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The homophonic quotient groups for French and English (i.e., the quotient of the free group generated by the French/English alphabet determined by relations representing standard pronunciation rules) were explicitly characterized by Mestre et al. (1993). We apply the same methodology to three different language systems: German, Korean, and Turkish. We argue that our results point to some interesting differences between these three languages (or at least their current script systems).

## 1. Introduction

Mestre et al. [1993] explicitly characterized the homophonic quotient groups for French and English (i.e., the quotient of the free group generated by the French/English alphabet determined by relations representing standard pronunciation rules). Some references mention an analogous characterization for Japanese, but that result does not seem to be easily accessible.

In this paper we apply the same methodology to three different language systems: German, Korean, and Turkish. The analysis for German was circulated in unpublished form for a while; the Korean and the Turkish analyses are new. As we suggest in the final section of this paper, our results may point to some interesting differences between these three languages (or at least their current script systems).

The paper is organized in a straightforward manner, with each numbered section presenting the analysis for one language. In particular [Section 2](#) presents our results for German, [Section 3](#) presents our results for Korean, and [Section 4](#) presents our results for Turkish. A final section brings together these analyses and offers some thoughts on what we might gain from this comparative study.

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## 2. German

In their phonetically calibrated paper [Mestre et al. 1993], Mestre, Schoof, Washington, and Zagier showed that the homophonic quotient of the free group on the 26 letters of the alphabet is trivial for both the French and the English languages. As already foreshadowed in that paper, we obtain the same answer for the German language.

Let  $G$  be the quotient of the free group on 26 letters  $a, b, c, \dots, z$  by the relations  $A = B$  provided there are words  $A$  and  $B$  in the German language whose pronunciations agree.

We justify the term “agree” by invoking standard dictionaries like [Duden 1986; 1990], whose name “Duden” has become synonymous with the official norm, as well as its online version <http://www.duden.de/suchen/dudenonline>. Alternatively, for most of the pairs of words below, we can use an automatic phonetic converter such as the one at <http://familientagebuch.de/rainer/2007/38.html#4>.

**Theorem 1.** *The group  $G$  is trivial.*

*Proof.* We successively eliminate letters using specific properties of spoken German. For homophonicity we need to distinguish in particular between long and short vowels as well as between voiced and unvoiced consonants.

**Vowels** (methods of idempotents, see [Mestre et al. 1993], and of vanishing with ‘h’).

(a) For instance, ‘aa’, ‘ah’ and ‘a’ may often be pronounced alike, in particular they often have the same length, like “Waage” [scales] and “wage” [(I) dare] or “Wahl” [choice] and “Wal” [whale].

(e) Similarly, ‘ee’, ‘eh’ and sometimes ‘e’ can sound the same: “Meer” [the sea] and “mehr” [more].

(o) Both ‘oo’ and ‘oh’ are often used within words, and can both be pronounced like a single ‘o’ (“Boot” [boat] and “bot” [(he) offered], and “hohle” [hollow (pl.)] vs. “hole” [(I) fetch]).

We note that for ‘i’ and ‘u’ the corresponding identifications do not work; e.g., while both ‘ie’ and ‘ih’ indicate a long ‘i’, the former can never occur at the beginning of a word where it is instead replaced by the second one (“ihnen” [(to) them], “ihr” [her]), and ‘ii’ in a word (like “liieren” [(to) liaise]) is pronounced with a glottal stop between the ‘i’s. Similarly, the ‘uu’ in words like “Kontinuum” or “Trauung” indeed comes across as two ‘u’s, and there aren’t any words with a ‘uu’ that would sound like a long ‘u’, say. Hence we need to treat these two vowels separately.

### **Consonants.**

(g/b/n) (voiceless in the end) At the end of a word, a voiced consonant is pronounced in the same way as the corresponding unvoiced one (like “Bug” [(nautical) bow]

and “buk” [(he) baked], or “Alb” and “Alp” [both for nightmare]. Similarly, an ‘nn’ at times sounds like a single ‘n’ (“Mann” [man] and “man” [one/you (pronoun)]).

