

TENSOR PRODUCT OF DUALIZING COMPLEXES OVER A FIELD

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ABSTRACT. Let \mathbb{k} be a field, and let X and Y be two locally noetherian \mathbb{k} -schemes (respectively, \mathbb{k} -formal schemes) with dualizing complexes R_X and R_Y , respectively. We show that $R_X \boxtimes_{\mathbb{k}} R_Y$ (respectively, its derived completion) is a dualizing complex over $X \times_{\mathbb{k}} Y$ if and only if $X \times_{\mathbb{k}} Y$ is locally noetherian of finite Krull dimension.

Introduction. Throughout this note, rings are assumed to be commutative and unital. Given a ring A , we denote by $D(\text{Mod } A)$ the derived category of A -modules, and by $D^b(\text{Mod } A)$ and $D_f^b(\text{Mod } A)$ its triangulated subcategories made of bounded complexes, as well as bounded complexes with coherent cohomology, respectively. We will also use commutative DG-algebras. Given such a DG-algebra A , we will denote the category of differential graded A -modules by $DG\text{Mod } A$, and its derived category by $D(DG\text{Mod } A)$.

Dualizing complexes, first introduced [7] half a century ago, are now a ubiquitous tool in commutative algebra and algebraic geometry. In this note, we are concerned with dualizing complexes over a fibre product of schemes or formal schemes over a field.

In the first section, we work with ordinary schemes. Our main result in Section 1 shows that, if two locally noetherian schemes X and Y , over a field \mathbb{k} , have dualizing complexes R_X and R_Y , then the only obstruction for $X \times_{\mathbb{k}} Y$ to possess a dualizing complex is the trivial one, namely, $X \times_{\mathbb{k}} Y$ must be locally noetherian and of finite Krull dimension. In that case, we show that the box tensor product $R_X \boxtimes_{\mathbb{k}} R_Y$ is a dualizing complex over $X \times_{\mathbb{k}} Y$. This is proven in Corollary 1.7

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below. If the schemes involved are of finite type over \mathbb{k} , then this is not new, and could easily be deduced from the results of [7]. In fact, in that case, \mathbb{k} can even be replaced by a Gorenstein ring; assume one of X, Y is flat over it, and replace the tensor product with the derived tensor product. However, we make no finiteness assumption on either of the maps $X \rightarrow \mathbb{k}, Y \rightarrow \mathbb{k}$.

One interesting consequence of this result in the affine case, given in Corollary 1.8 below, is the fact that, for such noetherian rings, the tensor product functor

$$- \otimes_{\mathbb{k}} - : D_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A) \times D_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} B) \longrightarrow D_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A \otimes_{\mathbb{k}} B)$$

preserves finite injective dimension.

In Section 2, we switch to the more difficult case of formal schemes. We are able to reproduce the above result in the formal case and prove that, if \mathfrak{X} and \mathfrak{Y} are two locally noetherian formal schemes over a field \mathbb{k} , with dualizing complexes $R_{\mathfrak{X}}$ and $R_{\mathfrak{Y}}$, respectively, and, if $\mathfrak{X} \times_{\mathbb{k}} \mathfrak{Y}$ is locally noetherian and of finite Krull dimension, then the derived completion and derived torsion of $R_{\mathfrak{X}} \boxtimes_{\mathbb{k}} R_{\mathfrak{Y}}$ are c-dualizing and t-dualizing complexes (notions that are recalled in Section 2) over $\mathfrak{X} \times_{\mathbb{k}} \mathfrak{Y}$. This is given in Theorem 2.9 below.

In order to understand why the formal case is much more involved, consider the simplest corresponding affine situation, where \mathbb{k} is a field, and A and B are two noetherian Gorenstein \mathbb{k} -algebras of finite Krull dimension, which are adically complete with respect to some ideals $\mathfrak{a} \subseteq A$ and $\mathfrak{b} \subseteq B$. The Gorenstein hypothesis implies that A and B are dualizing complexes over themselves; thus, we need to prove that, in this situation, the completed tensor product $A \widehat{\otimes}_{\mathbb{k}} B$ is also a Gorenstein ring whenever it is noetherian of finite Krull dimension. However, the ring $A \otimes_{\mathbb{k}} B$ is usually non-noetherian; hence, we do not know if the completion map $A \otimes_{\mathbb{k}} B \rightarrow A \widehat{\otimes}_{\mathbb{k}} B$ is flat, and thus, we do not know if, in general, the maps $A \rightarrow A \widehat{\otimes}_{\mathbb{k}} B$ and $B \rightarrow A \widehat{\otimes}_{\mathbb{k}} B$ are flat. This rules out attempts to prove such a result using the methods in the corresponding discrete case (i.e., when $\mathfrak{a} = 0$ and $\mathfrak{b} = 0$, so that $A \otimes_{\mathbb{k}} B$ is noetherian) given in [14] and other similar papers. As a replacement for flatness, we heavily rely on the theory of weakly proregular ideals of [4, 10, 12]. Using it as well as other homological and homotopical tools, we are able to prove the above-mentioned result concerning dualizing complexes over the fiber product of formal schemes.

1. Tensor product of dualizing complexes over ordinary schemes. We shall need the following result, contained in the proof of [8, Corollary 1.4].

Proposition 1.1. *Let A be a commutative noetherian ring. Assume that A has a dualizing complex. Then, there is a finite type A -algebra A' which is Gorenstein of finite Krull dimension, and such that there is a surjection $A' \twoheadrightarrow A$.*

Proof. From [8, Corollary 1.4], there is a Gorenstein ring A' of finite Krull dimension and a surjection $A' \twoheadrightarrow A$; thus, it is sufficient to verify that this ring is a finitely generated A -algebra. The first step in the proof of [8, Corollary 1.4] reduces to the case where the codimension function of A is constant on its associated primes. One way to do this is using [8, Lemma 5.5], which says that, if a ring A is noetherian, universally catenary, and has a codimension function (all these properties are satisfied by a ring possessing a dualizing complex), then there exist a finite type A -algebra B whose codimension function is constant on its associated primes, and such that there is a surjection $B \twoheadrightarrow A$. Hence, we may assume without loss of generality that the codimension function of A is constant on its associated primes. Next, for such an A , it is shown in [8] that there is some ideal $I \subseteq A$ of positive height such that the Rees algebra

$$A(I) = \bigoplus_{n \geq 0} (Ix)^n \subseteq A[x]$$

is Cohen-Macaulay. Note that, if $I = (f_1, \dots, f_m)$, then the map

$$A[x_1, \dots, x_m] \longrightarrow A(I)$$

given by $x_i \mapsto f_i \cdot x$ is surjective so that $A(I)$ is of finite type over A . Since A is a quotient of $A(I)$, we reduce to the case where A is Cohen-Macaulay. Then, in the final step of [8, Corollary 1.4], it is observed that, by [13, Theorem 4.3], there is some finitely generated A -module M such that the trivial extension ring $A' := A \times M$ (in the sense of [9, page 191, Chapter 25]) is a finite-dimensional Gorenstein ring. Since there is a surjection $A' \rightarrow A$, and since A' is clearly a finite type A -algebra, we are done. \square

Lemma 1.2. *Let \mathbb{k} be a field, and let A and B be two noetherian \mathbb{k} -algebras with dualizing complexes R_A and R_B , respectively, such that $A \otimes_{\mathbb{k}} B$ is a noetherian ring. Then, the canonical map*

$$A \otimes_{\mathbb{k}} B \longrightarrow \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(R_A \otimes_{\mathbb{k}} R_B, R_A \otimes_{\mathbb{k}} R_B)$$

is an isomorphism in $\mathrm{D}(\mathrm{Mod} A \otimes_{\mathbb{k}} B)$.

