REVERSE LEXICOGRAPHIC SQUAREFREE INITIAL IDEALS AND GORENSTEIN FANO POLYTOPES

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ABSTRACT. Via the theory of reverse lexicographic squarefree initial ideals of toric ideals, we give a new class of Gorenstein Fano polytopes (reflexive polytopes) arising from a pair of stable set polytopes of perfect graphs.

Introduction. Recall that an *integral* convex polytope is a convex polytope, all of whose vertices have integer coordinates. An integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is called a *Fano polytope* if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$. We say that a Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ is *Gorenstein* if the dual polytope $\mathcal{P}^\vee$ of $\mathcal{P}$ is again integral. (A Gorenstein Fano polytope is often called a reflexive polytope in the literature.) Gorenstein Fano polytopes are related with mirror symmetry and studied in many areas of mathematics, see, e.g., [**5**, subsection 8.3] and [**15**]. It is known that there are only finitely many Gorenstein Fano polytopes up to unimodular equivalence if the dimension is fixed. Classification results are known for low-dimensional cases [**16, 17**]. On the other hand, one of the most important problems is to construct new classes of Gorenstein Fano polytopes. In the case of Gorenstein Fano simplices, there are nice results on classifications and constructions, see, e.g., [**4, 14, 18**] and the references therein. In order to find classes of Gorenstein Fano polytopes of high dimension which are not necessarily simplices, integral convex polytopes arising from some combinatorial objects are studied in several papers. For example, the following classes are known:

- Gorenstein Fano polytopes arising from the order polytopes of graded posets (Hibi [**10**], revisited by Hegedüs-Kasprzyk [**7,** Lemma 5.10]);

- Gorenstein Fano polytopes arising from the Birkhoff polytopes (appearing in many papers, see, e.g., Stanley [**25,** I.13] and Athanasiadis [**1**]);

- Gorenstein Fano polytopes arising from directed graphs satisfying some conditions, Higashitani [**13**];

- Centrally symmetric configurations, Ohsugi and Hibi [**23**];

- The centrally symmetric polytope $\mathcal{O}(P)^{\pm}$ of the order polytope $\mathcal{O}(P)$ of a finite poset P, Hibi, et al. [**12**].

In the present paper, via the theory of Gröbner bases, we give a new class of Gorenstein Fano polytopes, which is not necessarily a simplex. For any pair of perfect graphs G_1 and G_2 (here, $G_1 = G_2$ is possible) on d vertices, we show that the convex hull of $\mathcal{Q}_{G_1} \cup -\mathcal{Q}_{G_2}$, where $\mathcal{Q}_{G_i}$ is the stable set polytope of G_i, is a Gorenstein Fano polytope of dimension d. Note that there are many pairs of perfect simple graphs on d vertices,[1] see Table 1.

TABLE 1. Number of perfect graphs/pairs of perfect graphs.

number of vertices	2	3	4	5	6	7	8
perfect graphs	2	4	11	33	148	906	8887
pairs of perfect graphs	3	10	66	561	11,026	410,871	39,493,828

Any Gorenstein Fano polytope $\mathcal{P}$ in our class is *terminal*, i.e., each integer point belonging to the boundary of $\mathcal{P}$ is a vertex of $\mathcal{P}$. In particular, if both of the two graphs are complete (respectively, empty) graphs on d vertices, then the Gorenstein Fano polytope has $2d$ (respectively, $2^{d+1} - 2$) vertices. Thus, our class has enough size and variety compared to the existing classes above.

Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}_{\geq 0}^{d \times n}$ and $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{Z}_{\geq 0}^{d \times m}$, where each $\mathbf{a}_i$ and each $\mathbf{b}_j$ is a nonzero column vector belonging to $\mathbb{Z}_{\geq 0}^d$. In Section 1, after reviewing basic material and notation on toric ideals, we introduce the concept that A and B are of *harmony*. Roughly speaking, Theorem 1.1 states that, if A and B are of harmony and if the toric ideal of each of A and

B possesses a reverse lexicographic squarefree initial ideal which enjoys certain properties, then the toric ideal of $[\mathbf{0}, -B, A] \in \mathbb{Z}^{d \times (n+m+1)}$ possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$. Working with the same situation as in Theorem 1.1, Corollary 1.3 guarantees that, if the integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ which is the convex hull of $\{-\mathbf{b}_1, \ldots, -\mathbf{b}_m, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a Fano polytope with

$$\mathcal{P} \cap \mathbb{Z}^d = \{\mathbf{0}, -\mathbf{b}_1, \ldots, -\mathbf{b}_m, \mathbf{a}_1, \ldots, \mathbf{a}_n\},$$

and if there is a $d \times d$ minor A' of $[-B, A]$ with $\det(A') = \pm 1$, then $\mathcal{P}$ is Gorenstein.