(v/w) (WVF?) The consonant ‘v’ is typically pronounced in one of two ways: like ‘f’ or like ‘w’, depending mostly on the etymological origin of the word (“viel” [many] vs. “fiel” [(he) fell] and “vage” [vague] vs. “wage” [(I) dare]).

(l/r/f/p/s) (idempotents) By combining certain consonants we can further minimize the influence of a single contributing consonant, so while it is hard to find the same sounds for ‘ll’ and ‘l’ at the end of a word, one can add a ‘t’ to it and succeed (“hallt” [(it) reverberates/echoes] vs. “Halt” [halt]). Similar comments apply to ‘rr’ and ‘r’ (“starrt” [(he) stares] vs. “Start” [start]), for ‘ff’ and ‘f’ (“schafft” [(he) manages] vs. “Schaft” [shaft]), as well as ‘pp’ and ‘p’ (“klappst” [(you) flap/fold] vs. ‘klapst’ [(he) claps lightly]; alternatively, “schnippst” and “schnipst”, both from schnipsen [(to) clip]). Furthermore, “fasst” [(he) catches] and “fast” [almost] are homophonic.

(t/d) (little ‘dt’ for ‘tt’) A related case is the combination ‘th’ which also often ensures that a preceding vowel is pronounced as a short one: e.g., “Zithern” [zithers] and “zittern” [(to) tremble] are pronounced the same way; another means to the same end is the use of ‘dt’ in place of ‘tt’, giving, e.g., that “Stadt” [city] and “statt” [instead of] are homophonic.

(m) A variant of the idempotent method, using also the voiced/unvoiced consonant at the end of a word, is “hemmt” [hinders] vs. “Hemd” [shirt].

(c) (departing of the ‘c’) Other constructs that make sure that a vowel is short are to follow it up with a ‘ck’ rather than a ‘k’; for example, the words “packt” [(he) packs] and “Pakt” [(a) pact] sound alike. Note, however, that in a very similar setting the words “hackt” [(he) hacks] and “hakt” [(he) hooks] are pronounced differently, as the latter ‘a’ then denotes a *long* vowel.

(z) A further peculiarity is the pronunciation of ‘z’, typically equivalent to the combination ‘t-s’ (with obvious exceptions for loanwords like “Jazz” where the educated citizen will make an attempt to sound more anglophonic), so we can identify the genitive “Kitts” of “Kitt” [glue] with “Kitz” [fawn].

(x) In the same vein as ‘z’, the letter ‘x’ is pronounced ‘k-s’ which is also the pronunciation of ‘chs’ (i.e., when ‘ch’ precedes ‘s’ it often becomes ‘k’), so we find “lax” [lax] to be homophonic to “Lachs” [salmon].

The remaining letters ‘k’, ‘u’, ‘i’, ‘y’, ‘j’ and ‘q’ are somewhat harder to trivialize, but modulo the above this is doable, albeit by using loanwords from different languages (English, Italian, Hungarian).

(k) The English word “Clip” for office equipment is often used and is homophonic to “klipp” (e.g., from “klipp und klar” [in no uncertain terms]).

(u) The Italian word “ciao” has been assimilated as “tschau”, both terms being used.

(i) The word “roien” [(to) row] is homophonic to “reuen” [(to) rue], the former being used mainly in “Niederdeutsch”, i.e., in the north of Germany. Alternatively, the loanword (from the English language) “beaten” [(to) make beat music] is acceptable according to [Duden 1990], and it is homophonic to “bieten” [(to) offer].

(y) The word “toi” from the saying “toi, toi, toi” [break a leg] sounds like “Toy” [sex toy]. Alternatively, a “Bayer” [Bavarian] can be spelled “Baier”. (We could also invoke the ambiguous spellings of “Yoghurt” and “Joghurt”. For yet another possibility, the Hungarian word “Gulyas” [goulash] has been assimilated also as “Gulasch”.)