Proof. Since R_A and R_B are dualizing complexes, they have finitely generated bounded cohomologies, and, in particular, we may assume that they are bounded. Hence, by [17, Lemma 8.4], there is an isomorphism

$$\mathrm{RHom}_A(R_A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_B(R_B, R_B) \cong \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(R_A \otimes_{\mathbb{k}} R_B, R_A \otimes_{\mathbb{k}} R_B)$$

in $\mathrm{D}(\mathrm{Mod} A \otimes_{\mathbb{k}} B)$. Again, the fact that R_A and R_B are dualizing complexes implies that

$$\mathrm{RHom}_A(R_A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_B(R_B, R_B) \cong A \otimes_{\mathbb{k}} B.$$

Composing these two isomorphisms, we deduce that there is some isomorphism

$$\mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(R_A \otimes_{\mathbb{k}} R_B, R_A \otimes_{\mathbb{k}} R_B) \cong A \otimes_{\mathbb{k}} B$$

in $\mathrm{D}(\mathrm{Mod} A \otimes_{\mathbb{k}} B)$. Hence, by an unpublished result of Foxby, given in [3, Proposition 2.3], the canonical map

$$A \otimes_{\mathbb{k}} B \longrightarrow \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(R_A \otimes_{\mathbb{k}} R_B, R_A \otimes_{\mathbb{k}} R_B)$$

is also an isomorphism. \square

The next lemma is probably well known. We reproduce its easy proof for the convenience of the reader:

Lemma 1.3. *Let A be a noetherian ring, and let R be a dualizing complex over A . A complex $M \in \mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A)$ has finite injective dimension over A if and only if the complex $\mathrm{RHom}_A(M, R)$ is perfect.*

Proof. Since $\mathrm{RHom}_A(M, R)$ has finitely generated cohomologies, by [2, Corollary 2.10.F], it is perfect if and only if it has finite flat dimension, and, by [7, Proposition V.2.6], this occurs if and only if

$$\mathrm{RHom}_A(\mathrm{RHom}_A(M, R), R) \cong M$$

has finite injective dimension over A . \square

Lemma 1.4. *Let \mathbb{k} be a field, and let A and B be two noetherian \mathbb{k} -algebras such that $A \otimes_{\mathbb{k}} B$ is also noetherian. Assume that there are dualizing complexes R_A over A and R_B over B such that $R_A \otimes_{\mathbb{k}} R_B$ is a dualizing complex over $A \otimes_{\mathbb{k}} B$. Then, for any dualizing complex S_A over A , and any dualizing complex S_B over B , the complex $S_A \otimes_{\mathbb{k}} S_B$ is a dualizing complex over $A \otimes_{\mathbb{k}} B$.*

Proof. By Lemma 1.2, the canonical map

$$A \otimes_{\mathbb{k}} B \longrightarrow \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(S_A \otimes_{\mathbb{k}} S_B, S_A \otimes_{\mathbb{k}} S_B)$$

is an isomorphism; thus, it is enough to show that $S_A \otimes_{\mathbb{k}} S_B$ has finite injective dimension over $A \otimes_{\mathbb{k}} B$. Since, by assumption, $R_A \otimes_{\mathbb{k}} R_B$ is a dualizing complex, from Lemma 1.3, it is sufficient to show that the complex

$$\mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(S_A, \otimes_{\mathbb{k}} S_B, R_A \otimes_{\mathbb{k}} R_B)$$

is perfect. However, this is clear since, by [17, Lemma 8.4], it is isomorphic to

$$\mathrm{RHom}_A(S_A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_B(S_B, R_B),$$

and since the (box) tensor product of two finitely generated projectives is a finitely generated projective. \square

In the next lemma, we must use differential graded algebras as well as dualizing DG-modules over them. We refer the reader to [16] for the terminology regarding DG-algebras used in this lemma. The lemma essentially states that, if, for a pair of \mathbb{k} -algebras A' and B' , our main theorem regarding tensor product of dualizing complexes holds, then it also holds for any pair of quotients $A' \twoheadrightarrow A$, $B' \twoheadrightarrow B$.

Lemma 1.5. *Let \mathbb{k} be a field, and let A' and B' be two noetherian \mathbb{k} -algebras with dualizing complexes R_A and R_B , respectively. Assume that $R_A \otimes_{\mathbb{k}} R_B$ is a dualizing complex over the noetherian ring $A' \otimes_{\mathbb{k}} B'$. Let A be an A' -algebra, and let B be a B' -algebra such that the structure maps $A' \rightarrow A$ and $B' \rightarrow B$ are surjective. Then:*

$$\mathrm{RHom}_{A'}(A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_{B'}(B, R_B)$$

is a dualizing complex over $A \otimes_{\mathbb{k}} B$.

Proof. Since the map $A' \otimes_{\mathbb{k}} B' \rightarrow A \otimes_{\mathbb{k}} B$ is finite, it follows from [7, Proposition V.2.4] that

$$\mathrm{R Hom}_{A' \otimes_{\mathbb{k}} B'}(A \otimes_{\mathbb{k}} B, R_A \otimes_{\mathbb{k}} R_B)$$

is a dualizing complex over $A \otimes_{\mathbb{k}} B$. From [17, Lemma 8.4], there is an isomorphism

$$\mathrm{R Hom}_{A'}(A, R_A) \otimes_{\mathbb{k}} \mathrm{R Hom}_{B'}(B, R_B) \cong \mathrm{R Hom}_{A' \otimes_{\mathbb{k}} B'}(A \otimes_{\mathbb{k}} B, R_A \otimes_{\mathbb{k}} R_B)$$

obtained by replacing A and B by projective resolutions over A' and B' , respectively. However, this isomorphism is only $A' \otimes_{\mathbb{k}} B'$ -linear, and it is not currently known how to show that there is such an $A \otimes_{\mathbb{k}} B$ -linear isomorphism. Instead, using [1, Proposition 2.2.8], let $A' \rightarrow \tilde{A} \cong A$ and $B' \rightarrow \tilde{B} \cong B$ be DG-algebra resolutions of $A' \rightarrow A$ and $B' \rightarrow B$, respectively, such that $\tilde{A}^0 = A'$ and $\tilde{B}^0 = B'$. For each $i < 0$, \tilde{A}^i is a finitely generated projective \tilde{A}^0 -module, \tilde{B}^i is a finitely generated projective \tilde{B}^0 -module, and, for each $i > 0$, $\tilde{A}^i = \tilde{B}^i = 0$. Then, as shown in the proof of [17, Lemma 8.4], the natural map

$$\mathrm{Hom}_{A'}(\tilde{A}, R_A) \otimes_{\mathbb{k}} \mathrm{Hom}_{B'}(\tilde{B}, R_B) \longrightarrow \mathrm{Hom}_{A' \otimes_{\mathbb{k}} B'}(\tilde{A} \otimes_{\mathbb{k}} \tilde{B}, R_A \otimes_{\mathbb{k}} R_B),$$

