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TORIC REPRESENTATIONS OF ALGEBRAS DEFINED BY CERTAIN NONSIMPLE POLYOMINOES

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ABSTRACT. In this paper, we give a toric representation of the associated ring of a polyomino which is obtained by removing a convex polyomino from its ambient rectangle.

Introduction. Polyominoes are two-dimensional objects which are obtained by joining squares of equal sizes edge-to-edge. They are originally rooted in recreational mathematics and combinatorics. For example, they have been studied in tiling problems of the plane. In combinatorial commutative algebra, polyominoes were first introduced in **[6]** by assigning each polyomino the ideal of inner 2-minors, or the *polyomino ideal*. The study of ideal of t-minors of an $m \times n$ matrix is a classical subject in commutative algebra. The class of polyomino ideals widely generalizes the class of ideals of 2-minors of the $m \times n$ matrix as well as the ideals of inner 2-minors attached to a one- or two-sided ladder.

Let $\mathcal{P}$ be a polyomino and K a field. We denote by $I_{\mathcal{P}}$, the polyomino ideal attached to $\mathcal{P}$, in a suitable polynomial ring over K. It is natural to investigate the algebraic properties of $I_{\mathcal{P}}$ depending on shape of $\mathcal{P}$. The classes of polyominoes whose polyomino ideal is prime have been discussed in many papers, including **[2, 3, 4, 6]**. The most outstanding result in these studies of polyomino ideals was given in **[7]**. It is proved that the polyomino ideals of simple polyominoes are prime by identifying their quotient rings with toric rings of the edge rings of graphs.

Recently, in **[4]**, it was shown that the polyomino ideal of the nonsimple polyomino, obtained by removing a convex polyomino from its ambient rectangle, is prime by using a localization argument. In the

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present paper, we give a toric representation of the quotient rings of the polyomino ideals of this class of nonsimple polyominoes.

1. Definitions and known results. We recall some definitions and notation from **[6]**. Given $a = (i, j)$ and $b = (k, l)$ in $\mathbb{N}^2$, we write $a \leq b$ if $i \leq k$ and $j \leq l$. We say that a and b are in *horizontal* (or *vertical*) *position* if $j = l$ (or $i = k$). The set

$$[a, b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}$$

is called an *interval*. If $i < k$ and $j < l$, then the vertices a and b are called *diagonal corners* and (i, l) and (k, j) are called *anti-diagonal corners* of $[a, b]$. The interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. The elements $a, a + (1, 0), a + (0, 1), a + (1, 1)$ are called *vertices* of C. We denote the set of vertices of C by $V(C)$. The sets

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\},$$

$$\{a + (1, 0), a + (1, 1)\},$$

and

$$\{a + (0, 1), a + (1, 1)\}$$

are called the *edges* of C. We denote the set of edges of C by $E(C)$.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. The vertex set of $\mathcal{P}$ is denoted by $V(\mathcal{P}) = \cup_{C \in \mathcal{P}} V(C)$. The edge set of $\mathcal{P}$ is denoted by $E(\mathcal{P}) = \cup_{C \in \mathcal{P}} E(C)$. Let C and D be two cells of $\mathcal{P}$. Then, C and D are said to be *connected* if there exists a sequence of cells $\mathcal{C} : C = C_1, \ldots, C_m = D$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. If, in addition, $C_i \neq C_j$ for all $i \neq j$, then $\mathcal{C}$ is called a *path* from C to D. The collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. For example, Figure 1 shows a polyomino.

Now, we recall the definition of polyomino ideals from **[6]**. Let $\mathcal{P}$ be a polyomino, and let K be a field. Let S be the polynomial ring over K with variables x_{ij} with $(i, j) \in V(\mathcal{P})$. A binomial $x_{ij}x_{kl} - x_{il}x_{kj}$ is called an *inner minor* of $\mathcal{P}$ if all of the cells $[(r, s), (r + 1, s + 1)]$ with $i \leq r \leq k - 1$ and $j \leq s \leq l - 1$ belong to $\mathcal{P}$. In that case, the interval $[(i, j), (k, l)]$ is called an *inner interval* of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of $\mathcal{P}$, called the *polyomino ideal* of $\mathcal{P}$. An interval $[a, b]$ with $a = (i, j)$ and $b = (k, l)$ is called a *horizontal*

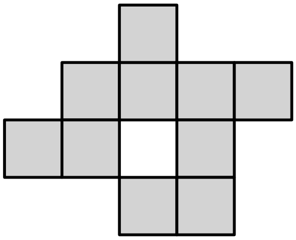

FIGURE 1. A polyomino.

edge interval of $\mathcal{P}$ if $j = l$ and the sets $\{(r, j), (r + 1, j)\} \in E(\mathcal{P})$ for $r = i, \ldots, k - 1$. Similarly, one defines *vertical edge interval.*

A polyomino $\mathcal{P}$ is called *simple* if, for any two cells C, D not belonging to $\mathcal{P}$, there exists a sequence of cells $C = C_1, \ldots, C_m = D$ such that each $C_i \notin \mathcal{P}$ and $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots m-1$. Roughly speaking, a simple polyomino is a polyomino with no "hole."

