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On the classification of Specht modules with one-dimensional summands

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This paper extends a result of James to a combinatorial condition on partitions for the corresponding Specht module to have a summand isomorphic to the unique one-dimensional $F\Sigma$-module over fields of characteristic 2. The work makes use of a recursively defined condition to reprove a result of Murphy and prove a new result for self-conjugate partitions. Finally we present a Python script which utilizes this work to test Specht modules for a one-dimensional summand.

1. Introduction

Specht modules are crucial to understanding the representation theory of the symmetric group; see [James 1978; James and Kerber 1981]. Gwendolen Murphy [1980] classified the decomposable Specht modules which correspond to hook partitions. Dodge and Fayers [2012] produced the first new examples of decomposable Specht modules since Murphy's work. More recently, Donkin and Geranios [2018] used analogous modules for the general linear groups and applied the Schur functor in order to find even broader families of decomposable Specht modules. Further work on the question has been addressed in the Iwahori–Hecke algebra in [Speyer 2014; Speyer and Sutton 2018].

Murphy additionally classified the Specht modules corresponding to hook partitions which have a one-dimensional summand.

Theorem 1.1 [Murphy 1980, Theorem 5.5]. *Let $\lambda = (n-r, 1^r)$ and F be a field of characteristic 2. Then there exists a nonzero $F\Sigma_n$-module M such that $S^\lambda \cong S^{(n)} \oplus M$ if and only if n is odd, r is even, and $\binom{n-1}{r}$ is odd.*

This theorem was a consequence of determining the endomorphism ring of Specht modules corresponding to hook partitions. In this paper we attempt to address the question of one-dimensional summands in the spirit of [Dodge and Fayers 2012] by constructing split exact sequences of $F\Sigma_n$-modules.

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In Section 2, we present the foundational definitions as well as construct the Specht modules in the style of [James 1978]. In Section 3, we work through the consequences of James's theorem [1978, Theorem 24.4] concerning $\mathrm{Hom}_{F\Sigma_n}(S^{(n)}, S^{\lambda})$. Utilizing this work we describe conditions sufficient for the Specht module to decompose as desired (Theorem 3.5). While this final sufficient condition is expressed as a sum of a recursively defined finite sequence, we can use it to provide a new proof of Murphy's result. Additionally in Section 4 we make use of Theorem 3.5 and introduce the notion of the directed graph of a partition in order to prove Specht modules corresponding to self-conjugate partitions cannot have a one-dimensional summand. Finally in Section 5 we present Python [van Rossum 2001] algorithms designed to apply Theorem 3.5 to determine which Specht modules have a one-dimensional summand.

2. Preliminaries and notation

For any positive integer n, let Σ_n denote the symmetric group on n letters, and $F\Sigma_n$ denote the group algebra of Σ_n over F. This paper builds greatly upon the foundations found in [James and Kerber 1981; James 1978], adopting much of the notation and constructions.

2A. ***Compositions, partitions, and Young diagrams.*** We say $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ is a *partition* of n, and write $\lambda \vdash n$, if $\sum \lambda_i = n$ and for all i, $\lambda_i \geq \lambda_{i+1}$. For a partition λ of n, the *Young diagram* of λ, denoted by $[\lambda]$, is the set

$$[\lambda] := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}.$$

Each of the elements in the Young diagram is referred to as a *node*. We call the set

$$R_\lambda(i) = \{(i, j) \mid 1 \leq j \leq \lambda_i\} \subseteq [\lambda]$$

the i-th row of $[\lambda]$ and

$$C_\lambda(j) = \{(i, j) \mid 1 \leq i \leq \lambda'_j\}$$

the j-th column of $[\lambda]$. Given a partition $\lambda \vdash n$, we define the *conjugate* of λ, denoted by λ', as the unique partition such that

$$[\lambda'] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid (j, i) \in [\lambda]\}.$$

2B. ***Tableau.*** For a partition λ of n, if $t : [\lambda] \to \{1, 2, \dots, n\}$ is a bijection, we call t a λ-*tableau*. Let $\mathcal{T}(\lambda)$ denote the set of all λ-tableaux. For a partition λ, a λ-tableau t is called *row standard* if for $(i, j), (i, k) \in [\lambda]$ with $j < k$, then $t(i, j) < t(i, k)$. We define *column standard* similarly. Moreover t is *standard* if it is both row standard and column standard and $\mathcal{T}_0(\lambda)$ will denote the set of standard λ-tableaux. We call

$\mathcal{R}_t(i) = t[R_\lambda(i)]$ the set of entries in the i-th row of t. Similarly, $\mathcal{C}_t(j) = t[C_\lambda(j)]$ is the set of entries in the j-th column of t.

Since each element $\sigma \in \Sigma_n$ is a bijection on the set $\{1, 2, 3, \dots, n\}$, there is a Σ_n-action on $\mathcal{T}(\lambda)$ defined by $\sigma t = \sigma \circ t$. From this action, we can define two significant subgroups of Σ_n given a λ-tableau t. Define the *row stabilizer* of t, denoted by R_t, to be the subset of Σ_n which fixes the sets $\mathcal{R}_t(i)$. Similarly define the *column stabilizer* of t, denoted by C_t, to be the subset of Σ_n which fixes the sets $\mathcal{C}_t(j)$. We define an equivalence relation on $\mathcal{T}(\lambda)$ by $t \sim s$ if and only if there exists $\pi \in R_t$ such that $\pi t = s$. We will use $X(\lambda)$ to denote the set of equivalence classes of $\mathcal{T}(\lambda)$ and call an equivalence class $\{t\} \in X(\lambda)$ a λ-*tabloid*. Notice that since there is a well-defined action of Σ_n on $\mathcal{T}(\lambda)$ we can define an action of Σ_n on $X(\lambda)$ by $\sigma\{t\} = \{\sigma t\}$ for all $\sigma \in \Sigma_n$ and $\{t\} \in X(\lambda)$.

