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ON 2-STABLY ISOMORPHIC FOUR-DIMENSIONAL AFFINE DOMAINS

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ABSTRACT. In this paper, we exhibit examples of four-dimensional seminormal domains A and B which are finitely generated over the field $\mathbb{C}$ (or $\mathbb{R}$) such that $A[X,Y] \cong B[X,Y]$ but $A[X] \not\cong B[X]$.

Introduction. For integral domains $C \subset A$, the notation $A = C^{[n]}$ will mean that $A = C[t_1, \ldots, t_n]$ for elements $t_1, \ldots, t_n \in A$ algebraically independent over C.

Now, let k be a field. For $m \geq 0$, two finitely generated k-algebras A and B are said to be m-stably isomorphic if $A^{[m]} \cong_k B^{[m]}$; $m = 0$ refers to the case $A \cong_k B$. A finitely generated k-algebra B is said to be m-cancelative if any k-algebra A which is m-stably isomorphic to B is necessarily isomorphic to B.

The cancelation problem investigates whether a specific class of rings has the m-cancelative property. There are examples of k-algebras A and B which are 1-stably isomorphic but not (0-stably) isomorphic (see [**1, 2, 3, 6**]). In fact, when $\operatorname{ch} k > 0$ and $r \geq 3$, it has been shown that $B = k^{[r]}$ is not 1-cancelative (see [**4, 5**]). A natural question in the context of the cancelation problem is whether (or when) m-stable isomorphism implies at least 1-stable isomorphism. In particular, we have the following question.

Question 1. *Let k be a field, and let A and B be finitely generated k-algebras. Suppose that $A^{[2]} \cong_k B^{[2]}$. Does it follow that $A^{[1]} \cong_k B^{[1]}$?*

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The first known counterexample to the above question was given by Jelonek ([**7**]). His rings are smooth algebras over $\mathbb{C}$ but are of dimension at least 10. In this note, we demonstrate a new class of counterexamples of smaller dimension. Our rings A and B are of dimension 4 and, although they are not smooth, they are seminormal and their integral closures are polynomial rings.

We shall first describe (Theorem 1.4) a general method of construction and then display concrete examples over $k = \mathbb{R}$ and $k = \mathbb{C}$ (Examples 1.5 and 1.6).

1. The examples. Let R be a commutative ring. We shall denote the group of units of R by R^* and the set of $n \times n$ matrices over R by $\mathcal{M}_n(R)$.

Any matrix M in $\mathrm{SL}_n(R)$ ($\subset \mathcal{M}_n(R)$) can be identified with the matrix $\left(\begin{smallmatrix} M & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_{n+1}(R)$. Hence, for any n, $\mathrm{SL}_n(R) \subset \mathrm{SL}_{n+1}(R)$, and $\bigcup_{n\geq 0} \mathrm{SL}_n(R)$ is a group denoted by $\mathrm{SL}(R)$. $\mathbf{SK}_1(R)$ denotes the abelian group $\mathrm{SL}(R)/[\mathrm{SL}(R), \mathrm{SL}(R)]$.

Let R be a subring of the integral domain D such that R and D have the same field of fractions and D is a finite R-module. Then, the conductor ideal $\mathcal{C}_{D|_R}$ of R in D is defined to be the largest ideal of D contained in R. It can be seen that

$$\mathcal{C}_{D|_R} := \{x \in R \mid xD \subseteq R\}.$$

The next result is easy to see.

Lemma 1.1. *Let D be an integral domain and $\Gamma : D \to D$ an automorphism of rings. Let A be a subring of D such that D is a finite A-module and A has the same field of fractions as D. If $B = \Gamma(A)$, then D is a finite B-module and $\mathcal{C}_{D|_B} = \Gamma(\mathcal{C}_{D|_A})$.*

The next result follows from [**13,** Theorem 3.10].

