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Nilpotent orbits for Borel subgroups of $\mathrm{SO}_5(k)$

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Let G be a quasisimple algebraic group defined over an algebraically closed field k and B a Borel subgroup of G acting on the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{b}$ via the adjoint representation. It is known that B has only finitely many orbits in only five cases: when G is type A_n for $n \leq 4$, and when G is type B_2. We elaborate on this work in the case when $G = \mathrm{SO}_5(k)$ (type B_2) by finding the defining equations of each orbit. We use these equations to determine the dimension of the orbits and the closure ordering on the set of orbits. The other four cases, when G is type A_n, can be approached the same way and are treated in a separate paper.

1. Introduction

Before specializing to $G = \mathrm{SO}_5(k)$, we make some general remarks in order to provide context and some motivation for our work. Let k be an algebraically closed field and G a quasisimple algebraic group over k. Fix a maximal torus T of G, and let Φ denote the root system of G relative to T (Φ is irreducible since G is quasisimple). Fix a set Δ of simple roots in Φ, with corresponding set of positive roots Φ^+, and let $B = TU$ (U is the unipotent radical of B) be the Borel subgroup of G determined by Φ^+. Write the one-dimensional unipotent root group corresponding to a root α as U_α. Denote the Lie algebra of G by $\mathfrak{g}$, that of T by $\mathfrak{h}$, and that of B by $\mathfrak{b}$. Then the nilradical $\mathfrak{n} = \mathfrak{n}(\mathfrak{b})$ of $\mathfrak{b}$ is in fact the Lie algebra of U, and we have decompositions $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, and $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ as vector spaces, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ corresponding to α and is also the Lie algebra of U_α. There is a corresponding decomposition of $U \approx \prod_{\alpha \in \Phi^+} U_\alpha$ as algebraic varieties over k. In particular, U is generated as a group by the root groups $U = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$, a fact we use repeatedly in our calculations in Section 2 below.

G acts on $\mathfrak{g}$ via the adjoint representation, and the orbits of this action have been intensely studied, partly because there are connections between the nilpotent orbit theory and the representation theory of G. It is known that there are only finitely many nilpotent G-orbits (a *nilpotent orbit* means the orbit of a nilpotent element

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of $\mathfrak{g}$). There are combinatorial indexing sets for these nilpotent orbits, and there are formulas to compute the dimension of each orbit. Also, it is known which orbits are in the Zariski closure of any given orbit (the *closure ordering*). Therefore, it is well understood how all the nilpotent orbits fit together to form a larger object, called the *nullcone* $\mathcal{N}$ of $\mathfrak{g}$, which is the union of the nilpotent orbits. For details of this classical theory, see [Collingwood and McGovern 1993] for the characteristic-0 case and [Carter 1985; Jantzen 2004] more generally.

The notion of a support variety of a module is one example of a connection between nilpotent G-orbits and representation theory, when the characteristic of k is $p > 0$. In this case recall that there is a *p-th power map* $x \mapsto x^{[p]}$ on $\mathfrak{g}$ that makes $\mathfrak{g}$ into a restricted Lie algebra, and there is a *Frobenius map* $F : G \to G$, whose kernel is denoted by G_1, which is an infinitesimal group scheme whose rational representation theory coincides with the representation theory of $\mathfrak{g}$. Similarly, denote the kernel of $F : B \to B$ by B_1. By results of [Friedlander and Parshall 1986], the *cohomology variety* of G_1 (the maximal ideal spectrum of the even-degree cohomology ring $H^{2\bullet}(G_1, k)$) identifies naturally with a subvariety $\mathcal{N}_1 = \{x \in \mathcal{N} \mid x^{[p]} = 0\}$ of the nullcone $\mathcal{N}$ of $\mathfrak{g}$, and furthermore, for any finite-dimensional $\mathfrak{g}$-module M, there is an important subvariety of $\mathcal{N}_1$ called the *support variety* of M, denoted by $V_{\mathfrak{g}}(M)$ or $V_{G_1}(M)$. If M is a rational G-module, then $V_{G_1}(M)$ is G-stable in $\mathcal{N}_1$ and is therefore a union of nilpotent G-orbits.

Aspects of the representation theory of G_1 are determined by this support variety (for example, M is a projective module if and only if $V_{G_1}(M) = \{0\}$), so one would like to be able to compute these support varieties, and knowing they are unions of nilpotent G-orbits may be useful.

