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# REPRESENTATIONS OF FINITE POSETS OVER THE RING OF INTEGERS MODULO A PRIME POWER

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ABSTRACT. The classical category $\mathrm{Rep}(S, \mathbb{Z}_p)$ of representations of a finite poset $S$ over the field $\mathbb{Z}_p$ is extended to two categories, $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ and $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$, of representations of $S$ over the ring $\mathbb{Z}_{p^m}$. A list of values of $S$ and $m$ for which $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ or $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$ has infinite representation type is given for the case that $S$ is a forest. Applications include a computation of the representation type for certain classes of abelian groups, as the category of sincere representations in ($\mathrm{uRep}(S, \mathbb{Z}_{p^m})$) $\mathrm{Rep}(S, Z_{p^m})$ has the same representation type as (homocyclic) $(S, p^m)$-groups, a class of almost completely decomposable groups of finite rank. On the other hand, numerous known lists of examples of indecomposable $(S, p^m)$-groups give rise to lists of indecomposable representations.

**1. Introduction.** Let $(S, \leq)$ be a finite poset and $p$ a prime. The category $\mathrm{Rep}(S, \mathbb{Z}_p)$ of representations of $S$ over the field $\mathbb{Z}_p$ has objects $U = (U_0, U_s \mid s \in S)$ such that $U_0 = \sum_{s\in S} U_s$ is a finite dimensional $\mathbb{Z}_p$–vector space, $U_s$ is a subspace of $U_0$ and $U_s$ is a subspace of $U_t$ if $s \leq t$, (called $I$-spaces in **[23]** with $I = S \cup \{0\}$). The representation type of $\mathrm{Rep}(S, \mathbb{Z}_p)$ (finite, tame or wild) is characterized in terms of $S$ by the classical Kleiner-Nazarova theorems, **[23]**.

Representations of a finite poset over a commutative ring are investigated in **[19, 20]**. In particular, posets $S$ such that $\mathrm{Rep}_{\mathrm{fg}}(S, \mathbb{Z}_{p^m})$ has finite representation type are characterized in **[20]**, where the objects of $\mathrm{Rep}_{\mathrm{fg}}(S, \mathbb{Z}_{p^m})$ are those $U = (U_0, U_s \mid s \in S)$ such that $U_0 = \sum_{s\in S} U_s$ is a finite $\mathbb{Z}_{p^m}$-module, $U_s$ is a submodule of $U_0$, and $U_s$ is a submodule of $U_t$ if $s \leq t$. The category $\mathrm{Rep}_{\mathrm{fg}}(S, \mathbb{Z}_{p^m})$ has the same representation type as $\mathrm{Rep}(\widehat{S}_m, \mathbb{Z}_p)$ for a finite poset $\widehat{S}_m$ constructed from $S$

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and $m$ ([**10**]). For most $S$ and $m$, $\mathrm{Rep}_{\mathrm{fg}}(S, \mathbb{Z}_{p^m})$ has infinite or wild representation type.

In this paper, we consider a natural extension of $\mathrm{Rep}(S, \mathbb{Z}_p)$, properly contained in $\mathrm{Rep}_{\mathrm{fg}}(S, \mathbb{Z}_{p^m})$, that has not been investigated in depth.

Let $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ be the category of representations $U = (U_0, U_s \mid s \in S)$ such that

- $U_0 = \sum_{s\in S} U_s$ is a finite $\mathbb{Z}_{p^m}$-module,
- for every $s \in S$, $U_s$ is a free $\mathbb{Z}_{p^m}$-submodule of $U_0$,
- for every $s \in S$, $\sum_{t<s} U_t$ is a free $\mathbb{Z}_{p^m}$-submodule of $U_s$.

Since the ring $\mathbb{Z}_{p^m}$ is self-injective (see Section 2), the free module $U_s$ is a summand of $U_0$, and the free module $\sum_{t<s} U_t$ is a summand of $U_s$. Indecomposables in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ have local endomorphism rings, whence a representation is uniquely a direct sum of indecomposables.

The category of uniform representations is denoted by $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$, where $U = (U_0, U_s \mid s \in S)$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is *uniform* if $U_0$ is a free $\mathbb{Z}_{p^m}$-module. Notice that $\mathrm{Rep}(S, \mathbb{Z}_p) = \mathrm{uRep}(S, \mathbb{Z}_p)$.

It is assumed herein that $S$ is a *forest*, i.e., for each $s \in S$, the subset $\{t \in S \mid t \leq s)$ is a chain, and $m \geq 2$. When $S$ is a forest and $U = (U_0, U_s) \in \mathrm{Rep}(S, \mathbb{Z}_{p^m})$, then $\sum_{t<s} U_t = U_{t_m}$ where $t_m = \max\{t \mid t < s\}$, so it suffices to require that $U_s$ is free for all $s$. It is not known how much of the theory for $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ carries over to posets that are not forests.

There is a bijection $[U] \to [M_U]$ from isomorphism classes of representations $U$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ to equivalence classes of $\mathbb{Z}_{p^m}$-matrices $M_U$ in $\mathrm{Mat}(S, p^m)$ such that $U$ is indecomposable in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ if and only if $M_U$ is indecomposable in $\mathrm{Mat}(S, p^m)$ and $U$ is uniform if and only if $M_U$ is uniform (Lemma 3.1). Hence, indecomposability of $U$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ can be determined by solving a "matrix problem" in $\mathrm{Mat}(S, p^m)$, in the sense of [**11, 23**] for representations over a field and [**19, 20**] for $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$.

Given $U = (U_0, U_s \mid s \in S) \in \mathrm{Rep}(S, \mathbb{Z}_{p^m})$, define

$$\dim(U) = \textstyle\sum_{s\in S} \mathrm{rk}\left(U_s \,/ \sum_{t<s} U_t\right).$$

Call $U$ *sincere* if $\sum_{t<s} U_t \neq U_s$ for each $s$. The category $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is *unbounded* (has infinite representation type) if for each positive

integer $n$, there is an indecomposable $U$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ with $\dim(U) \geq n$ and $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is *bounded* (has finite representation type) if it is not unbounded.

Let $S_n$ denote an antichain with $n$ elements, i.e., of width $n$. A disjoint union of $r$ chains of widths $n_1, \ldots, n_r$ is denoted by $(n_1, \ldots, n_r)$.

**Theorem I.** *Assume $S$ is a forest. Then* $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ *is unbounded if*:

(i) $S \supseteq S_3$, $m \geq 2$;
(ii) $S \supseteq (1, 2)$, $m \geq 6$;
(iii) $S \supseteq (1, 3)$, $m \geq 4$;
(iv) $S \supseteq (1, 5)$, $m \geq 3$;
(v) $S \supseteq (2, 2)$, $m \geq 3$;
(vi) $S \supseteq (3, 3)$, $m \geq 2$;
(vii) $S \supseteq (2, 5)$, $m \geq 2$.

Theorem I is proved in subsection 4.1 by finding arbitrarily large indecomposable matrices $M_U$. If $S$ is a disjoint union of chains, then the only unresolved values of $(S, m)$ for $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ to be bounded or unbounded are (Corollary 6.3 (a)):

(i) $S = (1, 2)$, $m = 5$;
(ii) $S = (1, 4)$, $m = 3$;
(iii) $S = (2, n)$, $3 \leq n \leq 4$, $m = 2$.

There is an analogous theorem for $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$, proved in subsection 4.2.

**Theorem II.** *Assume $S$ is a forest. Then* $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$ *is unbounded if*:

(i) $S \supseteq S_4$, $m \geq 1$;
(ii) $S \supseteq (1, 1, 2)$, $m \geq 2$;
(iii) $S \supseteq (1, 1, 1)$, $m \geq 3$;
(iv) $S \supseteq (1, 3)$, $m \geq 6$;
(v) $S \supseteq (2, 2)$, $m \geq 4$;
(vi) $S \supseteq (1, 4)$, $m \geq 4$;
(vii) $S \supseteq (1, 6)$, $m \geq 3$;
(viii) $S \supseteq (2, 3)$, $m \geq 3$;

(ix) $S \supseteq (3,3)$, $m \geq 2$;
(x) $S \supseteq (2,5)$, $m \geq 2$.

If $S$ is a disjoint union of chains, then the only unresolved values of $(S,m)$ for $\mathrm{uRep}(S,\mathbb{Z}_{p^m})$ to be bounded or unbounded are Corollary 6.3 (b):

(i) $S = (1,3)$, $m = 5$;
(ii) $S = (1,n)$, $4 \leq n \leq 5$, $m = 3$;
(iii) $S = (2,n)$, $3 \leq n \leq 4$, $m = 2$.

The category $\mathrm{Rep}(S,\mathbb{Z}_{p^m})$ is related to a category of almost completely decomposable abelian groups, called $(S,p^m)$-groups (Section 5), **[2]**. An $(S,p^m)$-group is, up to near-isomorphism, uniquely the direct sum of indecomposable $(S,p^m)$-groups, so that finding the indecomposable $(S,p^m)$-groups amounts to a classification of these groups up to near-isomorphism. Near-isomorphism is a weakening of isomorphism. There is evidence that a classification up to isomorphism is not feasible (see [**16,** Chapter 9]) and near-isomorphism is accepted as the proper equivalence relation for almost completely decomposable groups.

If $(S,\leq)$ is an *inverted forest* of $p$-locally free types, i.e., for each $s \in S$, the subset $\{t \in S \mid t \geq s)$ is a chain, then the opposite poset $S^{\mathrm{op}} = (S,\geq)$ is a forest. In this case, there is a bijection from near-isomorphism classes of $(S,p^m)$-groups $[G]$ to isomorphism classes $[\mathcal{D}_G]$ of sincere representations in $\mathrm{Rep}(S^{\mathrm{op}},\mathbb{Z}_{p^m})$ called *anti-representations*, such that $G$ is homocyclic (Section 5) if and only if $\mathcal{D}_G$ is uniform, and $G$ is indecomposable if and only if $\mathcal{D}_G$ is indecomposable (Theorem 5.1). Other applications of representations of finite posets to abelian groups include [**9, 21**].

**Corollary III.** *Assume $S$ is a disjoint union of chains.*

(a) $(S,p^m)$*-groups are unbounded if $S$ and $m$ satisfy one of the conditions of Theorem* I.
(b) Homocyclic $(S,p^m)$*-groups are unbounded if $S$ and $m$ satisfy one of the conditions of Theorem* II.