The topic of Section 2 is the incidence matrix A_Δ of a simplicial complex Δ on $[d] = \{1, \ldots, d\}$. It follows that, if Δ and Δ' are simplicial complexes on $[d]$, then A_Δ and $A_{\Delta'}$ are of harmony. Following Theorem 1.1, it is reasonable to study the problem when the toric ideal of A_Δ satisfies the required condition on initial ideals of Theorem 1.1. Somewhat surprisingly, Theorem 2.6 states that A_Δ satisfies the required condition on initial ideals of Theorem 1.1 if and only if Δ coincides with the set $S(G)$ of stable sets of a perfect graph G on $[d]$. A related topic on Gorenstein Fano polytopes arising from simplicial complexes will be studied (Theorem 2.8).

1. Reverse lexicographic squarefree initial ideals. Let K be a field and $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$ the Laurent polynomial ring in $d + 1$ variables over K. Given an integer $d \times n$ matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, where $\mathbf{a}_j = [a_{1j}, \ldots, a_{dj}]^\top$, the transpose of $[a_{1j}, \ldots, a_{dj}]$, is the jth column of A, the *toric ring* of A is the subalgebra $K[A]$ of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$, which is generated by the Laurent polynomials

$$\mathbf{t}^{\mathbf{a}_1} s = t_1^{a_{11}} \cdots t_d^{a_{d1}} s, \ldots, \mathbf{t}^{\mathbf{a}_n} s = t_1^{a_{1n}} \cdots t_d^{a_{dn}} s.$$

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K, and define the surjective ring homomorphism

$$\pi : K[\mathbf{x}] \longrightarrow K[A]$$

by setting $\pi(x_j) = \mathbf{t}^{\mathbf{a}_j} s$ for $j = 1, \ldots, n$. The *toric ideal* of A is the kernel I_A of π. Every toric ideal is generated by binomials. (Recall

that a polynomial $f \in K[\mathbf{x}]$ is a binomial if $f = u - v$, where

$$u = \prod_{i=1}^{n} x_i^{a_i} \quad \text{and} \quad v = \prod_{i=1}^{n} x_i^{b_i}$$

are monomials with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.) Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\mathrm{in}_<(I_A)$ the initial ideal of I_A with respect to $<$. We say that $\mathrm{in}_<(I_A)$ is *squarefree* if $\mathrm{in}_<(I_A)$ is generated by squarefree monomials. The reader is referred to [**11,** Chapters 1, 5] for information regarding Gröbner bases and toric ideals.

Let $\mathbb{Z}_{\geq 0}^d$ denote the set of integer column vectors $[a_1, \ldots, a_d]^\top$ with each $a_i \geq 0$. Given an integer vector $\mathbf{a} = [a_1, \ldots, a_d]^\top \in \mathbb{Z}^d$, let

$$\mathbf{a}^{(+)} = [a_1^{(+)}, \ldots, a_d^{(+)}]^\top, \mathbf{a}^{(-)} = [a_1^{(-)}, \ldots, a_d^{(-)}]^\top \in \mathbb{Z}_{\geq 0}^d,$$

where $a_i^{(+)} = \max\{0, a_i\}$ and $a_i^{(-)} = \max\{0, -a_i\}$. Note that $\mathbf{a} = \mathbf{a}^{(+)} - \mathbf{a}^{(-)}$ holds in general. Let $\mathbb{Z}_{\geq 0}^{d \times n}$ denote the set of $d \times n$ integer matrices $(a_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}}$ with each $a_{ij} \geq 0$. Furthermore, if no column of $A \in \mathbb{Z}_{\geq 0}^{d \times n}$ is the zero vector $\mathbf{0} = [0, \ldots, 0]^\top \in \mathbb{Z}^d$, then we introduce the $d \times (n+1)$ integer matrix $A^\sharp$, which is obtained by adding the column $\mathbf{0} \in \mathbb{Z}^d$ to A.

Now, given $A \in \mathbb{Z}_{\geq 0}^{d \times n}$ and $B \in \mathbb{Z}_{\geq 0}^{d \times m}$, we say that A and B are of harmony if the following condition is satisfied: let $\mathbf{a}$ be a column of $A^\sharp$ and $\mathbf{b}$ one of $B^\sharp$. Let $\mathbf{c} = \mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$. If $\mathbf{c} = \mathbf{c}^{(+)} - \mathbf{c}^{(-)}$, then $\mathbf{c}^{(+)}$ is a column vector of $A^\sharp$, and $\mathbf{c}^{(-)}$ is a column vector of $B^\sharp$.

Theorem 1.1. *Let* $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}_{\geq 0}^{d \times n}$ *and* $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{Z}_{\geq 0}^{d \times m}$, *where none of the* $\mathbf{a}_i$'s *or* $\mathbf{b}_j$'s *is* $\mathbf{0} \in \mathbb{Z}^d$, *be of harmony. Let* $K[z, \mathbf{x}] = K[z, x_1, \ldots, x_n]$ *and* $K[z, \mathbf{y}] = K[z, y_1, \ldots, y_m]$ *be the polynomial rings over a field* K. *Suppose that* $\mathrm{in}_{<_A}(I_{A^\sharp}) \subset K[z, \mathbf{x}]$ *and* $\mathrm{in}_{<_B}(I_{B^\sharp}) \subset K[z, \mathbf{y}]$ *are squarefree with respect to reverse lexicographic orders* $<_A$ *on* $K[z, \mathbf{x}]$ *and* $<_B$ *on* $K[z, \mathbf{y}]$, *respectively, satisfying the conditions that*

- $x_i <_A x_j$ *if* $\pi(x_i)$ *divides* $\pi(x_j)$;
- $z <_A x_k$ *for* $1 \leq k \leq n$, *where* z *corresponds to the column* $\mathbf{0} \in \mathbb{Z}^d$ *of* $A^\sharp$;

- $z <_B y_k$ for $1 \le k \le m$, where z corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of $B^\sharp$.