If H is a closed subgroup of G and M is a rational H-module, denote the rational G-module induced from M by $M|_H^G$. Now let $X(T)$ be the character group of T, and for $\lambda \in X(T)$, we also use the symbol λ to denote the one-dimensional T-module on which T acts via $\lambda : t \cdot v = \lambda(t)v$ for all $t \in T$. This rational T-module extends to a rational B-module by trivial U-action, also denoted by λ. The modules $\lambda|_B^G$ are (the duals of) the well-known *Weyl modules* of G. Another important class of modules are those of the form $Z(\lambda) = \lambda|_{B_1}^{G_1}$, sometimes called *baby Verma modules*. (The name comes from the fact that there is an alternate definition of $Z(\lambda)$ which is analogous to the definition of a Verma module for $\mathfrak{g}$, while the adjective "baby" can be interpreted as alluding to the infinitesimal subgroups in our definition using induction, or to the fact that the $Z(\lambda)$ are finite-dimensional and the usual Verma modules are not.)

One of the goals of the paper [Nakano et al. 2002, Theorem 6.21] was to calculate the support varieties $V_{G_1}(\lambda|_B^G)$ in order to prove the "Jantzen conjecture". A central strategy of that paper is to compare $V_{G_1}(\lambda|_B^G)$ to $V_{G_1}(Z(\lambda))$. Of course, $Z(\lambda)$ is not a G-module (only a G_1-module), so one would not expect $V_{G_1}(Z(\lambda))$ to be a

G-stable subset of $\mathcal{N}_1$. However, it turns out that this variety is stable under the action of B and therefore $V_{G_1}(Z(\lambda))$ is a union of nilpotent B-orbits [Nakano et al. 2002, Proposition 7.1.1].

That nilpotent B-orbits (as well as nilpotent G-orbits) have a connection to the representation theory of G provides a motivation for studying nilpotent B-orbits. Even without this explicit motivation, it is interesting to try to generalize what is known about nilpotent G-orbits to nilpotent B-orbits. However, the case of nilpotent B-orbits is not nearly as tidy as that of nilpotent G-orbits. One important difference is that most of the time there are infinitely many nilpotent B-orbits. Thus, finding a nice indexing set for the orbits could be difficult. In general, we would expect an indexing set to have a continuous piece as well as a discrete piece, as certain infinite families of orbits might be described by continuous parameters.

In this paper, our focus is on the case $G = SO_5(k)$, which is one of the five cases where there are finitely many nilpotent B-orbits. For each of the orbits, we find the polynomial defining equations of the orbit. From these calculations, it is easy to determine both the dimension of each orbit as well as the closure ordering for the set of orbits.

2. Nilpotent B-Orbits in $SO_5(k)$

Throughout this section we assume the characteristic of k is either 0 or a prime $p \neq 2$. The following proposition is a basic fact about algebraic group actions. A proof can be found in [Borel 1991; Humphreys 1975].

Proposition 1. *Let G be an algebraic group acting morphically on a nonempty variety V. Then each orbit is a locally closed*, *smooth variety*, *and its boundary is a union of orbits of strictly lower dimension.*

Thus, the orbit $G \cdot x$ is open and dense in its closure $\overline{G \cdot x}$, and hence has the same dimension as its closure.

We study the action of a Borel subgroup B of $G = SO_5(k)$ on the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{b} \subseteq \mathfrak{so}_5(k)$, via the adjoint representation. Our main results consist of finding the defining equations of each nilpotent B-orbit, which will exhibit each orbit explicitly as an intersection of an open set and a closed set. From there it will be an easy matter to determine the closure of each orbit, and thereby find the dimension of each orbit, as well as to determine which orbits comprise the boundary of a given orbit. That is, we will find the partial order determined by the orbit closures, which is defined as $G \cdot x \preceq G \cdot y$ if and only if $G \cdot x \subseteq \overline{G \cdot y}$.

Let f be a polynomial. We use the standard notation that the zero set of f is written as $Z(f)$ and that $Z(f, g) = Z(f) \cap Z(g)$ is the set of common zeros of polynomials f and g. If we have a finite set of polynomials $f_1, \ldots, f_r$, then $Z(f_1, f_2, \ldots, f_r)$ is a Zariski-closed set, that is, it is an affine variety. The notation

$V(f)$ denotes the complement of $Z(f)$, the set of elements that are not zeros of f, and so $V(f)$ is a Zariski-open set. A locally closed set is an intersection of an open set and a closed set, and in this section the orbits will turn out to be locally closed sets of the form $V = Z(f_1, f_2, \ldots, f_r) \cap V(g_1) \cap V(g_2) \cap \cdots \cap V(g_t)$ for polynomials f_i and $g_j \neq 0$ for all j. Observe that the closure of V is then $Z(f_1, f_2, \ldots, f_r)$ and V is open and dense in this closure, whence $\dim V = \dim Z(f_1, f_2, \ldots, f_r)$.