Corollary III is an application of Theorems I and II, since if $S \subset T$ are disjoint unions of chains, then sincere (uniform) indecomposables in

$\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ can be extended to sincere (uniform) indecomposables in $\mathrm{Rep}(T, \mathbb{Z}_{p^m})$ (Theorem 5.2). The applications of Theorem I to groups (Corollary III) are new except for (i)–(iii) and (v) that were settled in [**2**], the applications of Theorem II are new except for (i), (iii) published in [**6**].

Resolution of the unresolved cases following Theorems I and II await criteria for bounded other than a finite complete list of indecomposables. For instance, the list of Theorem I would be complete if the following conjecture were true, where $(S, m-1)$ is the disjoint union of $S$ and a chain of length $m-1$.

**Conjecture.** *If* $\mathrm{Rep}((S, m-1), \mathbb{Z}_p)$ *is bounded, then* $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ *is bounded.*

**2. Preliminaries.** It is well-known that the ring $\mathbb{Z}_{p^m}$ is *self-injective*, i.e., if $X$ is a free $\mathbb{Z}_{p^m}$-submodule of a finite $\mathbb{Z}_{p^m}$-module $Y$, then $X$ is a summand of $Y$. This is a consequence of the classical fact that an element of maximal order in a finite $p$-group generates a summand.

The following terminology is standard. A *homomorphism* in Rep $(S, \mathbb{Z}_{p^m})$ from $U = (U_0, U_s \mid s \in S)$ to $W = (W_0, W_s \mid s \in S)$ is a $\mathbb{Z}_{p^m}$-homomorphism $f : U_0 \to W_0$ with $f(U_s) \subseteq W_s$ for each $s \in S$. A homomorphism $f : U \to W$ is an *isomorphism* in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ if $f^{-1} : W \to U$ exists, equivalently, if $f : U \to W$ is an isomorphism and, for all $s \in S$, $f(U_s) = U_s'$. An *endomorphism* of $U$ is a homomorphism $f : U \to U$. The ring of endomorphisms of $U$ is denoted by $\mathrm{End}(U)$. The *direct sum* of $U$ and $W$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is

$$U \oplus W = (U_0 \oplus W_0, U_s \oplus W_s \mid s \in S).$$

It can be shown that *idempotents split* in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$, i.e., each idempotent $e$ of $\mathrm{End}(U)$ determines a decomposition $U = e(U) \oplus (1-e)(U)$. Hence, $U$ is indecomposable if and only if 0 and 1 are the only idempotents of $\mathrm{End}(U)$. If $U$ is indecomposable, then $\mathrm{End}(U) \subseteq \mathrm{End}(U_0)$ is finite, hence left artinian. In this case, $\mathrm{End}(U)$ is semi-perfect, [**1,** page 303], and so $\mathrm{End}(U)$ is a local ring, [**9,** Proposition 2.1.3]. As a consequence of the categorical version of the Krull, Schmidt and Azumaya theorem ([**1**] for the module version or

[**8,** Theorem 7.4]), a $U$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is uniquely a direct sum of indecomposable representations.

The following crucial lemma contains properties of representations in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ for a forest $S$. Part (a) is known to be false if $S$ is not a forest.

**Lemma 2.1.** *Assume $S$ is a forest.*

(a) *$U = (U_0, U_s \mid s \in S)$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ if and only if, for each $s \in S$, there is a finite free $\mathbb{Z}_{p^m}$-module $V_s$ with $U_0 = \sum_{s\in S} V_s$ and $U_s = \bigoplus_{t\leq s} V_t$.*

(b) *If $U = (U_0, U_s \mid s \in S)$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ with $U_0 = \sum_{s\in S} V_s$, $U_s = \bigoplus_{t\leq s} V_t$ a free module, and $U_*$ the kernel of the epimorphism $\pi_0 : \bigoplus_{s\in S} V_s \to U_0$ defined by $\pi_0(\oplus v_s) = \sum v_s$, then $U_* \cap \bigoplus_{t\leq s} V_t = 0$ for every $s$.*

(c) *$U = (U_0, U_s \mid s \in S)$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is in $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$ if and only if $U_*$ is a free $\mathbb{Z}_{p^m}$-module.*

*Proof.*

(a) Let $U = (U_0, U_s \mid s \in S)$ be in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$, i.e., $U_0 = \sum_{s\in S} U_s$ is a finite $\mathbb{Z}_{p^m}$-module, $U_s$ is a free submodule of $U_0$, and $\sum_{t<s} U_t$ is a free submodule, hence a summand, of $U_s$. Choose a free $\mathbb{Z}_{p^m}$-module $V_s$ with

$$U_s = \left(\sum_{t<s} U_t\right) \oplus V_s.$$

Since $\{t \in S \mid t \leq s\}$ is a chain, it follows by induction that $U_s = \bigoplus_{t\leq s} V_t$. Moreover, $U_0 = \sum_{s\in S} U_s = \sum_{s\in S} V_s$. The converse is clear as $U_0$ is finite and $U_s = \bigoplus_{t\leq s} V_t$ is a free module.

(b) Since $U_s = \bigoplus_{t\leq s} V_t \subseteq \bigoplus_{s\in S} V_s$, the map $\pi_0 \restriction_{U_s} : U_s \to U_s$ is one-to-one. Then $U_* \cap U_s = 0$ because $U_* = \ker(\pi_0)$.

(c) If $U$ is in $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$, then $U_0$ is a free $\mathbb{Z}_{p^m}$-module. Since $\pi_0 : \bigoplus_{s\in S} V_s \to U_0$ is onto, $U_* = \ker(\pi_0)$ is a summand of the free module $\bigoplus_{s\in S} V_s$, hence free.

Conversely, if $U_* = \ker(\pi_0)$ is free, then, since $\bigoplus_{s\in S} V_s$ is finite, $U_*$ is a summand of $\bigoplus_{s\in S} V_s$. Hence, $U_0 = \mathrm{image}(\pi_0)$ is a summand of the free module $\bigoplus_{s\in S} V_s$, and so $U_0$ is free. $\square$

Following Lemma 2.1, define $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ to be the category of representations $U=(U_0,U_s,U_* \mid s\in S)$ such that

- for each $s$, there is a finite free $\mathbb{Z}_{p^m}$-module $V_s$ with $U_0 = \bigoplus_{s\in S} V_s$ and $U_s = \bigoplus_{t\leq s} V_t$,
- $U_*$ a submodule of $U_0$ with $U_*\cap U_s = 0$.

Notice that $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ is a proper subcategory of $\mathrm{Rep}_{fg}(S^*,\mathbb{Z}_{p^m})$, where $S^* = S\cup\{*\}$ is a poset with $*$ incomparable to any element of $S$.

The subcategory of $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ with representations $U=(U_0,U_s, U_* \mid s\in S)$ such that $U_*$ is a free module is denoted by $\mathrm{hcdRep}(S,Z_{p^m})$, **[5]**.

We state an observation that will be used later.

**Lemma 2.2.** *Let $U=(U_0,U_s,U_* \mid s\in S)\in \mathrm{cdRep}(S,\mathbb{Z}_{p^m})$. Suppose that $U_s = W_s \oplus \sum_{t<s} U_t$ for each $s\in S$. Then $U_0 = \bigoplus_{s\in S} W_s$ and, for each $s\in S$, $U_s = \bigoplus_{t\leq s} W_t$.*

*Proof.* By definition of $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$, there exist free modules $V_s$ such that $U_0 = \bigoplus_{s\in S} V_s$ and $U_s = \bigoplus_{t\leq s} V_t$ for each $s\in S$. Hence,

$$V_s \cong \frac{U_s}{\sum_{t<s} U_t} \cong W_s.$$

It follows by induction that $U_0 = \sum_{s\in S} W_s$. This sum must be direct because of cardinalities:

$$|U_0| = \sum_s \left|\frac{U_s}{\sum_{t<s} U_t}\right| = \sum_{s\in S} |W_s|.$$

Similarly, $U_s = \bigoplus_{t\leq s} W_t$. $\square$

Both $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ and $\mathrm{hcdRep}(S,\mathbb{Z}_{p^m})$ are subcategories of $\mathrm{Rep}_{\mathrm{fg}}$ $(S\cup\{*\},\mathbb{Z}_{p^m})$. The category $\mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ is not a subcategory of $\mathrm{Rep}(S\cup\{*\},\mathbb{Z}_{p^m})$ because, in the latter, the $U_*$ must be free and it is easy to make examples of $(U_0,U_s,U_*)\in \mathrm{cdRep}(S,\mathbb{Z}_{p^m})$ where $U_*$ is not free. However, $\mathrm{hcdRep}(S,\mathbb{Z}_{p^m})$ evidently is a subcategory of $\mathrm{Rep}(S\cup\{*\},\mathbb{Z}_{p^m})$.

Recall that, for a forest $S$, the object $U = (U_0, U_s \mid s \in S)$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ if and only if $U_0 = \sum_{s \in S} U_s$ and, for every $s \in S$, $U_s$ is a finite free $\mathbb{Z}_{p^m}$-submodule of $U_0$.

We are now in a position to show that the categories $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ and $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ are equivalent (see **[22,** Theorem 5.3]).

**Lemma 2.3.** *Assume $S$ is a forest.*

(1) *There is an equivalence*

$$\mathcal{D} : \mathrm{cdRep}(S, \mathbb{Z}_{p^m}) \longrightarrow \mathrm{Rep}(S, \mathbb{Z}_{p^m}).$$

(2) $U \in \mathrm{hcdRep}(S, \mathbb{Z}_{p^m})$ *if and only if* $\mathcal{D}(U) \in \mathrm{uRep}(S, \mathbb{Z}_{p^m})$.

Consequently, the functor $\mathcal{D}$ induces a bijection $[U] \to [\mathcal{D}(U)]$ between isomorphism classes $[U]$ of indecomposable representations $U$ in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ and isomorphism classes $[D]$ of indecomposable representations $D$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ such that $U$ is in $\mathrm{hcdRep}(S, \mathbb{Z}_{p^m})$ if and only if $\mathcal{D}_U$ is in $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$.

*Proof.*

(1) (a) Let $U = (U_0, U_s, U_* \mid s \in S)$ and $(U'_0, U'_s, U'_* \mid s \in S)$ be objects of $\mathrm{Rep}_{\mathrm{fg}}(S \cup \{*\}, \mathbb{Z}_{p^m})$. Define

$$\mathcal{D}(U) = \left( D_0 = \frac{U_0}{U_*}, D_s = \frac{U_s + U_*}{U_*} \,\middle|\, s \in S \right)$$

and, for $f : (U_0, U_s, U_* \mid s \in S) \to (U'_0, U'_s, U'_* \mid s \in S) \in \mathrm{Rep}_{\mathrm{fg}}(S \cup \{*\}, \mathbb{Z}_{p^m})$, let $\mathcal{D}(f)$ be the induced map $U_0/U_* \to U'_0/U'_*$ that obviously maps $\mathcal{D}_s = U_s + U_*/U_*$ into $\mathcal{D}'_s = U'_s + U'_*/U'_*$. Then, clearly, $\mathcal{D}$ is a covariant additive functor.

