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A MINIMAL FREE COMPLEX ASSOCIATED TO THE MINORS OF A MATRIX

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ABSTRACT. This paper describes a construction of a minimal free resolution of a generic ideal defined by determinants in characteristic zero. It produces not only the free modules in the resolution, but it also defines the maps between them explicitly and in detail in terms of idempotents in the group algebra of the symmetric group.

This paper was originally written in 1976 and was an attempt to construct a generic minimal free resolution of an ideal defined by the determinants of a matrix. The complex was constructed in detail, and some evidence was given that it was indeed a resolution in the generic case. However, it was not proven that it was exact in the generic case. Meanwhile, a paper of Lascoux [**6**] did construct a generic resolution; as a result, the present paper was withdrawn. However, Lascoux omitted many details, whereas the complex presented here is constructed explicitly. Furthermore, the complex constructed by Lascoux is essentially the same as this one, and a later paper by Roberts [**10**] did provide a direct combinatorial proof that the complex constructed here is generically exact. The reason for reworking this paper now is that these results were used in a recent paper of Efremenko, et al. [**4**] so that it appears that the results contained here are still of interest.

We remark that, in the period shortly after this paper was written, several papers appeared giving various versions of the resolution; the one mentioned by Lascoux, another due to Nielsen [**7**] and a general construction of Akin, Buchsbaum and Weyman [**1**]. These papers used Schur functors in various ways, whereas here we use the closely related theory of idempotents in the group algebra of the symmetric group. All of these were defined for rings containing the field of rational numbers;

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there were also attempts to define such a complex over the ring of integers, but Hashimoto [**5**] showed that a generic minimal resolution over the integers could not exist since the Betti numbers depend on the characteristic.

The present version of the paper is actually not the same as that in the original paper of 1976. The original construction used a map f from one free module F to another free module G, and the boundary maps of the complex went from submodules of a tensor product of copies of F and G to another such submodule of a tensor product where certain of the factors F were replaced by G and the map was induced by f. The referee recommended replacing this by a complex defined by a map from F to the dual G^* with the boundary map now defined by contracting factors of the form $F \otimes G^*$ using the given map from F to G^*. This is basically the form of the complex given by Lascoux, and using this form makes a number of properties simpler. However, the computations with symmetric groups used in the original construction are essentially the same in the new version. It is the new version using a map from F to G^* that is presented here.

The earlier paper also contained a proof that it agreed with a construction of Poon [**9**] in the cases he proved, and it predicted the correct criteria for a determinantal ideal to be Gorenstein given by Svanes [**11**]. These were included to give evidence that it was a resolution; they are not so relevant at present; thus, we do not include them here.

1. Introduction. We assume throughout that R is a commutative ring that contains the field of rational numbers. Let (r_{ij}) be a matrix with entries in R, and let $t \geq 1$ be an integer. Suppose (r_{ij}) is an $(n+t-1) \times (m+t-1)$ matrix, where $m \geq n \geq 1$. Then, if F is a free module of rank $m+t-1$ and G is one of rank $n+t-1$ with given bases, (r_{ij}) defines a map ϕ from F to the dual $G^* = \mathrm{Hom}(G, R)$ of G. We will define a complex $C_\bullet(\phi, t)$ such that:

(1) $C_k(\phi, t)$ is free for all k.

(2) If R is local and r_{ij} is in the maximal ideal $\mathfrak{m}$ of R for all i and j, then the maps from $C_k(\phi, t)$ to $C_{k-1}(\phi, t)$ are defined by matrices with coefficients in $\mathfrak{m}$.

(3) $C_0(\phi, t) \cong R$, and the image of $C_1(\phi, t)$ in $C_0(\phi, t)$ is the ideal generated by the $t \times t$ minors of (r_{ij}).

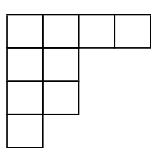
Let I be the ideal generated by the $t \times t$ minors of (r_{ij}). If $C_\bullet(\phi, t)$ is exact, it follows from (1), (2) and (3) that it is a minimal free resolution of R/I. In fact, it was shown by Roberts **[10]** that, if the r_{ij} are indeterminates over a subring of R, then $C_\bullet(\phi, t)$ is exact.

The complex $C_\bullet(\phi, t)$ will be made up of pieces that are constructed roughly as follows. If F and G are free modules as above, so that ϕ maps F to G^*, then the symmetric group S_k acts on $F^{\otimes k}$ and $G^{\otimes k}$; thus, the group algebra $\mathbb{Q}[S_k]$ acts on $F^{\otimes k}$ and $G^{\otimes k}$, for each k. The pieces which make up $C_\bullet(\phi, t)$ are of the type $(e \otimes e')(F^{\otimes k} \otimes G^{\otimes k})$, where e and e' are idempotent elements of $\mathbb{Q}[S_k]$. The boundary maps are essentially those maps induced by the map ψ defined from ϕ from $F \otimes G$ to R, that is, the map defined by $\psi(f \otimes g) = \phi(f)(g)$.

Since the construction involves idempotents of $\mathbb{Q}[S_k]$, we begin by presenting the necessary facts about representations of symmetric groups. These may all be found in Boerner **[2]**, to which we will refer for proofs when appropriate. In Section 2, we define $C_\bullet(\phi, t)$ and prove its most elementary properties. Much of the rest of the paper is comprised of proving two formulas involving idempotent elements of the group algebra which imply that $C_\bullet(\phi, t)$ is a complex.

2. Young tableaux.

Let S_n be the symmetric group on n elements, usually taken to be the numbers $1, 2, \ldots, n$. Let λ be a partition of n so that λ can be written as a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ with $\sum \lambda_i = n$. Associated to each such partition we have a *Young diagram*, also denoted λ, obtained by arranging n squares in r rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_r$, as in the following example, which is the Young diagram corresponding to the partition $4 \geq 2 \geq 2 \geq 1$ of 9.



A *Young tableau* is obtained by arranging the n elements on which S_n acts in the squares of a Young diagram in one of the $n!$ possible ways. Corresponding to the above Young diagram, two of the possible Young tableaux are:

$$\begin{array}{|c|c|c|c|}\hline 3 & 6 & 4 & 5 \\ \hline 7 & 1 \\ \cline{1-2} 2 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

When we are considering only one tableau we will often denote it also by λ; when we are considering different tableaux with the same diagram we will distinguish them by subscripts λ_α, λ_β, and so on.

Each Young tableau λ defines two subgroups P_λ and Q_λ of S_n. P_λ is the set of permutations that map each element to an element in the same row in λ, and Q_λ consists of permutations that map each element to one in the same column. Let

$$P = \sum_{\sigma \in P_\lambda} \sigma$$

and

$$Q = \sum_{\sigma \in Q_\lambda} (-1)^{\mathrm{sign}(\sigma)} \sigma;$$

P and Q are elements of the group algebra $\mathbb{Q}[S_n]$. Let $E_\lambda = PQ$. Then, [**2,** Theorem IV 3.1] E_λ is essentially idempotent, that is, there is a nonzero rational number κ_λ such that $E_\lambda^2 = \kappa_\lambda E_\lambda$. Let $e(\lambda)$ be the idempotent element $(1/\kappa_\lambda)E_\lambda$. Thus, to each tableau λ, we associate an idempotent $e(\lambda)$ in $\mathbb{Q}[S_n]$.

Now, let F be an R-module. S_n acts on the n-fold tensor product $F^{\otimes n}$ by permuting factors. More precisely, if $\sigma \in S_n$, we define

$$\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)},$$

it may be verified that, under this definition, we have

$$(\sigma\tau)(f_1 \otimes \cdots \otimes f_n) = \sigma(\tau(f_1 \otimes \cdots \otimes f_n)).$$

Thus, since R was assumed to contain $\mathbb{Q}$, the group algebra $\mathbb{Q}[S_n]$ acts on $F^{\otimes n}$, and each element of $\mathbb{Q}[S_n]$ acts as an R-module homomorphism. Hence, for each tableau λ, $e(\lambda)(F^{\otimes n})$ is an R-submodule of $F^{\otimes n}$. We also note that, if F is free, then $F \cong V \otimes_{\mathbb{Q}} R$ for some

$\mathbb{Q}$-module V, and there is an isomorphism of $F^{\otimes n}$ with $V^{\otimes n} \otimes_{\mathbb{Q}} R$ that preserves the action of $\mathbb{Q}[S_n]$. Hence, since R is necessarily flat over $\mathbb{Q}$, $e(\lambda)(F^{\otimes n}) = e(\lambda)(V^{\otimes n}) \otimes_{\mathbb{Q}} R$; thus, $e(\lambda)(F^{\otimes n})$ is also a free R-module.

We will use the abbreviated notation $e(\lambda)F$ to denote $e(\lambda)(F^{\otimes n})$.

3. Construction of $C_\bullet(\phi, t)$. Fix integers $t \geq 1$ and $m \geq n \geq 1$. F will denote a free module of rank $(m+t-1)$, and G will denote a free module of rank $n+t-1$. We denote by ϕ a map from F to G^* defined by a matrix (r_{ij}) with respect to a given basis of F and the dual basis of a given basis of G.