If $U\gamma$ is a root group of G, then $U_\gamma(t)$ denotes the image of t under the standard isomorphism $k_{\text{add}} \approx U_\gamma$. In classical groups, the adjoint action on the Lie algebra is simply conjugation of matrices. The matrix e_{ij} is the matrix with a 1 in the ij position and 0 everywhere else. Now $\mathfrak{g} = \mathfrak{so}_5(k)$, and we take T to be the set of diagonal matrices in G. More precisely, a typical element of the torus T has the form

$$T(s,t) = \operatorname{diag}(1, s, t, s^{-1}, t^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & 0 & t^{-1} \end{bmatrix}$$

for s and t nonzero in k.

For root systems we use the standard notation one finds in [Humphreys 1972; Bourbaki 2002]. In type B_2, the simple roots are $\Delta = \{\alpha_1, \alpha_2\}$ and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$, so $\mathfrak{n} \approx k^4$ as an affine variety and vector space. The root vectors in $\mathfrak{n}$ are the matrices

$$\begin{aligned} x_{\alpha_1} &= e_{23} - e_{54}, \\ x_{\alpha_2} &= e_{15} - e_{31}, \\ x_{\alpha_1+\alpha_2} &= e_{14} - e_{21}, \\ x_{\alpha_1+2\alpha_2} &= e_{25} - e_{34}. \end{aligned}$$

Now each root space has a coordinate function, which we denote by a capital X with the same subscript as the root space. In other words, a typical element of $\mathfrak{n}$ has the form of a linear combination of the four root vectors, $x = a_1 x_{\alpha_1} + a_2 x_{\alpha_2} + a_3 x_{\alpha_1+\alpha_2} + a_4 x_{\alpha_1+2\alpha_2}$, which we can write in coordinate form as (a_1, a_2, a_3, a_4) (showing explicitly $\mathfrak{n} \approx k^4$), and the coordinate function just selects the appropriate coordinate; so $X_{\alpha_1}(x) = a_1$ and $X_{\alpha_1+2\alpha_2}(x) = a_4$, for example. Thus, the B-orbits in which we are interested are locally closed sets in $\mathfrak{n}$ or k^4 which are defined by polynomials in the four variables of the polynomial ring $k[X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_1+\alpha_2}, X_{\alpha_1+2\alpha_2}]$.

To begin the calculations, let's determine the B-orbit of the highest root vector $x_{\alpha_1+2\alpha_2}$. Since the highest root vector has the highest weight of the adjoint representation, it is a maximal vector — it is fixed by U, the unipotent radical of B, and sent to a multiple of itself by the torus T. It follows that $B \cdot x_{\alpha_1+2\alpha_2} = T \cdot U \cdot x_{\alpha_1+2\alpha_2} = T \cdot x_{\alpha_1+2\alpha_2} \subseteq \mathfrak{g}_{\alpha_1+2\alpha_2}$. Thus, we need only compute the T-orbit of this weight vector, which is easy by direct calculation. Abbreviating the root vector by x, we have

$$\begin{aligned} T(s,t)\cdot x &= T(s,t)xT(s,t)^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & st \\ 0 & 0 & 0 & -st & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = stx. \end{aligned}$$

In particular, taking $s = 1$, we obtain all elements of the form tx, $t \neq 0$, as part of this orbit. Since 0 is in an orbit by itself, this shows the orbit is precisely the set of all nonzero multiples of x. In terms of linear combinations of root vectors, or coordinates in $\mathfrak{n} \approx k^4$, this says an element (a_1, a_2, a_3, a_4) belongs to the orbit $B \cdot x$ if and only if $a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$. In other words, this shows that the orbit is

$$B \cdot x = B \cdot x_{\alpha_1+2\alpha_2} = Z(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_1+\alpha_2}) \cap V(X_{\alpha_1+2\alpha_2}).$$

These are the defining equations of this orbit. Clearly, its closure is $\overline{B \cdot x} = Z(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_1+\alpha_2})$, the intersection of three coordinate hyperspaces in $n \approx k^4$, which is precisely the axis of the fourth coordinate $X_{\alpha_1+2\alpha_2}$. In particular, it is obvious that this orbit is one-dimensional (it is dense in the highest root space $\mathfrak{g}_{\alpha_1+2\alpha_2}$.)