(b) Suppose $U \in \mathrm{cdRep}(S, \mathbb{Z}_{p^m})$. Then $\mathcal{D}(U)$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ because $D_0 = \sum_{s \in S} D_s$ and $D_s = (U_s + U_*)/U_* \cong U_s/(U_* \cap U_s) = U_s$ is free.

(c) To show that $\mathcal{D}$ is dense, let $D = (D_0, D_s \mid s \in S)$ be in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$. By Lemma 2.1 (a), there are finite free $\mathbb{Z}_{p^m}$-modules $E_s$ with $D_s = \bigoplus_{t \leq s} E_t$ and $D_0 = \sum_{s \in S} E_s$.

Define

$$\mathcal{U}_D = \left(U_0 = \bigoplus_{s\in S} E_s, U_s = \bigoplus_{t\leq s} E_t, U_* \mid s \in S\right)$$

where, as in Lemma 2.1 (b), $U_*$ is the kernel of the epimorphism $\pi_0 : \bigoplus_{s\in S} E_s \to D_0$ defined by $\pi_0(\oplus e_s) = \sum e_s$. Then $\mathcal{U}_D$ is in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ since $U_0 = \bigoplus_{s\in S} E_s$, $U_s = \bigoplus_{t\leq s} E_t$ and $U_* \cap U_s = \ker(\pi_0) \cap (\bigoplus_{t\leq s} E_t) = 0$. It is straightforward to check that $\mathcal{D}_{(\mathcal{U}_D)} \cong D$.

(d) We show next that $\mathcal{D}$ is faithful. Suppose $f : U = (U_0, U_s, U_*) \to U' = (U_0', U_s', U_*')$ and $\mathcal{D}(f) = 0$. Then $f(U_0) \subseteq U_*'$, and hence, $f(U_s) \subseteq U_s' \cap U_* = 0$. Hence, $f(U_0) = 0$ as $U_0 = \sum_{s\in S} U_s$.

(e) Finally, we show that $\mathcal{D}$ is full. Let $U = (U_0, U_s, U_*)$ and $U' = (U_0', U_s', U_*')$ be objects in $\mathrm{cdRep}(S, Z_{p^m})$, and let $g : \mathcal{D}(U) \to \mathcal{D}(U')$ be a representation homomorphism, i.e., $g : U_0/U_* \to U_0'/U_*'$ is a module homomorphism with $g((U_s + U_*)/U_*) \subseteq (U_s' + U_*')/U_*'$. There exist free modules $V_s$ such that $U_0 = \bigoplus_{s\in S} V_s$ and $U_s = \bigoplus_{t\leq s} V_t$. Let $\{v_{s,i}\}$ be a basis of $V_s$. Then $g(v_{s,i} + U_*) = u_{s,i} + U_*'$ for a unique element $u_{s,i} \in U_s'$, uniqueness following from $U_s' \cap U_*' = 0$. The specification $f(v_{s,i}) = u_{s,i}$ determines a well-defined module homomorphism $f : U_0 \to U_0'$ with $f(U_s) \subseteq U_s'$. Furthermore, $g(v_{s,i}+U_*) = u_{s,i}+U_s' = f(v_{s,i})+U_*'$ showing that $f(U_*) \subseteq U_*'$, that $f$ is a representation homomorphism, and that $\mathcal{D}(f) = g$. So $\mathcal{D}$ is full.

(2) In view of Lemma 2.1 (c), $U$ is in $\mathrm{hcdRep}(S, \mathbb{Z}_{p^m})$ if and only if $\mathcal{D}(U)$ is in $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$. □

**3.** $\mathrm{Mat}(S, \mathbb{Z}_p^m)$**.** By Lemma 2.3, indecomposables in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ correspond to indecomposables in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$. Recall that $U = (U_0, U_s, U_* \mid s \in S)$ is in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ if there are finite free $\mathbb{Z}_{p^m}$-modules $V_s$ with $U_0 = \bigoplus_{s\in S} V_s$, $U_s = \bigoplus_{t\leq s} V_t$ and $U_*$ is a submodule of $U_0$ such that $U_* \cap U_s = 0$ for each $s$. Moreover, $U$ is isomorphic to $U'$ if and only if there is an isomorphism $f : U_0 \to U_0'$ with $f(U_s) = U_s'$ and $f(U_*) = U_*'$.

Choosing bases $\mathcal{H} = \{h\}$ of $U_*$ and $\mathcal{V} = \{v\}$ of $U_0$, we get

$$h = \textstyle\sum_{v\in\mathcal{V}} m_{hv} v,$$

$$M = M_{\mathcal{H}\mathcal{V}} := (m_{hv}), \text{ the } \textit{representing matrix of } U,$$

and $U_*$ is isomorphic to the row space $\operatorname{rsp}(M)$ of $M$, the rows constituting a basis of the row space.

Let $\mathcal{H}' = \{h'\}$ be another basis of $U_*$. Then $h' = \sum_{h\in\mathcal{H}} \alpha_{h'h} h$, and $(\alpha)_{\mathcal{V}} = (\alpha_{h'h})$ is the matrix of the automorphism $\alpha$ of $U_*$ given by $\alpha(h) = h'$. It is straightforward to check that

$$M_{\mathcal{H}'\mathcal{V}} = (\alpha_{h'h}) M_{\mathcal{H}\mathcal{V}}. \tag{3.1}$$

Let $\mathcal{V}' = \{v'\}$ be another basis of $U_0$. Then $v' = \sum_{v\in\mathcal{V}} \beta_{v'v} h$, and $(\beta)_{\mathcal{V}} = (\beta_{v'v})$ is the matrix of the automorphism $\beta$ of $U_0$ given by $\beta(v) = v'$. Again, it is straightforward to check that

$$M_{\mathcal{H}\mathcal{V}'} = M_{\mathcal{H}\mathcal{V}'}(\beta_{v'v}). \tag{3.2}$$

In order to account for the representation structure of $U$, the bases $\mathcal{V}$ need to be chosen as follows. Let $\mathcal{V}_s$ be a basis of $V_s$. Then $\mathcal{V} = \bigcup_{s\in S} \mathcal{V}_s$ is a basis of $U_0$ that we call *conforming*. Let $\mathcal{V} = \bigcup_{s\in S} \mathcal{V}_s$ be a conforming basis of $U$ and $h \in \mathcal{H}$. Then

$$h = \sum_{s\in S} \sum_{v\in\mathcal{V}_s} m_{hv} v, \tag{3.3}$$

$$M = M_{\mathcal{H}\mathcal{V}} := (\dots M_s \dots) \text{ where } M_s = (m_{hv} \mid v \in \mathcal{V}_s). \tag{3.4}$$

The preceding considerations motivate the following definition of $\operatorname{Mat}(S, \mathbb{Z}_p^m)$.

For any matrix $M$ let $\dim(M)$ denote the number of columns of $M$.

Define an element $M$ of $\operatorname{Mat}(S, \mathbb{Z}_{p^m})$ to be a $\mathbb{Z}_{p^m}$-matrix such that

(i) $M$ has a column block structure $M = (M_s \mid s \in S)$;
(ii) *Regulator condition*: the rows of $M$ form a basis of the row space $\operatorname{rsp}(M) \subseteq (\mathbb{Z}_{p^m})^N$ where $N = \dim(M)$;
(iii) for all $s \in S : \operatorname{rsp}(M) \cap \operatorname{rsp}(M_{\leq s}) = 0$ where

$$M_{\leq s} = (M'_t \mid t \in S) \quad \text{and} \quad M'_t = \begin{cases} M_t & \text{if } t \leq s \\ 0_{r\times n_t} & \text{otherwise.} \end{cases}$$

Given $M = (M_s \mid s \in S)$ in $\operatorname{Mat}(S, \mathbb{Z}_{p^m})$ satisfying (i)–(iii) above, the matrix $M$ is *sincere* if each $n_s = \dim(M_s) \neq 0$, and $M$ is *uniform* if $\operatorname{rsp}(M)$ is a free $\mathbb{Z}_{p^m}$-module. The collection of uniform matrices is denoted by $\operatorname{uMat}(S, \mathbb{Z}_{p^m})$.

The following definition reflects the choices of bases.

Two elements $M = (M_s \mid s \in S)$ and $M' = (M'_s \mid s \in S)$ of $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ are *equivalent* if $M$ can be transformed into $M'$ by a sequence of invertible $\mathbb{Z}_{p^m}$-row and column operations:

(a) Add a $\mathbb{Z}_{p^m}$-multiple of a row of $M$ of order $p^j$ to another row of $M$ of order $p^i$ if $j \leq i$;
(b) add a $p^{j-i}\mathbb{Z}_{p^m}$-multiple of a row of $M$ of order $p^j$ to another row of $M$ of order $p^i$ if $j > i$;
(c) multiply a row of $M$ by a unit of $\mathbb{Z}_{p^m}$;
(d) interchange any two rows of $M$;
(e) add a $\mathbb{Z}_{p^m}$-multiple of a column of $M_s$ to a column of $M_t$ if $s \geq t$ in $S$;
(f) multiply a column of $M$ by a unit of $\mathbb{Z}_{p^m}$;
(g) interchange any two columns of $M_s$.

Assume that $M = (M_s \mid s \in S)$ and $M' = (M'_s \mid s \in S)$ are in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$. The *direct sum* of $M$ and $M'$ is $M \oplus M' = (M_s \oplus M'_s \mid s \in S)$, where

$$M_s \oplus M'_s = \begin{pmatrix} M_s & 0 \\ 0 & M'_s \end{pmatrix}.$$

As condition (iii) holds, $M \oplus M'$ is in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$.

In $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ we set $M = 0$ if $r = N = 0$, i.e., if $M$ is the empty matrix. A matrix $M$ in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ is *indecomposable* if $M$ equivalent to $M' \oplus M''$ implies that $M' = 0$ or $M'' = 0$.