Let $\lambda(m,n)$ be a Young tableau consisting of a rectangle of m rows of n squares each. In our initial examples, we will take the tableau in which the ith row contains the numbers $(i-1)n+1$, $(i-1)n+2, \ldots, in$ in increasing order; however, in later computations we will use different notation. Let k be an integer such that $0 \leq k \leq mn$. Let λ be a partition of k into at most m parts, each of which is at most n, that is, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ where $\sum \lambda_i = k$ and $\lambda_i \leq n$ for each i (here some λ_i's could be zero). For each such partition λ we will define tableaux $\lambda_F(t)$ and $\lambda_G(t)$ for each integer $t \geq 1$.

We begin by assuming $t = 1$. Then, $\lambda_F = \lambda_F(1)$ is simply the subtableau of $\lambda(m,n)$ corresponding to the partition λ. We let λ_G be the transpose of λ_F, that is, the tableau obtained from λ_F by interchanging the rows and columns.

Example 3.1. Let $m = 4$ and $n = 3$, so $mn = 12$. Let $k = 5$, and let λ be the partition $2 \geq 2 \geq 1 \geq 0$. Then:

$$\lambda(m,n) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline \end{array}\,, \qquad \lambda_F = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 7 \\ \cline{1-1} \end{array}\,, \qquad \lambda_G = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}\,.$$

Next, let t be arbitrary. Define $\lambda(m+t-1, n)$ to be a tableau with $m+t-1$ rows of n squares; for example, we enter the numbers from 1 to $(m+t-1)n$ in the order described above for $\lambda(m,n)$. Let k be an integer with $0 \leq k \leq mn$ and λ a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ of k with $\lambda_i \leq n$ for each i.

Now, associate to each square of $\lambda(m,n)$ either a square or a set of t squares of $\lambda(m+t-1,n)$. To the square in the (i,j) position, we associate:

(1) The square in the (i,j) position of $\lambda(m+t-1,n)$ if $j>i$.
(2) The string of t squares from the (i,j) position to the $(i+t-1,j)$ position if $j=i$.
(3) The square in the $(i+t-1,j)$ position if $j<i$.

Let λ be a partition as above so that λ_F is defined. We let $\lambda_F(t)$ be the tableau obtained by replacing each square of λ_F by the associated square or string of squares of $\lambda(m+t-1,n)$, and we let $\lambda_G(t)$ be the tableau obtained from λ_G in the same way.

Example 3.2. We show what happens in Example 3.1 when $t=3$. We have:

$$\lambda(m,n) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline \end{array} \quad \text{and} \quad \lambda(m+t-1,n) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline 13 & 14 & 15 \\ \hline 16 & 17 & 18 \\ \hline \end{array}$$

where the squares of $\lambda(m,n)$ containing 1, 5 and 9 correspond to the 3 by 1 vertical rectangles of $\lambda(m+t-1,n)$ containing [1 4 7], [5 8 11], and [9 12 15], respectively. The set of squares in $\lambda(m,n)$ with $i=j$ is referred to as the *diagonal.*

Recall that we have:

$$\lambda_F = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 7 \\ \cline{1-1} \end{array} \quad \lambda_G = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \ .$$

The squares 1, 4, 7, 2, 5 of $\lambda(m,n)$ correspond in $\lambda(m+t-1,n)$ to

$$\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 7 \\ \hline \end{array} \qquad \boxed{10} \qquad \boxed{13} \qquad \boxed{2} \qquad \begin{array}{|c|}\hline 5 \\ \hline 8 \\ \hline 11 \\ \hline \end{array}$$

respectively; thus,

$$\lambda_F(t) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 7 & 8 \\ \hline 10 & 11 \\ \hline 13 \\ \cline{1-1}\end{array} \quad \text{and} \quad \lambda_G(t) = \begin{array}{|c|c|c|}\hline 1 & 10 & 13 \\ \hline 4 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 2 & 11 \\ \cline{1-2}\end{array} \; .$$

We note that the entries of $\lambda_F(t)$ are the same as those of $\lambda_G(t)$, although if $t > 1$, $\lambda_G(t)$ is not the transpose of $\lambda_F(t)$. In the construction of the complex, however, we will use tensor products with factors indexed by distinct sets of entries of $\lambda_F(t)$, and $\lambda_G(t)$ which will be in one-to-one correspondence. For an entry a in $\lambda_F(t)$, we will denote the corresponding entry in $\lambda_G(t)$ by $\widetilde{a}$. For example, we can write the λ_G of Example 3.1 as

$$\lambda_G = \begin{array}{|c|c|c|}\hline \widetilde{1} & \widetilde{4} & \widetilde{7} \\ \hline \widetilde{2} & \widetilde{5} \\ \cline{1-2}\end{array}$$

In addition, a permutation σ of the entries of $\lambda_F(t)$ defines a corresponding permutation of the entries of $\lambda_G(t)$, which we will denote $\widetilde{\sigma}$; if $\sigma(a) = b$, then $\widetilde{\sigma}(\widetilde{a}) = \widetilde{b}$.

We can now define $C_k(\phi, t)$ for $k = 0, 1, \dots, mn$. In the following definition, we let the notation $|\lambda| = k$ mean that λ is a partition of k. As above, we only allow partitions $\lambda_1 \geq \cdots \geq \lambda_m$ where $0 \leq \lambda_i \leq n$ for each i. Tensor products will be over R.

Definition 3.3. $C_k(\phi, t) = \bigoplus_{|\lambda|=k} e(\lambda_F(t))F \otimes e(\lambda_G(t))G$.

We now set about defining the boundary map d_k from $C_k(\phi, t)$ to $C_{k-1}(\phi, t)$. The boundary maps are defined using the map ϕ from F to G^*. As previously mentioned, the map ϕ defines a map from $F \otimes G$ to R, which takes $f \otimes g$ to $\phi(f)(g)$. This can be extended to a map from $(F^{\otimes n}) \otimes (G^{\otimes n})$ to R, and hence, if tensor powers $F^{\otimes n}$ and $G^{\otimes n}$ are indexed by sets in one-to-one correspondence, for any corresponding subsets of these sets with k elements, we have an associated map from

$(F^{\otimes n}) \otimes (G^{\otimes n})$ to $(F^{\otimes n-k}) \otimes (G^{\otimes n-k})$. These are the maps used in the definition of d_k.

Denote the summand $e(\lambda_F(t))F \otimes e(\lambda_G(t))G$ of $C_k(\phi, t)$ by $C_\lambda(t)$; it is a submodule of a tensor product $F^{\otimes r} \otimes G^{\otimes r}$, where the factors of $F^{\otimes r}$ and $G^{\otimes r}$ are indexed by the entries in the tableau $\lambda_F(t)$ and the corresponding entries of $\lambda_G(t)$. We will first define maps $d_{\mu\lambda}(t)$ from $C_\lambda(t)$ to $C_\mu(t)$ where $|\lambda| = k$ and $|\mu| = k-1$; d_k will be defined by combining the maps $d_{\mu\lambda}(t)$ with appropriate signs.

With λ and μ as above, λ_F consists of k squares and μ_F of $k-1$ squares. If there is an entry in μ_F that is not in λ_F, let $d_{\mu\lambda}(t) = 0$. If this is not true, μ_F can be obtained from λ_F by removing one square, say the square with entry a. We then denote μ by $\lambda - a$. Let $C_\lambda(t)$ be a submodule of $F^{\otimes r} \otimes G^{\otimes r}$ defined by an idempotent as above. Then, $C_\mu(t)$ is submodule of $F^{\otimes r'} \otimes G^{\otimes r'}$, where $F^{\otimes r'}$ and $G^{\otimes r'}$ are indexed over the same sets as $F^{\otimes r}$ and $G^{\otimes r}$ except that the entries or columns of t entries in $\lambda(m+t-1, n)$ corresponding to the entries a and $\widetilde{a}$ are deleted. The entry a must be on a corner of λ_F to be removed, and $\widetilde{a}$ will also be on a corner of λ_G so that $\lambda_G(t) - \widetilde{a}$ is also defined (we call a square of λ_F on the bottom of a column and the right end of a row a *corner* of λ_F). Whenever we have this situation, we will denote by ϕ_* the map from $F^{\otimes r} \otimes G^{\otimes r}$ to $F^{\otimes r'} \otimes G^{\otimes r'}$ defined by the map ϕ as described above.

We will use a fact about ϕ_* to reduce questions about the boundary maps to questions about elements of the group algebra of the symmetric group. We consider the involution on $\mathbb{Q}[S_n]$ induced by the map that sends a permutation σ to σ^{-1}, and we denote the image of an element r under this map by r^*.