In order to save space, we will eschew writing out the matrices from this point on, in favor of writing elements of $\mathfrak{n}$ as linear combinations of root vectors (or as ordered quadruples in k^4), and elements of B as products of elements in T and elements of the four one-dimensional root groups U_α for $\alpha \in \Phi^+$. Thus, the above calculation could be more compactly written as

$$T(s,t)U \cdot x = T(s,t) \cdot x = stx,$$

leaving the reader to check the actual matrix calculation. In subsequent calculations, it will be helpful to remember how the unipotent root groups act on weight vectors in rational G-modules:

Lemma 2 [Humphreys 1975, Proposition 27.2]. *Let $\alpha \in \Phi$, and let $v \in V_\lambda$ be a weight vector in any rational G-module. Then each element $u \in U_\alpha$ acts on v as* *follows*: $u \cdot v = v + \sum_{k>0} v_{\lambda+k\alpha}$, *where* $v_{\lambda+k\alpha}$ *is a weight vector of weight* $\lambda + k\alpha$, *and* k *is a positive integer.*

Next, consider the orbit of the root vector $x = x_{\alpha_1+\alpha_2}$ for the highest short root $\alpha_1 + \alpha_2$. If γ is a positive root, then by Lemma 2, $U_\gamma(r) \cdot x = x + w$, where w is a sum of root vectors for roots of the form $(\alpha_1 + \alpha_2) + k\gamma$ for some $k > 0$, but there are no roots of this form unless $k = 1$ and $\gamma = \alpha_2$. It follows that $w = 0$ for all positive roots γ except $\gamma = \alpha_2$. In particular, U_{α_1}, $U_{\alpha_1+\alpha_2}$, and $U_{\alpha_1+2\alpha_2}$ all fix the vector x, whence $U \cdot x = U_{\alpha_2} \cdot x$. Therefore, we have $B \cdot x = TU \cdot x = TU_{\alpha_2} \cdot x$. Now take arbitrary elements $T(s,t) \in T$ and $U_{\alpha_2}(r)$ and compute directly

$$T(s,t)U_{\alpha_2}(r) \cdot x = sx + rstx_{\alpha_1+2\alpha_2}. \tag{1}$$

It follows, since $s \neq 0$, that the $x = x_{\alpha_1+\alpha_2}$-coordinate is nonzero, while it is clear that for all $r \in k$ and $s, t \in k - \{0\}$ that the x_{α_1}- and x_{α_2}-coordinates are 0, whence

$$B \cdot x = T \cdot U_{\alpha_2} \cdot x \subseteq Z(X_{\alpha_1}, X_{\alpha_2}) \cap V(X_{\alpha_1+\alpha_2}).$$

To check the reverse containment, we start with an arbitrary element of the locally closed set on the right, and find an element of B that carries x to that element. So let $y \neq 0$ and $z \in k$ be arbitrary. Then the element $(0,0,y,z) = yx + zx_{\alpha_1+2\alpha_2}$ is an arbitrary element of our locally closed set. But substitute the element $T(y,1)U_{\alpha_2}(z/y)$ directly into (1) to obtain

$$T(y,1)U_{\alpha_2}\left(\frac{z}{y}\right) \cdot x = yx + y \cdot 1 \cdot \frac{z}{y} x_{\alpha_1+2\alpha_2} = (0,0,y,z).$$

This shows the reverse containment and so proves the orbit is $B \cdot x = B \cdot x_{\alpha_1+\alpha_2} = Z(X_{\alpha_1}, X_{\alpha_2}) \cap V(X_{\alpha_1+\alpha_2})$.

Next consider the orbit of $x = x_{\alpha_2}$. By Lemma 2, we only need consider the action of U_{α_1} and $U_{\alpha_1+\alpha_2}$. By direct calculation, we have

$$\begin{aligned} T(p,q)U_{\alpha_1}(s)U_{\alpha_1+\alpha_2}(r) \cdot x &= qx + psx_{\alpha_1+\alpha_2} - pqrx_{\alpha_1+2\alpha_2} \\ &= (0,q,ps,-pqr). \end{aligned} \tag{2}$$

Since $q \neq 0$, this shows $B \cdot x \subseteq Z(X_{\alpha_1}) \cap V(X_{\alpha_2})$. To see we have equality we again start with an arbitrary element $(0,w,y,z) \in Z(X_{\alpha_1}) \cap V(X_{\alpha_2})$ (so $w \neq 0$), and exhibit an element of B which carries $x = (0,1,0,0)$ to it. One such element is $T(1,w)U_{\alpha_1}(y)U_{\alpha_1+\alpha_2}(-z/w)$. Indeed by (2) we obtain

$$T(1,w)U_{\alpha_1}(y)U_{\alpha_1+\alpha_2}\left(-\frac{z}{w}\right) \cdot x = wx + yx_{\alpha_1+\alpha_2} + zx_{\alpha_1+2\alpha_2} = (0,w,y,z).$$

We have shown that $B \cdot x = B \cdot x_{\alpha_2} = Z(X_{\alpha_1}) \cap V(X_{\alpha_2})$.