Theorem 1.2. *Let $C \subseteq D$ be a finite birational extension and I the conductor ideal of C in D. Suppose that $C^* = D^*$ and $(C/I)^* = (D/I)^*$. Then, the sequence*

$$0 \longrightarrow \mathrm{Pic}\,(C) \xrightarrow{\Delta} \mathrm{Pic}\,(D) \times \mathrm{Pic}\,(C/I) \xrightarrow{\pm} \mathrm{Pic}\,(D/I)$$

is exact, where Δ denotes the diagonal map and $\pm$ denotes the difference map sending (L', L) to $L' \otimes_D D/I \otimes_{C/I} L^{-1}$.

The following result is well known (cf., [**11,** page 201]).

Lemma 1.3. *Let k be a field and A a k-subalgebra of a finitely generated k-algebra D. If D is integral over A, then A is a finitely generated k-algebra.*

We now prove our main result.

Theorem 1.4. *Let k be a field, $R := k[X, Y]$ a polynomial ring over k and $f \in R$ an irreducible polynomial. Set $S := R/(f)$, $\eta : R \to S$ the natural k-algebra surjection and $\eta_n : \mathcal{M}_n(R) \to \mathcal{M}_n(S)$ the induced ring homomorphism. Suppose that the following conditions hold*:

(i) *$S^* = k^*$.*
(ii) *There exists an invertible matrix $P \in \mathrm{SL}_2(S)$ whose image $[P]$ in $\mathbf{SK}_1(S)$ is not zero.*
(iii) *If $\phi : R[U, V, Z] \to R[U, V, Z]$ is a k-algebra automorphism such that $\phi(f) = \lambda f$ for some $\lambda \in k^*$, then $\phi(R) = R$.*

Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathcal{M}_2(R)$ be such that $\eta_2(M) = P$. Let $D = R[U, V]$,

$$A = R[U^2, V^2] + fR[U, V] \subseteq D$$

and

$$B = R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2] + fR[U, V] \subseteq D.$$

Then, A and B are affine domains such that

(I) *$A^{[1]} \ncong_k B^{[1]}$.*
(II) *$A^{[2]} \cong_k B^{[2]}$.*
(III) *A and B are seminormal rings.*

Proof. It is easy to see that D is finite integral over A. Hence, A is affine by Lemma 1.3.

We now show that B is affine. First, we show that

$$R[(\alpha U + \beta V), (\gamma U + \delta V)] + fR[U, V] = D. \tag{1.1}$$

Clearly, $R[(\alpha U + \beta V), (\gamma U + \delta V)] + fR[U, V] \subseteq D$.

Let $M' = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right) \in \mathcal{M}_2(R)$ be such that $\eta_2(M') = P^{-1}$. Then, $M'M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + fM''$ for some $M'' \in \mathcal{M}_2(R)$. Hence,

$$\begin{aligned}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} &= M'M\begin{pmatrix} U \\ V \end{pmatrix} - fM''\begin{pmatrix} U \\ V \end{pmatrix} \\ &= M'\begin{pmatrix} \alpha U + \beta V \\ \gamma U + \delta V \end{pmatrix} - fM''\begin{pmatrix} U \\ V \end{pmatrix}.\end{aligned}$$

Thus, $U, V \in R[(\alpha U + \beta V), (\gamma U + \delta V)] + fR[U, V]$, and (1.1) follows.

Now, note that $\alpha U + \beta V$ and $\gamma U + \delta V$ are integral over B. Therefore, by (1.1), D is finite integral over B, and thus, B is affine by Lemma 1.3.

(I) Suppose, if possible, that there exists a k-algebra isomorphism $\Phi : A[Z] \to B[Z]$. Note that D is the integral closure of A and B in their field of fractions. Hence, Φ extends to a k-algebra automorphism of the ring $D[Z]$ which we also denote by Φ.