Lastly we will make use of a dominance relation on the set of tabloids for a fixed composition λ, using the notation of [James 1978, Definition 3.11]. For $\{t\} \in X(\lambda)$ we let $m_{xy}(t) = |\{t(i, j) \le x \mid i \le y\}|$; that is $m_{xy}(t)$ is the number of entries less than or equal to x in the first y rows of $\{t\}$. Using this notation, we define the relation $\lhd$ on $X(\lambda)$ by $\{s\} \lhd \{t\}$ if and only if $m_{xy}(s) \le m_{xy}(t)$ for all positive integers x, y.

2C. *Permutation modules and Specht modules.* Define M^λ to be the free vector space over F generated by the set $X(\lambda)$. Additionally since $X(\lambda)$ is the basis of M^λ, we can define an $F\Sigma_n$-action on M^λ by extending the Σ_n-action on $X(\lambda)$ linearly. We will call M^λ with this module action the *permutation module* associated to λ.

Let $t \in \mathcal{T}(\lambda)$. Then we define $\kappa_t \in F\Sigma_n$ by

$$\kappa_t := \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\sigma,$$

where $\operatorname{sgn} : \Sigma_n \to \{1, -1\}$ is the signature function on the symmetric group. So for any $t \in \mathcal{T}(\lambda)$, we can define the element in M^λ called the *polytabloid* of t by $e_t := \kappa_t\{t\}$. Hence we can construct a submodule of M^λ called the *Specht module*, denoted by S^λ, which we define explicitly by

$$S^\lambda := \operatorname{Span}(\{e_t \mid t \in \mathcal{T}(\lambda)\}) \subseteq M^\lambda.$$

The following example describes two Specht modules which can be constructed for any positive integer n and will be a central to the focus of this paper.

Example 2.1. The Specht module $S^{(n)}$ for $F\Sigma_n$ is one-dimensional and for all $v \in S^\lambda$ and $\sigma \in \Sigma$, we have $\sigma v = v$. Alternatively, the Specht module $S^{(1^n)}$ is a one-dimensional $F\Sigma_n$-module, again spanned by any appropriate polytabloid. For all $w \in S^{(1^n)}$ and $\sigma \in \Sigma_n$

$$\sigma w = \begin{cases} w & \text{if } \sigma \text{ is an even permutation,} \\ -w & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

For all n, $S^{(n)}$ and $S^{(1^n)}$ form a complete list of the isomorphism classes of the one-dimensional modules of $F\Sigma_n$.

We conclude this section by stating several significant results concerning Specht modules that are relevant to our investigation.

Theorem 2.2. *Let* $\lambda \vdash n$.

(2.2.1) [James 1978, Proposition 4.5] S^λ *is a cyclic* $F\Sigma_n$*-module generated by every polytabloid.*

(2.2.2) [James 1978, Theorem 8.4] *The set* $\{e_t \mid t \in \mathcal{T}_0(\lambda)\}$ *is a basis for* S^λ.

(2.2.3) [James 1978, Theorem 8.15] *Let* $(S^\lambda)^*$ *denote the dual of* S^λ. *Then* $(S^\lambda)^* \cong S^{\lambda'} \otimes S^{(1^n)}$.

(2.2.4) [James 1978, Corollary 13.18] *Suppose* F *has characteristic* 2. *Then* S^λ *is indecomposable.*

The basis in (2.2.2) is often referred to as the *standard basis* of S^λ. The polytabloids will become very relevant to our discussion, in particular understanding which tabloids appear with nonzero coefficient in certain polytabloids, and so we introduce the following notation. If $s, t \in \mathcal{T}_0(\lambda)$ we say t *produces* s and write $t \to s$ if there exists $\pi \in C_t$ and $\sigma \in R_s$ such that $s = \sigma\pi t$. Note using this definition we see that s appears with nonzero coefficient in e_t if and only if $t \to s$. In order to further understand which standard tableaux produces other standard tableaux, we introduce the following results.

Theorem 2.3. *Let* t *be a* λ*-tableau and* $s, t \in \mathcal{T}_0(\lambda)$.

(2.3.1) [James 1978, Lemma 8.13] *If* $t \to s$ *then* $\{s\} \lhd \{t\}$.

(2.3.2) [James and Kerber 1981, Lemma 1.5.6] *Suppose* x, y *appear in the same column of* t. *If* $t \to s$ *then* x *and* y *appear in different rows of* s.

3. $F\Sigma_n$-module homomorphisms

Noting (2.2.4), we will assume F is a field of characteristic 2 for the remainder of the paper, and hence $S^{(n)} \cong S^{(1^n)}$. Therefore there is a unique one-dimensional $F\Sigma_n$-module up to isomorphism, namely $S^{(n)}$. Next we will introduce a theorem of James which will be fundamental to the remaining work of this paper. To this end, for $\lambda \vdash n$, let $l_d(\lambda)$ be the unique integer such that $2^{l_d - 1} \le \lambda_{d+1} < 2^{l_d}$. Using this notation we state the following theorem of James, specifically for the case when $p = 2$.