**Lemma 3.1.** *Let $S$ be a forest. There is a bijection $[U] \to [M_U]$ from isomorphism classes of representations $U$ in* $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ *to equivalence classes of $\mathbb{Z}_{p^m}$-matrices $M_U$ in* $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ *such that*:

(a) *$U$ is indecomposable if and only if $M_U$ is indecomposable.*
(b) *$U$ is in* $\mathrm{hcdRep}(S, \mathbb{Z}_{p^m})$ *if and only if $M_U$ is in* $\mathrm{uMat}(S, \mathbb{Z}_{p^m})$.

*Proof.* Assume $U = (U_0, U_s, U_* \mid s \in S)$ is in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ with

$$U_0 = \bigoplus_{s \in S} V_s, \qquad U_s = \bigoplus_{t \leq s} V_t,$$
$$U_* \subseteq U_0, \quad \text{and} \quad U_* \cap U_s = 0.$$

Let $B_s$ be a basis of $V_s$, $C_U = \{h_1, \dots, h_r\}$ a basis of $U_*$, and define $M_U = (M_s \mid s \in S)$ as in (3.3). Then $M_U$ is in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ with $\mathrm{rsp}(M_U) \cong U_*$.

On the other hand, let $M = (M_s \mid s \in S)$ be in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ of size $r \times N$, $M_s$ having size $r \times n_s$. Then

$$\mathrm{rsp}(M) \subseteq (\mathbb{Z}_{p^m})^N = \bigoplus_{s \in S} (\mathbb{Z}_{p^m})^{n_s} .$$

We assume that $B_s$ is the set of the canonical (unit) basis elements of $(\mathbb{Z}_{p^m})^{n_s}$, so that $B := \bigcup_{s \in S} B_s$ is the canonical basis of $(\mathbb{Z}_{p^m})^N$.

Define $U_M = (U_0, U_s, U_* \mid s \in S)$ by setting

$$U_0 = (\mathbb{Z}_{p^m})^N = \bigoplus_{b \in B} \mathbb{Z}_{p^m} b,$$
$$U_s = (\mathbb{Z}_{p^m})^{n_s} = \bigoplus_{b \in B_s} \mathbb{Z}_{p^m} b,$$

and $U_* = \mathrm{rsp}(M)$. Then $U_M$ is in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$, in particular, $U_* \cap U_s = 0$ by condition (iii) on $M$.

We outline an argument that $U$ is isomorphic to $U'$ in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ if and only if $M_U$ is equivalent to $M_{U'}$ in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$. Details are left to the reader.

Assume $U$ is isomorphic to $U'$, and reduce to the case that $U_0 = U_0'$, $U_s = U_s'$ and $U_* = U_*'$. In this case, $M_U$ and $M_{U'}$ represent matrices of $U$ with

$$U_* = \mathrm{rsp}(M_U) = \mathrm{rsp}(M_{U'}),$$

$B_s$ and $B_s'$ are bases of $V_s$ and $V_s'$, respectively, and $C$ and $C'$ are ordered bases of $U_*$. It follows from (3.2) and Appendix C that $M_U$ can be transformed into $M_{U'}$ by a series of invertible column operations (e)–(g) corresponding to basis changes in $U_0$.

Furthermore, by (3.1) and Appendix B, $M_U$ can be transformed by a sequence of invertible row transformations (a)–(d) corresponding to basis changes of $U_*$. It now follows that $M_U$ is transformed into $M_{U'}$ by a sequence of row and column operations (a)–(g), whence $M_U$ and $M_{U'}$ are equivalent.

Conversely, assume $M_U$ and $M_{U'}$ are equivalent. A sequence of invertible row and column operations (a)–(g) transforming $M_U$ into

$M_{U'}$ amounts to replacing a conforming basis $(B_s : s \in S)$ of $U_0$ by a conforming basis $(B'_s : s \in S)$ of $U'_0$ and a basis $C$ of $U_* = \mathrm{rsp}(M_U)$ by another basis $C'$ of $U'_*$ inducing an isomorphism $U \to U'$.

It is routine to verify that $M$ is equivalent to $M_{U_M}$ and $U$ is isomorphic to $U_{M_U}$, whence $[U] \to [M_U]$ is a bijection.

(a) If $M_U$ is indecomposable and $U = (U_0, U_s, U_* \mid s \in S)$ is isomorphic to $W \oplus Y = (W_0 \oplus Y_0, W_s \oplus Y_s, W_* \oplus Y_* \mid s \in S)$, then, from the definition of $M_U$ and equivalence of matrices, $M_U$ is equivalent to $M_W \oplus M_Y$. Hence, $M_W = 0$ or $M_Y = 0$, $W = 0$ or $Y = 0$, and so $U$ is indecomposable.

Conversely, if $U$ is indecomposable and $M_U = K \oplus L$, then, as a consequence of the definitions, $U$ is equivalent to $U_K \oplus U_L$. Since $U$ is indecomposable, $K = 0$ or $L = 0$, and so $M_U$ is indecomposable.

(b) is clear from the fact that $U_* = \mathrm{rsp}(M_U)$. $\square$

Following is a description of an endomorphism of

$$U = \left(U_0 = \bigoplus_{s\in S} V_s, U_s = \bigoplus_{t\leq s} V_t, U_* \mid s \in S\right) \in \mathrm{cdRep}(S, \mathbb{Z}_{p^m}),$$

in terms of $M_U$. Recall that an endomorphism of $U$ is a homomorphism $f : \bigoplus_{s\in S} V_s \to \bigoplus_{s\in S} V_s$ with $f(\bigoplus_{t\leq s} V_t) \subseteq \bigoplus_{t\leq s} V_t$ for each $s$ and $f(U_*) \subseteq U_*$. In particular, $f(V_t) \subseteq \bigoplus_{s\leq t} V_s = U_t$ and, for the maximal $t' < t$ in the forest $S$, $f(U_{t'} = \bigoplus_{s<t} V_s) \subseteq \bigoplus_{s<t} V_s = U_{t'}$. Let $\pi_s : U_0 \to V_s$ be the projection along $\bigoplus_{t\neq s} V_t$, and let $f \in \mathrm{End}(U)$. Let

$$f_s = \pi_s\,(f \restriction_{V_s}) : V_s \longrightarrow V_s$$

and

$$f_{ts} = \pi_t\,(f \restriction_{V_s}) : V_s \longrightarrow V_t \text{ for } t > s.$$

Then

$$f_s + \textstyle\sum_{t>s} f_{ts} : V_s \longrightarrow \bigoplus_{t\geq s} V_t : \left(f_s + \sum_{t>s} f_{ts}\right)(x) = f_s(x) + \sum_{t>s} f_{ts}(x)$$

describes $f$ in terms of its action on $V_s$, and we write

$$f = \oplus_{s\in S}(f_s + \textstyle\sum_{t>s} f_{ts}).$$

If $u \in \mathbb{Z}_{p^m}^r$, then $uM_U = (uM_s \mid s \in S) \in U_* = \operatorname{rsp}(M_U)$, and so

$$f(uM_U) = \oplus_s(f_s(uM_s) + \textstyle\sum_{t>s} f_{ts}(uM_t)) = wM_U \in U_* \quad \text{for some } w \in \mathbb{Z}_{p^m}^r.$$

Notice that $f_{ts}$ may be extended to a nilpotent endomorphism $h_{ts}$ of $U_0$ with $h_{ts}(V_r) = 0$ for all $r \neq t$.

## 4. Unbounded representation type.

**Theorem 4.1. [12].** *The category* $\operatorname{uRep}(S, \mathbb{Z}_p) = \operatorname{Rep}(S, \mathbb{Z}_p)$ *is unbounded if and only if:*

(i) $S \supseteq S_4$;
(ii) $S \supseteq (2, 2, 2)$;
(iii) $S \supseteq (1, 3, 3)$;
(iv) $S \supseteq (1, 2, 5)$;
(v) $S \supseteq (N, 4)$.

The next lemma is used in the proof of Theorem I. Part (a) of the lemma is well known for representations over fields as part of a more elaborate theory involving adjoint functors ([**13**]). Part (b) has no analogue for fields. We give simple direct proofs of the facts we need. Note that both proofs substantially use that $S$ is a forest.

**Lemma 4.2.** *Assume* $(S, \leq)$ *is a forest.*

(a) *If* $T \subseteq S$ *and* $\operatorname{Rep}(T, \mathbb{Z}_{p^m})$ *is unbounded, then* $\operatorname{Rep}(S, \mathbb{Z}_{p^m})$ *is unbounded.*
(b) *If* $k \leq m$ *and* $\operatorname{Rep}(S, \mathbb{Z}_{p^k})$ *is unbounded, then* $\operatorname{Rep}(S, \mathbb{Z}_{p^m})$ *is unbounded.*

*Proof.*
(a) Let $U = (U_0, U_t \mid t \in T)$ be in $\operatorname{Rep}(T, \mathbb{Z}_{p^m})$, i.e., $U_0 = \sum_{t\in T} U_t$, $U_t$ is a free submodule of $U_0$, and $\sum_{u<t} U_u$ is free. Define $W = (W_0, W_s \mid s \in S)$, where $W_0 = U_0, W_s = U_s$ if $s \in T$, $W_s = \sum\{U_t \mid t \in T, t < s\}$ if $s \notin T$ and there is $t \in T$ with $t < s$, and $W_s = 0$ if there is no $t \in T$ with $t < s$. Then $W$ is in $\operatorname{Rep}(S, \mathbb{Z}_{p^m})$ as

$$W_0 = U_0 = \sum_{t\in T} U_t = \sum_{s\in S} W_s$$

and

$$\sum_{v<s\in S} W_v = \sum\{U_t \mid t \in T, t < s\}$$

is a free submodule of

$$W_s = \sum\{U_t \mid t \in T, t \leq s\}.$$

It can readily be verified that $\mathrm{End}(U) = \mathrm{End}(W)$. Consequently, if $U$ is indecomposable, then $W$ is indecomposable with $\dim(U) \leq \dim(W)$. This shows that, if $\mathrm{Rep}(T, Z_{p^m})$ is unbounded, then $\mathrm{Rep}(S, Z_{p^m})$ is unbounded.

(b) Let $U = (U_0, U_s \mid s \in S)$ be in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ with $U_0 = \sum_{t\in S} U_s$, $U_s$ a free submodule of $U_0$, and $\sum_{t<s} U_t$ a free submodule of $U_s$. The free submodule $U_s$ is a summand of $U_0$, whence $p^k U_0 \cap U_s = p^k U_s$.