which is clearly $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$ -linear, is an isomorphism. Since \tilde{A} is K-projective over A' , \tilde{B} is K-projective over B' , and $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$ is K-projective over $A' \otimes_{\mathbb{k}} B'$, we deduce that there is an isomorphism

$$\mathrm{R Hom}_{A'}(\tilde{A}, R_A) \otimes_{\mathbb{k}} \mathrm{R Hom}_{B'}(\tilde{B}, R_B) \cong \mathrm{R Hom}_{A' \otimes_{\mathbb{k}} B'}(\tilde{A} \otimes_{\mathbb{k}} \tilde{B}, R_A \otimes_{\mathbb{k}} R_B)$$

in $\mathrm{D}(\mathrm{DGMod} \tilde{A} \otimes_{\mathbb{k}} \tilde{B})$. From [16, Proposition 7.5(1)], the right hand side is a dualizing DG-module over $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$ so that the left hand side is also a dualizing DG-module. As there are isomorphisms

$$\mathrm{R Hom}_{A'}(\tilde{A}, R_A) \cong \mathrm{R Hom}_{A'}(A, R_A)$$

and

$$\mathrm{R Hom}_{B'}(\tilde{B}, R_B) \cong \mathrm{R Hom}_{B'}(B, R_B)$$

over \tilde{A} and \tilde{B} , respectively, it follows that the DG-module

$$\mathrm{R Hom}_{A'}(A, R_A) \otimes_{\mathbb{k}} \mathrm{R Hom}_{B'}(B, R_B)$$

is a dualizing DG-module over $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$. Set $R = \mathrm{RHom}_{A'}(A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_{B'}(B, R_B) \in \mathrm{D}(\mathrm{Mod} A \otimes_{\mathbb{k}} B)$. Since

$$\tilde{A} \otimes_{\mathbb{k}} \tilde{B} \longrightarrow A \otimes_{\mathbb{k}} B$$

is a quasi-isomorphism, the fact that the image of R in the derived category over $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$ has a finite injective dimension implies that R has finite injective dimension over $A \otimes_{\mathbb{k}} B$. From Lemma 1.2, the canonical map $A \otimes_{\mathbb{k}} B \rightarrow \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(R, R)$ is an isomorphism. Hence, $R = \mathrm{RHom}_{A'}(A, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_{B'}(B, R_B)$ is a dualizing complex over $A \otimes_{\mathbb{k}} B$. \square

We now give the main result of this section.

Theorem 1.6. *Let \mathbb{k} be a field, and let A, B be commutative noetherian \mathbb{k} -algebras. Assume that A and B have dualizing complexes. Then, the ring $A \otimes_{\mathbb{k}} B$ has a dualizing complex if and only if $A \otimes_{\mathbb{k}} B$ is noetherian of finite Krull dimension. In that case, for every dualizing complex R_A over A and R_B over B , the complex $R_A \otimes_{\mathbb{k}} R_B$ is a dualizing complex over $A \otimes_{\mathbb{k}} B$.*

Proof. The only if part is well known. Assume that $A \otimes_{\mathbb{k}} B$ is noetherian of finite Krull dimension. Let $A' \twoheadrightarrow A$ and $B' \twoheadrightarrow B$ be the Gorenstein rings guaranteed to exist from Proposition 1.1. Since $A \otimes_{\mathbb{k}} B$ is noetherian of finite Krull dimension, and since A' (respectively B') is a finite type A (respectively, B)-algebra, it follows that $A' \otimes_{\mathbb{k}} B'$ is also noetherian of finite Krull dimension. Hence, by [14, Theorem 6(a)], the ring $A' \otimes_{\mathbb{k}} B'$ is also Gorenstein. Let $R := \mathrm{RHom}_{A'}(A, A')$, and let $S := \mathrm{RHom}_{B'}(B, B')$. As A' is Gorenstein of finite Krull dimension, A' is a dualizing complex over A' so that, by [7, Proposition V.2.4], R is a dualizing complex over A , and, in the same manner, S is a dualizing complex over B . Similarly, since $A' \otimes_{\mathbb{k}} B'$ is Gorenstein of finite Krull dimension, $A' \otimes_{\mathbb{k}} B'$ is a dualizing complex over $A' \otimes_{\mathbb{k}} B'$. Thus, the conditions of Lemma 1.5 are satisfied for $A' \rightarrow A$ and $B' \rightarrow B$ so that

$$\mathrm{RHom}_{A'}(A, A') \otimes_{\mathbb{k}} \mathrm{RHom}_{B'}(B, B') = R \otimes_{\mathbb{k}} S$$

is a dualizing complex over $A \otimes_{\mathbb{k}} B$. Hence, by Lemma 1.4, the same is true for the complex $R_A \otimes_{\mathbb{k}} R_B$. \square

Since the property of being a dualizing complex on an affine open cover can easily be verified (due to [7, Lemma II.7.16]), we obtain:

Corollary 1.7. *Let \mathbb{k} be a field, and let X and Y be two locally noetherian \mathbb{k} -schemes with dualizing complexes R_X and R_Y , respectively. If $X \times_{\mathbb{k}} Y$ is locally noetherian of finite Krull dimension, then $R_X \boxtimes_{\mathbb{k}} R_Y$ is a dualizing complex over $X \times_{\mathbb{k}} Y$.*

Corollary 1.8. *Let \mathbb{k} be a field, and let A and B be two \mathbb{k} -algebras. Assume that A and B have dualizing complexes R_A and R_B , respectively, and that $A \otimes_{\mathbb{k}} B$ is noetherian of finite Krull dimension. Given a complex $M \in D_{\mathbb{f}}^b(\text{Mod } A)$ and a complex $N \in D_{\mathbb{f}}^b(\text{Mod } B)$, if M has finite injective dimension over A , and N has finite injective dimension over B , then $M \otimes_{\mathbb{k}} N$ has finite injective dimension over $A \otimes_{\mathbb{k}} B$.*

Proof. From Lemma 1.3, the complexes

$$\mathbf{R} \text{Hom}_A(M, R_A) \quad \text{and} \quad \mathbf{R} \text{Hom}_B(N, R_B)$$

are perfect over A and B , respectively. Hence, the complex

$$\mathbf{R} \text{Hom}_A(M, R_A) \otimes_{\mathbb{k}} \mathbf{R} \text{Hom}_B(N, R_B)$$

is perfect over $A \otimes_{\mathbb{k}} B$. By [17, Lemma 8.4], there is an isomorphism

$$\mathbf{R} \text{Hom}_A(M, R_A) \otimes_{\mathbb{k}} \mathbf{R} \text{Hom}_B(N, R_B) \cong \mathbf{R} \text{Hom}_{A \otimes_{\mathbb{k}} B}(M \otimes_{\mathbb{k}} N, R_A \otimes_{\mathbb{k}} R_B),$$

and since, by Theorem 1.6, $R_A \otimes_{\mathbb{k}} R_B$ is a dualizing complex over $A \otimes_{\mathbb{k}} B$, we have that

$$M \otimes_{\mathbb{k}} N \cong \mathbf{R} \text{Hom}_{A \otimes_{\mathbb{k}} B}(\mathbf{R} \text{Hom}_{A \otimes_{\mathbb{k}} B}(M \otimes_{\mathbb{k}} N, R_A \otimes_{\mathbb{k}} R_B), R_A \otimes_{\mathbb{k}} R_B);$$

thus, the result follows from again applying Lemma 1.3. \square

Remark 1.9. The fact that Corollary 1.8 follows from the theorem regarding the tensor product of dualizing complexes was already observed in [17, Corollary 8.6], in a noncommutative situation. The result given there, in the commutative setting, makes the assumption that both A and B are finitely generated \mathbb{k} -algebras.