Next consider the orbit of $x = x_{\alpha_1}$. By Lemma 2, the U-orbit of x is the same as the U_{α_2}-orbit. Thus, direct calculation yields

$$T(s,t)U_{\alpha_2}(r)\cdot x = \frac{s}{t}x - rsx_{\alpha_1+\alpha_2} - \frac{r^2st}{2}x_{\alpha_1+2\alpha_2} = \left(\frac{s}{t}, 0, -rs, -\frac{r^2st}{2}\right). \quad (3)$$

Note that because of the 2 in the denominator, we must avoid characteristic-2 fields. Since $s/t \neq 0$, this shows $B\cdot x \subseteq Z(X_{\alpha_2}) \cap V(X_{\alpha_1})$. However, unlike the above orbits, we do not have an equality in this case due to algebraic dependence relations among the coordinates.

Indeed, define w, y, z by equating

$$\left(\frac{s}{t}, 0, -rs, -\frac{r^2st}{2}\right) = (w, 0, y, z),$$

and observe that $y^2 + 2wz = 0$ for every element of this form. This shows that, in fact, $B\cdot x \subseteq Z(X_{\alpha_2}, X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1}X_{\alpha_1+2\alpha_2}) \cap V(X_{\alpha_1})$. We now claim we have an equality. Indeed an arbitrary element of this locally closed set has the form $(w, 0, y, z)$ with $y^2 + 2wz = 0$ and $w \neq 0$ But it follows from (3) that

$$\begin{aligned} T(w,1)U_{\alpha_2}\left(-\frac{y}{w}\right)\cdot x &= wx + yx_{\alpha_1+\alpha_2} - \frac{y^2}{2w}x_{\alpha_1+2\alpha_2} \\ &= \left(w, 0, y, -\frac{y^2}{2w}\right) = (w, 0, y, z), \end{aligned}$$

where the last equality follows because $y^2 + 2wz = 0$. This shows that $B\cdot x = B\cdot x_{\alpha_1} = Z(X_{\alpha_2}, X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1}X_{\alpha_1+2\alpha_2}) \cap V(X_{\alpha_1})$.

So far we have determined the orbits of the four root vectors, but taken together they do not exhaust all of $\mathfrak{n}$. The remaining orbits can be taken to be orbits of certain sums of root vectors. For example, consider the element $x = x_{\alpha_1} + x_{\alpha_1+2\alpha_2}$. All the root groups U_γ of U fix x except for U_{α_2} by Lemma 2. By direct computation we have

$$T(s,t)U_{\alpha_2}(r)\cdot x = \frac{s}{t}x_{\alpha_1} - rsx_{a_1+\alpha_2} + st\left(1 - \frac{r^2}{2}\right)x_{\alpha_1+2\alpha_2} = \left(\frac{s}{t}, 0, rs, st\left(1 - \frac{r^2}{2}\right)\right).$$

Now $s/t \neq 0$, so the orbit is contained in $Z(X_{\alpha_2}) \cap V(X_{\alpha_1})$. But also

$$X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1}X_{\alpha_1+2\alpha_2} = (-rs)^2 + 2\,\frac{s}{t}\left(st\left(1 - \frac{r^2}{2}\right)\right) = 2s^2 \neq 0.$$

So $B\cdot x \subseteq Z(X_{\alpha_2}) \cap V(X_{\alpha_1}) \cap V(X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1}X_{\alpha_1+2\alpha_2})$. We now prove the reverse containment. Note that an arbitrary element of this locally closed set has the form $(w, 0, y, z)$ with $w \neq 0$, and $y^2 + 2wz \neq 0$. Since k is not of characteristic 2, the element $\frac{1}{2}(y^2+2wz)$ exists and is nonzero in k, and since k is algebraically closed, its square root exists in k and is also nonzero. Now by direct calculation we have

$$T\left(\sqrt{\frac{y^2+2wz}{2}},\ \frac{1}{w}\sqrt{\frac{y^2+2wz}{2}}\right)U_{\alpha_2}\left(-y\sqrt{\frac{2}{y^2+2wz}}\right)\cdot x=(w,0,y,z).$$

This proves $B\cdot x=B\cdot(x_{\alpha_1}+x_{\alpha_1+\alpha_2})=Z(X_{\alpha_2})\cap V(X_{\alpha_1})\cap V(X^2_{\alpha_1+\alpha_2}+2X_{\alpha_1}X_{\alpha_1+2\alpha_2})$.