Proposition 3.4. *Let ϕ_* be defined on a tensor product $F^{\otimes r} \otimes G^{\otimes r}$, where the two factors are indexed by sets in one-one correspondence as above. Let x and y be elements of $F^{\otimes r}$ and $G^{\otimes r}$ respectively, and let s and t be elements of $\mathbb{Q}[S_r]$. Then*:

$$\phi_*((sx) \otimes (\widetilde{t}y)) = \phi_*((t^*s)x \otimes y).$$

Proof. By linearity, it suffices to prove that, if σ and τ are elements of S_r, then

$$\phi_*((\sigma x) \otimes (\widetilde{\tau}y)) = \phi_*((\tau^{-1}\sigma)x \otimes y).$$

We have

$$\begin{aligned}\phi_*((\sigma x)\otimes(\widetilde{\tau}y)) &= \prod_i \phi((\sigma x)_i)((\widetilde{\tau}y)_{\widetilde{i}}) \\ &= \prod \phi(x_{\sigma^{-1}(i)})(y_{\widetilde{\tau}^{-1}(\widetilde{i})}).\end{aligned}$$

We now wish to write this product in terms of the original element y, that is, to index the second factor over its original index set. The factor $y_{\widetilde{j}}$ will appear for $\widetilde{j} = \widetilde{\tau}^{-1}(\widetilde{i})$, or $\widetilde{i} = \widetilde{\tau}(\widetilde{j})$, which means that $i = \tau(j)$. Thus, the above product is

$$\prod_j \phi(x_{\sigma^{-1}\tau(j)})(y_{\widetilde{j}}) = \phi_*(\tau^{-1}\sigma(x)\otimes y). \qquad \square$$

Therefore, when $\mu = \lambda - a$, we define $d_{\mu\lambda}(t)$ by first taking ϕ_* restricted to $C_\lambda(t)$, which lands in $F^{\otimes r'} \otimes G^{\otimes r'}$, then following with $e(\mu_F(t)) \otimes e(\mu_G(t))$ yielding $C_\mu(t)$. Hence,

$$d_{\mu\lambda}(t) = e(\mu_F(t)) \otimes e(\mu_G(t)) \cdot \phi_* \text{ restricted to } C_\lambda(t).$$

The final ingredient in the definition of the boundary map is an appropriate sign. For a partition λ and a corner a of λ, we let $s(\lambda, a)$ be the number of squares in the column above a plus the number of squares to the right of this column. For example, if λ_F is as in Example 3.1, $s(\lambda_F, 7) = 4$ and $s(\lambda_F, 5) = 1$.

We define d_k by letting the component from $C_\lambda(t)$ to $C_\mu(t)$ where $\mu = \lambda - a$ be

$$(-1)^{s(\lambda,a)} d_{\mu\lambda}(t).$$

To prove that this defines a complex, we must show that, for every λ_F with $|\lambda_F| = k$ and ν_F with $|\nu_F| = k-2$, the sum of all compositions $d_{\nu\mu}d_{\mu\lambda}$ with appropriate signs is zero. For there to be any nonzero contributions, ν must be of the form $\lambda - a - b$ for two squares a and b of λ_F, where a is a corner of λ_F and b becomes a corner of $\lambda_F - a$ or vice versa. There are two possible cases: either the squares a and b can be removed in only one order, in which case they are next to each other in one column or row, or they can be removed in either order, in which case they are on different corners of λ_F. In the first case, λ_F is of the form

$$\begin{ytableau} \; & & & \\ & & \\ & b & a \\ \; \end{ytableau} \quad \text{or} \quad \begin{ytableau} \; & & & \\ & & b \\ & & a \\ \; \end{ytableau}$$

In the second case λ_F is of the form

$$\begin{ytableau} \; & & & & \\ & & & a \\ & b \\ \; \end{ytableau} \quad \text{or} \quad \begin{ytableau} \; & & & & \\ & & & b \\ & a \\ \; \end{ytableau}$$

In the first case, there is a unique μ with $d_{\nu\mu} \neq 0$ and $d_{\mu\lambda} \neq 0$; therefore, we must have $d_{\nu\mu}d_{\mu\lambda} = 0$. In the second case, we have both $\mu = \lambda - a$ and $\mu' = \lambda - b$, and the condition that we need is:

$$(-1)^{s(\lambda-a,b)}d_{\nu\mu}(-1)^{s(\lambda,a)}d_{\mu\lambda} + (-1)^{s(\lambda-b,a)}d_{\nu\mu'}(-1)^{s(\lambda,b)}d_{\mu'\lambda} = 0.$$

We note that, in the second case, where b is below a, $s(\lambda - a, b) = s(\lambda, b) - 1$ and $s(\lambda - b, a) = s(\lambda, a)$. Hence, $(-1)^{s(\lambda-a,b)}(-1)^{s(\lambda,a)} = -(-1)^{s(\lambda-b,a)}(-1)^{s(\lambda,b)}$ (the case where a is below b is similar). Thus, we must show that

$$d_{\nu\mu}d_{\mu\lambda} = d_{\nu\mu'}d_{\mu'\lambda}.$$

Most of the remainder of the paper will be devoted to showing that $C_\bullet(\phi, t)$ is a complex, that is, that $d_{k-1}d_k = 0$. It follows from the discussion at the end of Section 1 that $C_k(\phi, t)$ is a free module for all k. In addition, if R is local and the r_{ij} are in the maximal ideal $\mathfrak{m}$ of R, then ϕ_* is also defined by a matrix with coefficients in $\mathfrak{m}$; thus, $[e(\mu_F)(t) \otimes e(\mu_G(t)]\phi_*$ maps C_λ into $\mathfrak{m}C_\mu$, and d_k is also defined by a matrix with coefficients in $\mathfrak{m}$. Hence, $C_\bullet(\phi, t)$ is a minimal free complex. It is shown in **[10]** that, in the generic case, $C_\bullet(\phi, t)$ is, in fact, a minimal free resolution of the ideal generated by the $t \times t$ minors of (r_{ij}).

In order to see that the map from C_0 to C_1 is defined by the minors of the matrix defining the map ϕ from F to G^*, we assume that we have a basis $f_1, \ldots, f_{m+t-1}$ of F and a basis $g_1, \ldots g_{n+t-1}$ of G and the (r_{ij}) is the matrix defining ϕ in terms of the given basis of F and the dual basis of the given basis of G. The module $C_0(t)$ is the tensor product of two tensor powers over the empty set, which is $R \otimes R = R$. The module $C_1(t)$ has as index set the partition given by the one square in the top left corner of $\lambda(m, n)$. Since this square in $\lambda(m, n)$ corresponds to a column of t squares in $\lambda(m + t - 1, n)$, $C_1(t) = e(\lambda)(F^{\otimes t}) \otimes e(\lambda)(G^{\otimes t})$, where λ is the partition defined by a column of t squares. This partition defines the tth exterior power; thus, the map d_1 from $C_1(t)$ to $C_0(t)$ is a map from $\bigwedge^t F \otimes \bigwedge^t G$ to R.

Let e be the idempotent defining the exterior product so that $e = (1/t!) \sum_{\sigma \in S_t} (-1)^{\mathrm{sign}(\sigma)} \sigma$. Then, a basis for $\bigwedge^t F \otimes \bigwedge^t G$ is given by the set of elements

$$e(f_{i_1} \otimes \cdots \otimes f_{i_t}) \otimes e(g_{j_1} \otimes \cdots \otimes g_{j_t}),$$

where

$$1 \le i_1 < \cdots < i_t \le m + t - 1 \quad \text{and} \quad 1 \le j_1 < j_2 < \cdots < j_t \le n + t - 1.$$

Using Proposition 3.4, since $e^2 = e$ and $e^* = e$, where * is the involution defined in that proposition, this can be replaced by

$$e(f_{i_1} \otimes \cdots \otimes f_{i_t}) \otimes (g_{j_1} \otimes \cdots \otimes g_{j_t}).$$

The map ϕ sends f_i to $\sum_j r_{ij} g_j^*$, and hence, we have

$$\begin{aligned}
&\phi_*(f_{i_1} \otimes \cdots \otimes f_{i_t}) \otimes (g_{j_1} \otimes \cdots \otimes g_{j_t}) \\
&\quad = \Big(\sum_j r_{i_1 j} g_j^*(g_{j_1})\Big)\Big(\sum_j r_{i_2 j} g_j^*(g_{j_2})\Big) \cdots \Big(\sum_j r_{i_t j} g_j^*(g_{j_t})\Big) \\
&\quad = r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_t j_t}.
\end{aligned}$$

If we now apply e, we obtain

$$(1/t!) \sum_{\sigma \in S_t} (-1)^{\mathrm{sign}(\sigma)} r_{\sigma(i_1) j_1} r_{\sigma(i_2) j_2} \cdots r_{\sigma(i_t) j_t},$$

which is $(1/t!)$ times the minor of (r_{ij}) corresponding to the rows $j_1, \ldots, j_t$ and the columns $i_1, \ldots, i_t$.

4. Proof that $C_\bullet(\phi, t)$ is a complex (Preliminaries). The proof that $C_\bullet(\phi, t)$ is a complex depends on certain identities involving the maps $d_{\mu\lambda}$, and the first step in proving these is to reduce them to identities involving elements of the group algebra $\mathbb{Q}[S_{(m+t-1)n}]$.