Theorem 3.1 [James 1978, Theorem 24.4 ($p{=}2$)]. *Suppose that* F *is a field of characteristic* 2 *and* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ *is a partition of* n. *If* $\lambda_d \equiv -1 \pmod{2^{l_d(\lambda)}}$ *for all* $1 \le d < s$, *then* $\mathrm{Hom}(S^{(n)}, S^\lambda)$ *is one-dimensional*; *otherwise* $\dim \mathrm{Hom}(S^{(n)}, S^\lambda) = 0$.

Recall that when F is a field of characteristic 2, we have $S^{(n)} \cong S^{(1^n)}$. Moreover we note that for all $F\Sigma_n$-modules M, we have $M \otimes S^{(n)} \cong M$. Thus when F is a field of characteristic 2, (2.2.3) gives us $(S^\lambda)^* \cong S^{\lambda'} \otimes S^{(n)} \cong S^{\lambda'}$. Hence from the previous theorem we have the following corollary.

Corollary 3.2. *Suppose that F is a field of characteristic* 2, $\lambda \vdash n$, *and* $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$. *Then* $\dim \mathrm{Hom}(S^\lambda, S^{(n)}) = 1$ *if and only if* $\lambda'_e \equiv -1 \pmod{2^{l_e(\lambda')}}$ *for all* $1 \le e < t$.

Proof. First note the following isomorphisms of homomorphism spaces:

$$\mathrm{Hom}(S^\lambda, S^{(n)}) \cong \mathrm{Hom}(S^\lambda, S^{(1^n)}) \cong \mathrm{Hom}((S^{(1^n)})^*, (S^\lambda)^*) \cong \mathrm{Hom}(S^{(n)}, S^{\lambda'}).$$

Hence the corollary follows from Theorem 3.1. □

3A. ***Composition of module homomorphisms.*** We can combine Theorem 3.1 and Corollary 3.2 to motivate the following definition about partitions under consideration.

Definition 3.3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$. We say that λ is *Lucas perfect* if $\lambda_d \equiv -1 \pmod{2^{l_d(\lambda)}}$ for all $1 \le d < s$ and $\lambda'_e \equiv -1 \pmod{2^{l_e(\lambda')}}$ for all $1 \le e < t$.

In order to further our discussion, let us consider an arbitrary Lucas perfect partition of n, λ. We may fix nonzero $F\Sigma_n$-module homomorphisms $i_\lambda : S^{(n)} \to S^\lambda$ and $p_\lambda : S^\lambda \to S^{(n)}$. Our goal will be to understand the composition $p_\lambda \circ i_\lambda : S^{(n)} \to S^{(n)}$. To that end, we first explore the image of i_λ expressed as a linear combination of polytabloids. Let $v \in S^{(n)}$ be nonzero. Since $S^\lambda \subset M^\lambda$, we may fix $a_{\{t\}} \in F$ such that

$$i_\lambda(v) = \sum_{\{t\} \in X(\lambda)} a_{\{t\}}\{t\}.$$

Now let $\sigma \in \Sigma$ be arbitrary. Observe by reindexing the λ-tabloids,

$$\sigma^{-1}\Bigg(\sum_{\{t\} \in X(\lambda)} a_{\{t\}}\{t\}\Bigg) = \sum_{\{t\} \in X(\lambda)} a_{\{\sigma t\}}\{t\}.$$

Additionally $i_\lambda(v) = i_\lambda(\sigma^{-1}v) = \sigma^{-1} i_\lambda(v)$, so

$$\sum_{\{t\} \in X(\lambda)} a_{\{t\}}\{t\} = \sum_{\{t\} \in X(\lambda)} a_{\{\sigma t\}}\{t\}.$$

Since $X(\lambda)$ is a basis for M^λ it follows that $a_{\{t\}} = a_{\{\sigma t\}}$ for all σ. Moreover since Σ_n acts transitively on $X(t)$, we conclude that

$$i_\lambda(v) = a \sum_{\{t\} \in X(\lambda)} \{t\}.$$

Moreover since $i_\lambda \neq 0$, we know $a \neq 0$. To understand the composition $p_\lambda \circ i_\lambda$, we will be making use of the image of a polytabloid under p_λ. To that end, we will focus on expressing $i_\lambda(v)$ as a linear combination of polytabloids. From our observations of i_λ, we note

$$\sum_{\{t\} \in X(\lambda)} \{t\} \in S^\lambda.$$

Therefore by (2.2.2), we may fix $x_t \in F$ such that

$$\sum_{t \in \mathcal{T}_0(\lambda)} x_t e_t = \sum_{\{s\} \in X(\lambda)} \{s\}. \tag{3-1}$$

Ideally we would be able to determine a closed formula for the coefficients x_t. For now we will settle on developing a recursive formula. To assist with this task, let ρ be the linear transformation defined by

$$\rho(\{s\}) = \begin{cases} \{s\} & \text{if } \{s\} \text{ is standard,} \\ 0 & \text{if } \{s\} \text{ is not standard.} \end{cases}$$

Therefore ρ is a linear projection from M^λ to the span of $\{\{t\} \mid t \in \mathcal{T}_0(\lambda)\}$. We note by (2.3.1) and (2.2.2) that $\{\rho(e_t) \mid t \in \mathcal{T}_0(\lambda)\}$ is linearly independent. By applying ρ to (3-1), we have

$$\sum_{t \in \mathcal{T}_0(\lambda)} x_t \rho(e_t) = \sum_{s \in \mathcal{T}_0(\lambda)} \{s\}. \tag{3-2}$$

Since both $\{e_t \mid t \in \mathcal{T}_0(\lambda)\}$ and $\{\rho(e_t) \mid t \in \mathcal{T}_0(\lambda)\}$ are linearly independent sets, (3-1) and (3-2) have unique solutions and therefore the same solution sets. Using these observations we prove the solution satisfies the following condition.