2. Tensor product of dualizing complexes for formal schemes.

We now turn to generalize Theorem 1.6 to formal schemes. In order to do so, we first recall some adic homological algebra. The reader is referred to [4, 5, 10, 11, 12] for a detailed treatment of the material below. By a preadic ring (A, \mathfrak{a}) , we shall mean a commutative ring

A equipped with an adic topology generated by some finitely generated ideal $\mathfrak{a} \subseteq A$. (It is important to note that we do not assume that A is noetherian.) Given a preadic ring (A, \mathfrak{a}) , there are functors $\Gamma_{\mathfrak{a}}(-) := \varinjlim \operatorname{Hom}_A(A/\mathfrak{a}^n, -)$ and $\Lambda_{\mathfrak{a}}(-) := \varprojlim A/\mathfrak{a}^n \otimes_A -$, called the \mathfrak{a} -torsion and \mathfrak{a} -completion functors. These are both additive functors

$$\operatorname{Mod} A \longrightarrow \operatorname{Mod} A.$$

The A -module $\widehat{A} := \Lambda_{\mathfrak{a}}(A)$ has a structure of a commutative ring, and there is a natural map $A \rightarrow \widehat{A}$. If this map is bijective, then we will call (A, \mathfrak{a}) an adic ring and say that A is \mathfrak{a} -adically complete. For any $M \in \operatorname{Mod} A$, the A -modules $\Gamma_{\mathfrak{a}}(M)$ and $\Lambda_{\mathfrak{a}}(M)$ naturally carry a \widehat{A} -module structure so that we obtain functors $\widehat{\Gamma}_{\mathfrak{a}}, \widehat{\Lambda}_{\mathfrak{a}} : \operatorname{Mod} A \rightarrow \operatorname{Mod} \widehat{A}$, defined by exactly the same formulae as $\Gamma_{\mathfrak{a}}$ and $\Lambda_{\mathfrak{a}}$. The derived functors

$$\operatorname{R}\Gamma_{\mathfrak{a}}, \operatorname{L}\Lambda_{\mathfrak{a}} : \operatorname{D}(\operatorname{Mod} A) \longrightarrow \operatorname{D}(\operatorname{Mod} A)$$

and

$$\operatorname{R}\widehat{\Gamma}_{\mathfrak{a}}, \operatorname{L}\widehat{\Lambda}_{\mathfrak{a}} : \operatorname{D}(\operatorname{Mod} A) \longrightarrow \operatorname{D}(\operatorname{Mod} \widehat{A})$$

exist. $\operatorname{R}\Gamma_{\mathfrak{a}}$ and $\operatorname{R}\widehat{\Gamma}_{\mathfrak{a}}$ are calculated using K-injective resolutions, while $\operatorname{L}\Lambda_{\mathfrak{a}}$ and $\operatorname{L}\widehat{\Lambda}_{\mathfrak{a}}$ are calculated using K-flat resolutions. See [4, Section 1] for a proof.

For any $M \in \operatorname{D}(\operatorname{Mod} A)$, there are canonical morphisms $\operatorname{R}\Gamma_{\mathfrak{a}}(M) \rightarrow M$ and $M \rightarrow \operatorname{L}\Lambda_{\mathfrak{a}}(M)$ in $\operatorname{D}(\operatorname{Mod} A)$. If these maps are isomorphisms, we say that M is cohomologically \mathfrak{a} -torsion and cohomologically \mathfrak{a} -adically complete, respectively. The collections of all cohomologically \mathfrak{a} -torsion and cohomologically \mathfrak{a} -adically complete complexes form two full triangulated subcategories of $\operatorname{D}(\operatorname{Mod} A)$. These are denoted by $\operatorname{D}(\operatorname{Mod} A)_{\mathfrak{a}\text{-tor}}$ and $\operatorname{D}(\operatorname{Mod} A)_{\mathfrak{a}\text{-com}}$, respectively.

Given a ring A and a finite sequence of elements \mathbf{a} , there is a bounded complex of free A -modules $\operatorname{Tel}(A; \mathbf{a})$, called the telescope complex associated to \mathbf{a} . See [10, Section 5] for its definition. If $A \rightarrow B$ is a ring homomorphism, and if \mathbf{b} is the image of \mathbf{a} under this map, then there is an isomorphism of complexes $\operatorname{Tel}(A; \mathbf{a}) \otimes_A B \rightarrow \operatorname{Tel}(B; \mathbf{b})$. Given an ideal $\mathfrak{a} \subseteq A$, and a finite sequence of elements $\mathbf{a} \subseteq A$ that generates \mathfrak{a} , there is a morphism of functors

$$\operatorname{R}\Gamma_{\mathfrak{a}}(-) \longrightarrow \operatorname{Tel}(A; \mathbf{a}) \otimes_A -.$$

If this morphism is a quasi-isomorphism, then \mathfrak{a} and \mathfrak{a} are said to be weakly proregular. See [12, Section 2], and, in particular, [12, Definition 2.3]. In a noetherian ring, every ideal and every finite sequence are weakly proregular. If the ideal \mathfrak{a} is weakly proregular, then the functors

$$\begin{aligned} \mathrm{R}\Gamma_{\mathfrak{a}} : \mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-com}} &\longrightarrow \mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-tor}}, \\ \mathrm{L}\Lambda_{\mathfrak{a}} : \mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-tor}} &\longrightarrow \mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-com}} \end{aligned}$$

are quasi-inverse to each other and induce an equivalence between these two triangulated categories, called the Matlis-Greenlees-May equivalence. If A is noetherian and \mathfrak{a} -adically complete, then

$$\mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A) \subseteq \mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-com}}.$$

In that case, the essential image of $\mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A)$ under the functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ is denoted by $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod} A)_{\mathfrak{a}\text{-cof}}$. This is a triangulated category, called the category of cohomologically \mathfrak{a} -adically cofinite complexes and is equivalent to the category $\mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(\mathrm{Mod} A)$. See [11] for a study of this category.

The following proposition, whose proof is immediate from the definitions, will be useful in the sequel.

Proposition 2.1. *Let A be a commutative ring, let $\mathfrak{a} \subseteq A$ be an ideal, and let $\widehat{A} := \Lambda_{\mathfrak{a}}(A)$. Let $Q : \mathrm{D}(\mathrm{Mod} \widehat{A}) \rightarrow \mathrm{D}(\mathrm{Mod} A)$ be the forgetful functor. Then, there are isomorphisms*

$$Q \circ \mathrm{L}\widehat{\Lambda}_{\mathfrak{a}}(-) \cong \mathrm{L}\Lambda_{\mathfrak{a}}(-)$$

and

$$Q \circ \mathrm{R}\widehat{\Gamma}_{\mathfrak{a}}(-) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(-)$$

of functors $\mathrm{D}(\mathrm{Mod} A) \rightarrow \mathrm{D}(\mathrm{Mod} A)$.