A product of the form $d_{\nu\mu}d_{\mu\lambda}$ can be written as the restriction to C_λ of

$$[e(\nu_F(t)) \otimes e(\nu_G(t))]\phi_*[e(\mu_F(t)) \otimes e(\mu_G(t))]\phi_*,$$

which, since $e(\lambda_F(t)) \otimes e(\lambda_G(t))$ is the identity on C_λ, can be written

$$[e(\nu_F(t)) \otimes e(\nu_G(t))]\phi_*[e(\mu_F(t)) \otimes e(\mu_G(t))]\phi_*[e(\lambda_F(t)) \otimes e(\lambda_G(t))]. \tag{4.1}$$

We next wish to remove ϕ_* and reduce the problem to identities in the group algebra. Consider the composition $[e(\mu_F(t))\otimes e(\mu_G(t))]\phi_*$. If $\mu_F = \lambda - a$, then ϕ_* contracts a and $\widetilde{a}$, which are not acted on by $e(\mu_F)$ or $e(\mu_G)$. Thus, we can apply the idempotent $[e(\mu_F(t)) \otimes e(\mu_G(t))]$ first and ϕ_* second. We then combine the two contractions ϕ_* to give a contraction on the union of the two sets. Finally, we apply the same argument on the next factor to the left and replace (4.1) by

$$\phi_*[e(\nu_F(t))\otimes e(\nu_G(t))][e(\mu_F(t))\otimes e(\mu_G(t))][e(\lambda_F(t))\otimes e(\lambda_G(t))]. \tag{4.2}$$

In order to simplify notation, let $\widehat{e}(\lambda)$ denote $e(\lambda_F(t))\otimes e(\lambda_G(t))$ for any partition λ. Let $k = (m+t-1)n$; we can then consider $\widehat{e}(\lambda)$ as an element of $\mathbb{Q}[S_{2k}]$, namely, as the product $e(\lambda_F(t))e(\lambda_G(t))$ in $\mathbb{Q}[S_{2k}]$.

The purpose of expressing $d_{\nu\mu}d_{\mu\lambda}$ in the form of equation (4.2) is that, now, identities involving $\widehat{e}(\nu)\widehat{e}(\mu)\widehat{e}(\lambda)$ become identities involving $d_{\nu\mu}d_{\mu\lambda}$ merely by adding ϕ_* on the left. Specifically, we have reduced the proof that $C_\bullet(\phi, t)$ is a complex to the following two lemmas (in which we denote the partition obtained by removing a corner a from λ_F by $\lambda - a$, as above).

Lemma 4.1. *Let b be a corner of λ_F, and let a be a square of λ_F directly above or to the left of b in such a way that it becomes a corner of $\lambda_F - b$. Then*:

$$\widehat{e}(\lambda - a - b)\widehat{e}(\lambda - b)\widehat{e}(\lambda) = 0.$$

Lemma 4.2. *Let a and b be distinct corners of λ_F with b below a. Then*:

$$\widehat{e}(\lambda - a - b)\widehat{e}(\lambda - a)\widehat{e}(\lambda) = \widehat{e}(\lambda - a - b)\widehat{e}(\lambda - b)\widehat{e}(\lambda).$$

We now examine products of the form $\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda)$ more closely. Writing $\lambda-b=\mu$ and $\lambda-a-b=\nu$, this product is:

$$[e(\nu_F(t))e(\nu_G(t))][e(\mu_F(t))e(\mu_G(t))][e(\lambda_F(t))e(\lambda_G(t))].$$

If two elements of $\mathbb{Q}[S_{2k}]$ involve only permutations that move disjoint sets of elements they commute, then, since the sets indexing powers of F and those indexing powers of G are disjoint, the above product is equal to

$$[e(\nu_G(t))e(\mu_G(t))e(\lambda_G(t))][e(\nu_F(t))e(\mu_F(t))e(\lambda_F(t))].$$

Consider the product $[e(\nu_F(t))e(\mu_F(t))e(\lambda_F(t))]$. Letting λ be the tableau $\lambda_F(t)$, it is of the form

$$e(\lambda-a-b)e(\lambda-b)e(\lambda)$$

and is an element of $\mathbb{Q}[S_k]$. In fact, it is an element of the subgroup of $\mathbb{Q}[S_k]$ consisting of permutations on the entries in $\lambda_F(t)$, and our computations will take place in this group ring. A similar formula holds for λ_G with a and b replaced by $\widetilde{a}$ and $\widetilde{b}$.

Products of this form will be calculated herein. The main idea is to express the products of idempotents involved in terms of a basis for the left ideal $\mathbb{Q}[S_k]$ and show that the terms all become zero. However, we first must go back and develop more properties of the idempotents $e(\lambda)$. The references in the remainder of this section will be to theorems in **[2]**.

We consider different tableaux with the same partition and denote them $\lambda_\alpha, \lambda_\beta$, and so on. One of these will be the above λ (either $\lambda_F(t)$ or $\lambda_G(t)$), which we denote λ_1. For each pair of tableaux λ_α and λ_β we denote $s_{\alpha\beta}$ as the permutation that takes λ_β to λ_α, that is, $s_{\alpha\beta}$ takes the entry in a given position in λ_β to the entry in that position in λ_α. We then have ([**2,** Theorem V.2.1]; this follows from the usual properties of conjugation):

$$s_{\alpha\beta}(e(\lambda_\beta)) = e(\lambda_\alpha)s_{\alpha\beta}.$$

There is one property of a pair of tableaux which will arise again and again; thus, we introduce notation for it. This property is that no two elements are in the same column in λ_α and in the same row in λ_β. We denote this by $\lambda_\alpha \ll \lambda_\beta$.

We have the following:

(1) the permutation $s_{\alpha 1}$ can be written as a product pq with $p \in P_\lambda$ and $q \in Q_\lambda$ (so $s_{\alpha 1}$ has a nonzero coefficient in $e(\lambda)$) if and only if $\lambda_\alpha \ll \lambda_1$ [**2,** Theorems IV.2.2, IV.2.3].
(2) If $\lambda_\alpha \not\ll \lambda_1$, then $e(\lambda_\alpha)e(\lambda_\beta) = 0$ [**2,** Theorem IV.2.3].

The proof of the second of these properties can be carried out as follows. Suppose there are elements x and y in the same column in λ_α and the same row in λ_β. Then, it may quite easily be shown that $e(\lambda_\alpha)$ is divisible on the right by $1 - (xy)$, where (xy) is the permutation which transposes x and y, and $e(\lambda_\beta)$ is divisible on the left by $1 + (xy)$. Further, in taking the product $e(\lambda_\alpha)e(\lambda_\beta)$, a factor $(1 - (xy))(1 + (xy)) = 0$ is obtained. The proofs of Lemmas 4.1 and 4.2 will use similar, although more complicated, techniques.

Suppose that the elements on which S_k acts are linearly ordered (for instance, they might be the integers $1, 2, \ldots, k$ with the usual ordering). A tableau λ_α in which the entries in each row and in each column are in increasing order will be called a *standard tableau.* We can then order the standard tableaux lexicographically by the rows and, denoting this order relation by $<$, by [**2,** Theorem IV.4.4], we have:

$$\text{If } \lambda_\alpha \ll \lambda_\beta, \text{ then } \lambda_\alpha < \lambda_\beta.$$

The number of standard tableaux is equal to the dimension over $\mathbb{Q}$ of the left ideal $\mathbb{Q}[S_k]e(\lambda)$ [**2,** Theorem IV.4.6]. In addition, it may be shown by the method of the proof of [**2,** Theorem IV.4.5] that the elements $s_{\alpha 1}e(\lambda) = e(\lambda_\alpha)s_{\alpha 1}$, where λ_α ranges over all the standard tableaux, are linearly independent over $\mathbb{Q}$; thus, they form a basis for $\mathbb{Q}[S_k]e(\lambda)$. Hence, in particular, we can write

$$e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda),$$

where the sum runs over the standard tableaux λ_α and the k_α are rational numbers defined uniquely by this equation.

In addition to the k_α, we define numbers j_α which we take to be κ_λ times the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$, where κ_λ is the number such that $E_\lambda^2 = \kappa_\lambda E_\lambda$. In good cases, it works out that $j_\alpha = k_\alpha$; however, this is not true in general since it can occur that we can write $s_{\alpha 1} = s_{\beta 1} pq$ with $p \in P_\lambda$ and $q \in Q_\lambda$ so that the term $(-1)^{\mathrm{sign}(q)} k_\beta s_{\beta 1} pq$ also contributes to the value of j_α.