A polyomino $\mathcal{P}$ is called *row convex* if any two cells $C = [(i, j), (i + 1, j + 1)]$, $D = [(k, l), (k + 1, l + 1)]$ of $\mathcal{P}$ with $i < k$ with $j = l$ for all cells $[(l, j), (l + 1, j + 1)] \in \mathcal{P}$ for $i \leq l \leq k$. Similarly, one defines *column convex* polyominoes. A polyomino is called *convex* if it is both row and column convex.

For the polyomino ideals, the following classification of primeness is known.

Proposition 1.1. *Let $\mathcal{P}$ be a polyomino with one of the following conditions. Then, $I_{\mathcal{P}}$ is a prime ideal.*

(1) *$\mathcal{P}$ is a one-sided ladder* [**5**].
(2) *$\mathcal{P}$ is a two-sided ladder* [**1**].
(3) *$\mathcal{P}$ is row or column convex* [**6**].
(4) *$\mathcal{P}$ is balanced* [**2**].
(5) *$\mathcal{P}$ is simple* [**3, 7**].
(6) *$\mathcal{P}$ is obtained by removing a rectangle from its ambient rectangle* [**8**].
(7) *$\mathcal{P}$ is obtained by removing a convex polyomino from its ambient rectangle* [**4**].

Note that the first four classes of polyominoes are simple. Recall from [**5**] that a simple polyomino is called a *one-sided ladder* if it is of the following type:

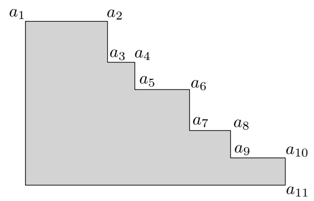

FIGURE 2. A one-sided ladder.

The sequence of vertices $a_1, \ldots, a_s$ of the corners of one-sided ladder $\mathcal{P}$ other than the opposite corner of the ladder is called the *defining sequence* of $\mathcal{P}$ if each a_i and a_{i+1} are in the horizontal or vertical position. For example, the sequence $a_1, a_2, \ldots, a_{11}$ in Figure 2 is the defining sequence of this one-sided ladder.

It is also known that there exist nonsimple polyominoes whose polyomino ideals are not prime. Figure 3 is one such example given in [**7**].

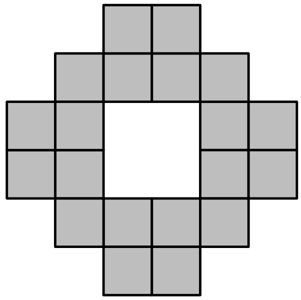

FIGURE 3. A polyomino with non-prime polyomino ideal.

2. The main result. The aim of this paper is to give a toric representation of the associated ring of a polyomino which is obtained by removing a convex polyomino from its ambient rectangle. In order to prove the main theorem, we give some properties of convex polyominoes.

Lemma 2.1. *Let $\mathcal{P}$ be a convex polyomino, and let $\mathcal{I}$ be the unique minimal interval such that $\mathcal{P} \subset \mathcal{I}$. Then*:

(a) *$\mathcal{I} \setminus \mathcal{P}$ consists of at most four connected components*;
(b) *each connected component of $\mathcal{I} \setminus \mathcal{P}$ contains exactly one corner vertex of $\mathcal{I}$*;
(c) *each connected component of $\mathcal{I} \setminus \mathcal{P}$ is a one-sided ladder.*

Let $\mathcal{P}$ be a convex polyomino. A vertex of $\mathcal{P}$ is called an *outside corner* if it belongs to exactly one cell of $\mathcal{P}$. On the other hand, a vertex of $\mathcal{P}$ is called an *inside corner* if it belongs to three cells of $\mathcal{P}$. A vertex is called an *interior vertex* if it belongs to four cells of $\mathcal{P}$. The *boundary vertices* are the vertices which are not interior vertices. A cell of $\mathcal{P}$ is called an *interior cell* if all of its four vertices are interior vertices. A cell of $\mathcal{P}$ is called an *boundary cell* if it is not an interior cell. We denote the set of boundary vertices of $\mathcal{P}$ by $\partial\mathcal{P}$.