2.1. A reduction lemma. The aim of this subsection is to prove Lemma 2.5, which allows us to reduce certain questions over \widehat{A} to questions over A .

The next lemma was inspired by a result of Yekutieli (private communication).

Lemma 2.2. *Let $A \rightarrow B$ be a quasi-isomorphism of commutative DG-algebras. Let C be a B -algebra, and let $Q : D(\text{DGMod } B) \rightarrow D(\text{DGMod } A)$ be the forgetful functor. Then, there is an isomorphism*

$$\text{RHom}_B(C, -) \cong \text{RHom}_A(C, Q(-))$$

of functors $D(\text{DGMod } B) \rightarrow D(\text{Mod } C)$.

Proof. Let $M \in D(\text{DGMod } B)$. Let $M \rightarrow I_B$ be a K-injective resolution of M over B , and let $Q(M) \rightarrow I_A$ be a K-injective resolution of M over A . The functor Q induces an isomorphism $I_B \cong I_A$ in $D(\text{DGMod } A)$. Since I_A is K-injective, there is some A -linear quasi-isomorphism $\alpha : I_B \rightarrow I_A$. Composition with α induces a map $\alpha' : \text{Hom}_B(C, I_B) \rightarrow \text{Hom}_A(C, I_A)$, and this map is clearly C -linear. It is sufficient to show that α' is a quasi-isomorphism. In order to see this, consider the map $\phi : I_B \rightarrow \text{Hom}_A(B, I_A)$, given by $\phi(x)(b) = \alpha(b \cdot x)$ for $b \in B$ and $x \in I_B$. This map fits into a commutative diagram

$$\begin{array}{ccc} I_B & \xrightarrow{\phi} & \text{Hom}_A(B, I_A) \\ \downarrow \alpha & & \downarrow \\ I_A & \xrightarrow{\quad} & \text{Hom}_A(A, I_A). \end{array}$$

Since $A \rightarrow B$ is a quasi-isomorphism, and I_A is K-injective, the right vertical map is a quasi-isomorphism. Hence, ϕ is also a quasi-isomorphism between two K-injective DG B -modules; thus, it is a homotopy equivalence. Hence, in the commutative diagram

$$\begin{array}{ccc} \text{Hom}_B(C, I_B) & \xrightarrow{\alpha'} & \text{Hom}_A(C, I_A) \\ \downarrow \text{Hom}_B(1_C, \phi) & & \downarrow \\ \text{Hom}_B(C, \text{Hom}_A(B, I_A)) & \xrightarrow{\quad} & \text{Hom}_A(C \otimes_B B, I_A), \end{array}$$

the left vertical arrow induced by this homotopy equivalence is a quasi-isomorphism, while the right vertical arrow and the bottom horizontal arrow are obviously isomorphisms. Hence, α' is a quasi-isomorphism, as claimed. \square

Lemma 2.3. *Let A be a commutative ring, let $\mathfrak{a} \subseteq A$ be a finitely generated weakly proregular ideal, and set $\hat{A} := \Lambda_{\mathfrak{a}}(A)$. Let $J \subseteq A$ be an*

ideal, and assume that there are integers m, n , such that $\mathfrak{a}^m \subseteq J \subseteq \mathfrak{a}^n$. Then, for all $i \neq 0$, we have that

$$\mathrm{Tor}_i^A(A/J, \widehat{A}) = 0.$$

Proof. Let $B = A/J$. Let \mathbf{a} be a finite sequence of elements that generates \mathfrak{a} , and let \mathbf{b} be its image in B . Note that, by assumption, each element of \mathbf{b} is nilpotent. Hence, by [10, Lemma 7.4], there is a B -linear homotopy equivalence $\mathrm{Tel}(B; \mathbf{b}) \rightarrow B$. By the base change property of the telescope complex, we deduce that there is an A -linear homotopy equivalence $\mathrm{Tel}(A; \mathbf{a}) \otimes_A B \rightarrow B$.

Since \mathfrak{a} is weakly proregular, by the Greenlees-May duality (specifically, by [4, Corollary (iv) following Theorem (0.3)*], or by [10, Lemma 7.6]), there is an isomorphism $\widehat{A} \otimes_A \mathrm{Tel}(A; \mathbf{a}) \cong \mathrm{Tel}(A; \mathbf{a})$ in $\mathrm{D}(\mathrm{Mod} A)$.

Combining these two isomorphisms, and the fact that $\mathrm{Tel}(A; \mathbf{a})$ is a K -flat complex, we obtain the following sequence of isomorphisms in $\mathrm{D}(\mathrm{Mod} A)$:

$$\begin{aligned} B &\cong \mathrm{Tel}(A; \mathbf{a}) \otimes_A^L B \cong (\widehat{A} \otimes_A^L \mathrm{Tel}(A; \mathbf{a})) \otimes_A^L B \cong \widehat{A} \otimes_A^L (\mathrm{Tel}(A; \mathbf{a}) \otimes_A^L B) \\ &\cong \widehat{A} \otimes_A^L B. \end{aligned}$$

As B is a complex concentrated in degree 0, the result follows. \square

Lemma 2.4. *Let A be a commutative ring, let $\mathfrak{a} \subseteq A$ be a finitely generated weakly proregular ideal, and set $\widehat{A} := \Lambda_{\mathfrak{a}}(A)$. Let $J \subseteq A$ be an ideal, and assume that there are integers m, n , such that $\mathfrak{a}^m \subseteq J \subseteq \mathfrak{a}^n$. Set $B = A/J$, and let $Q_A : \mathrm{D}(\mathrm{Mod} \widehat{A}) \rightarrow \mathrm{D}(\mathrm{Mod} A)$ be the forgetful functor. Then, there is an isomorphism*

$$\mathrm{RHom}_{\widehat{A}}(B, -) \cong \mathrm{RHom}_A(B, Q_A(-))$$

of functors $\mathrm{D}(\mathrm{Mod} \widehat{A}) \rightarrow \mathrm{D}(\mathrm{Mod} B)$.

Proof. Let $A \rightarrow \widetilde{A} \cong \widehat{A}$ be a commutative semi-free DG-algebra resolution of $A \rightarrow \widehat{A}$, and let $Q_{\widetilde{A}} : \mathrm{D}(\mathrm{Mod} \widehat{A}) \rightarrow \mathrm{D}(\mathrm{DGMod} \widetilde{A})$ be the corresponding forgetful functor. Given $M \in \mathrm{D}(\mathrm{Mod} \widehat{A})$, according to Lemma 2.2, there is an isomorphism of functors

$$\mathrm{RHom}_{\widehat{A}}(B, M) \cong \mathrm{RHom}_{\widetilde{A}}(B, Q_{\widetilde{A}}(M)).$$

Let $Q_{\tilde{A}}(M) \rightarrow I$ be a K-injective resolution of $Q_{\tilde{A}}(M)$ over \tilde{A} . Then, there is an obvious B -linear isomorphism

$$\mathrm{RHom}_{\tilde{A}}(B, Q_{\tilde{A}}(M)) \cong \mathrm{Hom}_{\tilde{A}}(B, I).$$

According to Lemma 2.3, we have that $\mathrm{Tor}_i^A(B, \hat{A}) = 0$ for all $i \neq 0$. Hence, the map $B \otimes_A \tilde{A} \rightarrow B \otimes_A \hat{A}$ induced by the map $\tilde{A} \rightarrow \hat{A}$ is a quasi-isomorphism.