Proposition 4.3. *We can write $s_{\alpha 1} = s_{\beta 1} pq$ as above if and only if $\lambda_\alpha \ll \lambda_\beta$.*

Proof. We have $s_{\alpha 1} = s_{\beta 1} pq$ if and only if $s_{\beta 1}^{-1} s_{\alpha 1} = pq$. Write $s_{\beta 1}^{-1} s_{\alpha 1}$ as $s_{\gamma 1}$ with corresponding tableau λ_γ; we then have $s_{\beta 1}^{-1} s_{\alpha 1} = pq$ if and only if $\lambda_\gamma \ll \lambda_1$. Now,

$$s_{\beta 1} s_{\gamma 1} s_{\beta 1}^{-1} = s_{\beta 1} s_{\beta 1}^{-1} s_{\alpha 1} s_{\beta 1}^{-1} = s_{\alpha 1} s_{\beta 1}^{-1} = s_{\alpha 1} s_{1\beta} = s_{\alpha\beta}.$$

Hence, if the element in the (i,j) position in λ_1 is in the (k,l) position in λ_γ, then the element in the (i,j) position in λ_β is in the (k,l) position in λ_α, and $\lambda_\gamma \ll \lambda_1$ if and only if $\lambda_\alpha \ll \lambda_\beta$. Thus, Proposition 4.3 is proven. $\square$

Proposition 4.4. *Suppose that there are two elements in the same column in λ_α and in the same row in λ_1, and neither of these is an entry of b. Then, the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$ is zero.*

Proof. Let x and y be these elements. Let u and v be the entries of λ in the positions that x and y occupy in λ_α. Then, the transposition (uv) is in Q_λ and, since neither x nor y is in b, (xy) is in $P_{\lambda - b}$. Hence:

$$(xy)e(\lambda - b)e(\lambda) = e(\lambda - b)e(\lambda)$$

and

$$e(\lambda - b)e(\lambda)(uv) = -e(\lambda - b)e(\lambda).$$

Since $s_{\alpha 1}$ sends u to x and v to y, we have

$$(xy)s_{\alpha 1} = s_{\alpha 1}(uv).$$

From these equalities, it follows that the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$ equals the coefficient of $(xy)s_{\alpha 1}$ equals the coefficient of $s_{\alpha 1}(uv)$ equals minus the coefficient of $s_{\alpha 1}$. Hence, this coefficient must be zero. $\square$

We can now reduce the problem of calculating the products $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ to manageable proportions. The next proposition shows that, in the expression of $e(\lambda - a)e(\lambda)$ as a sum of terms $k_\alpha s_{\alpha 1} e(\lambda)$, we only need consider permutations $s_{\alpha 1}$ that move only entries in a and b.

Proposition 4.5. *Let λ, a and b be as above. Assume that λ is a standard tableau in which the entries of $\lambda - a - b$ all precede those of a and b and, in $\lambda - a - b$, the entry in a given row precedes any element in any lower row. Write*

$$e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda),$$

as above. Then, if $s_{\alpha 1}$ moves any entry of $\lambda - a - b$, we have either

(i) $k_\alpha = 0$, *or*
(ii) $e(\lambda - a - b)s_{\alpha 1} e(\lambda) = 0$.

In addition, if $s_{\alpha 1}$ moves only elements of a and b, we have

$$j_\alpha = k_\alpha.$$

Proof. Let λ_α be a standard tableau in which an entry of $\lambda - a - b$ is in a different position than in λ_1. Let u be the lowest of these. Since the entries of $\lambda - a - b$ are arranged in λ_1 in increasing order along successive rows, there are two possibilities.

(i) u is in a higher row in λ_α than in λ_1.
(ii) u is the first entry in λ_α of the row below the one it occupies in λ_1.

If case (i) holds, there is a v in the same column as u in λ_1 and the same row as u in λ_α. Since neither u nor v is an entry of a or b, this implies that

$$e(\lambda - a - b)s_{\alpha 1} e(\lambda) = e(\lambda - a - b)e(\lambda_\alpha)s_{\alpha 1} = 0.$$

Suppose case (ii) holds; we will show this implies that $k_\alpha = 0$. Suppose not, and let u be minimal among all λ_α such that case (ii) holds and $k_\alpha \neq 0$. Then, if v is the first element of the row of u in λ_1, u and v are in the same column of λ_α; thus, $j_\alpha = 0$ by Proposition 4.4.

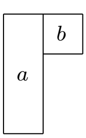
Since $k_\alpha \neq 0$, by Proposition 4.3 there must be a β with $\lambda_\alpha \ll \lambda_\beta$ and $k_\beta \neq 0$.

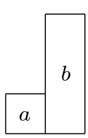
Let w be the lowest element which is moved by $s_{\beta 1}$. If $w < u$ and case (i) holds for λ_β, then two elements are in the same column of λ_α and in the same row of λ_β, contradicting $\lambda_\alpha \ll \lambda_\beta$. If case (ii) holds, this contradicts the minimality of u. Hence, $w \geq u$. If w is not moved by $s_{\beta 1}$, this again contradicts $\lambda_\alpha \ll \lambda_\beta$. Hence, the elements up to u are in exactly the same place in λ_β as in λ_α, and $j_\beta = 0$. Thus, we must have a γ with $\lambda_\beta \ll \lambda_\gamma$ and $k_\gamma \neq 0$. However, this then continues indefinitely, since there is a linear ordering $<$ on the standard tableaux such that $\lambda_\alpha \ll \lambda_\beta$ implies $\lambda_\alpha < \lambda_\beta$, this procedure must eventually stop, which is a contradiction.

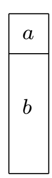
In order to prove the last statement of Proposition 4.5, suppose that $s_{\alpha 1}$ is a permutation which moves only elements of a and b. If $j_\alpha \neq k_\alpha$, we must have a standard tableau λ_β with $\lambda_\alpha \ll \lambda_\beta$ and $k_\beta \neq 0$. If $s_{\alpha 1}$ moves an entry of $\lambda - a - b$, either case (i) or case (ii) above is true; case (i) contradicts $\lambda_\alpha \ll \lambda_\beta$, and case (ii) contradicts $k_\beta \neq 0$. Since a and b each consist of squares in the same column, two different standard tableaux that move only elements of a and b must move some entry from a to b and vice versa. This implies that an entry from whichever is lower or to the left must move to a position higher and to the right in λ_β, and this again contradicts $\lambda_\alpha \ll \lambda_\beta$. Hence, we have $j_\alpha = k_\alpha$. $\square$

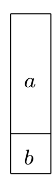
The above propositions provide enough machinery for most of the calculation of the expression $e(\lambda - a - b)e(\lambda - b)e(\lambda)$.

5. Proof of Lemma 4.1. Let a and b be two elements in a row or column of λ_F such that b is a corner and a becomes a corner after b is removed. Let $\lambda = \lambda_F(t)$. Then, there are four possible configurations for a and b in λ, depending upon whether a and b are in the same row or the same column in λ_F and depending upon whether a or b corresponds to a column of t elements in λ (we can consider the case where both a and b correspond to single squares as a special case of either of these). The four possible arrangements are:

1.


2.


3.


4.


We assume that the elements of λ are ordered as in Proposition 4.5, where we first have all the elements of $\lambda - a - b$ arranged row-by-row, and then the entries of a and b. Since we will only have to deal with the entries of a and b, however, we will denote them $1, 2, \ldots, t+1$.

Proposition 5.1. *In cases* 1 *and* 2 *above, the product* $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ *is a sum of terms, each of which is divisible on the left by* $1+(xy)$ *for some distinct* x *and* y *between* 1 *and* $t+1$.

Proof. We let $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$, and, since we are multiplying on the left by $e(\lambda - a - b)$, from Proposition 4.5 we need consider only those terms for which $s_{\alpha 1}$ moves only the elements $1, 2, \ldots, t+1$. For each such $s_{\alpha 1}$, two elements between 1 and $t+1$, say x and y, end up in a row in λ_α.

Hence, $s_{\alpha 1} e(\lambda) = e(\lambda_\alpha) s_{\alpha 1}$ is divisible on the left by $1 + (xy)$. Since $e(\lambda - a - b)$ does not involve any of the numbers between 1 and $t+1$, $1 + (xy)$ commutes with $e(\lambda - a - b)$; thus, $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ is a sum of terms divisible on the left by $1 + (xy)$ for various pairs x, y of elements between 1 and $t+1$, as was shown. □

We next prove a similar result for cases 3 and 4; however, since the main idea will be used again in a more general form, we prove it separately.

Proposition 5.2. *Let* λ *be any tableau, and let* c *be a column of consecutive squares of* λ, *all of which are at the right ends of their rows* (*which must therefore have the same length*), *and such that the bottom square of* c *is a corner. Then,* $e(\lambda - c)$ *is defined, and for any permutation* q *of the entries in* c, *we have*:

$$qe(\lambda - c)e(\lambda) = (-1)^{\mathrm{sign}(q)} e(\lambda - c)e(\lambda).$$

Similarly, if d *is another corner square of* λ, *we have*

$$qe(\lambda - c - d)e(\lambda) = (-1)^{\mathrm{sign}(q)} e(\lambda - c - d)e(\lambda).$$

Proof. Suppose that c is in the nth column of λ. For $i = 1, 2, \ldots, n-1$, define q_i to be the permutation which acts on the ith column in the same way as q acts on the nth column, that is, if q sends the entry in the (k, n) position to the entry in the (k', n) position, then q_i sends the entry in the (k, i) position to the entry in the (k', i) position. Let $\overline{q} = q_1 q_2 \cdots q_{n-1}$.