Define $U^* = (U_0/p^k U_0,\ (U_s + p^k U_0)/p^k U_0 \mid s \in S)$. Then $U^*$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^k})$ with $\dim(U^*) \leq \dim(U)$ as

$$U_0/p^k U_0 = \sum_{s\in S} (U_s + p^k U_0)/p^k U_0,$$

$(U_s + p^k U_0)/p^k U_0) \cong U_s/p^k U_s$ is a free $\mathbb{Z}_{p^k}$-submodule of $U_0/p^k U_0$, $\sum_{t<s}(U_t + p^k U_0)/p^k U_0$ is a free submodule of $(U_s + p^k U_0)/p^k U_0$, and, because $S$ is a forest, $\sum_{t<s} U_t = U_{t'}$ for some $t' < s$.

A routine argument shows that, if $W$ is in $\mathrm{Rep}(S, \mathbb{Z}_{p^k})$, there is some $U = (U_0, U_s \mid s \in S)$ in $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ with $U^* = W$. Moreover, if $W$ is indecomposable, then $U$ is indecomposable because if $U = X \oplus Y$, then $W = (X \oplus Y)^* = X^* \oplus Y^*$. Consequently, if $\mathrm{Rep}(S, \mathbb{Z}_{p^k})$ is unbounded, then $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ is unbounded. □

Notice that, in the proof of (a), $W$ need not be sincere, even if $U$ is sincere. By Theorem 5.2, there is a sincere extension $W$ of a sincere $U$ if $T \subset S$ are disjoint unions of chains.

**4.1. Proof of Theorem I.** In view of Lemmas 2.3, 3.1 and 4.2, it is sufficient to find indecomposable matrices $M$ in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ of arbitrarily large dimension for

(i) $(S, m) = (S_3, 2)$;
(ii) $(S, m) = ((1, 2), 6)$;

(iii) $(S, m) = ((1, 3), 4)$;
(iv) $(S, m) = ((1, 5), 3)$;
(v) $(S, m) = ((2, 2), 3)$;
(vi) $(S, m) = ((3, 3), 2)$;
(vii) $(S, m) = ((2, 5), 2)$.

Cases (i), (ii), (iii) and (v) are proved in **[2]**.

An the $n \times n$ $\mathbb{Z}_{p^m}$-matrix $A$ is *module indecomposable*, the minimal polynomial of $A(\mathrm{mod}\, p)$ is a power of an irreducible polynomial in $\mathbb{Z}_p[x]$. In this case, $\mathbb{Z}_p^n$ is an indecomposable $\mathbb{Z}_p[A(\mathrm{mod}\, p)]$-module.

The format of the proof for each of the remaining cases is as follows, see **[2]** for details of similar arguments. Given $M$ in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ with a module indecomposable matrix $A$ as a submatrix, there is a $U$ in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$ with $M = M_U$ by Lemma 3.1.

Let $f = (\oplus_{s \in S}(f_s + \sum_{t>s} f_{ts})) : U_0 \to U_0$ be an idempotent endomorphism of $U$ and $\overline{f} : U_0/pU_0 \to U_0/pU_0$ the idempotent endomorphism of $U_0/pU_0$ induced by $f$. The difficult part of the argument in each case (details not included) is to use equations arising from the condition that $f(\mathrm{rsp}(M_U)) \subseteq \mathrm{rsp}(M_U)$ to prove that $\overline{f} = (a, a, \ldots, a) + h$ for some nilpotent $h : U_0/pU_0 \to U_0/pU_0$ and idempotent $a \in \mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p^n)$ with $aA(\mathrm{mod}\, p) = A(\mathrm{mod}\, p)a$. Then $a \in \mathrm{End}_{\mathbb{Z}_p[A(\mathrm{mod}\, p)]}(\mathbb{Z}_p^n)$, and so $a = 0, 1$ because $A$ is module indecomposable.

If $a = 0$, then $\overline{f} = h = 0$, being a nilpotent idempotent. Hence, $f = pf' = 0$, again is a nilpotent idempotent. Similarly, if $a = 1$, then $f = 1$. This shows that $U$ is an indecomposable representation, whence $M = M_U$ is an indecomposable matrix. In the following, $A$ denotes an $n \times n$ module indecomposable matrix.

We adopt the following notation for matrices $M$ in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ such that $S = (n_1, \ldots, n_k) = C_1 \cup \cdots \cup C_k$ is a disjoint union of chains with $|C_i| = n_i$. Write

$$M = (M_{C_1} \| \cdots \| M_{C_k})$$

with $M_{C_k}$ in $\mathrm{Mat}(C_k, \mathbb{Z}_{p^m})$. If $C = \{1, \ldots, n\}$ is a chain, then write

$$M_C = (M_n \| \cdots \| M_1)$$

in $\mathrm{Mat}(C, \mathbb{Z}_{p^m})$. With this convention, column operations are allowed from left-to-right, i.e., from $M_i$ to $M_j$ if $i > j$.

(iv) $S \supseteq (1,5), m \geq 3$. The matrix

$$M = \left(\begin{array}{ccc||c|c|c|c|c} p^2 I_n & 0 & 0 & 0 & p^2 I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & p^2 I_n & pI_n & pI_n & 0 & I_n \\ 0 & 0 & I_n & 0 & pI_n & pA & I_n & 0 \end{array}\right)$$

is an indecomposable in $\mathrm{Mat}((1,5),\mathbb{Z}_{p^3})$ with dimension $8n$.

(vi) $S \supseteq (3,3), m \geq 2$. The matrix

$$M = \left(\begin{array}{c|c|c||c|c|c} pI_n & I_n & 0 & pI_n & 0 & I_n \\ pI_n & 0 & I_n & pA & I_n & 0 \end{array}\right)$$

is an indecomposable in $\mathrm{Mat}((3,3),\mathbb{Z}_{p^2})$ of dimension $6n$.

(vii) $S \supseteq (2,5)$, $m \geq 2$. The matrix

$$M = \left(\begin{array}{cc|cc||c|c|c|c|c} I_n & 0 & 0 & 0 & pI_n & 0 & 0 & 0 & I_n \\ 0 & pI_n & I_n & 0 & pI_n & pI_n & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n & pI_n & pA & I_n & 0 & 0 \end{array}\right)$$

is an indecomposable in $\mathrm{Mat}((2,5),\mathbb{Z}_{p^2})$ with dimension $9n$.

**4.2. Proof of Theorem II.** The proof is analogous to the proof of Theorem I. Assume $A$ is a module-indecomposable $n \times n$ $\mathbb{Z}_{p^m}$-matrix.

(i) $S \supseteq S_4$, $m \geq 2$,

(iii) $S \supseteq S_3$, $m = 3$.

By [**6**], $\mathrm{uRep}(S,\mathbb{Z}_{p^m})$ is unbounded for these cases.

(ii) $S \supseteq (1,1,2)$, $m \geq 2$. The matrix

$$M = \left(\begin{array}{c||c||c|c} I_n & I_n & 0 & I_n \\ 0 & I_n & pI_n & A \end{array}\right)$$

is an indecomposable in $\mathrm{uMat}((1,1,2),p^2)$ of dimension $4n$.

(iv) $S \supseteq (1,3)$, $m \geq 6$. The matrix

$$M = \left(\begin{array}{ccc||cc|cc|cc} I_n & 0 & 0 & p^2 I_n & 0 & I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & p^3 I_n & p^4 A & 0 & p^2 I_n & I_n & 0 \\ 0 & 0 & I_n & p^4 I_n & p^5 I_n & 0 & 0 & 0 & I_n \end{array}\right)$$

is an indecomposable in $\mathrm{uMat}((1,3),\mathbb{Z}_{p^6}))$ of dimension $9n$.

(v) $S \supseteq (2,2)$, $m \geq 4$. The matrix

$$M = \left(\begin{array}{cc|cc||cc|cc} I_n & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ pI_n & p^2A & I_n & 0 & 0 & p^2I_n & I_n & 0 \\ p^2I_n & p^3I_n & 0 & I_n & 0 & 0 & 0 & I_n \end{array}\right)$$

in $\mathrm{uMat}((2,2),p^4)$ is indecomposable of dimension $8n$.

(vi) $S \supseteq (1,4)$, $m \geq 4$. The matrix

$$M = \left(\begin{array}{cc||c|c|c|c} I_n & 0 & p^2I_n & pI_n & I_n & 0 \\ 0 & I_n & p^3I_n & p^2A & 0 & I_n \end{array}\right)$$

is an indecomposable in $\mathrm{uMat}((1,4),\mathbb{Z}_{p^4}))$ of dimension $6n$.

(vii) $S \supseteq (1,6)$, $m \geq 3$. The matrix

$$M = \left(\begin{array}{ccc||c|c|c|c|c|c} I_n & 0 & 0 & 0 & p^2I_n & 0 & 0 & 0 & I_n \\ 0 & I_n & 0 & p^2I_n & pI_n & pI_n & 0 & I_n & 0 \\ 0 & 0 & I_n & 0 & pI_n & pA & I_n & 0 & 0 \end{array}\right),$$

is an indecomposable in $\mathrm{uMat}((1,6),p^3)$ of dimension $9n$.

(viii) $S \supseteq (2,3)$, $m \geq 3$. The matrix

$$M = \left(\begin{array}{c|c||c|c|c} I_n & 0 & pI_n & I_n & 0 \\ pI_n & I & p^2A & 0 & I_n \end{array}\right)$$

is indecomposable in $\mathrm{uMat}(2,3),p^3)$ of dimension $5n$.

(ix) $S \supseteq (3,3)$, $m \geq 2$. The matrix

$$M = \left(\begin{array}{c|c|c||c|c|c} pI_n & I_n & 0 & pI_n & 0 & I_n \\ pI_n & 0 & I_n & pA & I_n & 0 \end{array}\right)$$

is an indecomposable in $\mathrm{uMat}((3,3),p^2)$ of dimension $6n$.

(x) $S \supseteq (2,5)$, $m \geq 2$. The matrix

$$M = \left(\begin{array}{cc|cc||c|c|c|c|c} I_n & 0 & 0 & 0 & pI_n & 0 & 0 & 0 & I_n \\ 0 & pI_n & I_n & 0 & pI_n & pI_n & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n & pI_n & pA & I_n & 0 & 0 \end{array}\right)$$

is an indecomposable in $\mathrm{uMat}((2,5),p^2)$ of dimension $9n$.

**5.** $(S,p^m)$**-groups.** Let $(S_G,\leq)$ denote the finite poset of critical types of an almost completely decomposable (acd) group $G$. The *regula-*

*tor* $\mathrm{R}(G)$ of $G$ defined by $\mathrm{R}(G) = \bigcap\{C \mid C$ a regulating subgroup of $G\}$ is a completely decomposable subgroup of finite index in $G$. If $S_G$ is an inverted forest, then $\mathrm{R}(G)$ is the unique regulating subgroup. These and other properties of almost completely decomposable groups, including near-isomorphism, may be found in [**9, 14, 15, 16**].

