

ON THE RINGS OF FRICKE CHARACTERS OF FREE ABELIAN GROUPS

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ABSTRACT. In this paper, we determine the structure of the rings $\mathfrak{X}_{\mathbf{Q}}(H)$ of Fricke characters over \mathbf{Q} of free abelian groups H of rank $n \geq 1$. In particular, we consider the ideal J in $\mathfrak{X}_{\mathbf{Q}}(H)$, generated by $\mathrm{tr} x - 2$ for any $x \in H$ and give a \mathbf{Q} -basis of each of the graded quotients $\mathrm{gr}^k(J) := J^k/J^{k+1}$ for $k \geq 1$. Then we introduce a weight for each element of $\mathfrak{X}_{\mathbf{Q}}(H)$. By using the concept of this weight we show that $\mathfrak{X}_{\mathbf{Q}}(H)$ is an integral domain.

1. Introduction. Let G be a group generated by x_1, \dots, x_n . We denote by $R(G)$ the set $\mathrm{Hom}(G, \mathrm{SL}(2, \mathbf{C}))$ of all $\mathrm{SL}(2, \mathbf{C})$ -representations of G . Let $\mathcal{F}(R(G), \mathbf{C})$ be the set $\{\chi : R(G) \rightarrow \mathbf{C}\}$ of all complex-valued functions on $R(G)$. Then we can regard $\mathcal{F}(R(G), \mathbf{C})$ to be a \mathbf{C} -algebra in a natural way. (See subsection 3.1 for details.) For an element $x \in G$, $\mathrm{tr} x \in \mathcal{F}(R(G), \mathbf{C})$ is defined to satisfy the equation

$$(\mathrm{tr} x)(\rho) = \mathrm{tr} \rho(x)$$

for any $\rho \in R(G)$. Here “tr” in the right hand side means the trace of 2×2 matrix $\rho(x) \in \mathrm{SL}(2, \mathbf{C})$. The element $\mathrm{tr} x$ in $\mathcal{F}(R(G), \mathbf{C})$ is called the *Fricke character* of x . Let $\mathfrak{X}_{\mathbf{Q}}(G)$ be the \mathbf{Q} -vector subspace in $\mathcal{F}(R(G), \mathbf{C})$, which is generated by all $\mathrm{tr} x$ for $x \in G$. Then $\mathfrak{X}_{\mathbf{Q}}(G)$ has a ring structure with the multiplication of $\mathcal{F}(R(G), \mathbf{C})$. We call $\mathfrak{X}_{\mathbf{Q}}(G)$ the ring of Fricke characters of G over \mathbf{Q} .

Classically, the study of Fricke characters was begun by Fricke for a free group F_n generated by x_1, \dots, x_n in connection with certain

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problems in the theory of Riemann surfaces. (See [3].) In the 1970s, Horowitz investigated algebraic properties of $\mathfrak{X}(G)$ by using the combinatorial group theory in [6, 7]. In particular, he showed in [6] that, for any $x \in G$, the Fricke character $\mathrm{tr} x$ can be written as a rational polynomial in $n + \binom{n}{2} + \binom{n}{3}$ characters $\mathrm{tr} x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq 3$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Hence, when we put \mathfrak{P} to be a polynomial ring

$$\mathbf{Q}[t_{i_1 \dots i_l} \mid 1 \leq l \leq 3, \quad 1 \leq i_1 < i_2 < \cdots < i_l \leq n],$$

then there exists a surjective homomorphism $\pi_G : \mathfrak{P} \rightarrow \mathfrak{X}_{\mathbf{Q}}(G)$ defined by $t_{i_1 \dots i_l} \mapsto \mathrm{tr} x_{i_1} x_{i_2} \cdots x_{i_l}$. The study of the ring structure of $\mathfrak{X}_{\mathbf{Q}}(G)$ is inextricably associated with to that of the ideal $\mathrm{Ker}(\pi_G)$. In general, however, it is quite a difficult problem to find a generating set of $\mathrm{Ker}(\pi_G)$ even in the case that G is a free group. (See also subsection 3.1.)

In the case that G is an *abelian* group, it is easy to see that any Fricke character of G is written as a rational polynomial in $n + \binom{n}{2}$ characters $\mathrm{tr} x_i$ for $1 \leq i \leq n$ and $\mathrm{tr} x_{i_1} x_{i_2}$ for $1 \leq i_1 < i_2 \leq n$. Hence, when we put \mathcal{P} as a polynomial ring

$$\mathbf{Q}[t_i, t_{i_1 i_2} \mid 1 \leq i \leq n, \quad 1 \leq i_1 < i_2 \leq n],$$

then there exists the surjective homomorphism $\bar{\pi}_G : \mathcal{P} \rightarrow \mathfrak{X}_{\mathbf{Q}}(G)$ defined by $t_i \mapsto \mathrm{tr} x_i$ and $t_{i_1 i_2} \mapsto \mathrm{tr} x_{i_1} x_{i_2}$. We denote by I the kernel of $\bar{\pi}_G$.

In the present paper, we consider the case where G is a free abelian group $H := H_1(F_n, \mathbf{Z})$ of rank n . First, we introduce a descending filtration in $\mathfrak{X}_{\mathbf{Q}}(H)$. Set $t'_{i_1 \dots i_l} := t_{i_1 \dots i_l} - 2 \in \mathcal{P}$. We also denote by $t'_{i_1 \dots i_l}$ its coset class in $\mathcal{P}/I \cong \mathfrak{X}_{\mathbf{Q}}(H)$ by abuse of language. Consider the ideal J_0 in \mathcal{P} generated by all $t'_{i_1 \dots i_l}$ s. That is,

$$J_0 = (t'_i, t'_{i_1 i_2} \mid 1 \leq i \leq n, \quad 1 \leq i_1 < i_2 \leq n) \subset \mathcal{P}.$$

Set $J := \bar{\pi}_H(J_0)$. Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of ideals in \mathcal{P}/I . (See subsection 3.1 for details.) Set $\mathrm{gr}^k(J) := J^k/J^{k+1}$ for $k \geq 1$. Each $\mathrm{gr}^k(J)$ is a \mathbf{Q} -vector space of finite dimension on which $\mathrm{Aut} H$ naturally acts. In general, to determine the structures of the graded quotients $\mathrm{gr}^k(J)$ plays an important role on various studies of

the ring \mathcal{P}/I . The first purpose of this paper is to give a basis of $\mathrm{gr}^k(J)$ for any $k \geq 1$. More precisely, we show the following theorem.

Theorem 1.1 (equivalent to Theorems 4.2 and 4.5).

(i) For each $k \geq 1$ and $0 \leq l \leq k$, set

$$T_l := \{t'_{p_1 q_1} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in J_0 \\ | 1 \leq p_1 < q_1 < \cdots < p_l < q_l \leq n, 1 \leq i_{l+1} \leq \cdots \leq i_k \leq n\}.$$

Then

$$S_k := \bigcup_{l=0}^k \bar{\pi}_H(T_l)$$

forms a basis of $\mathrm{gr}^k(J)$.