By Lemma 1.1, we have $\Phi(\mathcal{C}_{D[Z]|_{A[Z]}}) = \mathcal{C}_{D[Z]|_{B[Z]}}$ as $\Phi(A[Z]) = B[Z]$. Now,

$$\mathcal{C}_{D[Z]|_{A[Z]}} = fD[Z] = \mathcal{C}_{D[Z]|_{B[Z]}}.$$

Hence, $\Phi(f) = \lambda f$ for some $\lambda \in (D[Z])^*$. However, $(D[Z])^* = D^* = k^*$, and hence, $\lambda \in k^*$. Therefore, Φ is a k-algebra automorphism of the ring $D[Z]$ satisfying $\Phi(f) = \lambda f$ for some $\lambda \in k^*$. Hence, $\Phi(R) = R$ by hypothesis (iii). Set

$$\text{(1.2)} \qquad U_1 := \Phi(U), \qquad V_1 := \Phi(V) \quad \text{and} \quad Z_1 := \Phi(Z).$$

Let u, v, z, u_1, v_1, z_1 denote, respectively, the images of U, V, Z, U_1, V_1, Z_1 in $D[Z]/(f)$ and a, b, c, d the images of $\alpha, \beta, \gamma, \delta$ in S. Thus, we may make the following identifications:

$$\frac{D[Z]}{(f)} = S[u, v, z] = S^{[3]}, \qquad \frac{A[Z]}{fD[Z] \cap A[Z]} = S[u^2, v^2, z]$$

and

$$\frac{B[Z]}{fD[Z] \cap B[Z]} = S[(au + bv)^2, (cu + dv)^2, z].$$

Since $\Phi(f) = \lambda f$ and $\Phi(R) = R$, Φ induces a k-algebra automorphism ϕ of the ring $D[Z]/(f) = S[u, v, z]$ such that $\phi(S) = S$. Moreover, ϕ

restricts to a k-algebra isomorphism (which we again denote by ϕ)

$$\begin{aligned}\phi : \frac{A[Z]}{fD[Z] \cap A[Z]} = S[u^2, v^2, z] &\longrightarrow S[(au+bv)^2, (cu+dv)^2, z] \\ &= \frac{B[Z]}{fD[Z] \cap B[Z]}.\end{aligned}$$

Hence, by (1.2), we have

$$S[{u_1}^2, {v_1}^2, z_1] = S[(au+bv)^2, (cu+dv)^2, z]. \tag{1.3}$$

Therefore, the determinant of the Jacobian matrix

$$J := \det\left(\frac{\partial((au+bv)^2, (cu+dv)^2, z)}{\partial({u_1}^2, {v_1}^2, z_1)}\right) \in S^* = k^* \quad \text{(by (i))}.$$

Now,

$$\begin{aligned}J \cdot \det\left(\frac{\partial({u_1}^2, {v_1}^2, z_1)}{\partial(u, v, z)}\right) &= \det\left(\frac{\partial((au+bv)^2, (cu+dv)^2, z)}{\partial(u, v, z)}\right) \\ &= 4(au+bv)(cu+dv),\end{aligned}$$

as $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(S)$. Hence, setting $\mu := \det\left(\partial(u_1, v_1, z_1)/\partial(u, v, z)\right) \in S^* = k^*$, we have

$$\begin{aligned}4(au+bv)(cu+dv) &= J \cdot \det\left(\frac{\partial({u_1}^2, {v_1}^2, z_1)}{\partial(u, v, z)}\right) \\ &= J \cdot \det\left(\frac{\partial({u_1}^2, {v_1}^2, z_1)}{\partial(u_1, v_1, z_1)} \frac{\partial(u_1, v_1, z_1)}{\partial(u, v, z)}\right) \\ &= 4J\mu u_1 v_1.\end{aligned} \tag{1.4}$$

Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(S)$, we have $S[(au+bv), (cu+dv)] = S[u, v]$. As $au+bv$, $cu+dv$ and u_1, v_1 are prime elements of $S[u, v, z]$ $(= S[u_1, v_1, z_1] = S^{[3]})$ and $J, \mu \in k^*$, we have, by (1.4), either

$$u_1 = \epsilon_1(au+bv) \quad \text{and} \quad v_1 = (J\mu\epsilon_1)^{-1}(cu+dv) \tag{1.5}$$

or

$$u_1 = \epsilon_2(cu+dv) \quad \text{and} \quad v_1 = (J\mu\epsilon_2)^{-1}(au+bv) \tag{1.6}$$

for some $\epsilon_1, \epsilon_2 \in k^*$. Therefore, by (1.5) or (1.6),

$$S[u_1, v_1, z] = S[(au+bv), (cu+dv), z] = S[u, v, z] = S[u_1, v_1, z_1].$$

Hence, $z = \nu z_1 + g(u_1, v_1)$ for some polynomial $g \in S^{[2]}$ and $\nu \in S^* = k^*$. Since $S[{u_1}^2, {v_1}^2, z_1] = S[(au+bv)^2, (cu+dv)^2, z]$ by (1.3), we have $g(u_1, v_1) \in S[{u_1}^2, {v_1}^2]$.

Now, by (1.5) or (1.6), there exists an S-linear automorphism θ of the ring $D[Z]/(f) = S[u, v, z] = S[u_1, v_1, z_1]$ such that

$$\theta(u_1) = au + bv, \qquad \theta(v_1) = cu + dv, \qquad \theta(z_1) = z.$$

Clearly, θ can be lifted to an R-linear automorphism Θ of the ring $D[Z]$ satisfying $\Theta(B[Z]) = B[Z]$. Set $\Psi := \Theta\Phi$ (a k-algebra automorphism of $D[Z]$),

$$\begin{aligned} U_2 &:= \Theta(U_1) = \Psi(U), \\ V_2 &:= \Theta(V_1) = \Psi(V), \\ Z_2 &:= \Theta(Z_1) = \Psi(Z). \end{aligned} \tag{1.7}$$

Then, $\Psi(f) = \lambda f$, and hence, Ψ induces a k-algebra automorphism $\psi = \theta\phi$ of the ring $D[Z]/(f)$. Let u_2, v_2, z_2 denote the images of U_2, V_2, Z_2 in $D[Z]/(f)$. Then,

$$\begin{aligned} u_2 &= \psi(u) = au + bv, \\ v_2 &= \psi(v) = cu + dv, \\ z_2 &= \psi(z) = z. \end{aligned} \tag{1.8}$$

As Ψ is a k-algebra automorphism of $D[Z] = R[U, V, Z] = R[U_2, V_2, Z_2]$, we have

$$\begin{aligned} U_2 &= \delta_1 + \alpha_1 U + \beta_1 V + \gamma_1 Z + \text{ higher degree terms}, \\ V_2 &= \delta_2 + \alpha_2 U + \beta_2 V + \gamma_2 Z + \text{ higher degree terms}, \end{aligned}$$

and

$$Z_2 = \delta_3 + \alpha_3 U + \beta_3 V + \gamma_3 Z + \text{ higher degree terms}$$

for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in R$, $1 \le i \le 3$, such that

$$N := \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in \mathrm{GL}_3(R).$$

By (1.8), we have

$$\eta_3(N) = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\det(P) = 1$ and $\det(N) \in R^* = k^*$, we have $\det(N) = 1$. Thus, there exists a matrix N in $\mathrm{SL}_3(R)$ such that $\eta_3(N) = \left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By Suslin's stability theorem ([**9,** Theorem VI 4.5]), $\mathbf{SK}_1(R) = 0$, and so, $[N] = [0]$ in $\mathbf{SK}_1(R)$. Hence, $[P] = \overline{\eta}([N]) = [0]$ in $\mathbf{SK}_1(S)$, where $\overline{\eta}$ denotes the group homomorphism $\mathbf{SK}_1(R) \to \mathbf{SK}_1(S)$ induced by η. However, this contradicts hypothesis (ii). Thus, $A[Z] \ncong B[Z]$.