Let $[-B, A]$ denote the $d \times (n + m)$ integer matrix

$$[-\mathbf{b}_1, \ldots, -\mathbf{b}_m, \mathbf{a}_1, \ldots, \mathbf{a}_n].$$

Then, the toric ideal $I_{[-B,A]^\sharp}$ of $[-B, A]^\sharp$ possesses a squarefree initial ideal with respect to a reverse lexicographic order, whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of $[-B, A]^\sharp$.

Proof. Let

$$K[[-B, A]^\sharp] \subset K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$$

be the toric ring of $[-B, A]^\sharp$ and

$$I_{[-B,A]^\sharp} \subset K[\mathbf{x}, \mathbf{y}, z] = K[x_1, \ldots, x_n, y_1, \ldots, y_m, z]$$

the toric ideal of $[-B, A]^\sharp$. Recall that $I_{[-B,A]^\sharp}$ is the kernel of

$$\pi : K[\mathbf{x}, \mathbf{y}, z] \longrightarrow K[[-B, A]^\sharp]$$

with $\pi(z) = s$, $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i} s$ for $i = 1, \ldots, n$ and $\pi(y_j) = \mathbf{t}^{-\mathbf{b}_j} s$ for $j = 1, \ldots, m$.

Suppose that the reverse lexicographic orders $<_A$ and $<_B$ are induced by the orderings $z <_A x_n <_A \cdots <_A x_1$ and $z <_B y_m <_B \cdots <_B y_1$. Let $<_{\mathrm{rev}}$ be the reverse lexicographic order on $K[\mathbf{x}, \mathbf{y}, z]$ induced by the ordering

$$z < x_n < \cdots < x_1 < y_m < \cdots < y_1.$$

In general, if $\mathbf{a} = [a_1, \ldots, a_d]^\top \in \mathbb{Z}_{\ge 0}^d$, then $\mathrm{supp}(\mathbf{a})$ is the set of those $1 \le i \le d$ with $a_i \ne 0$.

Now, we introduce the following:

$$\mathcal{E} = \{(i, j) : 1 \le i \le n, \ 1 \le j \le m, \ \mathrm{supp}(\mathbf{a}_i) \cap \mathrm{supp}(\mathbf{b}_j) \ne \emptyset\}.$$

Let $\mathbf{c} = \mathbf{a}_i - \mathbf{b}_j$ with $(i, j) \in \mathcal{E}$. Then, $\mathbf{c}^{(+)} \ne \mathbf{a}_i$ and $\mathbf{c}^{(-)} \ne \mathbf{b}_j$. The hypothesis that A and B are of harmony guarantees that $\mathbf{c}^{(+)}$ is a column of $A^\sharp$ and $\mathbf{c}^{(-)}$ is a column of $B^\sharp$. It follows that $f = x_i y_j - u$

($\neq 0$) belongs to $I_{[-B,A]^{\sharp}}$, where

$$
u = \begin{cases}
x_k y_\ell & \text{if } \mathbf{c}^{(+)} = \mathbf{a}_k \text{ and } \mathbf{c}^{(-)} = \mathbf{b}_\ell, \\
z y_\ell & \text{if } \mathbf{c}^{(+)} = \mathbf{0} \text{ and } \mathbf{c}^{(-)} = \mathbf{b}_\ell, \\
x_k z & \text{if } \mathbf{c}^{(+)} = \mathbf{a}_k \text{ and } \mathbf{c}^{(-)} = \mathbf{0}, \\
z^2 & \text{if } \mathbf{c}^{(+)} = \mathbf{c}^{(-)} = \mathbf{0}.
\end{cases}
$$

If z divides u, then $\text{in}_{<_{\text{rev}}}(f) = x_i y_j$, where $\text{in}_{<_{\text{rev}}}(f)$ is the initial monomial of $f \in K[\mathbf{x}, \mathbf{y}, z]$. If z cannot divide u, then, since $\pi(x_k)$ divides $\pi(x_i)$, we have $x_k <_A x_i$ and $\text{in}_{<_{\text{rev}}}(f) = x_i y_j$. Hence,

$$
\{x_i y_j : (i,j) \in \mathcal{E}\} \subset \text{in}_{<_{\text{rev}}}(I_{[-B,A]^{\sharp}}).
$$