The last orbit we need to consider is the orbit of $x=x_{\alpha_1}+x_{\alpha_2}$. Only $U_{\alpha_1+2\alpha_2}$ fixes x, so we need to see how all three of the other root groups act. By direct matrix calculation we have

$$T(s,t)U_{\alpha_1}(a)U_{\alpha_2}(b)U_{\alpha_1+\alpha_2}(c)\cdot x=\left(\frac{s}{t},\ t,\ (a-b)s,\ -st\left(\frac{b^2}{2}+c\right)\right). \tag{4}$$

Since $s,t\neq 0$, we have $B\cdot x\subseteq V(X_{\alpha_1})\cap V(X_{\alpha_2})$. We now show the reverse containment. Let $(w,u,y,z)\in V(X_{\alpha_1})\cap V(X_{\alpha_2})$ be arbitrary (so $w,u\neq 0$). Then using (4), we have

$$T(wu,u)U_{\alpha_1}\left(\frac{y}{wu}\right)U_{\alpha_2}(0)U_{\alpha_1+\alpha_2}\left(-\frac{z}{wu^2}\right)\cdot x=(w,u,y,z).$$

This shows $B\cdot x=V(X_{\alpha_1})\cap V(X_{\alpha_2})$ is an open, dense orbit in $\mathfrak{n}$, called the *regular* orbit. We are nearly finished with the proof of our main result:

Theorem 3. *Let $G=\mathrm{SO}_5(k)$, where k is algebraically closed and not of characteristic 2, and let B be a Borel subgroup acting on $\mathfrak{n}$ via the adjoint action. Then B has just seven orbits as indicated in the following table along with their defining equations. The dimensions of these orbits are also indicated in the table, and the closure order is indicated by the Hasse diagram in Figure 1.*

element x of $\mathfrak{n}$	*defining equations for $B\cdot x$*	$\dim B\cdot x$
0	$Z(X_{\alpha_1},X_{\alpha_2},X_{\alpha_1+\alpha_2},X_{\alpha_1+2\alpha_2})$	0
$x_{\alpha_1+2\alpha_2}$	$Z(X_{\alpha_1},X_{\alpha_2},X_{\alpha_1+\alpha_2})\cap V(X_{\alpha_1+2\alpha_2})$	1
$x_{\alpha_1+\alpha_2}$	$Z(X_{\alpha_1},X_{\alpha_2})\cap V(X_{\alpha_1+\alpha_2})$	2
x_{α_1}	$Z(X_{\alpha_2},X^2_{\alpha_1+\alpha_2}+2X_{\alpha_1}X_{\alpha_1+2\alpha_2})\cap V(X_{\alpha_1})$	2
x_{α_2}	$Z(X_{\alpha_1})\cap V(\alpha_2)$	3
$x_{\alpha_1}+x_{\alpha_1+2\alpha_2}$	$Z(X_{\alpha_2})\cap V(X_{\alpha_1})\cap V(X^2_{\alpha_1+\alpha_2}+2X_{\alpha_1}X_{\alpha_1+2\alpha_2})$	3
$x_{\alpha_1}+x_{\alpha_2}$	$V(X_{\alpha_1})\cap V(X_{\alpha_2})$	4

In the Hasse diagram of the closure ordering in Figure 1, each orbit is indicated by its representative element from the first column of the table.

Proof. We have already verified the entries in the first two columns of the table. Note that the orbit closures are just the closed sets from the defining equations. For example, since $B\cdot x_{\alpha_1+\alpha_2}=Z(X_{\alpha_1},X_{\alpha_2})\cap V(X_{\alpha_1+\alpha_2})$, we have $\overline{B\cdot x_{\alpha_1+\alpha_2}}$

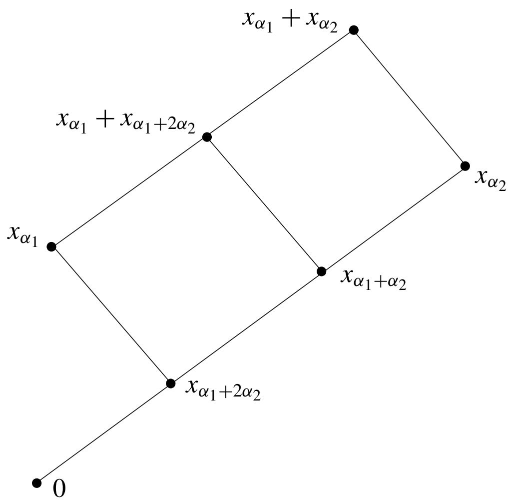

Figure 1. The closure order for nilpotent B-orbits in type B_2.