Given a prime $p$ and a finite poset $(S, \leq)$, an acd group $G$ is an $(S, p^m)$-*group* if $S = S_G$, each type in $S$ is $p$-locally free, and the exponent of $G/\mathrm{R}(G)$ is $p^m$. If $s \in S_G$, then $(\mathrm{R}(G))(s) = \{x \in \mathrm{R}(G) \mid \text{type}\,(x) \geq s\}$ is a summand of $\mathrm{R}(G)$, $G/p^mG$ and $\mathrm{R}(G)/p^m\mathrm{R}(G)$ are finite free $\mathbb{Z}_{p^m}$-modules, and each $((R(G))(s) + p^mG)/p^mG$ is a finite free $\mathbb{Z}_{p^m}$-module.

An $(S, p^m)$-group $G$ is *homocyclic* if $G/\mathrm{R}(G)$ is a free $\mathbb{Z}_{p^m}$-module [**4**].

**Theorem 5.1.** *If $S$ is an inverted forest of $p$-locally free types, then $S^{\mathrm{op}}$ is a forest and there is a bijection from near-isomorphism classes $[G]$ of $(S, p^m)$-groups to isomorphism classes $[\mathcal{D}_G]$ of sincere representations in* $\mathrm{Rep}(S^{\mathrm{op}}, \mathbb{Z}_{p^m})$ *given by*

$$G \longrightarrow D_G = \left( \frac{R(G)}{p^mG}, \frac{(R(G))(s) + p^mG}{p^mG} \,\middle|\, s \in S \right)$$

*such that*:

(a) *$G$ is homocyclic if and only if $D_G$ is in* $\mathrm{uRep}(S^{\mathrm{op}}, \mathbb{Z}_{p^m})$.
(b) *$G$ is indecomposable if and only $D_G$ is indecomposable.*

*Proof.* By [**2,** Lemma 4], there is a bijective correspondence from near-isomorphism classes of $(S, p^m)$-groups $[G]$ to isomorphism classes of sincere representations $[U_G]$ in $\mathrm{cdRep}(S^{\mathrm{op}}, \mathbb{Z}_{p^m})$ defined by

$$G \longrightarrow U_G = \left( \frac{\mathrm{R}(G)}{p^m\mathrm{R}(G)}, \frac{(\mathrm{R}(G))(s) + p^m\mathrm{R}(G)}{p^m\mathrm{R}(G)}, \frac{p^mG}{p^m\mathrm{R}(G)} \,\middle|\, s \in S^{\mathrm{op}} \right)$$

such that $G$ is indecomposable if and only if $U_G$ is indecomposable and $G$ is homocyclic if and only if $p^mG/p^m\mathrm{R}(G)$ is a free $\mathbb{Z}_{p^m}$-module. By Lemma 2.3, there is a bijection from isomorphism classes $[U_G]$ in $\mathrm{cdRep}(S^{\mathrm{op}}, \mathbb{Z}_{p^m})$ to isomorphism classes $[D_G]$ in $\mathrm{Rep}(S^{\mathrm{op}}, \mathbb{Z}_{p^m})$ given by

$$U_G \longrightarrow D_G = \left( \frac{R(G)}{p^mG}, \frac{(\mathrm{R}(G))(s) + p^mG}{p^mG} \,\middle|\, s \in S^{\mathrm{op}} \right),$$

observing that the natural epimorphism

$$\phi : \frac{\mathrm{R}(G)}{p^m\mathrm{R}(G)} \longrightarrow \frac{\mathrm{R}(G)}{p^m G}$$

has kernel

$$\frac{p^m G}{p^m\mathrm{R}(G)}$$

and

$$\phi\left(\frac{(\mathrm{R}(G))(s) + p^m\mathrm{R}(G)}{p^m\mathrm{R}(G)}\right) = \frac{(\mathrm{R}(G))(s) + p^m G}{p^m G}.$$

Moreover, $U_G$ is indecomposable if and only if $D_G$ is indecomposable and

$$G/\mathrm{R}(G) \cong (G/p^m G)/\mathrm{R}(G)/p^m G)$$

is a free $\mathbb{Z}_{p^m}$-module if and only if $\mathrm{R}(G)/p^m G$ is a free module because $G/p^m G$ is a free $\mathbb{Z}_{p^m}$-module. $\square$

**Theorem 5.2.** *Assume $S \subset T$ are disjoint unions of chains and $m \geq 2$.*

(a) *If sincere representations in* $\mathrm{Rep}(S, \mathbb{Z}_{p^m})$ *are unbounded, then sincere representations in* $\mathrm{Rep}(T, \mathbb{Z}_{p^m})$ *are unbounded.*
(b) *If sincere representations in* $\mathrm{uRep}(S, \mathbb{Z}_{p^m})$ *are unbounded, then sincere representations in* $\mathrm{uRep}(T, \mathbb{Z}_{p^m})$ *are unbounded.*

*Proof.*

(a) In view of Lemma 2.3 and Lemma 3.1, it is sufficient to prove that if sincere indecomposable matrices in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ are unbounded, then sincere indecomposable matrices in $\mathrm{Mat}(T, \mathbb{Z}_{p^m})$ are unbounded.

Assume, by way of induction on $|S|$, that $T = S \cup \{t\}$ with $t \notin S$. Write $S = C_1 \cup \cdots \cup C_k$ as a disjoint union of chains $C_i$.

*Case* I. $t$ is incomparable with every element of $S$.

(a) Let $n$ be a natural number and $M = (M_s \mid s \in S)$ a sincere indecomposable matrix in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ with $\dim(M) \geq n$. By Lemma 3.1, there is a sincere indecomposable $W = (W_0, W_s, W_*)$ in $\mathrm{cdRep}(S, \mathbb{Z}_{p^m})$

with $M = M_W$ and $W_* = \operatorname{rsp}(M)$. Write

$$M = \begin{pmatrix} p^{i_0} M_0 \\ p^{i_1} M_1 \\ \cdots \\ p^{i_r} M_r \end{pmatrix}$$

where

$$\operatorname{rsp}(M) \cong \mathbb{Z}_{p^m}^{l_0} \oplus \mathbb{Z}_{p^{m-i_1}}^{l_1} \oplus \cdots \oplus \mathbb{Z}_{p^{m-i_r}}^{l_r}, \quad i_0 = 0 < i_1 < \cdots < i_r.$$

Define

$$N = (M_s \| N_t \mid s \in S), \quad \text{where} \quad N_t = \begin{pmatrix} p^{i_0} I_{l_0} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & p^{i_r} I_{l_r} \end{pmatrix}.$$

Then $N$ is a sincere matrix in $\operatorname{Mat}(T, \mathbb{Z}_{p^m})$ with $\dim(N) > \dim(M) \geq n$.

We prove that $N$ is indecomposable. By Lemma 3.1, there is a sincere representation $U = (U_0, W_s, U_t, U_* \mid s \in S)$ in $\operatorname{cdRep}(T, \mathbb{Z}_{p^m})$ with $M_U = N$, $U_0 = W_0 \oplus U_t$, $\operatorname{rsp}(N_t) \subseteq U_t$, and $U_* = \operatorname{rsp}(N)$.

Assume $f : U_0 \to U_0$ is an idempotent endomorphism of $U$. Then $g = f \restriction_{W_0}: W_0 \to W_0$ is an idempotent endomorphism of $W$ as $t$ is incomparable to each $s \in S$. Hence, $g = 0$ or $g = 1$ because $W$ is indecomposable.

Assume $g = f \restriction_{W_0} = 0$. Then $f(W_0) = 0$, $f(U_0) = f(W_0 \oplus U_t) = f(U_t) \subseteq U_t$, and so $f(U_*) \subseteq U_* \cap U_t = 0$. Now $f(\operatorname{rsp}(N_t)) = 0$ since $f(W_0) = 0 = f(U_*)$, $N = (M \| N_t)$, and $U_* = \operatorname{rsp}(N)$. Write $U_t = U_{l_0} \oplus \cdots \oplus U_{l_r}$ with each $U_{l_j}$ corresponding to the matrix block $p^{i_j} I_{l_j}$ in $N_t$. Then $f(U_{l_0}) = 0$ because the first row block of $N_t$ is $(I_{l_0} \cdots 0) \subseteq \operatorname{rsp}(N_t)$. If $1 \leq j$, then the $j$th row block of $N_t$ is $(0 \cdots p^{i_j} I_{l_j} \cdots 0) \subseteq \operatorname{rsp}(N_t)$ so that $f(p^{i_j} U_{t_j}) = 0$ for $1 \leq j$. Then $f$ is nilpotent because $U_0 = W_0 \oplus U_t$, $f(W_0) = 0$, $f(U_{l_0}) = 0$, and $f(p^{i_j} U_{t_j}) = 0$ with $i_j > 0$, $1 \leq j$. As $f$ is also idempotent, $f = 0$. Similarly, if $f \restriction_W = 1$, then $f = 1$.

This shows that $U$ is indecomposable. By Lemma 3.1, $N = M_U$ is indecomposable. Hence, $\operatorname{Mat}(T, \mathbb{Z}_{p^m})$ contains indecomposables of rank $\geq n$.

*Case* II. For some $i$, $C_i \cup \{t\}$ is a chain contained in $T$, $n$ is a natural number and $M = (M_s \mid s \in S)$ is a sincere indecomposable in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ with $\dim(M) \geq n$ and $\dim(M_j) > 1$ for some $j \in C_i$.

Write $C_i = \{1 < \cdots < j < \cdots < v\}$. Since $\dim(M_j) > 1$, $M_j = M_{j_1} \cup M_{j_2}$ is the disjoint union of non-zero column blocks $M_{j_1}$, $M_{j_2}$ with $\dim(M_{j_i}) \geq 1$. Define $N = (N_r \mid r \in T)$, where $N_r = M_r$ if $r \neq j$, $N_j = M_{j_1}$, and $N_t = M_{j_2}$. Then $N$ is a sincere matrix in $\mathrm{Mat}(T, \mathbb{Z}_{p^m})$ with $\dim(N) = \dim(M) \geq n$. Moreover, $N$ is indecomposable since allowable row and column operations decomposing $N$ induce allowable row and column operation decomposing $M$ and $M$ is indecomposable.