(j) As to ‘j’, we use the word “Yak” [yak] (from the Tibetan “gyag”) and its similarity to “Jacke” [jacket], which are not homophonic as such, but their respective diminutives “Yäkchen” and “Jäckchen” (note the ensuing umlaut for either case) are.

(q) Finally, for the quite rare letter ‘q’, we can use the French word “clique” (which has been adapted into German with a short ‘i’), whose pronunciation agrees with that of “klicke” [(I) click]. Another possibility is to note that the letter “Q” itself can be used as a word (say, as the Q in a game of Scrabble) and is homophonic to “Kuh” [cow].

In the table below we successively eliminate the letters on the left using the homophonic ambiguity displayed on their right, completing the proof of the theorem:

a	Waage — wage	s	fasst — fast
h	Wahl — Wal	t	Zittern — Zithern
e	Meer — mehr	d	Stadt — statt
o	Boot — bot	m	hemmt — Hemd
g	Bug — buk	c	packt — Pakt
b	Alb — Alp	z	Kitz — Kitts
n	Mann — man	x	lax — Lachs
v	viel — fiel	k	klipp — Clip
w	wage — vage	u	tschau — ciao
l	gewallt — Gewalt	i	roien — reuen
r	startt — Start	y	Toy — toi
f	schaftt — Schaft	j	Jäckchen — Yäkchen
p	klappst — klapst	q	Clique — klicke

□

**Generalizations.** One can also try to include the umlaute ‘ä’, ‘ö’, ‘ü’, and the “sharp s” ß into these investigations. The result remains the same. Our suggestion for the corresponding trivializations are the following: For ‘ä’ we invoke that in combination with ‘u’ the diphthongs ‘äu’ and ‘eu’ sound alike, for instance in the words “häutig” [of a skinny texture] and “heutig” [contemporarily]; alternatively, we can use that a long ‘ä’ can sound like the ‘ai’ for certain loanwords from the English language, for example in “Fähre” [ferry] and “faire” [fair]. For ‘ö’ we use that certain words are spelled with both the original French ‘eu’ and the assimilated German ‘ö’, like “Frisör” and “Friseur”. Furthermore, the pronunciation of ‘ü’ is often the same as that of ‘y’, like in the Greek letter “My” [mu] and “müh” [(I) labor], or, a far better one due to Martin Brandenburg, “Mythen” [myths] and “mühten” [(they) labored]. Finally, a ‘sharp s’ at the end of a word is typically preceded by a long vowel, and hence it is not difficult to construct word pairs like “aß” [(I) ate] and “Aas” [(rotten) carcass]:

ä	häutig — heutig
ö	Frisör — Friseur
ü	müh — My
ß	aß — Aas

### 3. Korean

What differentiates Korean from the languages discussed in [Mestre et al. 1993] is the number of alphabets and some fundamental grammar structures. Nevertheless, there exist many rules regarding homophones, so the first natural assumption would be that the resulting quotient group shouldn’t have too many elements. It turns out that this is indeed the case.

Here we note that this mathematical analysis of Korean does not describe the entire structure of the Korean language. It takes the phonetic aspect of the language and restructures the alphabets into a free group with a very specific and somewhat restrictive equivalence relation. Using such a structure, we inevitably lose a lot of information about the Korean language, but are, however, rewarded with a unique finite group that characterizes it.

Now, let us begin with describing some necessary concepts about the Korean language.

**3.1. Some basics of Korean.** Korean characters, like English, consist of *vowels* and *consonants*. The alphabet contains 19 consonants and 21 vowels. The exact list is shown below in Table 1.