To each interval $[a, b]$ we attach a polyomino $\mathcal{P}_{[a,b]}$ in the obvious way. Such a polyomino is called a *rectangle*. Hereafter, let $\mathcal{P}$ be a polyomino which is obtained by removing a convex polyomino $\mathcal{Q}$ from its ambient rectangle $\mathcal{P}_{[a,b]}$. We assume that

$$\partial\mathcal{P}_{[a,b]} \cap \partial\mathcal{Q} = \emptyset;$$

otherwise, $\mathcal{P}$ is a simple polyomino, and its toric representation is well studied in **[7]**. Also, we assume that $a = (1, 1)$ and $b = (m, n)$.

We define two types of intervals of $\mathcal{P}$ as follows:

(i) For the lowest corner e among all most left outside corners of $\mathcal{Q}$, let $\mathcal{I}_e = [a, e]$.
(ii) The maximal vertical or horizontal intervals $\mathcal{I}$ of $\mathcal{P}$.

For example, for the given polyomino, the intervals of types (i) and (ii) are displayed in Figures 4 and 5.

We denote the set of intervals of types (i) and (ii) by Λ. We define a map

$$\alpha : V(\mathcal{P}) \longrightarrow K[\{u_{\mathcal{I}}\}_{\mathcal{I}\in\Lambda}]$$

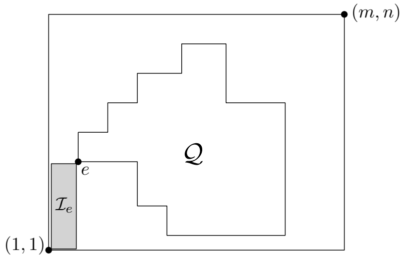

FIGURE 4. (i) Interval $\mathcal{I}_e$.

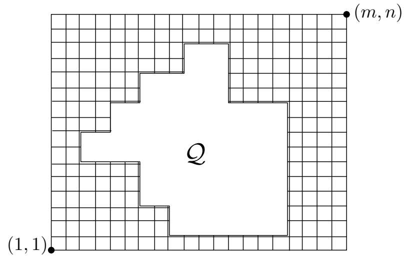

FIGURE 5. (ii) Edge intervals.

by

$$v \longmapsto \prod_{\substack{v \in \mathcal{I} \\ \mathcal{I} \in \Lambda}} u_{\mathcal{I}}.$$

Now, we define the toric ring and the toric ideal. The *toric ring* denoted by T is defined as

$$T = K[\alpha(v) \mid v \in V(\mathcal{P})] \subset K[\{u_{\mathcal{I}}\}_{\mathcal{I} \in \Lambda}].$$

Let $\varphi : S \to T$ be the surjective ring homomorphism with the setting $\varphi(x_{ij}) = \alpha((i, j))$. The *toric ideal* $J_{\mathcal{P}}$ is the kernel of φ. We claim that $J_{\mathcal{P}} = I_{\mathcal{P}}$. In order to prove this, we will repeatedly use the next lemma.

For any binomial $f = f^+ - f^- \in J_{\mathcal{P}}$, we let V_+ be the set of vertices v such that x_v appear in f^+. Similarly, one defines V_-. A binomial f in a binomial ideal $I \subset S$ is said to be *redundant* if it can be expressed as a linear combination of binomials in I of lower degree. Of course, a binomial is said to be *irredundant* if it is not redundant.

Lemma 2.2. *Let $f = f^+ - f^-$ be a binomial of degree ≥ 3 belonging to $J_{\mathcal{P}}$. If there exist three vertices $p, q \in V_+$ and $r \in V_-$ such that p, q are diagonal, respectively, anti-diagonal, corners of an inner interval and r is one of anti-diagonal, respectively, diagonal, corners of the inner interval, then f is redundant in $J_{\mathcal{P}}$.*

Proof. Let s be the other corner of the interval determined by p, q and r. Then:

$$\begin{aligned} f = f^+ - f^- &= x_p x_q \frac{f^+}{x_p x_q} - f^- \\ &= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r x_s \frac{f^+}{x_p x_q} - x_r \frac{f^-}{x_r} \\ &= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r \left(x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \right). \end{aligned}$$

Since $x_p x_q - x_r x_s$ is an inner minor of $\mathcal{P}$ and since $J_{\mathcal{P}}$ is a toric ideal, we have the desired conclusion. □