Now, let $\mathcal{M}_A$ (respectively, $\mathcal{M}_B$) be the minimal set of squarefree monomial generators of $\text{in}_{<_A}(I_{A^{\sharp}})$ (respectively, $\text{in}_{<_B}(I_{B^{\sharp}})$). Suppose that $\text{in}_{<_{\text{rev}}}(I_{[-B,A]^{\sharp}})$ cannot be generated by the set of squarefree monomials

$$
\mathcal{M} = \{x_i y_j : (i,j) \in \mathcal{E}\} \cup \mathcal{M}_A \cup \mathcal{M}_B \ (\subset \text{in}_{<_{\text{rev}}}(I_{[-B,A]^{\sharp}})).
$$

The following fact ([**22,** page 1914]) on Gröbner bases is well known:

> A finite set $\mathcal{G}$ of I_A is a Gröbner basis with respect to $<$ if and only if $\pi(u) \neq \pi(v)$ for any monomials $u \notin (\text{in}_<(g) : g \in \mathcal{G})$ and $v \notin (\text{in}_<(g) : g \in \mathcal{G})$ with $u \neq v$.

Since $\mathcal{G}$ with $\mathcal{M} = \{\text{in}_<(f) : f \in \mathcal{G}\}$ is not a Gröbner basis, it follows that there exists a nonzero irreducible binomial $g = u - v$ belonging to $I_{[-B,A]^{\sharp}}$ such that each of u and v can be divided by none of the monomials belonging to $\mathcal{M}$. Write

$$
u = \left(\prod_{p \in P} x_p^{i_p}\right)\left(\prod_{q \in Q} y_q^{j_q}\right), \qquad v = z^\alpha \left(\prod_{r \in R} x_r^{k_r}\right)\left(\prod_{s \in S} y_s^{\ell_s}\right),
$$

where P and R are subsets of $\{1, \ldots, n\}$, where Q and S are subsets of $\{1, \ldots, m\}$, where α is a nonnegative integer, and where each of i_p, j_q, k_r, ℓ_s is a positive integer. Since $g = u - v$ is irreducible, we have $P \cap R = Q \cap S = \emptyset$. Furthermore, the fact that each of $x_i y_j$ with

$(i, j) \in \mathcal{E}$ can divide neither u nor v guarantees that

$$\left(\bigcup_{p \in P} \operatorname{supp}(\mathbf{a}_p) \right) \cap \left(\bigcup_{q \in Q} \operatorname{supp}(\mathbf{b}_q) \right)$$
$$= \left(\bigcup_{r \in R} \operatorname{supp}(\mathbf{a}_r) \right) \cap \left(\bigcup_{s \in S} \operatorname{supp}(\mathbf{b}_s) \right) = \emptyset.$$

Since $\pi(u) = \pi(v)$, it follows that

$$\sum_{p \in P} i_p \mathbf{a}_p = \sum_{r \in R} k_r \mathbf{a}_r, \qquad \sum_{q \in Q} j_q \mathbf{b}_q = \sum_{s \in S} \ell_s \mathbf{b}_s.$$

Let $\gamma_P = \sum_{p \in P} i_p$, $\gamma_Q = \sum_{q \in Q} j_q$, $\gamma_R = \sum_{r \in R} k_r$, and $\gamma_S = \sum_{s \in S} \ell_s$. Then,

$$\gamma_P + \gamma_Q = \alpha + \gamma_R + \gamma_S.$$

Since $\alpha \geq 0$, it follows that either $\gamma_P \geq \gamma_R$ or $\gamma_Q \geq \gamma_S$. Let, say, $\gamma_P > \gamma_R$. Then,

$$h = \prod_{p \in P} x_p^{i_p} - z^{\gamma_P - \gamma_R} \left(\prod_{r \in R} x_r^{k_r} \right) \neq 0$$

belongs to $I_{[-B,A]^\sharp}$ and $\operatorname{in}_{<_{\mathrm{rev}}}(h) = \prod_{p \in P} x_p^{i_p}$ divides u, a contradiction. Hence, we may assume $\gamma_P = \gamma_R$. Then, the binomial

$$h_0 = \prod_{p \in P} x_p^{i_p} - \prod_{r \in R} x_r^{k_r}$$

belongs to $I_{[-B,A]^\sharp}$. If $h_0 \neq 0$, then either $\prod_{p \in P} x_p^{i_p}$ or $\prod_{r \in R} x_r^{k_r}$ must belong to $\operatorname{in}_{<_{\mathrm{rev}}}(I_{[-B,A]^\sharp})$. This contradicts the fact that each of u and v can be divided by none of the monomials belonging to $\mathcal{M}$. Hence, $h_0 = 0$ and $P = R = \emptyset$. Similarly, $Q = S = \emptyset$. Hence, $\alpha = 0$ and $g = 0$. This contradiction guarantees that $\mathcal{M}$ is the minimal set of squarefree monomial generators of $\operatorname{in}_{<_{\mathrm{rev}}}(I_{[-B,A]^\sharp})$, as desired. $\qquad \square$

Given an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$, we write $A_{\mathcal{P}}$ for the integer matrices whose column vectors are those $\mathbf{a} \in \mathbb{Z}^d$ belonging to $\mathcal{P}$. The toric ring $K[A_{\mathcal{P}}]$ is often called the toric ring of $\mathcal{P}$. A triangulation Δ of $\mathcal{P}$ using the vertices belonging to $\mathcal{P} \cap \mathbb{Z}^d$ is *unimodular* if the normalized volume ([**11**, page 253]) of each facet of Δ is equal to 1 and is *flag* if every minimal nonface of Δ is an edge. It follows

from [**26,** Chapter 8] that, if the toric ideal $I_{A_\mathcal{P}}$ of $A_\mathcal{P}$ possesses a squarefree initial ideal, then $\mathcal{P}$ possesses a unimodular triangulation. Furthermore, if $I_{A_\mathcal{P}}$ possesses an initial ideal generated by quadratic squarefree monomials, then $\mathcal{P}$ possesses a unimodular triangulation, which is flag.

An integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is called *Fano* if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$. We say that a Fano polytope $\mathcal{P}$ is *Gorenstein* if the dual polytope $\mathcal{P}^\vee$ of $\mathcal{P}$ is again integral [**2, 9**]. A *smooth* Fano polytope is a simplicial Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ for which the d vertices of each facet of $\mathcal{P}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Lemma 1.2. *Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d for which $\mathbf{0} \in \mathbb{Z}^d$ belongs to $\mathcal{P}$. Suppose that there is a $d \times d$ minor A' of $A_\mathcal{P}$ with $\det(A') = \pm 1$ and that $I_{A_\mathcal{P}}$ possesses a squarefree initial ideal with respect to a reverse lexicographic order, whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of $A_\mathcal{P}$. Then, for each facet $\mathcal{F}$ of $\mathcal{P}$ with $\mathbf{0} \notin \mathcal{F}$, we have $\mathbb{Z}\mathcal{F} = \mathbb{Z}^d$, where*

$$\mathbb{Z}\mathcal{F} = \sum_{\mathbf{a} \in \mathcal{F} \cap \mathbb{Z}^d} \mathbb{Z}\mathbf{a},$$

and the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathcal{F} \subset \mathcal{H}$ is of the form

$$a_1 z_1 + \cdots + a_d z_d = 1$$

with each $a_j \in \mathbb{Z}$. In particular, if $\mathcal{P}$ is a Fano polytope, then $\mathcal{P}$ is Gorenstein. Furthermore, if $\mathcal{P}$ is a simplicial Fano polytope, then $\mathcal{P}$ is a smooth Fano polytope.

Proof. Let Δ be the *pulling triangulation* ([**11,** page 268]) coming from a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of $A_\mathcal{P}$. A crucial fact is that the origin of $\mathbb{R}^d$ belongs to each facet of Δ. Let F be a facet of Δ with the vertices $\mathbf{0}, \mathbf{b}_1, \ldots, \mathbf{b}_d$ for which $\{\mathbf{b}_1, \ldots, \mathbf{b}_d\} \subset \mathcal{F}$. The existence of a $d \times d$ minor A' of $A_\mathcal{P}$ with $\det(A') = \pm 1$ guarantees that the normalized volume of F coincides with $|\det(B)|$, where $B = [\mathbf{b}_1, \ldots, \mathbf{b}_d]$. Since Δ is unimodular, we have $\det(B) = \pm 1$. Hence, $\{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$, and $\mathbb{Z}\mathcal{F} = \mathbb{Z}^d$

follows. Moreover, the hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with each $\mathbf{b}_j \in \mathcal{H}$ is of the form $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_j \in \mathbb{Z}$, as desired. $\qquad \square$

Corollary 1.3. *Work with the same situation as in Theorem* 1.1. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *be the integral convex polytope, which is the convex hull of* $\{-\mathbf{b}_1, \ldots, -\mathbf{b}_m, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$. *Suppose that* $\mathbf{0} \in \mathbb{Z}^d$ *belongs to the interior of* $\mathcal{P}$ *and that there is a* $d \times d$ *minor* A' *of* $A_{\mathcal{P}}$ *with* $\det(A') = \pm 1$. *Then,* $\mathcal{P}$ *is a Gorenstein Fano polytope. Furthermore, if* $\mathcal{P}$ *is a simplicial polytope, then* $\mathcal{P}$ *is a smooth Fano polytope.*

Example 1.4. Let A_1 and A_2 be the following matrices:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, A_i and A_j are of harmony and satisfy the condition in Theorem 1.1 for any $1 \leq i \leq j \leq 2$. From Corollary 1.3, we have three Gorenstein Fano polygons. It is known that there are exactly 16 Gorenstein Fano polygons ([**5,** page 382]).