Since $\mathfrak{a}^m \subseteq J$, we have that $B \otimes_A A/\mathfrak{a}^m \cong B$. On the other hand, since \mathfrak{a} is finitely generated, we have that $A/\mathfrak{a}^m \otimes_A \hat{A} \cong A/\mathfrak{a}^m$. Combining these two facts, we deduce that $B \otimes_A \hat{A} \cong B$. It follows that there is a quasi-isomorphism $B \otimes_A \tilde{A} \rightarrow B$, which is B -linear on the left, and \tilde{A} -linear on the right.¹

This, in turn, induces a quasi-isomorphism

$$\mathrm{Hom}_{\tilde{A}}(B, I) \rightarrow \mathrm{Hom}_{\tilde{A}}(B \otimes_A \tilde{A}, I),$$

which, by the hom-tensor adjunction is naturally isomorphic to

$$\mathrm{Hom}_A(B, I).$$

Since $A \rightarrow \tilde{A}$ is flat, we deduce that I is K-injective over A so that $\mathrm{Hom}_A(B, I) \cong \mathrm{RHom}_A(B, Q_A(M))$, which proves the claim. \square

Lemma 2.5. *Let A be a commutative ring, let $\mathfrak{a} \subseteq A$ be a finitely generated weakly proregular ideal, and set $\hat{A} := \Lambda_{\mathfrak{a}}(A)$. Let $J \subseteq A$ be an ideal, and assume that there are integers m, n , such that $\mathfrak{a}^m \subseteq J \subseteq \mathfrak{a}^n$. Set $B = A/J$. Then, there are isomorphisms*

$$\mathrm{RHom}_{\hat{A}}(B, \mathrm{R}\hat{\Gamma}_{\mathfrak{a}}(-)) \cong \mathrm{RHom}_{\hat{A}}(B, \mathrm{L}\hat{\Lambda}_{\mathfrak{a}}(-)) \cong \mathrm{RHom}_A(B, -)$$

of functors $\mathrm{D}(\mathrm{Mod} A) \rightarrow \mathrm{D}(\mathrm{Mod}(B))$.

Proof. Let $Q : \mathrm{D}(\mathrm{Mod} \hat{A}) \rightarrow \mathrm{D}(\mathrm{Mod} A)$ be the forgetful functor. According to Lemma 2.4, there are B -linear isomorphisms of functors

$$\mathrm{RHom}_{\hat{A}}(B, \mathrm{R}\hat{\Gamma}_{\mathfrak{a}}(-)) \cong \mathrm{RHom}_A(B, Q(\mathrm{R}\hat{\Gamma}_{\mathfrak{a}}(-)))$$

and

$$\mathrm{RHom}_{\hat{A}}(B, \mathrm{L}\hat{\Lambda}_{\mathfrak{a}}(-)) \cong \mathrm{RHom}_A(B, Q(\mathrm{L}\hat{\Lambda}_{\mathfrak{a}}(-)))$$

By Proposition 2.1, these are isomorphic in $D(\text{Mod } B)$ to

$$\mathbf{R} \operatorname{Hom}_A(B, \mathbf{R}\Gamma_{\mathfrak{a}}(-))$$

and

$$\mathbf{R} \operatorname{Hom}_A(B, \mathbf{L}\Lambda_{\mathfrak{a}}(-)),$$

respectively. In the proof of Lemma 2.3, we have seen that $B \cong \operatorname{Tel}(A; \mathfrak{a}) \otimes_A B$, which implies that B is cohomologically \mathfrak{a} -torsion. Hence, by the Greenlees-May duality ([4, Theorem 0.3], [10, Theorem 7.12]), there are natural isomorphisms

$$\mathbf{R} \operatorname{Hom}_A(B, \mathbf{R}\Gamma_{\mathfrak{a}}(-)) \cong \mathbf{R} \operatorname{Hom}_A(B, -) \cong \mathbf{R} \operatorname{Hom}_A(B, \mathbf{L}\Lambda_{\mathfrak{a}}(-)).$$

The isomorphisms constructed in [4, 10] are A -linear, but it is easy to verify that, in our situation, they actually are B -linear. This proves the claim. \square

2.2. The box tensor products over affine formal schemes.

Next, we obtain some general finiteness results concerning the adic box tensor products. For a moment, we drop the assumption that \mathbb{k} is a field as it does not produce additional difficulties, and it seems that this result might be of independent interest in this greater generality.

Proposition 2.6. *Let \mathbb{k} be a commutative ring, and let (A, \mathfrak{a}) and (B, \mathfrak{b}) be two noetherian adic rings which are flat \mathbb{k} -algebras. Let $I = \mathfrak{a} \otimes_{\mathbb{k}} B + A \otimes_{\mathbb{k}} \mathfrak{b}$ be the ideal of definition of the adic topology on $A \otimes_{\mathbb{k}} B$, let \widehat{I} be the ideal generated by its image in $\widehat{A \otimes_{\mathbb{k}} B}$, and assume that I is weakly proregular (if \mathbb{k} is a field this always holds), and that $\widehat{A \otimes_{\mathbb{k}} B}$ is noetherian. Given $M \in D_{\mathfrak{f}}^b(\text{Mod } A)$ and $N \in D_{\mathfrak{f}}^b(\text{Mod } B)$ with M having finite flat dimension over \mathbb{k} , we have that*

$$\mathbf{L}\widehat{\Gamma}_I(M \otimes_{\mathbb{k}}^{\mathbf{L}} N) \in D_{\mathfrak{f}}^b(\text{Mod } \widehat{A \otimes_{\mathbb{k}} B}),$$

and

$$\mathbf{R}\widehat{\Gamma}_I(M \otimes_{\mathbb{k}}^{\mathbf{L}} N) \in D^b(\text{Mod } \widehat{A \otimes_{\mathbb{k}} B})_{\widehat{I}\text{-cof}}.$$

Proof. We first show that both of these complexes have bounded cohomology. Let $Q : D(\text{Mod } \widehat{A \otimes_{\mathbb{k}} B}) \rightarrow D(\text{Mod } A \otimes_{\mathbb{k}} B)$ be the forgetful functor. Clearly, a complex X has bounded cohomology if and only if

the complex $Q(X)$ has bounded cohomology. In view of Proposition 2.1, it is enough to show that the complexes

$$\mathrm{L}\Lambda_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N)$$

and

$$\mathrm{R}\Gamma_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N)$$

have bounded cohomology; however, this immediately follows from the flat dimension assumption on M , combined with the fact that, when I is weakly proregular, the functors $\mathrm{L}\Lambda_I$ and $\mathrm{R}\Gamma_I$ have finite cohomological dimension (for example, by [10, Corollary 4.28, Corollary 5.27]).

Next, we show the claims regarding finiteness of the cohomologies. Let $P \rightarrow M$ and $Q \rightarrow N$ be bounded above resolutions made of finitely generated free modules. Since A is flat over \mathbb{k} , P is also flat over \mathbb{k} so that $M \otimes_{\mathbb{k}}^{\mathrm{L}} N \cong P \otimes_{\mathbb{k}} Q$, and the latter is also a bounded above complex made of finitely generated free modules; therefore,

$$\mathrm{L}\widehat{\Lambda}_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N) \cong \Lambda_I(P \otimes_{\mathbb{k}} Q).$$

Since the completion functor commutes with finite direct sums, it follows that $\Lambda_I(P \otimes_{\mathbb{k}} Q)$ is also a bounded above complex made of finitely generated free modules, which shows that the cohomologies of this complex are finitely generated over $\widehat{A \otimes_{\mathbb{k}} B}$.