(ii)

$$\bigcap_{k \geq 1} J^k = \{0\}.$$

The main purpose of this paper is to give an application to our previous work for the Fricke characters of free groups. Let us explain it. Needless to say, by an argument similar to the above, we can define the ideal J_{F_n} in the ring $\mathfrak{X}_{\mathbf{Q}}(F_n)$ of Fricke characters of the free group F_n generated by all $t'_{i_1 \dots i_l} := t_{i_1 \dots i_l} - 2$ for $1 \leq l \leq 3$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Then we have a descending filtration $J_{F_n} \supset J_{F_n}^2 \supset \cdots$ in $\mathfrak{X}_{\mathbf{Q}}(F_n)$ and the graded quotients $\mathrm{gr}^k(J_{F_n}) := J_{F_n}^k / J_{F_n}^{k+1}$ for each $k \geq 1$. Such graded quotients were originally studied by Magnus [13] to investigate the behavior of the action of $\mathrm{Aut} F_3$ on $\mathrm{gr}^k(J_{F_n})$. In [5], we gave bases of the graded quotients $\mathrm{gr}^k(J_{F_n}) := J_{F_n}^k / J_{F_n}^{k+1}$ for $k = 1$ and 2 . In general, however, it seems that no basis of $\mathrm{gr}^k(J_{F_n})$ is obtained for $k \geq 4$. From Theorem 1.1, we can give a lower bound on a dimension of $\mathrm{gr}^k(J_{F_n})$ since the natural projection $F_n \rightarrow H$ induces the surjective homomorphism $\mathrm{gr}^k(J_{F_n}) \rightarrow \mathrm{gr}^k(J)$. More precisely, we have

Corollary 1.2. For any $n \geq 2$,

$$\dim_{\mathbf{Q}}(\mathrm{gr}^k(J_{F_n})) \geq \dim_{\mathbf{Q}}(\mathrm{gr}^k(J)) = \sum_{l=0}^k \binom{n}{2l} \binom{n+k-l-1}{k-l}.$$

For any $k \geq 1$, let $\mathcal{E}_{F_n}(k)$ be the subgroup of $\text{Aut } F_n$, consisting of automorphisms which act on $J_{F_n}/J_{F_n}^{k+1}$ trivially. Then we have a descending filtration $\{\mathcal{E}_{F_n}(k)\}_{k \geq 1}$ of $\text{Aut } F_n$. This filtration is a Fricke character analogue of the Andreadakis-Johnson filtration $\{\mathcal{A}_{F_n}(k)\}_{k \geq 1}$ of $\text{Aut } F_n$. The Andreadakis-Johnson filtration was originally introduced by Andreadakis [2] in the 1960's. The name "Johnson" comes from Dennis Johnson who studied this type of filtration for the mapping class group of a surface in the 1980's. It is called the Johnson filtration of the mapping class group. The Johnson homomorphisms were originally introduced by Johnson in order to investigate the graded quotients of the Johnson filtration in a series of his pioneering works [8, 9, 10, 11]. In [14], Morita began to study the Johnson homomorphisms of the mapping class groups and $\text{Aut } F_n$ systematically. Today, together with the theory of the Johnson homomorphisms, the Andreadakis-Johnson filtration is one of the powerful tools to study the group structure of the automorphism group of a group. (For basic material for the Andreadakis-Johnson filtration and the Johnson homomorphisms of $\text{Aut } F_n$, see [15, 16], for example.) In our previous paper [5], we also studied the graded quotients $\text{gr}^k(\mathcal{E}_{F_n}) := \mathcal{E}_{F_n}(k)/\mathcal{E}_{F_n}(k+1)$, by introducing and using a Johnson homomorphism-like homomorphism

$$\eta_k : \text{gr}^k(\mathcal{E}_{F_n}) \longrightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(J_{F_n}), \text{gr}^{k+1}(J_{F_n}))$$

for each $k \geq 1$. In general, to determine the image of η_k is quite a difficult problem. Since η_k vanishes through the homomorphism

$$\text{Hom}_{\mathbf{Q}}(\text{gr}^1(J_{F_n}), \text{gr}^{k+1}(J_{F_n})) \longrightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(J_{F_n}), \text{gr}^{k+1}(J))$$

induced from the homomorphism $\text{gr}^{k+1}(J_{F_n}) \rightarrow \text{gr}^{k+1}(J)$ obtained from the natural projection $F_n \rightarrow H$, we can obtain information about the cokernel of η_k by Theorem 1.1.

Corollary 1.3. *For any $n \geq 3$ and $k \geq 2$,*

$$\dim_{\mathbf{Q}}(\text{Coker}(\eta_k)) \geq \left(n + \binom{n}{2} + \binom{n}{3} \right) \sum_{l=0}^{k+1} \binom{n}{2l} \binom{n+k-l}{k+1-l}.$$

In [5], we also showed that $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$ for each $k \geq 1$. Hence, the natural inclusion map and η_k define the homomorphism

$$\mathcal{A}_{F_n}(2k)/\mathcal{A}_{F_n}(2k+2) \rightarrow \text{gr}^k(\mathcal{E}_{F_n}) \xrightarrow{\eta_k} \text{Hom}_{\mathbf{Q}}(\text{gr}^1(J_{F_n}), \text{gr}^{k+1}(J_{F_n}))$$

for each $k \geq 1$. At the present stage, we do not know whether the above homomorphism is injective or not. To the best of our knowledge, other than this, it seems quite difficult to develop the connection of the ring of Fricke characters to the Lie algebra arising from the Andreadakis-Johnson filtration.

Now, by using Theorem 1.1, we can obtain the following.

Theorem 1.4 (equivalent to Theorem 4.3). *The ideal I is generated by*

$$\begin{aligned} & t'_{ir}t'_{js} - t'_{is}t'_{jr} - \{t'_it'_r + t'_jt'_s - t'_jt'_r - t'_it'_s\} \\ & + \{t'_it'_{jr} + t'_jt'_{is} + t'_rt'_{is} + t'_st'_{jr} - t'_jt'_{ir} - t'_it'_{js} - t'_rt'_{js} - t'_st'_{ir}\} \\ & + \frac{1}{2}\{t'_jt'_rt'_{is} + t'_it'_st'_{jr} - t'_it'_rt'_{js} - t'_jt'_st'_{ir}\} \end{aligned}$$

for any $1 \leq i, j, r, s \leq n$. Here we remark that, in the above notation, t'_{ij} should be read

$$\begin{cases} t'_{ji} & \text{if } i > j, \\ (t'_i)^2 + 4t'_i & \text{if } i = j. \end{cases}$$

In particular, I is finitely generated.

Furthermore, from Theorem 1.1, we see that the ideal I is generated by polynomials of degree greater than one. This fact enables us to define the concept of a weight for each element of $\mathcal{P}/I \cong \mathfrak{X}_{\mathbf{Q}}(H)$. Using this, we show

Theorem 1.5 (equivalent to Theorem 5.2). *The ring $\mathfrak{X}_{\mathbf{Q}}(H)$ of Fricke characters is an integral domain. That is, the ideal I is a prime ideal in \mathcal{P} .*

Finally, in Section 6, we will give some remarks on the natural action of $\text{Aut } H$ on J/J^{k+1} for $k \geq 1$. Let $\mathcal{E}_H(k)$ be the kernel of a natural homomorphism $\text{Aut } H \rightarrow \text{Aut}(J/J^{k+1})$ induced from the action of $\text{Aut } G$. Let $\iota \in \text{Aut } H$ be an automorphism of H defined by

$$x_i^\iota := x_i^{-1}, \quad 1 \leq i \leq n.$$

Then we show the following.