Because of the complications arising from the unique structure of Korean, from here on, each of the above symbols in the table will be called a *character*. To show why such clarification is crucial, let us take a look at a Korean word that stands for



we will examine the words  $\text{부엌}$  [kitchen] and  $\text{밖}$  [outside]. By the equivalence relation defined above,  $\text{부엌} = \text{부엌}$  and  $\text{밖} = \text{밖}$ . By rewriting these relations in ordered decomposition,  $\text{ㅂ} + \text{ㅍ} + \text{ㅇ} + \text{ㅣ} + \text{ㅋ} = \text{ㅂ} + \text{ㅍ} + \text{ㅇ} + \text{ㅣ} + \text{ㄱ}$  and  $\text{ㅅ} + \text{ㅈ} + \text{ㅍ} = \text{ㅅ} + \text{ㅈ} + \text{ㅍ}$ . Now it is clear that  $\text{ㄱ} = \text{ㅋ}$  and  $\text{ㄱ} = \text{ㅍ}$ . By the transitive property  $\text{ㅋ} = \text{ㅍ} = \text{ㄱ}$ .

For the second part we can examine the equivalence relations,  $\text{낫}$  [scythe] =  $\text{낫}$ ,  $\text{낮}$  [day] =  $\text{낫}$ ,  $\text{낫}$  [face] =  $\text{낫}$ ,  $\text{밭}$  [field] =  $\text{받}$ . Clearly  $\text{ㄷ} = \text{ㅅ} = \text{ㅈ} = \text{ㅊ} = \text{ㅌ}$ . Proving  $\text{ㄷ} = \text{ㅈ}$  is a bit more difficult as there are no single-letter words in Korean ending in  $\text{ㅈ}$ . To prove this we need to look at the two-letter word  $\text{불소}$  [fluorine]. By the equivalence relation  $\text{불소} = \text{불쏘}$ , we clearly have  $\text{ㅅ} = \text{ㅈ}$ . Since we already know  $\text{ㅅ} = \text{ㄷ}$ , by the transitive property,  $\text{ㄷ} = \text{ㅈ}$ , thus concluding our proof of the second equivalence relation.

For the last equivalence relation, we can look at  $\text{짚}$  [hay] =  $\text{집}$ , and can conclude that  $\text{ㅈ} = \text{ㅉ}$ .

By proving these relations, we've reduced the set of consonants to  $\{\text{ㄱ}, \text{ㄴ}, \text{ㄷ}, \text{ㄹ}, \text{ㄷ}, \text{ㄷ}, \text{ㄷ}, \text{ㅈ}, \text{ㅈ}, \text{ㅈ}\}$ .

**3.2.3. Consonants are trivial.** To further reduce the set of consonants let us look at the equivalence relation  $\text{앞마당}$  [lawn] =  $\text{암마당}$ . This shows that  $\text{ㄴ} = \text{ㅍ}$ , and  $\text{ㅍ} = \text{ㅂ}$ , so  $\text{ㄴ} = \text{ㅂ}$ . Additionally,  $\text{있는}$  [existing] =  $\text{인는}$ , and so  $\text{ㄴ} = \text{ㅈ} = \text{ㄷ}$ . Also,  $\text{국물}$  [soup] =  $\text{궁물}$ , and  $\text{놓는}$  [lay down] =  $\text{논는}$ , so  $\text{ㄱ}$  is trivial and  $\text{ㄷ} = \text{ㄴ} = \text{ㅎ}$ . Observe that  $\text{술하다}$  [to be in abundance] =  $\text{수타다}$ , which shows that  $\text{ㅎ}$  is also trivial. Since  $\text{ㄷ} = \text{ㅎ}$  and  $\text{ㅎ}$  is trivial,  $\text{ㄷ}$  is also trivial. Now, there only remain five nontrivial consonants,  $\{\text{ㄹ}, \text{ㄷ}, \text{ㅂ}, \text{ㅈ}, \text{ㅈ}\}$ .