(II) We now show that $A^{[2]} \cong B^{[2]}$. Let $P_1 = \left(\begin{smallmatrix} P & 0 \\ 0 & P^{-1} \end{smallmatrix}\right)$. Then, by Whitehead's lemma (cf., [**9,** page 44]), P_1 is an elementary matrix of S. Thus, there exists an elementary matrix $L \in \mathrm{SL}_4(R)$ such that $\eta_4(L) = P_1$. Let Λ be an R-algebra automorphism of the ring $R[U, V, Z, W]$, defined by

$$\Lambda \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} = L \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix}.$$

Recall that $\eta_4 \left(\begin{smallmatrix} M & 0 \\ 0 & M' \end{smallmatrix}\right) = P_1$. Since $\eta_4(L) = P_1$, we have

$$L \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} + f \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}$$

for some $g_1, g_2, g_3, g_4 \in R[U, V, Z, W]$. Now,

$$\begin{aligned}
&\Lambda(A[Z, W]) \\
&\quad = \Lambda\big((R[U^2, V^2] + fR[U, V])[Z, W]\big) \\
&\quad = \Lambda\big(R[U^2, V^2, Z, W] + fR[U, V, Z, W]\big) \\
&\quad = \Lambda\big(R[U^2, V^2, Z, W]\big) + f\Lambda\left(R[U, V, Z, W]\right) \\
&\quad = R[\Lambda(U)^2, \Lambda(V)^2, \Lambda(Z), \Lambda(W)] + fR[U, V, Z, W] \\
&\quad = R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2, (\alpha' Z + \beta' W), (\gamma' Z + \delta' W)] + fR[U, V, Z, W] \\
&\quad = R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2, Z, W] + fR[U, V, Z, W]
\end{aligned}$$

(since $\eta_2(M') = P^{-1}$)

$$= B[Z, W].$$

Thus, Λ restricts to an isomorphism: $A[Z, W] \to B[Z, W]$.

(III) It is sufficient to show that A is a seminormal ring. Let $I = fD = fD \cap A$. Since $A \subseteq D$ is a finite birational extension and I is the conductor ideal of A in D, with $A^* = D^* = k^*$ and $(D/I)^* = (A/I)^* = S^* = k^*$, by Theorem 1.2, we have the exact sequence

$$0 \longrightarrow \mathrm{Pic}\,(A) \xrightarrow{\Delta} \mathrm{Pic}\,(D) \times \mathrm{Pic}\,(A/I) \xrightarrow{\pm} \mathrm{Pic}\,(D/I). \tag{1.9}$$

Since $A/I \cong D/I \cong S^{[2]}$, we have $\mathrm{Pic}\,(A/I) \cong \mathrm{Pic}\,(D/I)$. Hence, by (1.9), we have $\mathrm{Pic}\,(A) \cong \mathrm{Pic}\,(D) = (0)$. Similarly, $\mathrm{Pic}\,(A^{[1]}) = 0$. Hence, $\mathrm{Pic}\,(A) = \mathrm{Pic}\,(A^{[1]})$, and thus, A is a seminormal ring by **[12]**. □

Example 1.5. Let $k = \mathbb{R}$ be the field of real numbers,

$$R = \mathbb{R}[X,Y], \qquad f = X^2 + Y^2 - 1 \in R \quad \text{and} \quad S = R/(f).$$

We show that conditions (i)–(iii) of Theorem 1.4 hold. Let x and y denote, respectively, the images of X and Y in S.

It is well known that $S^* = \mathbb{R}^*$. This can be seen by observing that

$$S^* \hookrightarrow (S \otimes_{\mathbb{R}} \mathbb{C})^* = \{\lambda(x+iy)^\ell \mid \lambda \in \mathbb{C}^*,\ \ell \in \mathbb{Z}\}$$

and that $\lambda(x+iy)^\ell \in S$ if and only if $\ell = 0$ and $\lambda \in \mathbb{R}^*$.

Let $P = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right) \in \mathrm{SL}_2(S)$. By **[10,** Example 13.5], P is not stably elementary.