Proposition 1.6 (equivalent to Proposition 6.1 and Corollary 6.2). *For any $k \geq 1$, the group $\mathcal{E}_H(k)$ is the cyclic group of order 2, generated by ι .*

2. Notation and conventions. Throughout the paper, we use the following notation and conventions. Let F_n be the free group of rank n with a basis x_1, \dots, x_n , and let H be its abelianization $H_1(F_n, \mathbf{Z})$. Then H is a free abelian group of rank n , and the coset classes of x_1, \dots, x_n form a basis of H as a free abelian group. We also use the following notation.

- Let G be a group. The automorphism group $\text{Aut } G$ of G acts on G from the right. For any $\sigma \in \text{Aut } G$ and $x \in G$, the action of σ on x is denoted by x^σ .
- Let N be a normal subgroup of a group G . For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion. Similarly, for a ring R , an element $f \in R$ and an ideal I of R , we also denote by f the coset class of f in R/I if there is no confusion.
- For elements x and y in G , the commutator bracket $[x, y]$ of x and y is defined to be $xyx^{-1}y^{-1}$.

For pairs (i_1, i_2, \dots, i_k) and (j_1, j_2, \dots, j_k) of natural numbers $i_r, j_s \in \mathbf{N}$, we denote the lexicographic order among them by $(i_1, i_2, \dots, i_k) \leq (j_1, j_2, \dots, j_k)$. Namely, this means $i_1 < j_1$, $i_1 = j_1$ and $i_2 < j_2$, or and so on.

3. A filtration on the ring of Fricke characters of H . In this section, we briefly review the ring of Fricke characters of a finitely generated group G .

3.1. The ring $\mathfrak{X}_{\mathbf{Q}}(G)$ of Fricke characters of a finitely generated group G . Let G be a group generated by x_1, \dots, x_n . We denote by $R(G)$ the set $\text{Hom}(G, \text{SL}(2, \mathbf{C}))$ of all $\text{SL}(2, \mathbf{C})$ -representations of G . Let $\mathcal{F}(R(G), \mathbf{C})$ be the set $\{\chi : R(G) \rightarrow \mathbf{C}\}$ of all complex-valued functions on $R(G)$. Then $\mathcal{F}(R(G), \mathbf{C})$ has a \mathbf{C} -algebra structure by the operations defined by

$$\begin{aligned} (\chi + \chi')(\rho) &:= \chi(\rho) + \chi'(\rho), \\ (\chi\chi')(\rho) &:= \chi(\rho)\chi'(\rho), \end{aligned}$$

$$(\lambda\chi)(\rho) := \lambda(\chi(\rho)),$$

for any $\chi, \chi' \in \mathcal{F}(R(G), \mathbf{C})$, $\lambda \in \mathbf{C}$ and $\rho \in R(G)$.

The automorphism group $\text{Aut } G$ of G naturally acts on $R(G)$ and $\mathcal{F}(R(G), \mathbf{C})$ from the right by

$$\rho^\sigma(x) := \rho(x^{\sigma^{-1}}), \quad \rho \in R(G) \quad \text{and} \quad x \in G$$

and

$$\chi^\sigma(\rho) := \chi(\rho^{\sigma^{-1}}), \quad \chi \in \mathcal{F}(R(G), \mathbf{C}) \quad \text{and} \quad \rho \in R(G),$$

for any $\sigma \in \text{Aut } G$.

For any $x \in G$, we define an element $\text{tr } x$ of $\mathcal{F}(R(G), \mathbf{C})$ to be

$$(\text{tr } x)(\rho) := \text{tr } \rho(x),$$

for any $\rho \in R(G)$. Here, “tr” in the right hand side means the trace of the 2×2 matrix $\rho(x)$. The element $\text{tr } x$ in $\mathcal{F}(R(G), \mathbf{C})$ is called the Fricke character of $x \in G$. The action of an element $\sigma \in \text{Aut } G$ on $\text{tr } x$ is given by $\text{tr } x^\sigma$. We have the following well-known formulae:

(3.1)

$$\text{tr } x^{-1} = \text{tr } x,$$

(3.2)

$$\text{tr } xy = \text{tr } yx,$$

(3.3)

$$\text{tr } xy + \text{tr } xy^{-1} = (\text{tr } x)(\text{tr } y),$$

(3.4)

$$\begin{aligned} \text{tr } xyz + \text{tr } yxz &= (\text{tr } x)(\text{tr } yz) + (\text{tr } y)(\text{tr } xz) \\ &\quad + (\text{tr } z)(\text{tr } xy) - (\text{tr } x)(\text{tr } y)(\text{tr } z), \end{aligned}$$

(3.5)

$$\text{tr } [x, y] = (\text{tr } x)^2 + (\text{tr } y)^2 + (\text{tr } xy)^2 - (\text{tr } x)(\text{tr } y)(\text{tr } xy) - 2,$$

(3.6)

$$\begin{aligned} 2\text{tr } xyzw &= (\text{tr } x)(\text{tr } yzw) + (\text{tr } y)(\text{tr } zwx) + (\text{tr } z)(\text{tr } wxy) \\ &\quad + (\text{tr } w)(\text{tr } xyz) + (\text{tr } xy)(\text{tr } zw) - (\text{tr } xz)(\text{tr } yw) + (\text{tr } xw)(\text{tr } yz) \\ &\quad - (\text{tr } x)(\text{tr } y)(\text{tr } zw) - (\text{tr } y)(\text{tr } z)(\text{tr } xw) - (\text{tr } x)(\text{tr } w)(\text{tr } yz) \\ &\quad - (\text{tr } z)(\text{tr } w)(\text{tr } xy) + (\text{tr } x)(\text{tr } y)(\text{tr } z)(\text{tr } w), \end{aligned}$$

for any $x, y, z, w \in G$. The equations (3.4) and (3.6) are due to Vogt [17]. (For details, see [12, subsection 3.4], for example.)

Let $\mathfrak{X}_{\mathbf{Q}}(G)$ be the \mathbf{Q} -vector subspace of $\mathcal{F}(R(G), \mathbf{C})$ generated by all $\text{tr } x$ for $x \in G$. The set $\mathfrak{X}_{\mathbf{Q}}(G)$ naturally has a ring structure from (3.3). We call $\mathfrak{X}_{\mathbf{Q}}(G)$ the ring of Fricke characters of G over \mathbf{Q} . Let \mathfrak{P} be a rational polynomial ring

$$\mathbf{Q}[t_{i_1 \dots i_l} \mid 1 \leq l \leq 3, 1 \leq i_1 < i_2 < \dots < i_l \leq n]$$

of $n + \binom{n}{2} + \binom{n}{3}$ indeterminates. Consider a ring homomorphism $\pi = \pi_G : \mathfrak{P} \rightarrow \mathcal{F}(R(G), \mathbf{C})$ defined by

$$\pi(1) := \frac{1}{2}(\text{tr } 1_G), \quad \pi(t_{i_1 \dots i_l}) := \text{tr } x_{i_1} \cdots x_{i_l}.$$

We see $\text{Im}(\pi) \subset \mathfrak{X}_{\mathbf{Q}}(G)$. By a classical result due to Horowitz, we have