Let's look at the equivalence relation  $\text{웃다}$  [smile] =  $\text{운따}$ , which in ordered decomposition is  $\text{ㅇ} + \text{ㅍ} + \text{ㅅ} + \text{ㄷ} + \text{ㅣ} = \text{ㅇ} + \text{ㅍ} + \text{ㄷ} + \text{ㅉ} + \text{ㅣ}$ . We know  $\text{ㅇ}, \text{ㅅ} = \text{ㄷ}$  are trivial, so  $\text{ㅍ} + \text{ㅣ} = \text{ㅍ} + \text{ㅉ} + \text{ㅣ}$ . Hence  $\text{ㅉ} = 1$  and so  $\text{ㅉ}$  is also trivial.  $\text{약지}$  [ring finger] =  $\text{약찌}$ ; hence  $\text{ㅈ} = \text{ㅊ} = \text{ㄷ} = 1$  and  $\text{ㅈ}$  is trivial.  $\text{막론}$  [whether] =  $\text{망논}$ , which can be rewritten as  $\text{ㅁ} + \text{ㅣ} + \text{ㄱ} + \text{ㄷ} + \text{ㄴ} = \text{ㅁ} + \text{ㅣ} + \text{ㅇ} + \text{ㄴ} + \text{ㄱ} + \text{ㄴ}$ , and since  $\text{ㄱ}, \text{ㄴ} = \text{ㄷ}$  are both trivial,  $\text{ㅁ} + \text{ㅣ} + \text{ㄷ} + \text{ㄱ} = \text{ㅁ} + \text{ㅣ} + \text{ㄱ}$ , and so  $\text{ㄷ}$  is trivial.  $\text{국밥}$  [soup and rice] =  $\text{국빱}$  implies that  $\text{ㅂ} = \text{ㅃ}$ . There remains only one nontrivial consonant,  $\text{ㅂ}$ .

Lastly we need to examine a word with a letter of the form  $c + v + c + c$ .  $\text{넓다}$  [wide] =  $\text{널따}$ , and we know that  $\text{ㄴ}, \text{ㄷ}, \text{ㄷ}, \text{ㅉ}$  are all trivial. So, after canceling both sides, we have  $\text{ㅣ} + \text{ㅂ} + \text{ㅣ} = \text{ㅣ} + \text{ㅣ}$ , and so  $\text{ㅂ}$  is trivial. Hence we've proved the triviality of all Korean consonants.

**3.3. Vowels have two nontrivial elements: vowels =  $\{\text{ㅣ}, \text{ㅓ}\}$ .** While in examining consonants we only needed to look at a single equivalence relation, vowels are not so easy. There are multiple equivalence relations between three or more vowels, so we need to sort through these relations to see how they can be reduced. Furthermore,



- Recall that from (7) and (13), we have  $\lrcorner + \lrcorner = \lrcorner\lrcorner = \lrcorner + \lrcorner$ . However,  $\lrcorner = \lrcorner + \lrcorner = \lrcorner$ , so  $\lrcorner + \lrcorner = \lrcorner + \lrcorner$ , and by canceling  $\lrcorner$ , we have that  $\lrcorner = \lrcorner$ .
- (8) states that  $\lrcorner = \lrcorner$ , so with the above result,  $\lrcorner = \lrcorner$ .
- Since  $\lrcorner = \lrcorner + \lrcorner$ , we have  $\lrcorner$  is generated by  $\lrcorner$  and  $\lrcorner$ .

**Conclusion.** The homophonic quotient of the Korean language can be written in terms of two generators  $\{ \lrcorner, \lrcorner \}$ .

In some sense, we have identified the two most fundamental characters in Korean as their pronunciations are not discarded in any Korean word they appear in. Furthermore because we have allowed for the equivalence of words and their reduced forms, it is unlikely that the set of distinct Korean characters under homophonic quotients can be further reduced. However, distinctions between pronunciations of certain vowels are becoming more obscure; hence it is possible that after appropriately adjusting the formal pronunciation rules to accommodate such trends, the homophonic quotient group of Korean is further reduced.