We then have that $q\overline{q}$ commutes with $e(\lambda)$. In fact, if we let $q\overline{q} = s_{\alpha 1}$, then $s_{\alpha 1} e(\lambda) = e(\lambda_\alpha) s_{\alpha 1}$, and since λ_α is derived from λ by interchanging entire rows, $e(\lambda_\alpha) = e(\lambda)$. We note that q also commutes with $\overline{q}$ and with $e(\lambda - c)$. Thus,

$$\begin{aligned} qe(\lambda - c)e(\lambda) &= e(\lambda - c)qe(\lambda) \\ &= e(\lambda - c)\overline{q}^{-1} q\overline{q} e(\lambda) \\ &= [e(\lambda - c)\overline{q}^{-1}][e(\lambda) q\overline{q}]. \end{aligned}$$

Now, $\overline{q}^{-1} \in Q_{\lambda - c}$ and $q\overline{q} \in Q_\lambda$. Hence,

$$\begin{aligned} &[e(\lambda - c)\overline{q}^{-1}][e(\lambda) q\overline{q}] \\ &\quad = [e(\lambda - c)(-1)^{\operatorname{sign}(q^{-1})}][e(\lambda)(-1)^{\operatorname{sign}(q\overline{q})}] \\ &\quad = (-1)^{\operatorname{sign}(q)} e(\lambda - c)e(\lambda). \end{aligned}$$

If d is another corner square of λ, then it cannot be in any of the rows containing entries in the column c, and the same proof shows that

$$qe(\lambda - c - d)e(\lambda) = (-1)^{\operatorname{sign}(q)} e(\lambda - c - d)e(\lambda). \qquad \square$$

Remark 5.3. It is not necessarily true that $qe(\lambda) = (-1)^{\operatorname{sign}(q)} e(\lambda)$ in the situation of Proposition 5.2. For example, if

$$\lambda = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\ ,$$

then $(34)e(\lambda) \neq e(\lambda)$. It follows from Proposition 5.2, however (and can easily be directly verified) that

$$(34)[(1 - (12))e(\lambda)] = -[(1 - (12))e(\lambda)].$$

Proposition 5.4. *In cases* 3 *and* 4 *above,* $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ *is divisible on the left by* $1 - (xy)$ *for all pairs* x, y *of numbers between* 1 *and* $t + 1$.

Proof. Again, we only need to consider $s_{\alpha 1}$, which move only the elements $1, 2, \ldots, t + 1$, and since, in this case, they must remain in increasing order since they are all in one column, we have

$$e(\lambda-a-b)e(\lambda-b)e(\lambda)=k_1e(\lambda-a-b)e(\lambda),$$

and we must show that $e(\lambda-a-b)e(\lambda)$is divisible on the left by $1-(xy)$. This follows immediately from Proposition 5.2, letting c be a and b together and letting $q=(xy)$. □

We can now show that

$$\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda)=0.$$

Let $\lambda=\lambda_F(t)$ and $\mu=\lambda_G(t)$. We can write this product as

$$e(\lambda-a-b)e(\lambda-b)e(\lambda))e(\mu-\widetilde{a}-\widetilde{b})e(\mu-\widetilde{b})e(\mu).$$

Since λ_G is the transpose of λ_F, a and b are in a column in λ_F if and only if they are in a row in λ_G and vice versa; thus, case 1 or 2 holds for λ if and only if case 3 or 4 holds for μ (and vice versa). We assume that case 1 or 2 holds for λ and case 3 or 4 holds for μ; the proof in the other case is essentially the same. From Proposition 5.1, we can write

$$e(\lambda-a-b)e(\lambda-b)e(\lambda)=\sum_{x,y}(1+(xy))A_{x,y},$$

where $A_{x,y}$ are elements of $\mathbb{Q}[S_k]$ indexed over pairs out of the set $\{1,2,\ldots,t+1\}$. By Proposition 5.4, for each x and y we can find a $B_{x,y}$ with

$$e(\mu-\widetilde{a}-\widetilde{b})e(\mu-\widetilde{a})e(\mu)=(1-(\widetilde{x}\widetilde{y}))B_{x,y}.$$

Hence,

$$\begin{aligned}
&\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda)\\
&\quad=e(\lambda-a-b)e(\lambda-b)e(\lambda)e(\mu-\widetilde{a}-\widetilde{b})e(\mu-\widetilde{a})e(\mu)\\
&\quad=\sum_{x,y}(1+(xy))A_{x,y}e(\mu-\widetilde{a}-\widetilde{b})e(\mu-\widetilde{a})e(\mu)\\
&\quad=\sum_{x,y}(1+(xy))A_{x,y}(1-(\widetilde{x}\widetilde{y}))B_{x,y}.
\end{aligned}$$

We now apply ϕ_* and Proposition 3.4, using the fact that $(1-(xy))^* = 1-(xy)$ for all x and y.

$$\begin{aligned}
&\phi_*(\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda)) = \\
&\quad = \phi_*\left(\sum_{x,y}(1+(xy))A_{x,y}(1-(\widetilde{x}\widetilde{y}))B_{x,y}\right) \\
&\quad = \phi_*\left(\sum_{x,y}(1-(xy))(1+(xy))A_{x,y}B_{x,y}\right) \\
&\quad = 0.
\end{aligned}$$

This completes the proof of Lemma 4.1.

6. Proof of Lemma 4.2. We now let a and b be two distinct corners of λ_F, and we assume that b is below a. Now, we must calculate the products:

$$\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-a)\widehat{e}(\lambda)$$

and

$$\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda).$$

As before, we let $\lambda = \lambda_F(t)$, and let a and b denote the squares or columns of squares of $\lambda_F(t)$ corresponding to a and b in λ_F. We begin by calculating the products:

$$e(\lambda-a-b)e(\lambda-a)e(\lambda)$$

and

$$e(\lambda-a-b)e(\lambda-b)e(\lambda).$$

The method is similar to that used to prove Lemma 4.1 in that we will represent each of these products in the form

$$e(\lambda-a-b)\left(\sum k_\alpha s_{\alpha 1}e(\lambda)\right)$$

where λ_α ranges over the standard tableaux. We assume that $\lambda = \lambda_1$ is a standard tableau of the type in Proposition 4.5 so that the entries of $\lambda-a-b$ precede those of a and b and, in $\lambda-a-b$, the entries in a given row precede the entries on any lower row. We will also generally assume that either a or b is a column of t squares since the situation in which both are single squares is a special case of this.

Proposition 6.1. *We have* $k_1 = 1$.

Proof. From Proposition 4.5, we have $k_1 = j_1$, where j_1 is κ_λ times the coefficient of the identity in $e(\lambda - a)e(\lambda)$ (or $e(\lambda - b)e(\lambda)$).

We can write

$$e(\lambda - a)e(\lambda) = \frac{1}{\kappa_\lambda \kappa_{\lambda-a}} E_{\lambda-a}E_\lambda.$$

If $pqp'q' = 1$ with $p \in P_{\lambda-a}, q \in Q_{\lambda-a}, p' \in P_\lambda$, and $q' \in Q_\lambda$, then pq leaves the entries in a fixed; thus, $p'q'$ also does, and, since p' can only permute elements in the same row and q' can only permute elements in the same column, p' and q' must leave the entries in a fixed. Thus, $p' \in P_{\lambda-a}$ and $q' \in Q_{\lambda-a}$, and the coefficient of 1 in $E_{\lambda-a}E_\lambda$ is the same as in $E_{\lambda-a}E_{\lambda-a}$, which is $\kappa_{\lambda-a}$. Hence, the coefficient of 1 in $e(\lambda - a)e(\lambda)$ is

$$\frac{1}{\kappa_\lambda \kappa_{\lambda-a}} \cdot \kappa_{\lambda-a} = \frac{1}{\kappa_\lambda},$$

and $j_1 = \kappa_\lambda(1/\kappa_\lambda) = 1$. $\square$

Proposition 6.2. *We have*

$$e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)e(\lambda).$$

Proof. Writing $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$, by Proposition 4.5, it is again only necessary to consider $s_{\alpha 1}$, which move only $1, 2, \ldots, t+1$. If $\lambda_\alpha \neq \lambda_1$, at least one entry from a must go to a position in b in λ_α; since b is below a and to the left of a, this means that two elements in a row in λ_1 are in a column in λ_α; thus, $j_\alpha = 0$ by Proposition 4.4. Since $j_\alpha = k_\alpha$, we therefore have $k_\alpha = 0$.

Hence,

$$e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)[k_1 e(\lambda)] = e(\lambda - a - b)e(\lambda),$$

since $k_1 = 1$. $\square$

The calculation of $e(\lambda - a - b)e(\lambda - a)e(\lambda)$ is considerably more complicated since, in this case, there are nontrivial permutations $s_{\alpha 1}$ of $1, 2, \dots, t+1$ with $k_\alpha \neq 0$, and we must find the value of k_α.

In the next few propositions we will let 1 be the entry in whichever of a or b is a single square.