Theorem 3.1 (Horowitz, [6]). *For any group G generated by x_1, \dots, x_n , the homomorphism $\pi : \mathfrak{P} \rightarrow \mathfrak{X}_{\mathbf{Q}}(G)$ is surjective.*

More precisely, Horowitz obtained a generating set of the ring of Fricke characters of G over \mathbf{Z} in [6]. Using this and (3.6), we can obtain the above theorem. We should remark that, in general, the structure of an ideal

$$\text{Ker}(\pi_G) = \{f \in \mathfrak{P} \mid f(\text{tr } \rho(x_{i_1} \cdots x_{i_l})) = 0 \text{ for any } \rho \in R(G)\}$$

is quite difficult. For example, its generating set is not obtained even in the case where G is a free group F_n in general. Horowitz [6] showed that $\text{Ker}(\pi_{F_n}) = (0)$ for $n = 1$ and 2 , and that $\text{Ker}(\pi_{F_3})$ is a principal ideal generated by a quadratic polynomial

$$t_{123}^2 - P_{123}(t)t_{123} + Q_{123}(t),$$

where

$$\begin{aligned} P_{abc}(t) &:= t_{ab}t_c + t_{ac}t_b + t_{bc}t_a - t_at_bt_c, \\ Q_{abc}(t) &:= t_a^2 + t_b^2 + t_c^2 + t_{ab}^2 + t_{ac}^2 + t_{bc}^2 - t_at_bt_{ab} \\ &\quad - t_at_ct_{ac} - t_bt_ct_{bc} + t_abt_bct_{ac} - 4. \end{aligned}$$

For $n \geq 4$, Whittemore [18] showed that the ideal $\text{Ker}(\pi_{F_n})$ is not principal. However, very little is known about $\text{Ker}(\pi_{F_n})$ for general $n \geq 4$.

3.2. The ring $\mathfrak{X}_{\mathbf{Q}}(G)$ for an abelian group G . Now, in the following, we always consider the case where G is *abelian*. In this case, we see

(3.7)

$$2\operatorname{tr}xyz = (\operatorname{tr}x)(\operatorname{tr}yz) + (\operatorname{tr}y)(\operatorname{tr}xz) + (\operatorname{tr}z)(\operatorname{tr}xy) - (\operatorname{tr}x)(\operatorname{tr}y)(\operatorname{tr}z),$$

from (3.4). This shows that, if G is abelian and is generated by x_1, \dots, x_n , it turns out that $\mathfrak{X}_{\mathbf{Q}}(G)$ is generated by $\operatorname{tr} 1_G$,

$$\operatorname{tr} x_i \text{ for } 1 \leq i \leq n \quad \text{and} \quad \operatorname{tr} x_i x_j \text{ for } 1 \leq i < j \leq n,$$

by Theorem 3.1. In other words, consider a polynomial ring

$$\mathcal{P} := \mathbf{Q}[t_i, t_{i_1 i_2} \mid 1 \leq i \leq n, 1 \leq i_1 < i_2 \leq n],$$

and a ring homomorphism $\bar{\pi} = \bar{\pi}_G : \mathcal{P} \rightarrow \mathfrak{X}_{\mathbf{Q}}(G)$ defined by

$$\bar{\pi}(1) := \frac{1}{2}(\operatorname{tr} 1_G), \quad \bar{\pi}(t_{i_1 \dots i_l}) := \operatorname{tr} x_{i_1} \cdots x_{i_l}.$$

Then $\bar{\pi} : \mathcal{P} \rightarrow \mathfrak{X}_{\mathbf{Q}}(G)$ is surjective. Set $I := \operatorname{Ker}(\bar{\pi})$. In this paper, we always identify \mathcal{P}/I with $\mathfrak{X}_{\mathbf{Q}}(G)$ as a ring under the isomorphism induced from $\bar{\pi}$ and also call each of them the ring of Fricke characters of G over \mathbf{Q} .

Next, set $t'_{i_1 \dots i_l} := t_{i_1 \dots i_l} - 2 \in \mathcal{P}$. We also denote by $t'_{i_1 \dots i_l}$ its coset class in \mathcal{P}/I by abuse of language. An element in $\mathfrak{X}_{\mathbf{Q}}(G)$ corresponding to $t'_{i_1 \dots i_l} \in \mathcal{P}/I$ is

$$\begin{aligned} \operatorname{tr}' x_{i_1} \cdots x_{i_l} &:= (\operatorname{tr} x_{i_1} \cdots x_{i_l}) - 2 \\ &= (\operatorname{tr} x_{i_1} \cdots x_{i_l}) - \operatorname{tr} 1_G \in \mathfrak{X}_{\mathbf{Q}}(G). \end{aligned}$$

Consider an ideal

$$J_0 = (t'_i, t'_{i_1 i_2} \mid 1 \leq i \leq n, 1 \leq i_1 < i_2 \leq n) \subset \mathcal{P}$$

generated by all $t'_{i_1 \dots i_l}$'s in \mathcal{P} , and

$$J := \bar{\pi}(J_0) = (t'_i, t'_{i_1 i_2} \mid 1 \leq i \leq n, 1 \leq i_1 < i_2 \leq n) \subset \mathcal{P}/I.$$

Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of ideals of \mathcal{P}/I . Set

$$\operatorname{gr}^k(J) := J^k/J^{k+1}.$$

Then each of $\text{gr}^k(J)$ is a finite dimensional \mathbf{Q} -vector space. In the present paper, for the case where G is a free abelian group H of rank n , we determine the \mathbf{Q} -vector space structures of $\text{gr}^k(J)$ for any $k \geq 1$. More precisely, we give a basis of each of $\text{gr}^k(J)$ in Section 4.

3.3. Basic formulae among $\text{tr}'x$. In this subsection, we summarize basic and useful formulae among $\text{tr}'x$ for $x \in G$. To begin with, we confirm the following.

$$(3.8) \quad \text{tr}'x^{-1} = \text{tr}'x,$$

$$(3.9) \quad \text{tr}'xy = \text{tr}'yx,$$

$$(3.10) \quad \text{tr}'xy + \text{tr}'xy^{-1} = 2\text{tr}'x + 2\text{tr}'y + (\text{tr}'x)(\text{tr}'y),$$