Lemma 3.4. *If $\lambda \vdash n$ is Lucas perfect and $X_t = \{s \in \mathcal{T}_0(\lambda) \mid s \to t \text{ and } s \neq t\}$, then*

$$x_t = 1 + \sum_{s \in X_t} x_s$$

is the unique solution to

$$\sum_{t \in \mathcal{T}_0(\lambda)} x_t e_t = \sum_{\{s\} \in X(\lambda)} \{s\}.$$

Proof. It follows from Theorem 3.1 that (3-1) has a unique solution. Thus we complete the proof by demonstrating the solution proposed by the lemma satisfies (3-2). Suppose $t \in \mathcal{T}_0(\lambda)$, and define the sets

$$\begin{aligned} X_t &= \{s \in \mathcal{T}_0(\lambda) \mid s \to t \text{ and } s \neq t\}, \\ Y_t &= \{s \in \mathcal{T}_0(\lambda) \mid t \to s \text{ and } s \neq t\}, \\ W(\lambda) &= \{(u, v) \mid u \in \mathcal{T}_0(\lambda),\ v \in Y_t\} = \{(u, v) \mid v \in \mathcal{T}_0(\lambda),\ u \in X_v\}. \end{aligned}$$

Using this notation it follows that $\rho(e_t) = \{t\} + \sum_{s\in Y_t}\{s\}$. Therefore

$$\sum_{t\in\mathcal{T}_0(\lambda)} x_t\rho(e_t) = \sum_{t\in\mathcal{T}_0(\lambda)} x_t\Bigl(\{t\}+\sum_{s\in Y_t}\{s\}\Bigr) = \sum_{t\in\mathcal{T}_0(\lambda)} x_t\{t\} + \sum_{t\in\mathcal{T}_0(\lambda)}\sum_{s\in Y_t} x_t\{s\}.$$

Now observe

$$\sum_{t\in\mathcal{T}_0(\lambda)}\sum_{s\in Y_t} x_t\{s\} = \sum_{(t,s)\in W(\lambda)} x_t\{s\} = \sum_{s\in\mathcal{T}_0(\lambda)}\sum_{t\in X_s} x_t\{s\}.$$

Therefore by reindexing we have

$$\begin{aligned}\sum_{t\in\mathcal{T}_0(\lambda)} x_t\rho(e_t) &= \sum_{s\in\mathcal{T}_0(\lambda)} x_s\{s\} + \sum_{s\in\mathcal{T}_0(\lambda)}\sum_{t\in X_s} x_t\{s\}\\ &= \sum_{s\in\mathcal{T}_0(\lambda)}\biggl(\Bigl(1+\sum_{t\in X_s} x_t\Bigr)\{s\} + \sum_{t\in X_s} x_t\{s\}\biggr) = \sum_{s\in\mathcal{T}_0(\lambda)}\{s\}. \qquad\square\end{aligned}$$

Now we will use the work thus far to demonstrate how understanding the solution to (3-1) can be used to answer our question of decomposability.

Theorem 3.5. *Let λ be a Lucas perfect partition of n such that $\lambda \neq (n), (1^n)$, and let the coefficients $x_t \in F$ be as in Lemma 3.4. Then there exists a nonzero $F\Sigma_n$-module M such that $S^\lambda \cong S^{(n)} \oplus M$ if and only if $\sum_{t\in\mathcal{T}_0(\lambda)} x_t \neq 0$.*

Proof. Fix nonzero $F\Sigma_n$-module homomorphisms $p_\lambda : S^\lambda \to S^{(n)}$ and $i_\lambda : S^{(n)} \to S^\lambda$. Since $S^{(n)}$ is a simple $F\Sigma_n$-module, p_λ and i_λ span their respective homomorphism spaces, and S^λ is not one-dimensional, S^λ is decomposable with summand isomorphic to $S^{(n)}$ if and only if $p_\lambda \circ i_\lambda \neq 0$. Let $\{r\}$ be the unique tabloid in $M^{(n)} = S^{(n)}$. Notice for all $t, t' \in \mathcal{T}(\lambda)$, we have $p_\lambda(e_t) = p_\lambda(e_{t'}) = \alpha\{r\}$ for some $\alpha \neq 0$ since Σ_n acts transitively on the polytabloids and as the identity on $S^{(n)}$. To complete the proof we need only to observe if

$$i_\lambda(\{r\}) = \beta\sum_{\{s\}\in X(\lambda)}\{s\} = \beta\sum_{t\in\mathcal{T}_0(\lambda)\}} x_t e_t$$

for some nonzero $\beta \in F$ then

$$p_\lambda \circ i_\lambda(\{r\}) = \alpha\beta\Bigl(\sum_{t\in\mathcal{T}_0(\lambda)} x_t\Bigr)\{r\}.$$

Hence $p_\lambda \circ i_\lambda \neq 0$ if and only if $\sum_{t\in\mathcal{T}_0(\lambda)} x_t \neq 0$. $\square$

It is worth noting that this result has a simpler interpretation. Since we know the coefficients of tabloids in polytabloids are either 1 or 0 in characteristic 2, it follows that the x_t are either 0 or 1. The sum of coefficients in Theorem 3.5 is congruent to the number of nonzero coefficients modulo 2. Hence we can say for a Lucas perfect partition, the Specht module is decomposable with a one-dimensional summand if and only if the sum of all tabloids can be expressed as a sum of an odd number of standard polytabloids. In fact we do not need to insist in expressing the sum using standard polytabloids, but rather any polytabloids.