Proposition 6.3. *Let s and s' be permutations of* $1, 2, \dots, t+1$ *such that* $s(1) = s'(1)$. *Then*:

(i) $e(\lambda - a - b)se(\lambda) = (-1)^{\operatorname{sign}(s)+\operatorname{sign}(s')}e(\lambda - a - b)s'e(\lambda)$.
(ii) *If j is the coefficient of s in $e(\lambda - a)e(\lambda)$ and j' is the coefficient of s', then*
$$j = (-1)^{\operatorname{sign}(s)+\operatorname{sign}(s')}j'.$$

Proof. Since $s(1) = s'(1)$, $s^{-1}s' = q$ permutes only the numbers 2 through $t+1$. Since $\operatorname{sign}(s) + \operatorname{sign}(s') = \operatorname{sign}(q)$ and s and s' commute with $e(\lambda - a - b)$, statement (i) is equivalent to

$$qe(\lambda - a - b)e(\lambda) = (-1)^{\operatorname{sign}(q)}e(\lambda - a - b)e(\lambda),$$

which follows from Proposition 5.2.

In order to prove the second statement, since $sq = s'$, it suffices to show that
$$e(\lambda - a)e(\lambda)q = (-1)^{\operatorname{sign}(q)}e(\lambda - a)e(\lambda).$$

In fact, q is in Q_λ, so, in this case, we even have $e(\lambda)q = (-1)^{\operatorname{sign}(q)}e(\lambda)$; thus, this equality is clear. □

We will use Proposition 6.3 to replace the permutations $s_{\alpha 1}$ with λ_α a standard tableau by other permutations which make calculation easier. Specifically, let $s_{\alpha 1}$ be a permutation that moves only $1, 2, \dots, t+1$ with λ_α a standard tableau. Let r be the number in λ_α in the place of 1 (thus, $s_{\alpha 1}(1) = r$). Then, the other numbers must be arranged in increasing order in the remaining squares since they are in a column. For each r, we can construct such a tableau with r in place of 1; thus, we see that there is one such λ_α for each $r = 1, 2, \dots, t+1$, and it is characterized by the property that $s_{\alpha 1}(1) = r$. We replace $s_{\alpha 1}$ by the transposition $(1r)$. Then, we can write

$$e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)\left[\sum_{r=1}^{t+1} k_r(1r)e(\lambda)\right],$$

where k_r is κ_λ times the coefficient of $(1r)$ in the product $e(\lambda - a)e(\lambda)$.

Proposition 6.4. *If r and r' are numbers between 2 and $t+1$, then $k_r = k_{r'}$.*

Proof. We have

$$(1r') = (rr')(1r)(rr').$$

We note that (rr') transposes two elements in the same column at the ends of rows of the same length. Similarly to Proposition 5.2, we let $\overline{q}$ be the product of the transposition of the elements in the same rows in each column to the left of that containing r and r'. Then, $\overline{q}(rr')$ commutes with $e(\lambda)$. In addition, $\overline{q}(rr')$ commutes with $e(\lambda - a)$; if a is the square containing 1, $\overline{q}(rr')$ commutes with $e(\lambda - a)$ for the same reason it commutes with $e(\lambda)$, and, if a is the column containing r and r', the same reasoning shows that $\overline{q}$ commutes with $e(\lambda - a)$. We thus deduce that $\overline{q}(rr')$ commutes with $e(\lambda - a)$ since $e(\lambda - a)$ does not involve either r or r'.

Since $\overline{q}$ commutes with (rr') and $(1r)$ and $\overline{q}^2 = 1$, we have

$$(1r') = \overline{q}(rr')(1r)\overline{q}(rr').$$

Furthermore,

$$\begin{aligned}&\overline{q}(rr')[e(\lambda - a)e(\lambda)]\overline{q}(rr')\\ &\quad = e(\lambda - a)e(\lambda)[\overline{q}(rr')]^2\\ &\quad = e(\lambda - a)e(\lambda).\end{aligned}$$

From these two equations, it follows that the coefficient of $(1r')$ in $e(\lambda - a)e(\lambda)$ is the same as that of $(1r)$; thus, $k_r = k_{r'}$. ☐

We wish to reduce the calculation of k_r to the case where a and b are single squares. If a is a single square, this is already done since it is enough to calculate the coefficient of the transposition $(1, t+1)$ in $e(\lambda - a)e(\lambda)$. If a is a column of t squares, we need one more result.

Proposition 6.5. *Suppose that a is a column of t squares, and let λ' be the tableau obtained from λ by removing the squares containing $3, \dots, t+1$ so that it leaves only one square a' of a, and a' contains the number 2. The κ_λ times the coefficient of (12) in $e(\lambda - a)e(\lambda)$ is equal to $\kappa_{\lambda'}$ times the coefficient of (12) in $e(\lambda - a')e(\lambda')$.*

Proof. We note that $\lambda' - a' = \lambda - a$. Hence, the desired equality can be written

$$\begin{aligned}\kappa_{\lambda-a}\kappa_\lambda&[\text{coefficient of } (12) \text{ in } e(\lambda - a)e(\lambda)]\\ &= \kappa_{\lambda'-a}\kappa_{\lambda'}[\text{coefficient of } (12) \text{ in } e(\lambda' - a')e(\lambda')].\end{aligned}$$

Since $\kappa_\mu e(\mu) = E_\mu$ for any of the tableaux μ, this equality is equivalent to the statement that the coefficient of (12) in $E_{\lambda-a}E_\lambda$ is equal to the coefficient of (12) in $E_{\lambda'-a'}E_{\lambda'}$.

Now, $E_{\lambda-a} = E_{\lambda'-a'}$, and $E_{\lambda'}$ is the sum of those terms of E_λ that leave 3 through $t+1$ fixed. Suppose that $(12) = pqp'q'$ with $p \in P_{\lambda-a}$, $q \in Q_{\lambda-a}$, $p' \in P_\lambda$ and $q' \in Q_\lambda$. Then, pq and (12) leave 3 through $t+1$ fixed; thus, so does $p'q'$. Hence, the terms of E_λ that contribute to the coefficient of E_λ are all in $E_{\lambda'}$, which proves the result. $\square$

Thus, the problem of finding the coefficients k_r is reduced to the following.

Proposition 6.6. *Let a and b be two corner squares of λ, with b below a. Then, the coefficient of (ab) in $e(\lambda - a)e(\lambda)$ is $1/(\kappa_\lambda \eta)$, where η is the axial distance from a to b in λ. Hence, we have $k_{(ab)} = j_{(ab)} = 1/\eta$.*

Proof. Let n be the number of entries of λ. The proof will use Young's semiformal representation of $\mathbb{Q}[S_n]$ (cf., Boerner **[2,** IV.5]. References to theorems in this proof will be to theorems in this book).

If we have any isomorphism of $\mathbb{Q}[S_n]$ with a product of matrix rings

$$\mathbb{Q}[S_n] \cong \bigoplus_{i=1}^{k} M_{f_i}(\mathbb{Q}),$$

then the coefficient of 1 of any element $x \in \mathbb{Q}[S_n]$ is equal to

$$\frac{1}{n!}\sum_{i=1}^{k} f_i \operatorname{Tr}(x_i),$$

where x_i is the ith component of the image of x in $\prod_{i=1}^{k} M_{f_i}(\mathbb{Q})$ by the isomorphism above and $\operatorname{Tr}(x_i)$ is its trace. This may be seen by calculating the trace of x acting on $\mathbb{Q}[S_n]$ in two ways. If we consider $x = (x_i)$ as an element of $\prod_{i=1}^{k} M_{f_i}(\mathbb{Q})$, this trace is $\sum_{i=1}^{k} f_i \operatorname{Tr}(x_i)$, and, if we consider it as an element of $\mathbb{Q}[S_n]$ and use the elements of S_n as a basis, it is equal to $n!$ times the coefficient of 1 in x. Setting these equal gives the above formula.

Since the coefficient of (ab) in $e(\lambda - a)e(\lambda)$ is the same as the coefficient of 1 in $(ab)e(\lambda - a)e(\lambda)$, we can use this formula. Furthermore, the component of $e(\lambda)$ is zero in every factor of $\prod_{i=1}^{k} M_{f_i}(\mathbb{Q})$ except that of the irreducible representation corresponding to the Young diagram λ; thus, the coefficient we want is equal to $(f/n!)\operatorname{Tr}(A)$, where A is the matrix representing $(ab)e(\lambda - a)e(\lambda)$ in the semiformal representation of $\mathbb{Q}[S_n]$ corresponding to λ, and where f is the degree of this representation.