It remains to show that

$$\mathrm{R}\widehat{\Gamma}_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N) \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod} \widehat{A \otimes_{\mathbb{k}} B})_{\widehat{I}\text{-cof}}.$$

As we already established that this complex is bounded, and as it is clearly cohomologically \widehat{I} -torsion, by [11, Theorem 3.10], it is sufficient to show that the complex

$$\mathrm{RHom}_{\widehat{A \otimes_{\mathbb{k}} B}}(\widehat{A \otimes_{\mathbb{k}} B} / \widehat{I}, \mathrm{R}\widehat{\Gamma}_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N))$$

has finitely generated cohomologies. By Lemma 2.5, there is an isomorphism

$$\begin{aligned} \mathrm{RHom}_{\widehat{A \otimes_{\mathbb{k}} B}}(\widehat{A \otimes_{\mathbb{k}} B} / \widehat{I}, \mathrm{R}\widehat{\Gamma}_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N)) \\ \cong \mathrm{RHom}_{\widehat{A \otimes_{\mathbb{k}} B}}(\widehat{A \otimes_{\mathbb{k}} B} / \widehat{I}, \mathrm{L}\widehat{\Lambda}_I(M \otimes_{\mathbb{k}}^{\mathrm{L}} N)); \end{aligned}$$

thus, the result follows from the first claim in this proposition. \square

Remark 2.7. It may be wondered why, in the above proof, we had to invoke the rather difficult theorem of [11], instead of deducing the finiteness condition in the torsion case directly from the identity $\widehat{R\Gamma}_I(-) \cong R\Gamma_{\widehat{I}} \circ L\widehat{\Lambda}_I(-)$. The reason for doing so is that we do not know if this identity holds when $A \otimes_{\mathbb{k}} B \rightarrow \widehat{A \otimes_{\mathbb{k}} B}$ is not flat.

2.3. Tensor product of dualizing complexes for formal schemes.

In this subsection, we will prove Theorem 2.9, the main result of this section. First, we recall the definitions of dualizing complexes over affine formal schemes. See [5, Section 2.5], [15, Section 5] for details (keeping in mind [11, Theorem 3.10]). Let (A, \mathfrak{a}) be an adic noetherian ring. A complex $R \in D(\text{Mod } A)$, which has finite injective dimension over A , and such that the canonical map $A \rightarrow R\text{Hom}_A(R, R)$ is an isomorphism is called a *c-dualizing complex* if $R \in D_{\mathfrak{f}}^b(\text{Mod } A)$ and is called a *t-dualizing complex* if $R \in D^b(\text{Mod } A)_{\mathfrak{a}\text{-cof}}$.

The next lemma allows us to reduce the problem of determining whether a complex over the completed tensor product is dualizing to a problem over discrete rings. We will then use Theorem 1.6 to obtain the required result.

Lemma 2.8. *Let \mathbb{k} be a field, and let (A, \mathfrak{a}) and (B, \mathfrak{b}) be two noetherian adic rings which are \mathbb{k} -algebras, such that the completed tensor product $\widehat{A \otimes_{\mathbb{k}} B}$ is noetherian of finite Krull dimension. Let I be the ideal of definition of the adic topology on $\widehat{A \otimes_{\mathbb{k}} B}$, and let $M \in D_{\mathfrak{f}}^b(\text{Mod } \widehat{A \otimes_{\mathbb{k}} B})$ (respectively, $M \in D(\text{Mod } \widehat{A \otimes_{\mathbb{k}} B})_{I\text{-cof}}$). Then, M is a c-dualizing (respectively, t-dualizing) complex over $\widehat{A \otimes_{\mathbb{k}} B}$ if and only if, for each $n > 0$, the complex*

$$R\text{Hom}_{\widehat{A \otimes_{\mathbb{k}} B}}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, M) \in D(\text{Mod } A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n)$$

is a dualizing complex over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$.

Proof. Consider the sequence of ideals $J_n = \ker(\widehat{A \otimes_{\mathbb{k}} B} \rightarrow A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n)$. For every $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$, such that $J_n \subseteq I^m$, and likewise; for every $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$, such that $I^n \subseteq J_m$. Hence,

$$\varprojlim (\widehat{A \otimes_{\mathbb{k}} B} / J_n) \cong \widehat{A \otimes_{\mathbb{k}} B},$$

and moreover, the two functors $\Gamma_I(-)$ and $\varinjlim \operatorname{Hom}_{\widehat{A \otimes_{\mathbb{k}} B}}(\widehat{A \otimes_{\mathbb{k}} B}/J_n, -)$ are canonically isomorphic. With these observations, the result now follows from the proof of [5, Lemma 2.5.10] (see also [6, Satz 2]). \square

We now arrive at the main result of this section, an adic generalization of Theorem 1.6.

Theorem 2.9. *Let \mathbb{k} be a field, and let (A, \mathfrak{a}) and (B, \mathfrak{b}) be two noetherian adic rings which are \mathbb{k} -algebras. Let I be the ideal of definition of the adic topology on $A \otimes_{\mathbb{k}} B$. Let R_A be a c -dualizing complex over (A, \mathfrak{a}) , and let R_B be a c -dualizing complex over (B, \mathfrak{b}) . Then, the ring $\widehat{A \otimes_{\mathbb{k}} B}$ has dualizing complexes if and only if it is noetherian of finite Krull dimension. In that case, $\widehat{L\hat{\Lambda}}_I(R_A \otimes_{\mathbb{k}} R_B)$ is a c -dualizing complex over $\widehat{A \otimes_{\mathbb{k}} B}$, and $\widehat{R\hat{\Gamma}}_I(R_A \otimes_{\mathbb{k}} R_B)$ is a t -dualizing complex over $\widehat{A \otimes_{\mathbb{k}} B}$.*

Proof. Let \hat{I} be the ideal generated by the image of I in $\widehat{A \otimes_{\mathbb{k}} B}$. According to [10, Example 4.35], the ideal I is weakly proregular. Hence, by Proposition 2.6, we have that

$$\widehat{L\hat{\Lambda}}_I(R_A \otimes_{\mathbb{k}} R_B) \in D_{\mathfrak{f}}^b(\operatorname{Mod} \widehat{A \otimes_{\mathbb{k}} B}),$$

and

$$\widehat{R\hat{\Gamma}}_I(R_A \otimes_{\mathbb{k}} R_B) \in D^b(\operatorname{Mod} \widehat{A \otimes_{\mathbb{k}} B})_{\hat{I}\text{-cof}}.$$