3B. ***New proof of Murphy's result.*** Our work thus far allows us to provide a new proof of the result of Murphy, Theorem 1.1. First we will need a quick lemma concerning hook partitions.

Lemma 3.6. *Let $\lambda = (n-r, 1^r)$ be a hook partition. Then for all $t \in \mathcal{T}_0(\lambda)$,*

$$X_t = \{s \mid s \to t \text{ and } s \neq t\} = \varnothing.$$

Proof. Let t be an arbitrary standard λ-tableau. Assume for contradiction there is a nonidentity element $\pi \in C_t$ such that $\{\pi t\} = \{s\}$ for some $s \in \mathcal{T}_0(\lambda)$. Suppose x is the largest integer not fixed by π. Let $y = \pi^{-1}(x)$ and $z = \pi(x)$, so $y, z < x$ by our assumption of x. Since the first column is the only one with multiple entries, $x, y, z \in \mathcal{C}_t(1)$. We will consider two cases.

Case 1: Suppose $y = 1$. Then 1 is not fixed by π. So $1 \in \mathcal{R}_{\pi t}(j)$ for $j > 1$; thus πt is not row equivalent to a standard tableau. So we have reached a contradiction.

Case 2: Suppose $y \neq 1$. Then $y \in \mathcal{R}_t(i)$ and $x \in \mathcal{R}_t(j)$ with $1 < i < j$. Hence $\pi(y) = x \in \mathcal{R}_{\pi t}(i)$ and $\pi(x) = z \in \mathcal{R}_{\pi t}(j)$ with $1 < i < j$. Hence πt is not row equivalent to a standard tableau and we again have reached a contradiction. $\square$

In order to reproduce Murphy's result suppose $\lambda = (n-r, 1^r)$. First we note that if n is even or r is odd then λ or λ' is not Lucas perfect, so S^λ does not have $S^{(n)}$ as a summand by Theorem 3.5. Now it suffices to consider the case when n is odd, r even, and $0 < r < n$, so λ is Lucas perfect. Notice that a standard tableau is uniquely determined by the choice of r entries not appearing in the first row; thus $|\mathcal{T}_0(\lambda)| = \binom{n-1}{r}$, since the entries can be any subset of $\{2, 3, \dots, n\}$ of size r. Now by Lemmas 3.6 and 3.4 we have a solution to (3-1), $x_t = 1$ for all $t \in \mathcal{T}_0(\lambda)$ since $X_t = \varnothing$. Therefore

$$\sum_{t \in \mathcal{T}_0(\lambda)} x_t = \binom{n-1}{r}.$$

Hence Murphy's result follows from Theorem 3.5.

4. The directed graph of λ

Let Γ_λ be the graph whose vertex set is $\mathcal{T}_0(\lambda)$. The graph Γ_λ has a directed edge from t to s if and only if $t \to s$ and $t \neq s$. To illustrate the definition we will construct the graph $\Gamma_{(4,3)}$ in Figure 1 using the notation of [James 1978, Definition 3.6] to represent a tableau t by a Young diagram where $t(i, j)$ is the (i, j)-node of $[\lambda]$. We will define a *path* γ on Γ_λ to be a sequence $\gamma = (t_1, t_2, \dots, t_l)$, such that $t_i \to t_{i+1}$

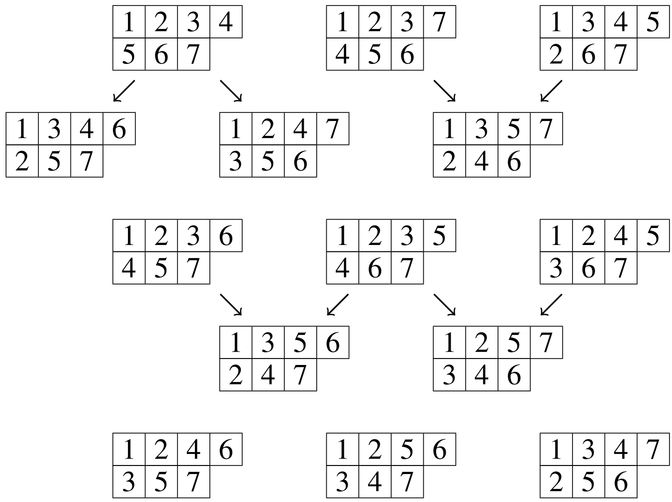

Figure 1. The directed graph of (4, 3).

and $t_i \neq t_{i+1}$ for $1 \leq i < l$. If $\gamma = (t_1, t_2, \ldots, t_l)$ we will say γ has *length* $l-1$ and *terminates* at t_l. We consider $\gamma = (t_1)$ to be a path on Γ_λ of length zero. Let P^λ be the set of all paths on Γ_λ and $\Omega_t = \{\gamma \in P^\lambda \mid \gamma \text{ terminates at } t\}$. We note that $\{\Omega_t \mid t \in \mathcal{T}_0(\lambda)\}$ partitions P^λ. Using this notation we discover that $|\Omega_t|$ satisfies a familiar relationship.