*Case* III. $S_3 \subseteq S$ and, for some $i$, $C_i \cup \{t\}$ is a chain contained in $T$.

Let $n$ be a natural number. By **[2]**, there is a sincere indecomposable

$$M = \begin{pmatrix} I_{n_T} & \Big\| & I_{n_T} & \Big\| & A \\ 0 & \Big\| & pI_{n_T} & \Big\| & pI_{n_T} \end{pmatrix}$$

in $\mathrm{Mat}(S_3, \mathbb{Z}_{p^2})$ with $\dim(M) = 3n \geq n$. Repeated applications of Cases I and II yield a sincere indecomposable $N$ in $\mathrm{Mat}(T, \mathbb{Z}_{p^m})$ with $\dim(N) \geq \dim(M) \geq n$.

*Case* IV. $S = C_1 \cup C_2$ is the disjoint union of two chains and, for some $i$, $C_i \cup \{t\}$ is a chain contained in $T$.

Let $n > |S|^2$ be a natural number and $M = (M_s \mid s \in S)$ a sincere indecomposable in $\mathrm{Mat}(S, \mathbb{Z}_{p^m})$ with $r$ rows and $\dim(M) \geq n$.

Assume, by way of contradiction, that $\dim(M_j) = 1$ for each $j \in C_i$. Then $\sum_{j \in C_i} \dim(M_j) = |C_i|$ is the number of elements in the chain $C_i$. Given $s \in S$, column operations on $M_s$ reduce $M_s$ to a column echelon form

$$M_s = \begin{pmatrix} * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & x_k & 0 \\ * & * & * & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & x_3 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ * & x_2 & 0 & \cdots & 0 & 0 \\ * & 0 & 0 & \cdots & 0 & 0 \\ x_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

To see this, choose a non-zero element $x_1$ of least $p$-height in the first non-zero row from the bottom of $M_s$ and annihilate to the right to obtain a matrix of the form

$$\begin{pmatrix} * & * & \cdots & * \\ x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Induction leads to the desired form. It now follows that $\dim(M_s) \leq r$ because $M_s$ has no zero columns.

Moreover, $r \leq |C_i|$ since, if $r > |C_i|$, then row and column operations on $M$ lead to a zero row in $(M_j \mid j \in C_i)$, contradicting condition (iii) on $M$. Consequently, $\dim(M) = \sum_{s \in S} \dim(M_s) \leq |S|r \leq |S||C_i| \leq |S|^2 < n$, a contradiction.

Hence, Case II applies to yield a sincere indecomposable $N$ in $\mathrm{Mat}(T, \mathbb{Z}_{p^m})$ with $\dim(N) \geq n$.

Induction on $|S|$ now completes the proof.

(b) For Case III, $\mathrm{uMat}(S_3, p^2)$ is bounded but there is an $M$ in $\mathrm{uMat}(S_3, p^3)$ with $\dim(M) = 3n$, **[6]**. Clearly, in Cases I and II, if $M$ is a sincere indecomposable in $\mathrm{uMat}(S, p^m)$, then $N$ is a sincere indecomposable in $\mathrm{uMat}(T, \mathbb{Z}_{p^m})$. □

**Example 5.3.**

(a) (Case I). Let $S = (1, 3) \subset T = (1, 3, 1)$, and let $A$ be an $n \times n$ module indecomposable matrix. Then

$$M = \left(\begin{array}{cc||c|c|c} I_n & 0 & p^2 I_n & pI_n & I_n \\ 0 & p^2 I_n & p^3 A & p^2 I_n & 0 \end{array}\right)$$

is a sincere indecomposable in $\mathrm{Mat}((1, 3), \mathbb{Z}_{p^4})$ with dimension $5n$ and

$$N = \left(\begin{array}{cc||c|c|c||cc} I_n & 0 & p^2 I_n & pI_n & I_n & I_n & 0 \\ 0 & p^2 I_n & p^3 A & p^2 I_n & 0 & 0 & p^2 I_n \end{array}\right)$$

is a sincere indecomposable in $\mathrm{Mat}((1, 3, 1), \mathbb{Z}_{p^4})$ with dimension $7n$.

(b) (Case II). Let $S = (1, 3) = (1 \| 2 < 3 < 4) \subset T = (1, 4) = \{1 \| 2 < 3 < 5 < 4\}$. If $A$ is an $n \times n$ module indecomposable matrix, then

$$M = (M_1 \| M_4 | M_3 | M_2) = \left(\begin{array}{cc||c|c|c} I_n & 0 & p^2 I_n & pI_n & I_n \\ 0 & p^2 I_n & p^3 A & p^2 I_n & 0 \end{array}\right)$$

is a sincere indecomposable in $\mathrm{Mat}((1,3),\mathbb{Z}_{p^4})$ with dimension $5n$. Write

$$M_3 = \begin{pmatrix} pI_n \\ p^2 I_n \end{pmatrix} = (M_{31} \big| M_{32}) = \left(\begin{array}{c|c} pI_{n_1} & 0 \\ 0 & pI_{n_2} \\ p^2 I_{n_1} & 0 \\ 0 & p^2 I_{n_2} \end{array}\right)$$

with $n = n_1 + n_2$. Then

$$N = (N_1 \big\| N_4 \big| N_5 \big| N_3 \big| N_2) = (M_1 \big\| M_4 \big| M_{31} \big| M_{32} \big| M_2)$$

is a sincere indecomposable in $\mathrm{Mat}((1,4),\mathbb{Z}_{p^4})$ with dimension $5n$.

## 6. Bounded representation type.

**Theorem 6.1.**
(a) $\mathrm{Rep}(S,\mathbb{Z}_{p^m})$ *is bounded in the following cases*:
  (i) **[18]**. $S = (1,2)$, $3 \le m \le 4$.
  (ii) **[3]**. $S = (1,3)$, $m = 3$.
  (iii) **[24]**. $S = (2,2)$, $m = 2$.
(b) $\mathrm{uRep}(S,\mathbb{Z}_{p^m})$ *is bounded in the following cases*:
  (i) **[6]**. $S = (1,1,1)$, $m = 2$.
  (ii) [**9,** Example 5.3.3]. $S = (1,2)$, $2 \le m$.
  (iii) **[5]**. $S = (1,n)$, $n \ge 3$, $m = 2$.
  (iv) **[4]**. $S = (1,3)$, $2 \le m \le 4$.
  (v) **[7]**. $S \subseteq (2,2)$, $2 \le m \le 3$.

*In each case, there is a complete list, up to isomorphism, of indecomposable sincere representations.*

*Proof.* Each reference in (a) and (b) for the specified $S$ and $m$, contains a complete finite list of indecomposable $(S,p^m)$-groups, respectively indecomposable homocyclic $(S,p^m)$-groups. By Theorem 5.1, indecomposable $(S,p^m)$-groups correspond to sincere indecomposable representations in $\mathrm{Rep}(S,\mathbb{Z}_{p^m})$, and homocyclic indecomposable $(S,p^m)$-groups correspond to sincere indecomposable representations in uRep $(S,\mathbb{Z}_{p^m})$. $\square$

**Theorem 6.2.** *Up to equivalence, the indecomposables in* $\mathrm{Mat}((1,n),\mathbb{Z}_{p^2})$ *are* $(1\big\|1)$ *and* $(1\big\|p\big|1)$. *Consequently,* $\mathrm{Rep}((1,n),\mathbb{Z}_{p^2})$ *is bounded.*

*Proof.* Let $M = (M_1 \| M_2)$ be a sincere indecomposable in $\mathrm{Mat}((1,n), \mathbb{Z}_{p^2})$ with $\mathrm{rsp}(M) \cong \mathbb{Z}_{p^2}^{l_1} \oplus \mathbb{Z}_p^{l_2}$. Use row and column operations to reduce to

$$M = \left(\begin{matrix} I_{l_1} & 0 \\ 0 & pI_{l_2} \end{matrix} \;\middle\|\; \begin{matrix} M_1 \\ pN \end{matrix}\right).$$

A row or column transformation is called *restorable* if the entries that became nonzero in the process can be made zero again by transformations that do not affect the achievements of the restorable transformation.

As in **[5]**, restorable row and column operations on the top row block of $M$ can be used to show that $M_1$ has an embedded identity matrix, a rest matrix $pB$ that is a permutation of a matrix of the form

$$\begin{pmatrix} pI & 0 \\ 0 & 0 \end{pmatrix},$$

and the columns of $pB \cup I_n$ are a permutation of the columns of $M_1$. Then

$$M = \left(\begin{matrix} I_{l_1} & 0 \\ 0 & pI_{l_2} \end{matrix} \;\middle\|\; \begin{matrix} pB \cup I \\ pC \cup pD \end{matrix}\right)$$

where the columns of $\begin{pmatrix} pB & pC \end{pmatrix}^{\mathrm{tr}} \cup \begin{pmatrix} I & pD \end{pmatrix}^{\mathrm{tr}}$ are permutations of the columns of $\begin{pmatrix} M_1 & pN \end{pmatrix}^{\mathrm{tr}}$.

If the bottom row block of $M$ is empty, then $M$ is equivalent to $(1\|1)$ or $(1\|p|1)$ because $M$ is indecomposable.

Assume the bottom row block of $M$ is not empty and use restorable row operations from $I$ in the top row block to $pD$ in the bottom row block of $M$ to get

$$M = \left(\begin{matrix} I & 0 \\ 0 & pI \end{matrix} \;\middle\|\; \begin{matrix} pB \cup I \\ pE \cup 0 \end{matrix}\right).$$

The matrix $pE$ has an embedded identity matrix $pI$ with 0 as a rest matrix. By the Regulator condition on $M$, $\begin{pmatrix} pI & 0 \end{pmatrix}$ has no zero rows so that

$$M = \left(\begin{matrix} I & 0 \\ 0 & pI \end{matrix} \;\middle\|\; \begin{matrix} (pF_1 \; pF_2) \cup I \\ (pI \; 0) \cup 0 \end{matrix}\right).$$

Use restorable row operations from the bottom row block to the upper row block of $M$ to obtain

$$M = \left(\begin{matrix} I & 0 \\ 0 & pI \end{matrix} \;\middle\|\; \begin{matrix} (0\ pF_2) \cup I \\ (pI\ 0) \cup 0 \end{matrix}\right).$$

Thus, $M$ is equivalent to $\big(I \| pF_2 | I\big) \oplus \big(pI \| pI | 0)\big)$. Since $M$ is indecomposable, $M$ is equivalent to $(1\|1)$ or $(1\|p|1)$. By Lemma 2.3 and Lemma 3.1, $\operatorname{Rep}((1,n),\mathbb{Z}_{p^2})$ is bounded. $\square$

**Corollary 6.3.** *Assume $S$ is a disjoint union of chains.*

(a) *The only cases for which* $\operatorname{Rep}(S,\mathbb{Z}_{p^m})$ *is not known to be bounded or unbounded are*:
  (i) $S = (1,2)$, $m = 5$.
  (ii) $S = (1,4)$, $m = 3$.
  (iii) $S = (2,n)$, $3 \le n \le 4$, $m = 2$.