### 4. Turkish

In this section we determine the homophonic quotient group for Turkish. There are several Turkic languages, and alphabets encoding them have many commonalities. We will exclusively focus on the modern Turkish alphabet.

**4.1. The sounds of Turkish.** The modern Turkish alphabet was introduced in 1928 along with a wide-reaching literacy campaign. The Latin-based script was developed to replace the use of the Arabic script, and contains a total of 29 letters (8 vowels and 21 consonants) as seen in Table 2.

This set of letters was specifically selected to represent the sounds present in the spoken language of the time, taking the Istanbul dialect as the standard. Each letter is supposed to represent a unique sound of the spoken language (except the so-called “soft g”, ğ, which tends to extend the vowel before it and blends it to the following vowel if there is one, but is otherwise completely silent; see [Logacev et al. 2014] for more on the “soft g”). For more on the sound system of modern Turkish, see [Yavuz and Balci 2011].

To this day the modern Turkish script retains most of its phonetic representativeness [Kopkalli-Yavuz 2010]. Indeed many hold that there are no homophones in

consonants	b c ç d f g ğ h j k l m n p r s ş t v y z
vowels	a e ı i o ö u ü

**Table 2.** Letters of the modern Turkish script (only lowercase letters are given).

Turkish; see for instance [Raman and Weekes 2005], where Turkish is described as a “completely transparent writing system” with “invariant and context-independent one-to-one mappings between orthography and phonology”.

This suggests that the free group generated by the 29 sound representatives will not shrink much if at all when we try introducing homophonic equivalences. Nonetheless there are indeed some relations we might use if we consider “how words are actually pronounced by real live people”.<sup>1</sup>

**4.2. The “soft g” disappears.** As noted above the “soft g” is often not a distinctly pronounced consonant but instead helps to accentuate or blend the surrounding vowels. Most native speakers would agree that we can identify the following encodings of the male name meaning “Khan”:

$$\text{Kaan} = \text{Kağan}.$$

Thus in the quotient group we would identify the “soft g” with identity.

**4.3. Other disappearing acts: ‘h’ and ‘t’.** The standard pronunciation of the word “dershane” [classroom] overlaps with the pronunciation of “dersane”, thus allowing us to conclude that ‘h’ too is trivial in the quotient. Similarly the double ‘t’s in the words “Hacettepe” and “Anıttepe” [two location names in Ankara] are most commonly pronounced as if they were written as “Hacetepe” and “Anitepe” respectively. Thus we can identify ‘tt’ with ‘t’, trivializing ‘t’.

**4.4. Vowel confusion: the transformations of ‘a’ and ‘e’ into ‘ı’ and ‘i’ and two final disappearing acts.** The Turkish language captures the phrase “let me look” in the single word “bakayım”. The native speaker pronounces the latter in the same way that she would read the letter collection “bakıyım”. This allows us to identify  $a = ı$ . Similarly the phrase “içecek” [drink] is pronounced the same way that one would read “içicek” and so we identify  $e = i$ .

Finally the word “ağabey” [older brother] has an almost universally accepted informal spelling, “abi”, representing the way people actually pronounce the word. Together with the sound equivalence of “ağa” [master, land owner] and “ağ” [network], this gives us two additional trivializations, of ‘a’ and ‘y’.

Putting the above reductions together we conclude that the homophonic quotient group for Turkish is a free group on 22 generators:

$$b, c, \check{c}, d, e (= i), f, g, j, k, l, m, n, o, \ddot{o}, p, r, s, \check{s}, u, \ddot{u}, v, z$$

<sup>1</sup>In his MathSciNet review (MR1273406), James Wiegold notes that the authors of [Mestre et al. 1993] “have [perhaps deliberately?] neglected all considerations of how words are actually pronounced by real live people.” Clearly if we were to take into consideration each native speaker’s distinct pronunciation patterns, the homophonic quotients problem would become quite intractable. However we will indeed introduce some of this complication into our analysis of Turkish. This may be justified by the fact that there is deemed to be a standard spoken Turkish, and it is indeed distinct from most formal descriptions of the orthography/phonology correspondence for the language.