$$(3.11) \quad \begin{aligned} \text{tr}'xyz + \text{tr}'yxz &= -2\{\text{tr}'x + \text{tr}'y + \text{tr}'z\} \\ &\quad + 2\{\text{tr}'xy + \text{tr}'yz + \text{tr}'xz\} \\ &\quad + (\text{tr}'x)(\text{tr}'yz) + (\text{tr}'y)(\text{tr}'xz) + (\text{tr}'z)(\text{tr}'xy), \\ &\quad - 2\{(\text{tr}'x)(\text{tr}'y) + (\text{tr}'y)(\text{tr}'z) + (\text{tr}'z)(\text{tr}'x)\} - (\text{tr}'x)(\text{tr}'y)(\text{tr}'z), \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} 2\text{tr}'xyzw &= 2(\text{tr}'x + \text{tr}'y + \text{tr}'z + \text{tr}'w) \\ &\quad - 2(\text{tr}'xy + \text{tr}'xz + \text{tr}'xw + \text{tr}'yz + \text{tr}'yw + \text{tr}'zw) \\ &\quad + 2(\text{tr}'xyz + \text{tr}'xyw + \text{tr}'xzw + \text{tr}'yzw) \\ &\quad + 2\{(\text{tr}'x)(\text{tr}'y) + (\text{tr}'x)(\text{tr}'w) \\ &\quad + (\text{tr}'y)(\text{tr}'z) + (\text{tr}'z)(\text{tr}'w) + 2(\text{tr}'x)(\text{tr}'z) + 2(\text{tr}'y)(\text{tr}'w)\} \\ &\quad - 2\{(\text{tr}'x)(\text{tr}'yz) + (\text{tr}'x)(\text{tr}'zw) \\ &\quad + (\text{tr}'y)(\text{tr}'xw) + (\text{tr}'y)(\text{tr}'zw) \\ &\quad + (\text{tr}'z)(\text{tr}'xy) + (\text{tr}'z)(\text{tr}'xw) \\ &\quad + (\text{tr}'w)(\text{tr}'xy) + (\text{tr}'w)(\text{tr}'yz)\} \\ &\quad + \{(\text{tr}'x)(\text{tr}'yzw) + (\text{tr}'y)(\text{tr}'xzw) \\ &\quad + (\text{tr}'z)(\text{tr}'xyw) + (\text{tr}'w)(\text{tr}'xyz)\} + \{(\text{tr}'xy)(\text{tr}'zw) \\ &\quad - (\text{tr}'xz)(\text{tr}'yw) + (\text{tr}'xw)(\text{tr}'yz)\} \\ &\quad - \{(\text{tr}'x)(\text{tr}'y)(\text{tr}'zw) + (\text{tr}'y)(\text{tr}'z)(\text{tr}'xw) \end{aligned}$$

$$\begin{aligned}
& + (\text{tr}'x)(\text{tr}'w)(\text{tr}'yz) + (\text{tr}'z)(\text{tr}'w)(\text{tr}'xy)\} \\
& + (\text{tr}'x)(\text{tr}'y)(\text{tr}'z)(\text{tr}'w) \\
& + 2\{(\text{tr}'x)(\text{tr}'y)(\text{tr}'z) + (\text{tr}'x)(\text{tr}'y)(\text{tr}'w) \\
& + (\text{tr}'x)(\text{tr}'z)(\text{tr}'w) + (\text{tr}'y)(\text{tr}'z)(\text{tr}'w)\}.
\end{aligned}$$

These formulae hold for any group G and $x, y, z, w \in G$. For details, see our previous paper [5, Section 4].

Lemma 3.2. *For any $x \in G$ and $\alpha \in \mathbf{Z}$,*

$$\text{tr}'x^\alpha \equiv \alpha^2 \text{tr}'x \pmod{J^2}.$$

Proof. It is obvious when $\alpha = 0, 1$. It suffices to show $\alpha > 0$. We show this by induction on α .

Assume $\alpha \geq 2$. Then, substituting $x^{\alpha-1}$ and x to x and y in (3.10), respectively, we have

$$\text{tr}'x^\alpha + \text{tr}'x^{\alpha-2} = 2\text{tr}'x^{\alpha-1} + 2\text{tr}'x + (\text{tr}'x^{\alpha-1})(\text{tr}'x),$$

and, hence, by the inductive hypothesis, we obtain

$$\text{tr}'x^\alpha \equiv 2(\alpha-1)^2 \text{tr}'x - (\alpha-2)^2 \text{tr}'x + 2\text{tr}'x \equiv \alpha^2 \text{tr}'x \pmod{J^2}.$$

This completes the proof of Lemma 3.2. \square

Here, we consider additional relations among $\text{tr}'x$ for abelian group G .

Lemma 3.3. *For an abelian group G and any $x, y, z, w \in G$, we have*

$$\begin{aligned}
& (\text{tr}'xz)(\text{tr}'yw) = (\text{tr}'xw)(\text{tr}'yz) \\
& + \{(\text{tr}'x)(\text{tr}'z) + (\text{tr}'y)(\text{tr}'w) - (\text{tr}'y)(\text{tr}'z) - (\text{tr}'x)(\text{tr}'w)\} \\
& - \{(\text{tr}'x)(\text{tr}'yz) + (\text{tr}'y)(\text{tr}'xw) + (\text{tr}'z)(\text{tr}'xw) + (\text{tr}'w)(\text{tr}'yz) \\
& - (\text{tr}'y)(\text{tr}'xz) - (\text{tr}'x)(\text{tr}'yw) - (\text{tr}'z)(\text{tr}'yw) - (\text{tr}'w)(\text{tr}'xz)\} \\
& - \frac{1}{2}\{(\text{tr}'y)(\text{tr}'z)(\text{tr}'xw) + (\text{tr}'x)(\text{tr}'w)(\text{tr}'yz) \\
& - (\text{tr}'x)(\text{tr}'z)(\text{tr}'yw) - (\text{tr}'y)(\text{tr}'w)(\text{tr}'xz)\}
\end{aligned}$$

Proof. In order to obtain the equation above, it suffices to calculate $2\text{tr}'xyzw - 2\text{tr}'yxzw$ with equation (3.12). The calculation is straightforward. We leave it to the reader for an exercise. \square

As special cases of Lemma 3.3, we see the following corollaries.

Corollary 3.4. *For an abelian group G and $x, y, z \in G$, we have*

$$\begin{aligned} (tr'xy)(tr'xz) = & -3(tr'x)(tr'y) - 3(tr'x)(tr'z) - (tr'x)^2 - (tr'y)(tr'z) \\ & + 2(tr'x)(tr'yz) + (tr'x)(tr'xy) \\ & + (tr'x)(tr'xz) + (tr'y)(tr'xz) + (tr'z)(tr'xy) \\ & + \frac{1}{2}(tr'x)^2(tr'yz) - (tr'x)^2(tr'y) \\ & - (tr'x)^2(tr'z) - (tr'y)(tr'xz) \\ & - (tr'x)(tr'yz) - (tr'z)(tr'yz) - (tr'w)(tr'xz) \\ & - \frac{1}{2}(tr'x)^2(tr'y)(tr'z) - 2(tr'x)(tr'y)(tr'z) \\ & + \frac{1}{2}(tr'x)(tr'z)(tr'xy) + \frac{1}{2}(tr'x)(tr'y)(tr'xz). \end{aligned}$$

Corollary 3.5. *For an abelian group G and $x, y \in G$, we have*

$$\begin{aligned} (tr'xy)^2 = & -(tr'x)^2 - (tr'y)^2 \\ & + 2\{(tr'x)(tr'y) + (tr'x)(tr'xy) + (tr'y)(tr'xy)\} \\ & + (tr'x)(tr'y)(tr'xy). \end{aligned}$$

4. The structure of a \mathbf{Q} -vector space $\mathbf{gr}^k(J)$ for $G = H$. The goal of this section is to give a basis of $\mathbf{gr}^k(J)$ for any $k \geq 1$ as a \mathbf{Q} -vector space.

Proposition 4.1. *For any $k \geq 1$, consider a polynomial*

$$f := t'_{p_1 q_1} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in \mathcal{P},$$

for $0 \leq l \leq k$, $(p_1, q_1) \leq \cdots \leq (p_l, q_l)$ and $1 \leq i_{l+1} \leq \cdots \leq i_k \leq n$. Then, under the modulo I , the monomial f can be written as a sum of monomials of type

$$t'_{p'_1 q'_1} \cdots t'_{p'_m q'_m} t'_{j_{m+1}} \cdots t'_{j_k} \in \mathcal{P},$$

such that

- $0 \leq m \leq k$,

- $1 \leq p'_1 < q'_1 < \cdots < p'_m < q'_m \leq n$,
- $1 \leq j_{m+1} \leq \cdots \leq j_k \leq n$,
- $\{p'_1, q'_1, \dots, p'_m, q'_m\} \subset \{p_1, q_1, \dots, p_l, q_l\}$.