$= Z(X_{\alpha_1}, X_{\alpha_2})$. Using the closures, we can easily determine the dimensions in the third column as well as the closure ordering. Note that for polynomials f_i in r variables, the dimension of $Z(f_1, f_2, \dots, f_k)$ is just $r-k$ provided that the f_i are all algebraically independent. It should be clear that we found all the algebraic dependencies when we worked out the defining equations, so that the f_i are algebraically independent in the closed sets in the second column of the table. Thus, since $r = \dim \mathfrak{n} = 4$, the dimensions in the third column are equal to $4-k$, where k is the number of polynomials whose zero sets define the orbit closure.

The only nontrivial containment for the closure ordering is $B \cdot x_{a_1+2\alpha_2} \subseteq \overline{B \cdot x_{\alpha_1}}$, which happens if and only if $\overline{B \cdot x_{\alpha_1+2\alpha_2}} \subseteq \overline{B \cdot x_{\alpha_1}}$. So take an arbitrary element $x \in \overline{B \cdot x_{\alpha_1+2\alpha_2}} = Z(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_1+\alpha_2})$. Then, since both $X_{\alpha_1} = 0$ and $X_{\alpha_1+\alpha_2} = 0$, it follows that both $X^2_{\alpha_1+\alpha_2} = 0$ and $X_{\alpha_1} X_{\alpha_1+2\alpha_2} = 0$ when evaluated at x. Therefore, $X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1} X_{\alpha_1+2\alpha_2} = 0$ as well, so $x \in Z(X_{\alpha_2}, X^2_{\alpha_1+\alpha_2} + 2X_{\alpha_1} X_{\alpha_1+2\alpha_2}) = \overline{B \cdot x_{\alpha_1}}$, showing the desired containment. The other containments shown in the Hasse diagram follow similarly.

All that remains to show is that we have exhausted all the nilpotent orbits in $\mathfrak{n}$. So let $n = wx_{\alpha_1} + xx_{\alpha_2} + yx_{\alpha_1+\alpha_2} + zx_{\alpha_1+2\alpha_2} = (w, x, y, z)$ be an arbitrary element of $\mathfrak{n}$. We must show n lies in one of these seven orbits. We will distinguish cases according to how many and which of the four coordinates are 0. If both w and x are nonzero, then n is in $V(X_{\alpha_1}) \cap V(X_{\alpha_2}) = B \cdot (x_{\alpha_1} + x_{\alpha_2})$, the regular orbit. So it only remains to consider cases when one or both of w, x are 0. First, suppose $w = 0$ but $x \neq 0$. Then $n = (0, x, y, z) \in Z(X_{\alpha_1}) \cap V(X_{\alpha_2}) = B \cdot x_{\alpha_2}$. On the other hand, suppose $w \neq 0$ and $x = 0$, so $n = (w, 0, y, z) \in Z(X_{\alpha_2}) \cap V(X_{\alpha_1})$. But then $n \in B \cdot (x_{\alpha_1} + x_{\alpha_1+2\alpha_2})$ or $n \in B \cdot x_{\alpha_1}$, depending on whether or not $y^2 + 2wz = 0$.

Lastly, we consider cases where $w = 0 = x$. In this case, $n = (0, 0, y, z) \in Z(X_{\alpha_1}) \cap Z(X_{\alpha_2})$. If $y \neq 0$, then $n \in Z(X_{\alpha_1}, X_{\alpha_2}) \cap V(X_{\alpha_1+\alpha_2}) = B \cdot x_{\alpha_1+\alpha_2}$. On the other hand, if $y = 0$, then $n = (0, 0, 0, z)$, which belongs to either $B \cdot 0$ or $B \cdot x_{\alpha_1+2\alpha_2}$, depending on whether or not $z = 0$. This covers all possible cases, and in each case, n was in one of the above-mentioned orbits, whence the union of the seven orbits is all of $\mathfrak{n}$. $\square$

3. Conclusions

The result that there are only finitely many nilpotent B-orbits for $\mathrm{SO}_5(k)$ can be phrased in terms of a general concept for algebraic group actions called modality (see [Popov and Röhrle 1997] for example).