2. Convex polytopes arising from simplicial complexes. Let
$[d] = \{1, \ldots, d\}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard coordinate unit vectors of $\mathbb{R}^d$. Given a subset $W \subset [d]$, we have

$$\rho(W) = \sum_{j \in W} \mathbf{e}_j \in \mathbb{R}^d.$$

In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. Let Δ be a simplicial complex on the vertex set $[d]$. Thus, Δ is a collection of subsets of $[d]$ with $\{i\} \in \Delta$ for each $i \in [d]$ such that, if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. In particular, $\emptyset \in \Delta$. The *incidence matrix* A_Δ of Δ is the matrix whose columns are those $\rho(F)$ with $F \in \Delta$. We write $\mathcal{P}_\Delta \subset \mathbb{R}^d$ for the $(0,1)$-polytope, which is the convex hull of $\{\rho(F) : F \in \Delta\}$ in $\mathbb{R}^d$. We have $\dim \mathcal{P}_\Delta = d$. It follows from the definition of simplicial complexes that

Lemma 2.1. *Let* Δ *and* Δ' *be simplicial complexes on* $[d]$. *Then,* A_Δ *and* $A_{\Delta'}$ *are of harmony.*

Following Lemma 2.1, together with Theorem 1.1, it is reasonable to study the problem when the toric ideal I_{A_Δ} of a simplicial complex

Δ possesses a squarefree initial ideal with respect to a reverse lexico-graphic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ.

Let Δ be a simplicial complex on $[d]$. Since $\{i\} \in \Delta$ for each $i \in [d]$, the $d \times d$ identity matrix is a $d \times d$ minor of A_Δ. It then follows from Lemma 1.2 that

Corollary 2.2. *Let Δ be a simplicial complex on $[d]$. Suppose that I_{A_Δ} possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ. Then, for each facet $\mathcal{F}$ of $\mathcal{P}_\Delta$ with $\mathbf{0} \notin \mathcal{F}$, we have $\mathbb{Z}\mathcal{F} = \mathbb{Z}^d$, and the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathcal{F} \subset \mathcal{H}$ is of the form $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_j \in \mathbb{Z}$.*

Let G be a finite simple graph on $[d]$ and $E(G)$ the set of edges of G. (Recall that a finite graph is simple if G possesses no loop and no multiple edges.) A subset $W \subset [d]$ is called *stable* if, for all i and j belonging to W with $i \neq j$, we have $\{i, j\} \notin E(G)$. Let $S(G)$ denote the set of stable sets of G. We have $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. Clearly, $S(G)$ is a simplicial complex on $[d]$. The *stable set polytope* $\mathcal{Q}_G \subset \mathbb{R}^d$ of G is the $(0, 1)$-polytope $\mathcal{P}_{S(G)} \subset \mathbb{R}^d$ arising from the simplicial complex $S(G)$. A finite simple graph is said to be *perfect* ([**3**]) if, for any induced subgraph H of G including G itself, the chromatic number of H is equal to the maximal cardinality of cliques of H. (A chromatic number of G is the smallest integer t for which there exist stable sets $W_1, \ldots, W_t$ of G with $[d] = W_1 \cup \cdots \cup W_t$, and a clique of G is a subset $W \subset [d]$, which is a stable set of the complementary graph $\overline{G}$ of G.) A complementary graph of a perfect graph is perfect ([**3**]).

Recall that an integer matrix A is *compressed* ([**21, 27**]) if the initial ideal of the toric ideal I_A is squarefree with respect to any reverse lexicographic order.

Example 2.3. Let G be a perfect graph on $[d]$. Then, A_Δ, where $\Delta = S(G)$, is compressed ([**21**, Example 1.3(c)]). Let G and G' be perfect graphs on $[d]$ and $\mathcal{Q} \subset \mathbb{R}^d$ the Fano polytope, which is the convex hull of $\mathcal{Q}_G \cup (-\mathcal{Q}_{G'})$. It then follows from Corollary 1.3 together with Lemma 2.1 that $\mathcal{Q}$ is Gorenstein.

Lemma 2.4. *Let* Δ *be one of the following simplicial complexes:*

(i) *the simplicial complex on* $[e]$ *with the facets* $[e] \setminus \{i\}$, $1 \leq i \leq e$, *where* $e \geq 3$;

(ii) $S(G)$, *where* G *is an odd hole of length* $2\ell + 1$, *where* $\ell \geq 2$;

(iii) $S(G)$, *where* G *is an odd antihole of length* $2\ell + 1$, *where* $\ell \geq 2$.

Let $<$ *be any reverse lexicographic order whose smallest variable corresponds to the column* $\mathbf{0}$ *of* A_Δ. *Then, the initial ideal* $\mathrm{in}_<(I_{A_\Delta})$ *cannot be squarefree.* (*Recall that an odd hole is an induced odd cycle of length* ≥ 5 *and an odd antihole is the complementary graph of an odd hole.*)

Proof. By virtue of Corollary 2.2, we find a supporting hyperplane $\mathcal{H}$ of $\mathcal{P}_\Delta$ with $\mathbf{0} \notin \mathcal{H}$ for which $\mathcal{H} \cap \mathcal{P}_\Delta$ is a facet of $\mathcal{P}_\Delta$ such that the equation of $\mathcal{H}$ cannot be of the form $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_j \in \mathbb{Z}$. In each of (i), (ii) and (iii), the equation of a desired hyperplane $\mathcal{H}$ is as follows:

(i) $\sum_{i=1}^{e} z_i = e - 1$;

(ii) $\sum_{i=1}^{2\ell+1} z_i = \ell$;

(iii) $\sum_{i=1}^{2\ell+1} z_i = 2$.