(b) *The only cases for which* $\operatorname{uRep}(S,\mathbb{Z}_{p^m})$ *is not known to be bounded or unbounded are*:
  (i) $S = (1,3)$, $m = 5$.
  (ii) $S = (1,n)$, $4 \le n \le 5$, $m = 3$.
  (iii) $S = (2,n)$, $3 \le n \le 4$, $m = 2$.

*Proof.*

(a) The cases of $(S,m)$ left unresolved by Theorem I are:
$S = (1,n)$, $2 \le n$, $m = 2$;
$S = (1,2)$, $3 \le m \le 5$;
$S = (1,3)$, $m = 3$;
$S = (1,4)$, $m = 3$;
$S = (2,2)$, $m = 2$;
$S = (2,n)$, $3 \le n \le 4$, $m = 2$.

By Lemma 4.2, Theorem 6.2 and Theorem 6.1 (a), $\operatorname{Rep}(S,\mathbb{Z}_{p^m})$ is bounded for $S \subseteq (1,n)$, $m = 2$; $S \subseteq (1,2)$, $3 \le m \le 4$; $S \subseteq (1,3)$, $m = 3$; and $S \subseteq (2,2)$, $m = 2$. The only remaining unresolved cases are those listed in (a), (i)–(iii).

(b) The cases of $(S,m)$ left unresolved in Theorem II are*:*
$S = (1,1,1)$, $m = 2$;
$S = (1,2)$, $2 \le m$;
$S = (1,n)$, $n \ge 3$, $m = 2$;
$S = (1,3)$, $2 \le m \le 5$;
$S = (1,n)$, $4 \le n \le 5$, $m = 3$;

$S = (2, 2)$, $2 \le m \le 3$;
$S = (2, n)$, $3 \le n \le 4$, $m = 2$.

In view of Lemma 4.2, Theorem 6.1 (b) and Theorem 6.2, uRep $(S, \mathbb{Z}_{p^m})$ is bounded for $S \subseteq (1, n)$, $m = 2$; $S \subseteq (1, 1, 1)$, $m = 2$; $S \subseteq (1, 2)$, $2 \le m$; $S \subseteq (1, n)$, $n \ge 3$, $m = 2$; $S \subseteq (1, 3)$, $2 \le m \le 4$; $S \subseteq (2, 2)$, $2 \le m \le 3$. The only remaining unresolved cases are as listed in (b), (i)–(iii). □

**APPENDIX.** We show here that a matrix representing an automorphism of a finite abelian $p$-group or a representation is a product of elementary matrices and therefore its action is a sequence of elementary transformations.

**(a)** First, let $A$ be a homocyclic group of exponent $p^m$, or equivalently a finite free $\mathbb{Z}_{p^m}$-module. Let $M = \big(m_{ij}\big)$ be the matrix of $\alpha \in \operatorname{Aut} A$ with respect to some basis of $A$. Depending on conventions, we are allowed arbitrary row transformations or arbitrary column transformation. We choose row transformations, the case of column transformations being analogous or settled by transposition. Since $M$ is invertible, $\det(M) = 1$. There must be a unit in the first column that we can move to the first row and turn into a 1. By adding suitable multiples of the first row to rows below, we get the matrix $\left(\begin{smallmatrix} 1 & m \\ 0 & M' \end{smallmatrix}\right)$. Now $\det M' = 1$, and we may assume that there is an entry 1 in position $(2, 2)$. With row transformation, we obtain $\left(\begin{smallmatrix} 1 & 0 & m \\ 0 & 1 & m \\ 0 & 0 & M'' \end{smallmatrix}\right)$. Continuing in this fashion, we get the identity matrix which means that the original matrix $M$ is a product of elementary matrices corresponding to the elementary transformations used.

**(b)** Next, let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$ be an arbitrary finite abelian $p$-group where $A_i = 0$ or $A_i$ is homocyclic of exponent $p^i$, $1 \le i \le m$. By [**17,** Theorem 3.2], every automorphism of $A$ can be identified with a matrix

$$M = \begin{pmatrix} \mu_{11} & \mu_{21} & \cdots & \mu_{m1} \\ \mu_{12} & \mu_{22} & \cdots & \mu_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{1m} & \mu_{2m} & \cdots & \mu_{mm} \end{pmatrix}$$

where $\mu_{ij} \in \operatorname{Hom}(A_i, A_j)$. The action is given by $M\,a^{\downharpoonleft}$ where $a^{\downharpoonleft} = (a_1, \ldots, a_m)^{\mathrm{tr}}$ for $a_i \in A_i$. This action can be interpreted as matrix multiplication and the composite of two endomorphism is the product matrix. The endomorphism $M = (\mu_{ij})$ is an automorphism if and only if the $\mu_{ii} \in \mathrm{Hom}\,(G_i, G_i)$ are automorphisms for all $i$.

Let such a matrix $M$ be given. Then $\mu_{11}$ is invertible and can be transformed to the identity $I_1$ by multiplying row 1 by $\mu_{11}^{-1}$. By elementary row transformations or left multiplication by elementary matrices we get (note that maps act on the left, so that $\mu_{jk}\mu_{ij} \in \mathrm{Hom}\,(A_i, A_k)$)

$$\begin{pmatrix} I_1 & 0 & \cdots & 0 \\ -\mu_{12} & I_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m \end{pmatrix} \begin{pmatrix} I_1 & \mu_{21} & \cdots & \mu_{m1} \\ \mu_{12} & \mu_{22} & \cdots & \mu_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{1m} & \mu_{2m} & \cdots & \mu_{mm} \end{pmatrix}$$
$$= \begin{pmatrix} I_1 & \mu_{21} & \cdots & \mu_{m1} \\ 0 & \mu_{22} - \mu_{12}\mu_{21} & \cdots & \mu_{m2} - \mu_{12}\mu_{m1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{1m} & \mu_{2m} & \cdots & \mu_{mm} \end{pmatrix}$$

Repeating this process for the other rows we obtain the form

$$\begin{pmatrix} I_1 & \mu_{21} & \cdots & \mu_{m1} \\ 0 & \mu'_{22} & \cdots & \mu'_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \mu'_{2m} & \cdots & \mu'_{mm} \end{pmatrix}.$$

This matrix represents again an automorphism and therefore $\mu'_{22}$ is invertible and can be transformed to the identity matrix $I_2$. Using it, the remaining entries in the second column can be made 0 and, continuing in this fashion, we get the identity matrix. It remains to observe that by introducing bases in the summands $A_i$ the matter can be reduced to numerical matrices.

**(c)** Let $U' = (U_0, U_s, U_* \mid s \in S) \in \mathrm{cdRep}(S, \mathbb{Z}_{p^m})$, and consider $U = (U_0, U_s \mid s \in S)$. Here $U_0 = \bigoplus_{s \in S} V_s$ is a free module. Let $\mathcal{B}_s$ be a basis of $V_s$. Then $\mathcal{B} = \bigoplus_{s \in S} \mathcal{B}_s$ is a basis of $U_0$. For $x \in U_0$, $x = \sum_{v \in \mathcal{B}} m_v v$ with unique coefficients $m_v \in \mathbb{Z}_{p^m}$. Then the row vector $(x)_{\mathcal{B}} = (\cdots m_v \cdots)$ is the coordinate vector of $x$ with respect to the basis $\mathcal{B}$. Let $\alpha \in \mathrm{Aut}\, U$. Then $(\alpha)_{\mathcal{B}} = \begin{pmatrix} \cdots \\ (\alpha(v))_{\mathcal{B}} \\ \cdots \end{pmatrix}_{v \in \mathcal{B}}$ is the matrix of $\alpha$ with respect to $\mathcal{B}$. The equation $(\alpha(x))_{\mathcal{B}} = (x)_{\mathcal{B}}(\alpha)_{\mathcal{B}}$ displays the connection between automorphisms and their matrices. So $(\alpha)_{\mathcal{B}}$ acts on row vectors by right multiplication. Also $(\alpha\beta)_{\mathcal{B}} = (\beta)_{\mathcal{B}}(\alpha)_{\mathcal{B}}$.

The group $\operatorname{Aut} U$ is a proper subgroup of $\operatorname{Aut} U_0$ and the elementary matrices whose product is $(\alpha)$ must themselves be the matrices of representation automorphisms.

We will look at matrices $(\alpha)_{\mathcal{B}}$ in terms of the column blocks $M_s$ and row blocks

$$(\alpha(\mathcal{B}_s)) = \begin{pmatrix} \cdots \\ (\alpha(v))_{\mathcal{B}} \\ \cdots \end{pmatrix}_{v\in\mathcal{B}_s} = (\cdots m_s \cdots m_t \cdots), \tag{6.1}$$

where $m_s$ and $m_t$ are the intersections of the row block $(\alpha(\mathcal{B}_s))$ with the column blocks $M_s, M_t$, respectively. The fact that $\alpha(U_s) \subseteq U_s$ implies that

$$m_t = 0 \quad \text{if } s \not\leq t, \tag{6.2}$$

or, equivalently,

$$m_t \neq 0 \quad \text{implies } s \leq t.$$

Now suppose that $s$ is minimal in $S$. It then follows from the minimality of $s$ and (6.2) that in $(\alpha)_{\mathcal{B}}$ there are only zeros above and below the block $m_s$. Then $\det(m_s)$ must be a unit and $m_s$ can be transformed into an identity matrix $I_s$ by allowed elementary transformations by the special case considered above. All non-zero entries $m_t$ in the new row block $(\cdots I_s \cdots m_t \cdots)$ must have $t \geq s$ and therefore can be made to 0 by allowed column transformations. Thus, for all minimal $s \in S$, we have crosses

$$\begin{pmatrix} & 0 & \\ 0 & I_s & 0 \\ & 0 & \end{pmatrix}.$$

Ignoring these rows and columns, we can proceed as before with the rest of the matrix that is indexed by $S$ less its minimal elements. By iteration, we arrive at an identity matrix, which means that the original matrix $(\alpha)_{\mathcal{B}}$ is a product of elementary matrices.

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