Proof. We prove this proposition by induction on $l \geq 0$. If $l = 0$ or 1 , it is clear. Assume $l \geq 2$.

First, assume that some elements in $\{p_1, q_1, \dots, p_l, q_l\}$ are equal. For simplicity, we consider three cases:

- (i) $p_1 = p_2$ and $q_1 \neq q_2$,
- (ii) $p_1 = q_2$ and $q_1 \neq p_2$,
- (iii) $p_1 = p_2$ and $q_1 = q_2$.

For parts (i) and (ii), by using Corollary 3.4, we see that under the modulo I , the monomial f can be written as a sum of monomials

$$t'_{p_1 q_1} \cdots t'_{p_r q_r} t'_{i_{r+1}} \cdots t'_{i_k},$$

such that $r < l$. Hence, by applying the inductive hypothesis to each of such monomials, we obtain the required result. We can discuss an argument similar to the above for case (iii) by using Corollary 3.5. Therefore, we assume that $p_1, q_1, \dots, p_l, q_l$ are distinct.

Now, by the assumption $(p_1, q_1) \leq (p_2, q_2)$, we have $p_1 < p_2$. If $q_1 < p_2$, then we have the inequalities:

$$\begin{array}{ccccccc} p_1 & < & q_1 & < & p_2 & < & p_3 & < & \cdots & < & p_l \\ & & & & \wedge & & \wedge & & & & \wedge \\ & & & & q_2 & & q_3 & & & & q_l. \end{array}$$

Hence, if we set $f' := t'_{p_2 q_2} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in \mathcal{P}$, then we can apply the inductive hypothesis to f' and obtain the required result.

Next, if $p_2 < q_1$, we have the inequalities:

$$\begin{array}{ccccccc} p_1 & < & p_2 & < & p_3 & < & \cdots & < & p_l \\ & & & & \wedge & & \wedge & & & & \wedge \\ & & & & q_1, q_2 & & q_3 & & & & q_l. \end{array}$$

Then, by using Lemma 3.3, we can write f as a sum of monomials

$$t'_{p_1 p_2} t'_{q_1, q_2} t'_{p_3 q_3} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k}$$

and

$$t'_{p_1 q_1} \cdots t'_{p_r q_r} t'_{i_{r+1}} \cdots t'_{i_k},$$

such that $r < l$. If we apply the inductive hypothesis to $t'_{q_1, q_2} t'_{p_3 q_3} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k}$ and $t'_{p_1 q_1} \cdots t'_{p_r q_r} t'_{i_{r+1}} \cdots t'_{i_k}$, we obtain the required result. This completes the proof of Proposition 4.1. \square

Theorem 4.2. *For each $k \geq 1$ and $0 \leq l \leq k$, set*

$$T_l := \{t'_{p_1 q_1} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in J_0 \\ | 1 \leq p_1 < q_1 < \cdots < p_l < q_l \leq n, 1 \leq i_{l+1} \leq \cdots \leq i_k \leq n\}.$$

Then

$$S_k := \bigcup_{l=0}^k \pi(T_l)$$

forms a basis of $\text{gr}^k(J)$.

Proof. By Proposition 4.1, we see that S_k generates $\text{gr}^k(J)$. In order to show the linear independence of the elements of S_k , set

$$f := \sum_{l=0}^k \sum_{\substack{p_1 < q_1 < \cdots < p_l < q_l \\ i_{l+1} \leq \cdots \leq i_k}} a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_k} t'_{p_1 q_1} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in J_0^k,$$

for $a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_k} \in \mathbf{Q}$, and assume $\pi(f) \in J^{k+1}$.

Consider the interior

$$D := \{z \in \mathbf{C} \mid z\bar{z} < 1\}$$

of the unit disk in \mathbf{C} . For any $s_1, \dots, s_n \in D$, we define a representation $\rho: H \rightarrow \text{SL}(2, \mathbf{C})$ by

$$\rho(x_i) := \begin{pmatrix} 1 - s_i & 0 \\ 0 & (1 - s_i)^{-1} \end{pmatrix}$$

for any $1 \leq i \leq n$. If we consider the power series expansion

$$\frac{1}{1 - s_i} = 1 + s_i + s_i^2 + s_i^3 + \cdots$$

at the origin on D , we can write each of $\text{tr}'\rho(x_i)$ and $\text{tr}'\rho(x_i x_j)$ as a convergent power series:

$$\begin{aligned}\text{tr}'\rho(x_i) &= \frac{s_i^2}{1-s_i} = s_i^2 + s_i^3 + s_i^4 + \cdots, \\ \text{tr}'\rho(x_i x_j) &= s_i^2 + 2s_i s_j + s_j^2 + (\text{terms of degree} \geq 3).\end{aligned}$$

Then we have

$$\begin{aligned}\bar{\pi}(f)(\rho) &= \sum_{l=0}^k \sum_{\substack{p_1 < q_1 < \cdots < p_l < q_l \\ i_{l+1} \leq \cdots \leq i_k}} \left\{ a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_k} (s_{p_1}^2 + 2s_{p_1} s_{q_1} + s_{q_1}^2) \cdots \right. \\ &\quad \left. (s_{p_2}^2 + 2s_{p_2} s_{q_2} + s_{q_2}^2) s_{i_{l+1}}^2 \cdots s_{i_k}^2 \right\} \\ &\quad + (\text{terms of degree} \geq 2k+2).\end{aligned}$$

Since $\bar{\pi}(f) \in J^{k+1}$, if we regard $\bar{\pi}(f)(\rho)$ as a polynomial on $s_i s$, its degree must be greater than or equal to $2k+2$. Hence, all coefficients of degree $2k$ are zero.

To begin with, for any $1 \leq p_1 < q_2 < \cdots < p_k < q_k \leq n$, by observing the coefficients of $s_{p_1} s_{q_1} \cdots s_{p_k} s_{q_k}$, we see $a_{p_1 q_1, \dots, p_k q_k} = 0$. For any $0 \leq l \leq k$, assume that

$$a_{p_1 q_1, \dots, p_m q_m, i_{m+1}, \dots, i_k} = 0$$

for any $l \leq m$. Then, for any $1 \leq p_1 < q_2 < \cdots < p_{l-1} < q_{l-1} \leq n$ and $1 \leq i_l \leq \cdots \leq i_k \leq n$, we see

$$a_{p_1 q_1, \dots, p_{l-1} q_{l-1}, i_l, \dots, i_k} = 0.$$

Therefore, by the inductive argument, we verify that all coefficients of f are equal to zero. This shows that elements in S_k are linearly independent. This completes the proof of Theorem 4.2. \square

By observing the proof of Theorem 4.2, we have the following.