In (i), it is easy to see that $\mathcal{H} \cap \mathcal{P}_\Delta$ is a facet of $\mathcal{P}_\Delta$. In each of (ii) and (iii), it is known [**24, 28**] that $\mathcal{H} \cap \mathcal{P}_\Delta$ is a facet of $\mathcal{P}_\Delta$. $\square$

Let $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{Z}^{d \times m}$ be a submatrix of $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$. Then, $K[B]$ is called a *combinatorial pure subring* of $K[A]$ if the convex hull of $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is a face of the convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. For any combinatorial pure subring $K[B]$ of $K[A]$, it is known that, if the initial ideal of I_A is squarefree, then so is the corresponding initial ideal of I_B. See [**19, 20**] for details.

Lemma 2.5. *Let* Δ *be a simplicial complex on* $[d]$ *and* Δ' *an induced subcomplex of* Δ *which is one of* (i), (ii) *and* (iii) *of Lemma* 2.4. *Let* $<$ *be any reverse lexicographic order whose smallest variable corresponds to the column* $\mathbf{0} \in \mathbb{Z}^d$ *of* A_Δ. *Then, the initial ideal* $\mathrm{in}_<(I_{A_\Delta})$ *cannot be squarefree.*

Proof. Let Δ' be the induced subcomplex of Δ on V, where $V \subset [d]$, and $<'$ the reverse lexicographic order induced by $<$. Lemma 2.4 states that $\mathrm{in}_{<'}(I_{A_{\Delta'}})$ cannot be squarefree. Since Δ' is an induced

subcomplex of Δ, it follows that $\mathcal{P}_{\Delta'}$ is a face of $\mathcal{P}_\Delta$. Thus, $K[A_{\Delta'}]$ is a combinatorial pure subring of $K[A_\Delta]$, and hence, $\mathrm{in}_<(I_{A_\Delta})$ cannot be squarefree, as required. $\square$

We are now in a position to state a combinatorial characterization of simplicial complexes Δ on $[d]$ for which the toric ideal I_{A_Δ} possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ.

Theorem 2.6. *Let Δ be a simplicial complex on $[d]$. Then, the following conditions are equivalent*:

(i) *there exists a perfect graph G on $[d]$ with $\Delta = S(G)$*;
(ii) *A_Δ is compressed*;
(iii) *I_{A_Δ} possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ.*

Proof. In [**21**, Example 1.3 (c)], (i) $\Rightarrow$ (ii) is proven. (See also [**6**, Section 4].) Moreover, (ii) $\Rightarrow$ (iii) is trivial. Now, in order to show (iii) $\Rightarrow$ (i), we fix a reverse lexicographic order $<$ whose smallest variable corresponds $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ.

First step. Suppose that there is *no* finite simple graph G on $[d]$ with $\Delta = S(G)$. Given a simplicial complex Δ on $[d]$, there is a finite simple graph G on $[d]$ with $\Delta = S(G)$ if and only if Δ is flag, i.e, every minimal nonface of Δ is an edge of Δ. (See, e.g., [**8**, Lemma 9.1.3]. Note that $S(G)$ is the clique complex of the complement graph of G.) Let Δ be a simplicial complex which is not flag and $V \subset [d]$, where $|V| \geq 3$, a minimal nonface of Δ. We have $V \setminus \{i\} \in \Delta$ for all $i \in V$. Thus, the induced subcomplex Δ' of Δ on V coincides with the simplicial complex (i) of Lemma 2.4.

Second step. Let G be a nonperfect graph on $[d]$ with $A_\Delta = S(G)$. The strong perfect graph theorem [**3**] guarantees that G possesses either an odd hole or an odd antihole. Thus, Δ contains an induced subcomplex Δ' which coincides with either (ii) or (iii) of Lemma 2.4.

As a result, Lemma 2.5 states that I_{A_Δ} possesses no squarefree initial ideal with respect to a reverse lexicographic order whose smallest

variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of A_Δ. This completes the proof of (iii) $\Rightarrow$ (i). $\square$

Example 2.7. Let $A \in \mathbb{Z}^{d \times n}$ for which each entry of A belongs to $\{0, 1\}$ and $I_{A^\sharp}$ the toric ideal of $A^\sharp$. In general, even if $I_{A^\sharp}$ possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0} \in \mathbb{Z}^d$ of $A^\sharp$, the matrix $A^\sharp$ may not be compressed. For example, if

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

then $I_{A^\sharp}$ is generated by $x_1 x_3 x_5 x_7 - x_2 x_4^2 x_6$. Thus, the initial ideal of $I_{A^\sharp}$, with respect to the reverse lexicographic order induced by the ordering

$$z < x_2 < x_1 < x_3 < x_4 < x_5 < x_6 < x_7,$$

are generated by $x_1 x_3 x_5 x_7$, while the initial ideal of $I_{A^\sharp}$, with respect to the reverse lexicographic order induced by the ordering

$$z < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7,$$

is generated by $x_2 x_4^2 x_6$. Although $A^\sharp$ satisfies condition (iii) of Theorem 2.6, the integer matrix $A^\sharp$ cannot be compressed.