Theorem 2.3. *Let $\mathcal{P} = \mathcal{P}_{[(1,1),(m,n)]} \setminus \mathcal{Q}$ be a polyomino where $\mathcal{Q} \subset \mathcal{P}_{[(1,1),(m,n)]}$ is a convex polyomino. Then, $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. First, we show $I_{\mathcal{P}} \subset J_{\mathcal{P}}$. Let $x_p x_q - x_r x_s$ be an inner minor belonging to $I_{\mathcal{P}}$. Assume that p is the lower left corner, q the upper right corner, r the lower right corner and s is the upper left corner of $\mathcal{P}$. Since $[p, q]$ is an inner interval, it is clear that p and r, and q and s belong to the same maximal horizontal intervals. It is also clear that p and s, q and r belong to the same maximal vertical intervals. In order to show that $f = f^+ - f^- = x_p x_q - x_r x_s \in J_{\mathcal{P}}$, it suffices to show that the number of vertices in $V_+ = \{p, q\}$ belonging to $\mathcal{I}_e$ is equal to the number of vertices in $V_- = \{r, s\}$ belonging to $\mathcal{I}_e$. If $p \notin \mathcal{I}_e$, we see that $q, r, s \notin I_e$, and we are done in this case. Suppose that $p \in \mathcal{I}_e$. Since $[p, q]$ is an inner interval, if $r \in \mathcal{I}_e$, then we have that either both q and s belong to $\mathcal{I}_e$, or both q and s do not belong to $\mathcal{I}_e$. In these cases, we see that $x_p x_q - x_r x_s \in J_{\mathcal{P}}$. Similarly, if $r \notin \mathcal{I}_e$, then the only possibility is that $s \in \mathcal{I}_e$ and $q \notin \mathcal{I}_e$. Thus, we have $I_{\mathcal{P}} \subset J_{\mathcal{P}}$.

Next, in order to prove that $J_{\mathcal{P}} \subset I_{\mathcal{P}}$, it suffices to show that every binomial of degree 2 in $J_{\mathcal{P}}$ belongs to $I_{\mathcal{P}}$ and that every irredundant binomial in $J_{\mathcal{P}}$ is of degree 2. First, we show that every binomial $f \in J_{\mathcal{P}}$ of degree 2 belongs to $I_{\mathcal{P}}$. Suppose that $f = x_p x_q - x_r x_s \in J_{\mathcal{P}}$ is a binomial such that $\{p, q\} \neq \{r, s\}$.

Since $\varphi(x_p x_q) = \varphi(x_r x_s)$, we may assume that $[p, q]$ is an interval which has r and s as its anti-diagonal corners. Assume that the pair p and r and the pair s and q belong to the same horizontal edge interval. Then, we see that the pair p and s and the pair r and q belong to the same vertical edge interval. If $[p,q]$ is an inner minor of $\mathcal{P}$, then we are done. Suppose that $[p,q]$ is not an inner interval. Then, we either have $\mathcal{Q} \subset \mathcal{P}_{[p,q]}$ or $\mathcal{Q} \not\subset \mathcal{P}_{[p,q]}$ and $\mathcal{Q} \cap \mathcal{P}_{[p,q]} \neq \emptyset$. Suppose that $\mathcal{Q} \subset \mathcal{P}_{[p,q]}$. We see that $p \in \mathcal{I}_e$ and $q, r, s \notin \mathcal{I}_e$, where $\mathcal{I}_e$ is the interval given in Figure 4. Then, we have $u_{\mathcal{I}_e} \mid \varphi(x_p)$ and $u_{\mathcal{I}_e} \mid \varphi(x_r x_s)$, which contradicts $x_p x_q - x_r x_s \in J_{\mathcal{P}}$. Hence, this case is not possible.

Suppose that $\mathcal{Q} \not\subset \mathcal{P}_{[p,q]}$ and $[p,q]$ is not an inner interval of $\mathcal{P}$. We see that at least one of $[p,r]$, $[p,s]$, $[s,q]$ and $[r,q]$ is not an edge interval in $\mathcal{P}$. For example, say $[p,r]$ is not an edge interval in $\mathcal{P}$.

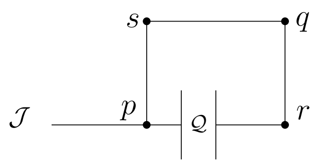

FIGURE 6. The maximal interval.