Lemma 4.1. *Let $\lambda \vdash n$ and $X_t = \{s \mid s \to t \text{ and } s \neq t\}$. If*

$$\Omega_t = \{\gamma \in P^\lambda \mid \gamma \text{ terminates at } t\}$$

then

$$|\Omega_t| = 1 + \sum_{s \in X_t} |\Omega_s|.$$

Proof. Let $\gamma = (t_1, t_2, \ldots, t_{l-1}, t) \in \Omega_t - \{(t)\}$. Define $F : \Omega_t - \{(t)\} \to \bigcup_{s \in X_t} \Omega_s$ by $F(\gamma) = (t_1, t_2, \ldots, t_{l-1})$. We will complete the proof by demonstrating that F is a bijection. If $F(\gamma) = (t_1, t_2, \ldots, t_{l-1}) = F(\gamma')$ for $\gamma, \gamma' \in \Omega_t - \{(t)\}$ then $\gamma = (t_1, t_2, \ldots, t_{l-1}, t) = \gamma'$. If $\tau = (t_1, t_2, \ldots, t_{l-1}, t_l)$ for some $t_l \neq t$ such that $t_l \to t$, then $\gamma = (t_1, t_2, \ldots, t_{l-1}, t_l, t) \in \Omega_t$ and $F(\gamma) = \tau$. Therefore F is a bijection. $\square$

Through Lemma 3.4, we establish an important connection between the directed graph of λ and our question of $S^{(n)}$ appearing as a submodule S^λ. We summarize this fact with the following theorem.

Theorem 4.2. *Suppose $\lambda \vdash n$ is Lucas perfect. Then there exists an $F\Sigma_n$-module M such that $S^\lambda \oplus M$ if and only if $|P^\lambda|$ is odd.*

Proof. Observe that $x_t \equiv |\Omega_t| \pmod 2$ is a solution to (3-1) by Lemmas 3.4 and 4.1. Moreover

$$|P^\lambda| = \sum_{\mathcal{T}_0(\lambda)} |\Omega_t| \equiv \sum_{\mathcal{T}_0(\lambda)} x_t \pmod 2.$$

Therefore the result follows immediately from Theorem 3.5. □

4A. *Self-conjugate partitions.* We say a partition λ is *self-conjugate* if $\lambda = \lambda'$. For the remainder of the section we will assume that λ is self-conjugate and Lucas perfect. In this circumstance, we are able to define an involution on $\mathcal{T}(\lambda)$. Suppose $t \in \mathcal{T}(\lambda)$, and define $\bar{t} \in \mathcal{T}(\lambda)$ by $\bar{t}(i, j) := t(j, i)$ for all $(i, j) \in [\lambda]$. Since the action of Σ_n is relevant to our discussion we note that $\overline{\sigma t} = \sigma \bar{t}$. From this fact we can conclude the following lemma.

Lemma 4.3. *For $t, s \in \mathcal{T}_0(\lambda)$, if $s \to t$, then $\bar{t} \to \bar{s}$.*

Proof. Suppose there exists $\sigma \in C_s$ and $\pi \in R_{\sigma s} = R_t$ such that $\pi\sigma s = t$. Then $\pi\sigma\bar{s} = \overline{\pi\sigma s} = \bar{t}$. So $\sigma^{-1}\pi^{-1}\bar{t} = \bar{s}$; moreover $\pi^{-1} \in C_{\bar{s}}$ and $\sigma^{-1} \in R_{\bar{t}}$. □

This lemma allows us to induce an involution on P^λ where if $\gamma = (t_1, t_2, \ldots, t_l) \in P^\lambda$ then $\bar{\gamma} = (\bar{t}_l, \bar{t}_{l-1}, \ldots, \bar{t}_1) \in P^\lambda$. Further, we wish to demonstrate that this involution fixes no paths. In order to prove this, we will need the following corollary of (2.3.2).

Corollary 4.4. *Suppose $\lambda \vdash n > 1$ is self-conjugate. Then for all $t \in \mathcal{T}(\lambda)$, we have $t \not\to \bar{t}$.*

Proof. Let $\lambda \vdash n > 1$ be self-conjugate. Then $(2, 1), (1, 2) \in [\lambda]$. Let $a = t(1, 1)$ and $b = t(2, 1)$. Since a, b are in the same column of t, they are in the same row of $\bar{t}$. Thus $t \not\to \bar{t}$ by (2.3.2). □

Now we have the tools needed to prove that the involution on P^λ fixes no elements.

Lemma 4.5. *Suppose $\lambda \vdash n > 1$ is self-conjugate and $\gamma \in P^\lambda$. Then $\gamma \neq \bar{\gamma}$.*

Proof. Let $\gamma = (t_1, t_2, \ldots, t_l)$. Assume for the sake of contradiction that $\gamma = \bar{\gamma}$. We will consider two cases.

<u>Case 1</u>: Assume γ has odd length. Then l is even and $t_{l/2+1} = \bar{t}_{l/2}$; thus $t_{l/2} \to \bar{t}_{l/2}$, which contradicts Corollary 4.4.