Apart from Theorem 2.6, we can formulate the problem when the convex polytope $\mathcal{P} \subset \mathbb{R}^d$ which is the convex hull of $\mathcal{P}_\Delta \cup (-\mathcal{P}_{\Delta'})$, where Δ and Δ' are simplicial complexes on $[d]$, is a Gorenstein Fano polytope.

Theorem 2.8. *Let Δ and Δ' be simplicial complexes on $[d]$ and $\mathcal{P} \subset \mathbb{R}^d$ the convex polytope which is the convex hull of $\mathcal{P}_\Delta \cup (-\mathcal{P}_{\Delta'})$. Then, $\mathcal{P}$ is a Gorenstein Fano polytope if and only if there exist perfect graphs G and G' on $[d]$ with $\Delta = S(G)$ and $\Delta' = S(G')$.*

Proof. The "if" part follows from Example 2.3. In order to see why the "only if" part is true, suppose that either Δ is not flag or there is

a nonperfect graph G with $\Delta = S(G)$. Since $\mathcal{P} \subset \mathbb{R}^d$ is a Gorenstein Fano polytope, the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ for which $\mathcal{H} \cap \mathcal{P}$ is a facet of $\mathcal{P}$ is of the form $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_j \in \mathbb{Z}$.

Let Δ not be flag and $V \subset [d]$ with $|V| \geq 3$ for which $V \setminus \{i\} \in \Delta$ for all $i \in V$ and $V \notin \Delta$. Let, say, $V = [e]$ with $e \geq 3$. Then, the hyperplane $\mathcal{H}' \subset \mathbb{R}^d$, defined by the equation $z_1 + \cdots + z_e = e - 1$, is a supporting hyperplane of $\mathcal{P}$. Let $\mathcal{F}$ be a facet of $\mathcal{P}$ with $\mathcal{H}' \cap \mathcal{P} \subset \mathcal{F}$ and $a_1 z_1 + \cdots + a_d z_d = 1$, with each $a_j \in \mathbb{Z}$ the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathcal{F} \subset \mathcal{H}$. Since $\rho(V \setminus \{i\}) \in \mathcal{H}$ for all $i \in V$, we have $\sum_{j \in [e] \setminus \{i\}} a_j = 1$. Thus, $(e - 1)(a_1 + \cdots + a_e) = e$. Hence, $a_1 + \cdots + a_e \notin \mathbb{Z}$, a contradiction.

Let $\Delta = S(G)$, where G possesses an odd hole C of length $2\ell+1$ with vertices, say, $1, \ldots, 2\ell+1$, where $\ell \geq 2$. Then, the hyperplane $\mathcal{H}' \subset \mathbb{R}^d$, defined by the equation $z_1 + \cdots + z_{2\ell+1} = \ell$, is a supporting hyperplane of $\mathcal{P}$. Let $\mathcal{F}$ be a facet of $\mathcal{P}$ with $\mathcal{H}' \cap \mathcal{P} \subset \mathcal{F}$ and $a_1 z_1 + \cdots + a_d z_d = 1$, with each $a_j \in \mathbb{Z}$ the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathcal{F} \subset \mathcal{H}$. The maximal stable sets of C are

$$\{1, 3, \ldots, 2\ell - 1\}, \{2, 4, \ldots, 2\ell\}, \ldots, \{2\ell + 1, 2, 4, \ldots, 2\ell - 2\},$$

and each $i \in [2\ell - 2]$ appears ℓ times in the above list. Since, for each maximal stable set U of C, we have $\sum_{i \in U} a_i = 1$, it follows that $\ell(a_1 + \cdots + a_{2\ell+1}) = 2\ell + 1$. Hence, $a_1 + \cdots + a_e \notin \mathbb{Z}$, a contradiction.

Let $\Delta = S(G)$, where G possesses an odd antihole C with vertices, say, $1, \ldots, 2\ell + 1$, where $\ell \geq 2$. Then, the hyperplane $\mathcal{H}' \subset \mathbb{R}^d$, defined by the equation $z_1 + \cdots + z_{2\ell+1} = 2$, is a supporting hyperplane of $\mathcal{P}$. Let $\mathcal{F}$ be a facet of $\mathcal{P}$ with $\mathcal{H}' \cap \mathcal{P} \subset \mathcal{F}$ and $a_1 z_1 + \cdots + a_d z_d = 1$, with each $a_j \in \mathbb{Z}$ the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathcal{F} \subset \mathcal{H}$. The maximal stable sets of C are

$$\{1, 2\}, \{2, 3\}, \ldots, \{2\ell + 1, 1\},$$

and each $i \in [2\ell - 2]$ appears twice in the above list. Since, for each maximal stable set U of C, we have $\sum_{i \in U} a_i = 1$, it follows that $2(a_1 + \cdots + a_{2\ell+1}) = 2\ell + 1$. Hence, $a_1 + \cdots + a_e \notin \mathbb{Z}$, a contradiction. $\quad\square$

ENDNOTES

1. See A052431 in "The On-Line Encyclopedia of Integer Sequences," at `http://oeis.org`.

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