By Lemma 2.8, it is sufficient to show that, for all n , the complexes

$$\operatorname{RHom}_{\widehat{A \otimes_{\mathbb{k}} B}}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, \widehat{L\hat{\Lambda}}_I(R_A \otimes_{\mathbb{k}} R_B))$$

and

$$\operatorname{RHom}_{\widehat{A \otimes_{\mathbb{k}} B}}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, \widehat{R\hat{\Gamma}}_I(R_A \otimes_{\mathbb{k}} R_B))$$

are dualizing complexes over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$. From Lemma 2.5, both of these complexes are isomorphic as objects in $D(\operatorname{Mod} A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n)$, and moreover, both of them are isomorphic to the complex

$$\operatorname{RHom}_{A \otimes_{\mathbb{k}} B}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, R_A \otimes_{\mathbb{k}} R_B).$$

Note that, as the maps $A \rightarrow A/\mathfrak{a}^n$ and $B \rightarrow B/\mathfrak{b}^n$ are finite, the complexes

$$\mathrm{RHom}_A(A/\mathfrak{a}^n, R_A)$$

and

$$\mathrm{RHom}_B(B/\mathfrak{b}^n, R_B)$$

are dualizing complexes over A/\mathfrak{a}^n and B/\mathfrak{b}^n , respectively. Since the ring $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$ is noetherian of finite Krull dimension (being a quotient of the noetherian ring of finite Krull dimension $\widehat{A \otimes_{\mathbb{k}} B}$), it follows from Theorem 1.6 that

$$\mathrm{RHom}_A(A/\mathfrak{a}^n, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_B(B/\mathfrak{b}^n, R_B)$$

is a dualizing complex over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$. We now use the same trick as in the proof of Lemma 1.5. Thus, let $A \rightarrow \tilde{A} \cong A/\mathfrak{a}^n$ and $B \rightarrow \tilde{B} \cong B/\mathfrak{b}^n$ be DG-algebra resolutions of $A \rightarrow A/\mathfrak{a}^n$ and $B \rightarrow B/\mathfrak{b}^n$, respectively, as in Lemma 1.5. Then, there is a $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$ -linear isomorphism

$$\begin{aligned} \mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, R_A \otimes_{\mathbb{k}} R_B) \\ \cong \mathrm{RHom}_A(A/\mathfrak{a}^n, R_A) \otimes_{\mathbb{k}} \mathrm{RHom}_B(B/\mathfrak{b}^n, R_B). \end{aligned}$$

The right hand side is a dualizing complex over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$, and hence, also a dualizing DG-module over $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$. Thus, the left hand side, which is a priori a complex over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$, is also a dualizing DG-module over $\tilde{A} \otimes_{\mathbb{k}} \tilde{B}$. Hence, by the argument used in the proof of Lemma 1.5, we deduce that

$$\mathrm{RHom}_{A \otimes_{\mathbb{k}} B}(A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n, R_A \otimes_{\mathbb{k}} R_B)$$

is a dualizing complex over $A/\mathfrak{a}^n \otimes_{\mathbb{k}} B/\mathfrak{b}^n$, which establishes the theorem. \square

Again, as in Corollary 1.7, this immediately generalizes to formal schemes. As an immediate corollary, we obtain an adic generalization of [14, Theorem 6(a)].

Corollary 2.10. *Let \mathbb{k} be a field, and let (A, \mathfrak{a}) and (B, \mathfrak{b}) be two adic noetherian Gorenstein \mathbb{k} -algebras of finite Krull dimension such that*

$\widehat{A \otimes_{\mathbb{k}} B}$ is also noetherian of finite Krull dimension. Then, $\widehat{A \otimes_{\mathbb{k}} B}$ is also a Gorenstein ring.

Remark 2.11. As far as is known, all similar results in the literature concerning the conservation of homological properties of commutative noetherian rings under the tensor product operation involve a flatness assumption. In that sense, Corollary 2.10 is different since, to our knowledge, it is not known whether, in the above situation, the maps $A \rightarrow \widehat{A \otimes_{\mathbb{k}} B}$ and $B \rightarrow \widehat{A \otimes_{\mathbb{k}} B}$ are flat (since it is not known whether the completion map $A \otimes_{\mathbb{k}} B \rightarrow \widehat{A \otimes_{\mathbb{k}} B}$ is flat when $A \otimes_{\mathbb{k}} B$ is non-noetherian), although, flatness is known to hold if A/\mathfrak{a} is essentially of finite type over \mathbb{k} (see [5, Proposition 7.1(b)]). We thus view this result as another example of the fact that weak proregularity of the ideal of the definition of adic topology can serve as a replacement for flatness of the completion map in many interesting situations.

The Cohen structure theorem may be stated as follows: given a noetherian local ring (A, \mathfrak{m}) , its completion $\Lambda_{\mathfrak{m}}(A)$ is a quotient of a regular local ring. Our final corollary is a weak variation of this for the tensor product of local rings. It states that the completion of a tensor product of local rings is a quotient of a Gorenstein ring.

Corollary 2.12. *Let \mathbb{k} be a field, and let (A, \mathfrak{m}) and (B, \mathfrak{n}) be two noetherian local \mathbb{k} -algebras. Let $I = \mathfrak{m} \otimes_{\mathbb{k}} B + A \otimes_{\mathbb{k}} \mathfrak{n}$, and assume that*

$$\widehat{A \otimes_{\mathbb{k}} B} := \Lambda_I(A \otimes_{\mathbb{k}} B)$$

is noetherian of finite Krull dimension. Then, $\widehat{A \otimes_{\mathbb{k}} B}$ has dualizing complexes, so it is a quotient of a Gorenstein ring of finite Krull dimension.

Proof. From the Cohen structure theorem, the rings \widehat{A} and \widehat{B} have dualizing complexes. Since there are isomorphisms $A/\mathfrak{m}^n \cong \widehat{A}/(\mathfrak{m} \cdot \widehat{A})^n$ and $B/\mathfrak{n}^n \cong \widehat{B}/(\mathfrak{n} \cdot \widehat{B})^n$, we see, as in the proof of Lemma 2.8, that

$$\widehat{A \otimes_{\mathbb{k}} B} \cong \varprojlim (A/\mathfrak{m}^n \otimes_{\mathbb{k}} B/\mathfrak{n}^n) \cong \varprojlim (\widehat{A}/(\mathfrak{m} \cdot \widehat{A})^n \otimes_{\mathbb{k}} \widehat{B}/(\mathfrak{n} \cdot \widehat{B})^n) \cong \widehat{A \otimes_{\mathbb{k}} B},$$

where

$$\widehat{A \otimes_{\mathbb{k}} B} := \Lambda_J(\widehat{A} \otimes_{\mathbb{k}} \widehat{B}), \quad J := (\mathfrak{m} \cdot \widehat{A}) \otimes_{\mathbb{k}} \widehat{B} + \widehat{A} \otimes_{\mathbb{k}} (\mathfrak{n} \cdot \widehat{B}).$$

From Theorem 2.9, the ring $\widehat{A \otimes_{\mathbb{k}} B}$ has dualizing complexes; thus, the isomorphic ring $\widehat{A \otimes_{\mathbb{k}} B}$ also has dualizing complexes. Hence, by Kawasaki's theorem, it is a quotient of a Gorenstein ring of finite Krull dimension. \square

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ENDNOTES

1. The main reason we need to take the DG-algebra resolution $A \rightarrow \widetilde{A} \cong \widehat{A}$ is in order to obtain these linearity conditions on this quasi-isomorphism. These allow us now to use the hom-tensor adjunction. The fact that there is such an A -linear isomorphism is already proven in Lemma 2.3, but this fact is not enough to use adjunction in the next step of the proof.

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