<u>Case 2</u>: Assume γ has even length. Then s is odd and $t_{(l+1)/2} = \bar{t}_{(l+1)/2}$, which is impossible. □

Finally we can conclude the following result for self conjugate partitions.

Theorem 4.6. *Suppose $\lambda \vdash n > 1$ is a self-conjugate partition. Then there does not exist a nonzero $F\Sigma_n$-module M such that $S^\lambda \cong S^{(n)} \oplus M$.*

Proof. By Lemma 4.5 there is an involution on the finite set P^λ with no fixed points; hence $|P^\lambda|$ is even. Therefore the theorem follows from Theorem 4.2. □

5. Computing the sum of polytabloid coefficients

In this section, we present an algorithm to compute the sum of the polytabloid coefficients described in Lemma 3.4. For the remainder of the section fix a particular Lucas perfect partition λ of n. For simplicity of notation, we define $m = |\mathcal{T}_0(\lambda)|$ to be the dimension of the Specht module $S^{(\lambda)}$. Let $t_1, \ldots, t_m$ be an enumeration of all standard λ-tableaux which preserves the dominance order, that is, $\{t_j\} \lhd \{t_k\}$ only if $k < j$. Next, for the sake of convenience, we adopt the notation

$$X_j = X_{t_j} = \{s \in \mathcal{T}_0(\lambda) \mid s \to t_j \text{ and } s \neq t_j\}.$$

Additionally for $1 \leq j \leq m$, let $Z_j = \{t_1, t_2, \ldots t_j\}$ be the set of the first j standard tableaux under our chosen ordering, using the convention that $Z_0 = \varnothing$.

Define the $m \times (m+1)$ matrix $\boldsymbol{V} = [\boldsymbol{v}^0, \boldsymbol{v}^1, \boldsymbol{v}^2, \ldots, \boldsymbol{v}^m]$ by

$$\boldsymbol{v}_j^i = 1 + \sum_{s \in X_j \cap Z_i} x_s$$

for all $0 \leq i \leq m$ and $1 \leq j \leq m$. Since $Z_0 = \varnothing$ we have that $v_j^0 = 1$ for all $1 \leq j \leq m$. Also if $t_i \to t_j$ and $i \neq j$ then $t_i \rhd t_j$ and hence $i < j$. So we conclude for $k \geq j$, $X_j \cap Z_k = X_j$. Therefore

$$\boldsymbol{v}_j^m = 1 + \sum_{s \in Z_j} x_s = x_{t_j}.$$

Hence the sum of the coefficients of $\boldsymbol{v}^m$ will be congruent to the sum of polytabloid coefficients from Theorem 3.5. In this final section we develop the algorithm to compute v_j^m for all $1 \leq j \leq m$ in order to compute the sum of those coefficients. To this end, observe if $i = j$ or $t_i \not\to t_j$ then $X_j \cap Z_i = X_j \cap Z_{i-1}$, so $v_j^i = v_j^{i-1}$. Additionally if $i \neq j$ and $t_i \to t_j$, then $i < j$ and $X_j \cap Z_i = (X_j \cap Z_{i-1}) \cup \{t_i\}$. Thus for all $1 \leq i \leq m$,

$$\begin{aligned} \boldsymbol{v}_j^i &= 1 + \sum_{s \in X_j \cap Z_i} x_s = x_{t_i} + 1 + \sum_{s \in X_j \cap Z_{i-1}} x_s \\ &= \left(1 + \sum_{s \in X_i} x_s\right) + \left(1 + \sum_{s \in X_j \cap Z_{i-1}} x_s\right) = \boldsymbol{v}_i^{i-1} + \boldsymbol{v}_j^{i-1} \end{aligned}$$

since $X_i \cap Z_{i-1} = X_i$. We summarize our observations by noting for all $1 \leq i, j \leq m$,

$$\boldsymbol{v}_j^i \equiv \begin{cases} \boldsymbol{v}_i^{i-1} + \boldsymbol{v}_j^{i-1} \pmod 2 & \text{if } t_i \to t_j \text{ and } i \neq j, \\ \boldsymbol{v}_j^{i-1} \pmod 2 & \text{otherwise.} \end{cases}$$

We can now see that in order to develop an algorithm which will compute the desired vectors, it is necessary for our algorithm to determine if $t \to s$ for all $t, s \in \mathcal{T}_0(\lambda)$.

5A. ***Algorithm for testing the production of tabloids from polytabloids.*** For $\lambda \vdash n$, let $t, s \in \mathcal{T}_0(\lambda)$ be such that $\{s\} \lhd \{t\}$. By definition, $t \to s$ if and only if there exist $\pi \in C_t$ and $\sigma \in R_s$ such that $\sigma\pi t = s$. If such a $\pi \in C_t$ exists, the image of $t(i, j)$ under π is uniquely determined as $t(i_0, j)$, where $t(i, j) \in \mathcal{R}_s(i_0)$. The algorithm defined below will attempt to build the permutation π, defining it by necessity. It is possible that such a function does not exist, depending on the shape of λ. Additionally even if such a function exists, it may not define a bijection. The following algorithm tests to see if a permutation π can be defined.