Let G be an arbitrary algebraic group acting morphically on a nonempty variety V. The *modality* of the action is

$$\operatorname{mod}(G, V) = \max_{Z} \min_{z \in Z} \operatorname{codim}_Z(G^0 \cdot z), \tag{5}$$

where Z runs through all irreducible G^0-invariant subvarieties of V. Here, G^0 is the connected component of the identity in G. Informally, the modality is the maximum number of (continuous) parameters on which a family of G-orbits may depend.

Although we are mainly interested in nilpotent orbits for a Borel subgroup B of G, much of the literature is written in terms of the more general case of orbits for a *parabolic* subgroup P, which is any closed subgroup containing a Borel subgroup. If P is parabolic, denote its Lie algebra by $\mathfrak{p}$, and the nilradical of $\mathfrak{p}$ by $\mathfrak{n}(\mathfrak{p})$. Then P acts on $\mathfrak{n}(\mathfrak{p})$ via the adjoint representation, and the *modality of* P is defined to be $\operatorname{mod}(P, \mathfrak{n}(\mathfrak{p}))$. Thus the modality of P is 0 precisely when there are only finitely many nilpotent P-orbits in $\mathfrak{n}(\mathfrak{p})$.

When $P = G$, the nilradical of $\mathfrak{g}$ is trivial since $\mathfrak{g}$ is simple, so the modality of G is trivially 0. At the other extreme, the modality of B is almost never 0. So one consequence of Theorem 3 is that if $p \neq 2$, then B has modality 0 in type B_2. Of course, this is a well-known result. Based on earlier work in [Bürgstein and Hesselink 1987], in [Kashin 1990] all the Borel subgroups of modality 0 were determined in characteristic 0:

Theorem 4 [Kashin 1990]. *Let G be quasisimple over k, where k has characteristic* 0, *and suppose B is a Borel subgroup of G. The number of orbits of B on $\mathfrak{n}$ is finite* (*that is*, *B has modality* 0) *if and only if G is type A_n for $n \leq 4$, or G is type B_2.*

Aside from the consequences of this theorem for our investigation on nilpotent B-orbits, Kashin's result launched an investigation into the modality of parabolic subgroups in general. For example, see [Röhrle 1996; 1999; Popov 1997; Popov and Röhrle 1997; Hille and Röhrle 1999; Brüstle et al. 1999]. In fact, Theorem 1.1 in [Hille and Röhrle 1999] shows there is a strong connection between the modality of a parabolic subgroup and the length of a descending central series of $R_u(P)$, the unipotent radical of P, also called the *nilpotency class* of $R_u(P)$. Using this theorem, one can easily recover Kashin's original theorem, with the added benefit that the proof is valid in good prime characteristics for G as well as for characteristic 0. In type A, all primes are good, and in type B, all primes are good except $p = 2$.

In a previous version of this paper, using similar techniques as here, we showed directly that there are finitely many nilpotent B-orbits for G of types A_1, A_2, A_3 and A_4 without any restrictions on p, and used that information to determine the dimensions of the orbits and the closure ordering. A referee pointed out to us that the closure orderings for the four type-A cases were already discussed in [Brüstle et al. 1999], making a lot of our work seem redundant. Note that the techniques used in that paper are quite different than ours — they are much more sophisticated than our matrix calculations. Their approach has some advantages, such as both being more elegant than our approach and also being closer in spirit to the way nilpotent G-orbits are classified. A possible advantage of our techniques, though, is that they yield the explicit polynomial defining equations of each orbit. It may be an advantage to knowing these defining equations in applying this work, perhaps to computing support varieties of baby Verma modules as discussed in Section 1, or perhaps for other applications. For this reason, we have uploaded our type-A calculations [Burkhart and Vella 2017] on the arXiv so that despite the overlap with [Brüstle et al. 1999], our tables for these orbits are publicly available. Here we conclude by simply reminding the reader how many orbits there are in each case: two orbits in type A_1, five orbits in type A_2, 16 orbits in type A_3, and 61 orbits in type A_4. For the details of the defining equations, etc., consult [Burkhart and Vella 2017], and for the dimensions of each orbit and the Hasse diagrams of the closure order in these cases, valid for all characteristics, consult either [Brüstle et al. 1999] or [Burkhart and Vella 2017].

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