative Artinian rings whose unitary Cayley graphs are toroidal ([**3,** Corollary 4.5]). However, the proof of [**3,** Theorem 4.2] contains several incorrect arguments, and it turns out that their list in [**3,** Corollary 4.5] is incorrect. To be precise, the proof of [**3,** Theorem 4.2] contains the following wrong arguments:

(1) Case 3.1 for $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$. The claim that "$\Gamma(R)$ is a subgraph of $K_{4,4}$ and so $g(\Gamma(R)) = 1$" is wrong. Indeed, it can easily be verified that $\Gamma(R, S)$ is a cube; thus, $\Gamma(R, S)$ is planar.

(2) Case 3.2 for $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S = \{(1, 1, 1)\}$. The contracted technique is used to obtain a minor subgraph whose genus is greater

than one (see [**3,** Figure 5]), and this was accomplished by merging the three vertices 002, 121 and 001. However, this is wrong since 002 is not adjacent to either of 121 or 001.

(3) Case 3.2 for $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$. A similar argument was used to claim that the graph as shown in [**3,** Figure 7] has genus at least 2. However, this claim is wrong since $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$ can be embedded into $\mathbb{S}_1$ as shown in Figure 5 (so $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$ has genus 1).

As a matter of fact, we can also embed the graph $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_6)$ ($\cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$) into $\mathbb{S}_1$, as shown in Figure 6. Therefore, the complete list in [**3,** Corollary 4.5] should include the two rings: $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{F}_4$, and exclude the ring $\mathbb{Z}_2 \times \mathbb{F}_4$.

Acknowledgments. We would like to thank the referee for a careful reading of the paper and for his/her valuable comments.

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