Theorem 4.3. *The ideal I is generated by*

$$\begin{aligned}(4.1) \quad & t'_{ir} t'_{js} - t'_{is} t'_{jr} - \{t'_i t'_r + t'_j t'_s - t'_j t'_r - t'_i t'_s\} \\ & + \{t'_i t'_{jr} + t'_j t'_{is} + t'_r t'_{is} + t'_s t'_{jr} - t'_j t'_{ir} - t'_i t'_{js} - t'_r t'_{js} - t'_s t'_{ir}\} \\ & + \frac{1}{2} \{t'_j t'_r t'_{is} + t'_i t'_s t'_{jr} - t'_i t'_r t'_{js} - t'_j t'_s t'_{ir}\},\end{aligned}$$

for any $1 \leq i, j, r, s \leq n$. Here we remark that, in the above notation, t'_{ij} should be read

$$\begin{cases} t'_{ji} & \text{if } i > j, \\ (t'_i)^2 + 4t'_i & \text{if } i = j. \end{cases}$$

In particular, I is finitely generated.

Proof. Let I' be an ideal of \mathcal{P} generated by elements (4.1). From Lemma 3.3, we have $I' \subset I$. For any $f \in I$, observing the proof of Proposition 4.1, we see that f can be written as

$$(4.2) \quad f \equiv \sum_{k \geq 0} \sum_{l=0}^k \sum_{\substack{p_1 < q_1 < \dots < p_l < q_l \\ i_{l+1} \leq \dots \leq i_k}} a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_k} t'_{p_1 q_1} \cdots \\ t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \in J_0^k$$

modulo I' . Here, in the sum of the right hand side of the equation above, k runs over finitely many non-negative integers. Hence, by Theorem 4.2, we see that all coefficients of f are zero and obtain $f \in I'$. This shows $I = I'$. \square

Remark 4.4. We remark that, if $n = 1$, we have $I = (0)$.

Theorem 4.5.

$$\bigcap_{k \geq 1} J^k = \{0\}.$$

Proof. For any $f \in \bigcap_{k \geq 1} J^k$, we can write f as (4.2). Then, observing the coset class of f in $\text{gr}^1(J)$, we see that $a_{p_1 q_1} = a_{i_1} = 0$ for any $1 \leq p_1 < q_1 \leq n$ and $1 \leq i_1 \leq n$. Next, observing the coset class of f in $\text{gr}^2(J)$, we see that all coefficients of f of degree 2 are zero. By repeating this argument inductively, we obtain $f = 0$. This completes the proof of Theorem 4.5. \square

5. The primeness of the ideal I . In this section, we show that the ideal I is a prime ideal of \mathcal{P} ; in other words, \mathcal{P}/I is an integral domain. In order to show this, we introduce a weight of an element $f \in \mathcal{P}/I$. From Theorem 4.5, for any $f \in \mathcal{P}/I \setminus \{0\}$, there exists some

integer $k \geq 0$ such that $f \in J^k \setminus J^{k+1}$. Then we call k the weight of f , and denote it by $\text{wt}(f)$.

Proposition 5.1. *For any $f, g \in \mathcal{P}/I \setminus \{0\}$, $\text{wt}(fg) = \text{wt}(f) + \text{wt}(g)$.*

Proof. It is obvious if $\text{wt}(f) = 0$ or $\text{wt}(g) = 0$. Hence, we may assume $\text{wt}(f), \text{wt}(g) \geq 1$. Set $k_1 := \text{wt}(f)$ and $k_2 := \text{wt}(g)$. Since it is clear that $\text{wt}(fg) \geq k_1 + k_2$, assume $\text{wt}(fg) > k_1 + k_2$.

For any integers $\alpha_1, \dots, \alpha_n \in \mathbf{Z}$, consider a group homomorphism $\rho_{(\alpha_1, \dots, \alpha_n)} : H \rightarrow \mathbf{Z} = \langle x_1 \rangle$ defined by $x_i \mapsto x_1^{\alpha_i}$ for any $1 \leq i \leq n$. Then $\rho_{(\alpha_1, \dots, \alpha_n)}$ induces a ring homomorphism $\bar{\rho}_{(\alpha_1, \dots, \alpha_n)} : \mathfrak{X}_{\mathbf{Q}}(H) \rightarrow \mathfrak{X}_{\mathbf{Q}}(\mathbf{Z})$ defined by

$$\text{tr}' x \longmapsto \text{tr}' \rho_{(\alpha_1, \dots, \alpha_n)}(x).$$

Namely, we have

$$\bar{\rho}_{(\alpha_1, \dots, \alpha_n)}(\text{tr}' x_i) = \text{tr}' x_1^{\alpha_i}, \quad \bar{\rho}_{(\alpha_1, \dots, \alpha_n)}(\text{tr}' x_i x_j) = \text{tr}' x_1^{\alpha_i + \alpha_j},$$

Now, set

$$\begin{aligned} f &:= \sum_{l=0}^{k_1} \sum_{\substack{p_1 < q_1 < \dots < p_l < q_l \\ i_{l+1} \leq \dots \leq i_{k_1}}} a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_{k_1}} t'_{p_1 q_1} \cdots \\ &\quad t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_{k_1}} + (\text{terms of degree} > k_1), \\ g &:= \sum_{m=0}^{k_2} \sum_{\substack{p'_1 < q'_1 < \dots < p'_m < q'_m \\ i'_{m+1} \leq \dots \leq i'_{k_2}}} a_{p'_1 q'_1, \dots, p'_m q'_m, i'_{m+1}, \dots, i'_{k_2}} t'_{p'_1 q'_1} \cdots \\ &\quad t'_{p'_m q'_m} t'_{i'_{m+1}} \cdots t'_{i'_{k_2}} + (\text{terms of degree} > k_2), \end{aligned}$$

and set $F := \bar{\rho}_{(\alpha_1, \dots, \alpha_n)}(f)$ and $G := \bar{\rho}_{(\alpha_1, \dots, \alpha_n)}(g)$. Then $\text{wt}(FG) > k_1 + k_2$ in $\mathfrak{X}_{\mathbf{Q}}(\mathbf{Z})$. Hence, the coefficient $P(\alpha_1, \dots, \alpha_n)$ of FG of degree $k_1 + k_2$ is equal to zero. Here, using Lemma 3.2, we have $P(\alpha_1, \dots, \alpha_n) = P_1 P_2$ for

$$\begin{aligned} P_1 &:= \left(\sum_{l=0}^{k_1} \sum_{\substack{p_1 < q_1 < \dots < p_l < q_l \\ i_{l+1} \leq \dots \leq i_{k_1}}} a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_{k_1}} (\alpha_{p_1} + \alpha_{q_1})^2 \cdots \right. \\ &\quad \left. (\alpha_{p_l} + \alpha_{q_l})^2 \alpha_{i_{l+1}}^2 \cdots \alpha_{i_{k_1}}^2 \right) \end{aligned}$$

$$P_2 := \left(\sum_{m=0}^{k_2} \sum_{\substack{p'_1 < q'_1 < \dots < p'_m < q'_m \\ i'_{m+1} \leq \dots \leq i'_{k_2}}} a_{p'_1 q'_1, \dots, p'_m q'_m, i'_{m+1}, \dots, i'_{k_2}} (\alpha_{p'_1} + \alpha_{q'_1})^2 \dots \right. \\ \left. (\alpha_{p'_m} + \alpha_{q'_m})^2 \alpha_{i'_{m+1}}^2 \dots \alpha_{i'_{k_2}}^2 \right).$$

Consider $P(\alpha_1, \dots, \alpha_n)$ as a polynomial in $\mathbf{Q}[\alpha_1, \dots, \alpha_n]$. Since $P(\alpha_1, \dots, \alpha_n) = 0$ if $\alpha_1, \dots, \alpha_n$ runs over all integers, we see that $P(\alpha_1, \dots, \alpha_n) = 0$ as a polynomial in $\mathbf{Q}[\alpha_1, \dots, \alpha_n]$. Since $\mathbf{Q}[\alpha_1, \dots, \alpha_n]$ is a domain, we have $P_1 = 0$ or $P_2 = 0$ in $\mathbf{Q}[\alpha_1, \dots, \alpha_n]$.