Suppose that $\mathcal{J} \in \Lambda$ is the maximal horizontal edge interval to which p belongs. Since $x_p x_q - x_r x_s \in J_{\mathcal{P}}$, we see that $u_{\mathcal{J}} \mid \varphi(x_p)$, and hence, $u_{\mathcal{J}} \mid \varphi(x_r x_s)$. This is a contradiction to the fact that neither r nor s belongs to $\mathcal{J}$ (see Figure 6). Hence this case is not possible. Thus, every binomial $f \in J_{\mathcal{P}}$ of degree 2 belongs to $I_{\mathcal{P}}$.

Now, we show that every binomial $f \in J_{\mathcal{P}}$ with $\deg f \geq 3$ is redundant. Suppose that $f = f^+ - f^-$ is an irredundant binomial with $\deg f \geq 3$.

First, we show that there does not exist any vertex $v \in V_+ \cup V_-$ such that $v \in \mathcal{I}_e$, where $\mathcal{I}_e$ is the interval shown in Figure 4. In order to show this, on the contrary, suppose that there exist $v_1 \in V_+ \cap \mathcal{I}_e$. Since $\varphi(f^+) = \varphi(f^-)$, we have a vertex $v_1' \in V_-$ such that $v_1' \in \mathcal{I}_e$. Also, we have a vertex v_2' such that v_1 and v_2' belong to the same maximal vertical edge interval. We see that there exists a vertex $v_2 \in V_+$ such that v_2 and v_1' belong to the same horizontal edge interval of $\mathcal{P}$.

If v_1' and v_1 are in the same horizontal interval, then, by applying Lemma 2.2 to the vertices v_1, v_1', v_2', we obtain that f is redundant, a contradiction. By using the same argument, we see that $v_2 \notin \mathcal{I}_e$.

Suppose that v_1' and v_1 are in the same vertical interval. Assume that the v is lower than v_1'. By using Lemma 2.1 (c), we observe that

v_1, v_2, v_1' are three corners of an inner interval. By applying Lemma 2.2, we see that f is redundant. Similarly, if v_1' is lower than v_1, we obtain that f is redundant. Hence, this case is not possible.

Finally, assume that v_1 and v_1' are not in the same edge intervals. If v_1 and v_2 belong to the same vertical edge interval, then, by applying Lemma 2.2 to the vertices v_2, v_1', v_2', we are done. Assume that the second coordinate of v_1 is less than that of v_1'. Let g and h be the other corners of the inner interval defined by v_1' and v_2'. Assume that v_1,v_2' and g belong to the same vertical edge interval. Then, we have $x_{v_1'}x_{v_2'} - x_g x_h \in J_{\mathcal{P}}$ and

$$\begin{aligned} f = f^+ - f^- &= f^+ - x_{v_1'}x_{v_2'}\frac{f^-}{x_{v_1'}x_{v_2'}} \\ &= f^+ - (x_{v_1'}x_{v_2'} - x_g x_h)\frac{f^-}{x_{v_1'}x_{v_2'}} - x_g x_h \frac{f^-}{x_{v_1'}x_{v_2'}}. \end{aligned}$$

Let

$$f' = f'^+ - f'^- = f^+ - x_g x_h \frac{f^-}{x_{v_1'}x_{v_2'}},$$

and let V_+' and V_-' be the vertices appearing in f'^+ and f'^-. Note that, since f and $x_{v_1'}x_{v_2'} - x_g x_h$ are binomials belonging to $J_{\mathcal{P}}$, $f' \in J_{\mathcal{P}}$. Then, by applying Lemma 2.2 to the vertices $v_1, v_2 \in V_+'$ and $g \in V_-'$, we obtain that f' is redundant, which implies that f is redundant. Thus, the vertices appearing in f do not belong to $\mathcal{I}_e$, in other words, we have

$$f \in J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e].$$

Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ which consists of all cells of $\mathcal{P}$ having no vertices belonging to $\mathcal{I}_e$. Then, we have

$$I_{\mathcal{P}'} = I_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e].$$

We observe that $\mathcal{P}'$ is a simple polyomino. Next, notice that $\alpha(v)$ for each $v \in \mathcal{P} \setminus \mathcal{I}_e$ is a monomial of degree 2 determined by the maximal horizontal and vertical intervals to which v belongs. Then, it is known from [**7,** Theorem 2.2] that

$$I_{\mathcal{P}'} = I_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e] = J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e].$$

Note that, if f is irredundant in $J_{\mathcal{P}}$, then it is also irredundant in $J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$ since we have

$$J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e] \subset J_{\mathcal{P}}.$$

We know that

$$J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$$

is generated by binomials of degree 2 since we have

$$I_{\mathcal{P}'} = J_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$$

is generated by binomials of degree 2. This is a contradiction. Hence, we have the desired conclusion. $\square$

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