## 5. Final words: bringing the three threads together

In this paper we investigated three different languages and their writing systems. We believe that our results offer an interesting example of applied algebra. We explored how the writing system of a modern language and its correspondence with the sounds of that language can be encoded in group theory. Other algebraic structures have been identified in various symmetrical constructions of nature such as crystals, as well as in a range of sociological and anthropological contexts such as the kinship structure of the Warlpiri of Australia.<sup>2</sup>

It is important to note that our methods do not address the full phonetic structure of any single language. Our work only pertains to the relationship between orthography and phonology of a language, that is, the extent to which a single symbol may represent a multiplicity of sounds of a given language. A simplistic interpretation of our method would suggest that if the generating set for the resulting quotient group is small, there are, on average, more sounds represented by a single symbol.

We should also note that the complexity of the resulting group may be correlated not directly with the complexity of the sound system of a given language but perhaps more with the maturity of the particular writing system associated to it. Languages evolve, and oral traditions evolve much faster than written ones. Thus a young script like modern Turkish might be naturally more representative of the phonetical structure of the language and equivalently offer fewer homophones than a script which is more mature, such as the Korean one, which in turn may offer fewer homophones than an even older script such as the German one.

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### References

- [Ascher 1991] M. Ascher, *Ethnomathematics: a multicultural view of mathematical ideas*, Chapman & Hall, New York, 1991. MR Zbl
- [Duden 1986] K. Duden, *Rechtschreibung der deutschen Sprache und der Fremdwörter*, edited by D. Berger and W. Scholze, Der Duden in 10 Bänden 1, Duden, Mannheim, 1986.
- [Duden 1990] K. Duden, *Duden Aussprachewörterbuch: Wörterbuch der deutschen Standardaussprache*, edited by M. Mangold, Der Duden in 10 Bänden 6, Duden, Mannheim, 1990.
- [KLI] “Romanization of Korean”, website, The National Institute of Korean Language, available at <http://www.mcst.go.kr/english/koreaInfo/language/romanization.jsp>.

<sup>2</sup>As ethnomathematician Marcia Ascher describes in detail in her book, the kinship structure of the Warlpiri, an indigenous people in Australia, can be accurately and succinctly represented by the dihedral group of order 8. See Chapter 3 of [Ascher 1991] for details.

- [Kopkalli-Yavuz 2010] H. Kopkalli-Yavuz, “The sound inventory of Turkish: consonants and vowels”, in *Communication disorders in Turkish*, edited by S. Topbaş and M. S. Yavas, Communication disorders across languages **4**, Multilingual Matters, Bristol, 2010.
- [Logacev et al. 2014] O. U. Logacev, S. Fuchs, and M. Żygis, “Soft ‘g’ in Turkish: evidence for sound change in progress”, pp. 437–440 in *Proceedings of the 10th International Seminar on Speech Production* (Cologne, 2014), edited by S. Fuchs et al., University of Cologne, 2014.
- [Mestre et al. 1993] J.-F. Mestre, R. Schoof, L. Washington, and D. Zagier, “Quotients homophones des groupes libres = Homophonic quotients of free groups”, *Experiment. Math.* **2**:3 (1993), 153–155. [MR](#) [Zbl](#)
- [Raman and Weekes 2005] I. Raman and B. S. Weekes, “Deep disgraphia in Turkish”, *Behavioural Neurology* **16**:2-3 (2005), 59–69.
- [Yavuz and Balci 2011] H. Yavuz and A. Balci, *Turkish phonology and morphology*, edited by Z. Balpınar, Anadolu University, Eskişehir, 2011.

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