Python Algorithm 5.1.

```
def produces(t, s):
  is_mapped_to = {}
  for val in t.vals:
  # For each positive integer less than n, attempt to find
  # the necessary image for that integer.
    (i, j), (i0, j0) = t.coords_of(val), s.coords_of(val)
    # Identify the column of t and row of s containing
    # the current value.
    try:
       target = t[i0][j]
       # Identify the necessary image of value by the function.
    except IndexError:
       # Return False since the target node is not in the
       # young diagram, hence the function cannot be defined.
       return False
    if is_mapped_to.get(target):
    # If the target is already an image of a previous value,
    # the function cannot be a bijection, so we return False.
       return False
    is_mapped_to[target] = True
    # After identifying the target for the value, record
    # that the target has been used.
  return True
```

The function produces(t, s) returns True if and only if the desired bijection $\pi \in C_t$ exists, and hence $t \to s$. Now we can use our observations and this function to write an algorithm which will compute the desired sum of polytabloid coefficients.

5B. ***Algorithm to sum polytabloid coefficients.*** We note that produces(t, s) is computationally demanding. This is not surprising as determining which s are produced from t is inherently tied to generating the coefficients of the polytabloid.

In an effort to be more computationally efficient we note since $v_i^{i-1} + v_j^{i-1} \equiv v_j^{i-1} \equiv v_j^i \pmod 2$ whenever $v_i^{i-1} \equiv 0 \pmod 2$, we have $v^i \equiv v^{i-1} \pmod 2$. Thus it is not necessary to determine if $t_i \to t_j$ for such i. Hence our final algorithm will not evaluate produces(t_i, t_j) in these cases.

Python Algorithm 5.2.

```
def sum_of_coefficients(standard_tableaux):
# standard_tableaux contains a list of all standard tableaux
# for a fixed partition lambda of n, ordered with respect to
# the dominance relation, with the least dominant first.
 standard=standard_tableaux[::-1]
 # this creates a second list of standard tableaux with
 # the order reversed, so most dominant is first.
 v = [1] * len(standard)
 # Define initial vector
 for i, t in enumerate(standard_tableaux):
  if v[i] == 0:
  # Skip the evaluation of produce function since the
  # entry is congruent to 0.  The next vector in the
  # sequence is congruent to the current vector.
    continue
  for j, s in enumerate(standard_tableaux[:i]):
  # Create the next vector in the sequence adjusting
    if produces(t, s):
    # If the corresponding tableau produces the
    # second, adjust the vector entry accordingly,
    # otherwise leave it the same.
     v[j] = (v[j] + 1) % 2
return sum(v) % 2
```

6. Conclusions

Using Python Algorithm 5.2, we are invoking Theorem 3.5 in order to determine if a particular Specht module has a one-dimensional summand. We exhausted the computational power available to us evaluating Lucas perfect partitions up to $n = 19$ excluding the partitions of the form (n) and (1^n). Table 1 records the output of our sum_of_coefficients() function for various partitions. We note in our results that the partitions corresponding to the Specht module having a one-dimensional summand were previously known Specht modules corresponding to hook partitions. Our results again confirm Murphy's result (Theorem 1.1) for these partitions.

n	λ	Σ
5	(3,1,1)	0
7	(3,1,1,1,1)	1
7	(5,1,1)	1
8	(**3,3,2**)	0
9	(3,1,1,1,1,1,1)	0
9	(5,1,1,1,1)	0
9	(7,1,1)	0
9	(**3,3,3**)	0
11	(3,1,1,1,1,1,1,1,1)	1
11	(5,1,1,1,1,1,1)	0
11	(7,1,1,1,1)	0
11	(9,1,1)	1
13	(3,1,1,1,1,1,1,1,1,1,1)	0
13	(5,1,1,1,1,1,1,1,1)	1
13	(7,1,1,1,1,1,1)	0
13	(**3,3,3,1,1,1,1**)	0
13	(9,1,1,1,1)	1
13	(11,1,1)	0
13	(**7,3,3**)	0
15	(3,1,1,1,1,1,1,1,1,1,1,1,1)	1
15	(5,1,1,1,1,1,1,1,1,1,1)	1
15	(7,1,1,1,1,1,1,1,1)	1
15	(9,1,1,1,1,1,1)	1
15	(11,1,1,1,1)	1
15	(13,1,1)	1
17	(**3,3,3,1,1,1,1,1,1,1,1**)	0
17	(**11,3,3**)	0
17	(9,1,1,1,1,1,1,1,1)	0
17	(11,1,1,1,1,1,1)	0
17	(7,1,1,1,1,1,1,1,1,1,1)	0
17	(**7,3,3,1,1,1,1**)	0
17	(13,1,1,1,1)	0
17	(5,1,1,1,1,1,1,1,1,1,1,1,1)	0
17	(15,1,1)	0
17	(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	0
19	(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	1
19	(17,1,1)	1
19	(5,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	0
19	(15,1,1,1,1)	0
19	(7,1,1,1,1,1,1,1,1,1,1,1,1)	0
19	(13,1,1,1,1,1,1)	0
19	(9,1,1,1,1,1,1,1,1,1,1)	0
19	(11,1,1,1,1,1,1,1,1)	0

Table 1. The partitions which are not hook partitions are noted in bold.

We see that in the collection of partitions within our computational limits, there are very few such nonhook partitions. Moreover a handful of these partitions are self-conjugate, so based on Theorem 4.6, we know these partitions would not have a one-dimensional summand. Perhaps with additional computational power or a more refined algorithm, we may discover a nonhook Specht module with a one-dimensional summand. It is worth noting that such an example would be the first decomposable Specht module associated to a partition that is not 2-quotient separated (see [James and Mathas 1996, Section 2] and [Dodge and Fayers 2012, Section 8.2]) ever discovered.

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