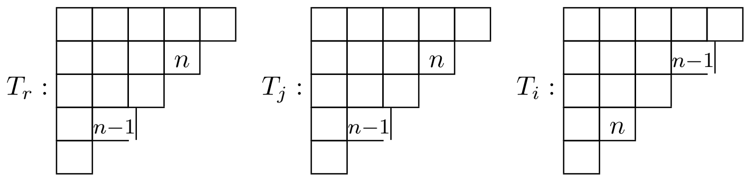
Let the entries of λ be the integers $1, 2, \dots, n$ arranged as a standard tableau in such a way that $a = n$ and $b = n - 1$. Using the notation of **[2,** IV.5], we arrange the standard tableaux $T_1, \dots, T_f$ in last letter sequence and let $\lambda = T_r$. Let $(g_{ij}), (h_{ij}^{\lambda-a})$ and (h_{ij}^{λ}) be the matrices representing (ab), $e(\lambda - a)$ and $e(\lambda)$, respectively. Then (Theorems 5.4 and 5.5), $(h_{ij}^{\lambda-a})$ and (h_{ij}^{λ}) have 1 at the (r,r) position, and the entry at the (i,j) position is zero if $i > r$ or $j < r$. In addition, $h_{ij}^{\lambda-a} = 0$ unless both T_i and T_j with n in the same position as in T_r (by Theorem 5.4). Let $(k_{ij}) = (h_{ij}^{\lambda-a})(h_{ij}^{\lambda})$; then, $k_{rr} = 1$, $k_{ij} = 0$ if $i > r$ or $j < r$, and $k_{ij} = 0$ unless T_i has n in the same position as in T_r.

Our aim is to calculate

$$\operatorname{Tr}((g_{ij})(k_{ij})) = \sum_{i=1}^{f}\sum_{j=1}^{f} g_{ij}k_{ji}.$$

We will show, in fact, that $g_{ij}k_{ji} = 0$ unless $i = j = r$, and that $g_{rr}k_{rr} = 1/\eta$, where η is the axial distance from a to b.

Suppose that $g_{ij}k_{ji} \neq 0$. If $i = j$, then $k_{ij} = 0$ unless $i = j = r$. Suppose $i \neq j$. Since (g_{ij}) represents $(ab) = (n, n-1)$, this can only occur when $(n, n-1)T_i = T_j$ (Corollary 1–Theorem 5.6). Furthermore, since $k_{ji} \neq 0$, we have $j \leq r \leq i$, and n is in the same position in T_j as in T_r. Since $j \leq r$, the position of $n-1$ cannot be higher in T_j than in T_r. Since $n-1$ is below n in T_r, the three tableaux T_r, T_j, and T_i look like:



However, this implies that the position of n in T_j is higher than in T_i, so $j > i$, which contradicts $j \leq r \leq i$. Thus, we cannot have $g_{ij}k_{ij} \neq 0$ for $i \neq j$.

Hence, $\mathrm{Tr}((g_{ij})(k_{ji})) = g_{rr}k_{rr} = g_{rr}$. From Corollary 2–Theorem 5.6, $g_{rr} = 1/\eta$, where η is the axial distance from n to $n-1$, or equivalently from a to b, in $T_r = \lambda$. Therefore, the coefficient of (ab) in $e(\lambda - a)e(\lambda)$ is $(f/n!) \cdot (1/\eta)$. Since $f/n! = 1/\kappa_\lambda$ ([**2,** IV.3, equation (3.1)]), this proves the result. $\square$

We can now finish the proof of Lemma 4.2. Let $\lambda = \lambda_F(t)$ and $\widetilde{\lambda} = \lambda_G(t)$. We have

$$\begin{aligned}&\widehat{e}(\lambda - a - b)\widehat{e}(\lambda - a)\widehat{e}(\lambda)\\ &\qquad = e(\lambda - a - b)e(\lambda - a)e(\lambda)e(\widetilde{\lambda} - \widetilde{a} - \widetilde{b})e(\widetilde{\lambda} - \widetilde{a})e(\widetilde{\lambda})\end{aligned}$$

and

$$\begin{aligned}&\widehat{e}(\lambda - a - b)\widehat{e}(\lambda - b)\widehat{e}(\lambda)\\ &\qquad = e(\lambda - a - b)e(\lambda - b)e(\lambda)e(\widetilde{\lambda} - \widetilde{a} - \widetilde{b})e(\widetilde{\lambda} - \widetilde{b})e(\widetilde{\lambda}).\end{aligned}$$

Since b is below a in λ, $\widetilde{a}$ is below $\widetilde{b}$ in $\widetilde{\lambda}$. Thus,

$$e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)e(\lambda)$$

and

$$e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}-\widetilde{a})e(\widetilde{\lambda}) = e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}).$$

We first assume that a and b correspond to single squares, and we let η be the axial distance from a to b, which is the same as the axial distance from $\widetilde{b}$ to $\widetilde{a}$. Denoting the transposition of a and b by (ab), we have from the results above that

$$e(\lambda-a-b)e(\lambda-a)e(\lambda) = e(\lambda-a-b)[1+(1/\eta)(ab)]e(\lambda)$$

and

$$e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}-\widetilde{b})e(\widetilde{\lambda}) = e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})[1+(1/\eta)(\widetilde{a}\widetilde{b})]e(\widetilde{\lambda}).$$

Since the permutation (ab) commutes with $e(\lambda-a-b)$ and $(\widetilde{a}\widetilde{b})$ commutes with $e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})$, we have

$$e(\lambda-a-b)e(\lambda-a)e(\lambda) = [1+(1/\eta)(ab)]e(\lambda-a-b)e(\lambda)$$

and

$$e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}-\widetilde{b})e(\widetilde{\lambda}) = [1+(1/\eta)(\widetilde{a}\widetilde{b})]e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}).$$

Thus, we have

$$\begin{aligned}&\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-a)\widehat{e}(\lambda)\\ &\qquad\qquad= [1+(1/\eta)(ab)]e(\lambda-a-b)e(\lambda)e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda})\end{aligned}$$

and

$$\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda) = e(\lambda-a-b)e(\lambda)[1+(1/\eta)(\widetilde{a}\widetilde{b})]e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}).$$

Hence, using Proposition 3.4, we have

$$\begin{aligned}&\phi_*(\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-b)\widehat{e}(\lambda))\\ &\quad= \phi_*(e(\lambda-a-b)e(\lambda)[1+(1/\eta)(\widetilde{a}\widetilde{b})]e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda}))\\ &\quad= \phi_*[1+(1/\eta)(ab)](e(\lambda-a-b)e(\lambda)e(\widetilde{\lambda}-\widetilde{a}-\widetilde{b})e(\widetilde{\lambda})\\ &\quad= \phi_*(\widehat{e}(\lambda-a-b)\widehat{e}(\lambda-a)\widehat{e}(\lambda)).\end{aligned}$$

The proof for the case where one of a or b is a column of squares is basically the same using the above results. However, since the situation is more complicated, it is necessary to check that the coefficients which occur are in fact the same. We will work out the case in which a is a column of squares; the other case is similar.

We let $\lambda = \lambda_F(t)$ and $\widetilde{\lambda} = \lambda_G(t)$. As above, the factors in which we remove b first from λ and $\widetilde{a}$ first from $\widetilde{\lambda}$ do not involve extra factors from the group ring. We now consider the other factors.

We have

$$e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)\sum_r k_r(1r)e(\lambda),$$

where r runs over the entries in column a. From Proposition 6.4, the coefficients k_r are all equal, and from Proposition 6.5, they are equal to the coefficient we would have if we had begun with the partition obtained by removing the bottom $t-1$ squares of the column given by a and computed the coefficient in that case. By Proposition 6.6, this is equal to $1/\eta$, where η is the axial distance from the top square of the column to b.

Similarly, we have

$$e(\widetilde{\lambda} - \widetilde{a} - \widetilde{b})e(\widetilde{\lambda} - \widetilde{b})e(\widetilde{\lambda}) = e(\widetilde{\lambda} - \widetilde{a} - \widetilde{b})\sum_{\widetilde{r}} k_{\widetilde{r}}(1\widetilde{r})e(\widetilde{\lambda}).$$

In this case, the coefficients $k_{\widetilde{r}}$ are equal to one over the axial distance from $\widetilde{b}$ to the corner square of the column $\widetilde{a}$, which is the bottom square of the column. This is the same as the axial distance from the top square of a in λ to b; thus, again, we have that the coefficients are equal to $1/\eta$.

We illustrate this with a simple case, where $t = 2$.

$$\lambda_F(t) = \begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & a \\ \ & \ & a \\ b \end{ytableau} \quad \text{and} \quad \lambda_G(t) = \begin{ytableau} \ & \ & \ & \widetilde{b} \\ \ & \ & \ \\ \ & \ & \widetilde{a} \\ \ & \ & \widetilde{a} \\ \ & \ \end{ytableau}\ .$$

It is clear that the axial distance from the top a to b in $\lambda_F(t)$ is equal to the axial distance from $\widetilde{b}$ to the bottom $\widetilde{a}$ in $\lambda_G(t)$; both are equal to 4. In general, if the axial distance from a to b in λ_F (not $\lambda_F(t)$) is η_0, then η_0 is also the axial distance from $\widetilde{a}$ to $\widetilde{b}$ in the transpose λ_G, and the axial distances we have been considering here are both equal to $\eta_0 + t - 1$.

The remainder of the proof uses Proposition 3.4, as in the previous case. This concludes the proof that $C_\bullet(\phi, t)$ is a complex. $\square$

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