Assume $P_1 = 0$. Then each coefficient of monomials in $\alpha_1, \dots, \alpha_n$ in P_1 is equal to zero. First, by observing the coefficient of $\alpha_{p_1} \alpha_{q_1} \dots \alpha_{p_{k_1}} \alpha_{q_{k_1}}$, we see

$$a_{p_1 q_1, \dots, p_{k_1} q_{k_1}} = 0.$$

Furthermore, by an argument similar to that in the proof of Theorem 4.2, we obtain that all $a_{p_1 q_1, \dots, p_l q_l, i_{l+1}, \dots, i_{k_1}}$ s are equal to zero. Hence, $f \in J^{k_1+1}$. This contradicts $\text{wt}(f) = k_1$. By the same argument, if $P_2 = 0$, we have $g \in J^{k_2+1}$, and a contradiction. Thus, we conclude that $\text{wt}(fg) = k_1 + k_2$. This completes the proof of Proposition 5.1. \square

As a corollary, we obtain

Theorem 5.2. *The quotient ring \mathcal{P}/I is an integral domain. That is, the ideal I is a prime ideal in \mathcal{P} .*

6. Some remarks. Finally, we give some remarks about an action of the automorphism group $\text{Aut } H$ of H on the ring $\mathfrak{X}_{\mathbf{Q}}(H)$. In general, for any group G , the automorphism group $\text{Aut } G$ of G naturally acts on $\mathfrak{X}_{\mathbf{Q}}(G)$ from the right. (See our previous paper [5] for details.) In particular, for any $\sigma \in \text{Aut } G$ and $x \in G$, the action of $\sigma \in \text{Aut } G$ on $\text{tr}'x \in \mathfrak{X}_{\mathbf{Q}}(G)$ is given by

$$(\text{tr}'x)^{\sigma} = \text{tr}'x^{\sigma}.$$

Clearly, we see that the ideal J generated by all $\text{tr}'x$ for $x \in G$ is an $\text{Aut } G$ -invariant ideal of $\mathfrak{X}_{\mathbf{Q}}(G)$. Hence, $\text{Aut } G$ naturally acts on $\text{gr}^k(J)$ for each $k \geq 1$.

Let $\mathcal{E}_G(k)$ be the kernel of a natural homomorphism $\text{Aut } G \rightarrow \text{Aut } (J/J^{k+1})$ induced from the action of $\text{Aut } G$. Then the groups $\mathcal{E}_G(k)$ define a descending filtration

$$\mathcal{E}_G(1) \supset \mathcal{E}_G(2) \supset \cdots \supset \mathcal{E}_G(k) \supset \cdots$$

of $\text{Aut } G$. In [5], we have shown that this filtration is central. That is, $[\mathcal{E}_G(k), \mathcal{E}_G(l)] \subset \mathcal{E}_G(k+l)$ for any $k, l \geq 1$. This is a Fricke character analogue of the Andreadakis-Johnson filtration of $\text{Aut } G$. (For details on the Andreadakis-Johnson filtration, see [15, 16], for example.)

Here, we determine $\mathcal{E}_H(1)$ and show $\mathcal{E}_H(k) = \mathcal{E}_H(1)$ for any $k \geq 1$. Let $\iota \in \text{Aut } H$ be an automorphism of H defined by

$$x_i^\iota := x_i^{-1}, \quad 1 \leq i \leq n.$$

Then we have

Proposition 6.1. $\mathcal{E}_H(1) = \langle \iota \rangle$. Namely, $\mathcal{E}_H(1)$ is the cyclic group of order 2, generated by ι .

Proof. Clearly, we see $\mathcal{E}_H(1) \supset \langle \iota \rangle$. For any $\sigma \in \mathcal{E}_H(1)$, set

$$x_i^\sigma := x_1^{e_1(i)} x_2^{e_2(i)} \cdots x_n^{e_n(i)}, \quad 1 \leq i \leq n.$$

It suffices to show

- $e_j(i) = 0$ if $j \neq i$,
- $e_1(1) = \cdots = e_n(n) = \pm 1$.

For any $1 \leq i \leq n$ and any $s_i \in D$, consider a homomorphism $\rho_i : H \rightarrow \text{SL}(2, \mathbf{C})$, defined by

$$\rho_i(x_j) := \begin{cases} \begin{pmatrix} 1 - s_i & 0 \\ 0 & (1 - s_i)^{-1} \end{pmatrix}, & \text{if } j = i, \\ E_2, & \text{if } j \neq i. \end{cases}$$

Then, from

$$\text{tr}' x_i \equiv \text{tr}' x_i^\sigma = \text{tr}' x_1^{e_1(i)} x_2^{e_2(i)} \cdots x_n^{e_n(i)} \pmod{J^2},$$

by substituting ρ_i , we see

$$s_i^2 \equiv e_i(i)^2 s_i^2 \pmod{(s_i^3)},$$

and hence $e_i(i) = \pm 1$. On the other hand, by substituting ρ_j for $j \neq i$, we see

$$0 \equiv e_j(i)^2 s_j^2 \pmod{(s_j^3)},$$

and hence $e_j(i) = 0$.

In order to show $e_1(1) = \cdots = e_n(n)$, assume that $e_i(i) \neq e_j(j)$ for some i and j . This means $(e_i(i), e_j(j)) = (\pm 1, \mp 1)$. For any $s_i, s_j \in D$, consider a homomorphism $\rho_{ij} : H \rightarrow \mathrm{SL}(2, \mathbf{C})$ defined by

$$\rho_{ij}(x_r) := \begin{cases} \rho_r(x_r), & \text{if } r = i, j, \\ E_2, & \text{if otherwise.} \end{cases}$$

Then, by substituting ρ_{ij} for equation $\mathrm{tr}' x_i x_j \equiv \mathrm{tr}' (x_i x_j)^\sigma = \mathrm{tr}' x_i^{e_i(i)} x_j^{e_j(j)} \pmod{J^2}$, we obtain

$$s_i^2 + 2s_i s_j + s_j^2 \equiv s_i^2 - 2s_i s_j + s_j^2 \pmod{(s_i^3, s_i^2 s_j, s_i s_j^2, s_j^3)}.$$

This is a contradiction. Therefore, we obtain the required result. This completes the proof of Proposition 6.1. \square

Corollary 6.2. *For any $k \geq 2$, $\mathcal{E}_H(k) = \mathcal{E}_H(1)$.*

Proof. In general, we have $\mathcal{E}_H(k) \subset \mathcal{E}_H(1)$. On the other hand, $\mathcal{E}_H(k) \supset \mathcal{E}_H(1)$ immediately follows from Proposition 6.1. \square

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