Now, let $\phi : R[U,V,W] \to R[U,V,W]$ be an $\mathbb{R}$-algebra automorphism such that $\phi(f) = \lambda f$ for some $\lambda \in \mathbb{R}^*$. We show that $\phi(\mathbb{R}[X,Y]) = \mathbb{R}[X,Y]$. Let

$$r = \deg_U \phi(X) \quad \text{and} \quad s = \deg_U \phi(Y).$$

Then, $\deg_U \phi(X)^2 = 2r$ and $\deg_U \phi(Y)^2 = 2s$. Since $a^2 + b^2 \neq 0$ for any two non-zero elements $a, b \in \mathbb{R}$, we have $\deg_U(\phi(X)^2 + \phi(Y)^2) = \max\{2r, 2s\}$. Now, since $\phi(X)^2 + \phi(Y)^2 - 1 = \lambda(X^2 + Y^2 - 1)$ and $\lambda \in \mathbb{R}^*$, we have $r = s = 0$. Similarly, $\deg_V \phi(X) = 0$, $\deg_V \phi(Y) = 0$, $\deg_W \phi(X) = 0$ and $\deg_W \phi(Y) = 0$. Thus, $\phi(X), \phi(Y) \in \mathbb{R}[X,Y]$.

Now, construct A and B as defined in Theorem 1.4. Then, A and B are seminormal affine domains such that $A^{[2]} \cong B^{[2]}$ but $A^{[1]} \ncong B^{[1]}$.

Example 1.6. Let $k = \mathbb{C}$ be the field of complex numbers,

$$R = \mathbb{C}[X, Y], \qquad f = Y^2 - X^3 - XY \in R \quad \text{and} \quad S = R/(f).$$

We show that conditions (i)–(iii) of Theorem 1.4 hold. Let x and y denote, respectively, the images of X and Y in S.

Since $S \cong \mathbb{C}[T^2 - T, T^3 - T^2] \hookrightarrow \mathbb{C}[T] = \mathbb{C}^{[1]}$, we have $S^* = \mathbb{C}^*$.

By [**8,** Section 4], $\mathbf{SK}_1(S) \cong \mathbf{K}_2(\mathbb{C})$, and by [**10,** Theorem 11.10], $\mathbf{K}_2(\mathbb{C})$ is uncountable. Hence, $\mathbf{SK}_1(S)$ is an uncountable group. Since S is a one-dimensional affine variety, the natural map $\mathrm{SL}_2(S) \to \mathbf{SK}_1(S)$ is surjective (cf., [**9,** Theorem III 3.7]). Hence, there exists a matrix $P \in \mathrm{SL}_2(S)$ such that P is not stably elementary.

Now, let $\phi : R[U, V, W] \to R[U, V, W]$ be a $\mathbb{C}$-algebra automorphism such that $\phi(f) = \lambda f$ for some $\lambda \in \mathbb{C}^*$. Let $X_1 = \phi(X)$, $Y_1 = \phi(Y)$. Then, $X_1, Y_1 \in R[U, V, W]$.

Suppose, if possible, that $X_1 \notin R$ and $Y_1 \notin R$. Let $K = \mathbb{C}(X, Y)$ be the field of fractions of R. Then, $X_1, Y_1 \notin K$. Since $\phi(f) = \lambda f$, we have

$$Y_1^2 - X_1^3 - X_1Y_1 - \lambda f = 0. \tag{1.10}$$

Thus, $\dim(K[X_1, Y_1]) = 1$, and the integral closure of $K[X_1, Y_1]$ in $K[U, V, W]$ is a polynomial ring in one variable over K. However, (1.10) is an equation of a non-rational curve which cannot have a polynomial parameterization. This is a contradiction. Hence, either $X_1 \in R$ or $Y_1 \in R$.

Suppose that $X_1 \in R$. Then, from (1.10), we have Y_1 is integral over $\mathbb{C}[X_1, f] \subseteq R$. As R is integrally closed in $R[U, V, W]$, we have $Y_1 \in R$. Similarly, if $Y_1 \in R$, then $X_1 \in R$. Hence, both $X_1, Y_1 \in R$.

Now, defining A and B as in Theorem 1.4, we have A and B are seminormal affine domains such that $A^{[2]} \cong B^{[2]}$ but $A^{[1]} \ncong B^{[1]}$.

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