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Toeplitz subshifts with trivial centralizers and positive entropy

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Given a dynamical system (X, G), the centralizer $C(G)$ denotes the group of all homeomorphisms of X which commute with the action of G. This group is sometimes called the automorphism group of the dynamical system (X, G). We generalize the construction of Bułatek and Kwiatkowski (1992) to $\mathbb{Z}^d$-Toeplitz systems by identifying a class of $\mathbb{Z}^d$-Toeplitz systems that have trivial centralizers. We show that this class of $\mathbb{Z}^d$-Toeplitz systems with trivial centralizers contains systems with positive topological entropy.

1. Introduction

Toeplitz dynamical systems were first introduced by Jacobs and Keane [1969]. They provided a classical definition for a Toeplitz sequence over $\{0, 1\}$. Markley [1975] studied these sequences and showed the equivalence of various definitions of them. The orbit closure of a Toeplitz sequence is regarded as a Toeplitz flow. Markley and Paul [1979] showed that these flows were exactly almost one-to-one extensions of odometers, or the group of p-adic integers. See [Hewitt and Ross 1979] for a general discussion of the group-theoretic properties of the group of p-adic integers. For a general survey of symbolic dynamics, we refer the reader to [Kitchens 1998]. For a good survey on $\mathbb{Z}$-odometers and Toeplitz flows, the reader is referred to [Downarowicz 2005]. Recently the definition of Toeplitz flows was extended to flows over $\mathbb{Z}^d$ by Cortez [2006], and then to flows over general groups in [Cortez and Petite 2008; Krieger 2010].

The centralizer of a dynamical system is the group of all homeomorphisms of the system which commute with the group action. Sometimes called the *automorphism group* of the dynamical system in the literature, the centralizer of a dynamical system has an intricate relationship with its parent dynamical system. For example,

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in [Boyle et al. 1988], Boyle, Lind and Rudolph studied the centralizers of shifts of finite type and showed that they are countable, residually finite and contain the free group on two generators. Several results have been shown by Cyr and Kra [2015; 2016a; 2016b] which relate varying levels of complexity of symbolic dynamical systems to algebraic properties of their centralizers. We notice that systems with positive entropy tend to have very large centralizers. For example, the centralizer of the full shift contains every finite group and the free group on two generators. On the other hand, Donoso, Durand and Petite [Donoso et al. 2016] showed that some classes of low complexity symbolic dynamical systems have very small centralizers, in the sense that they consist only of powers of T. Bułatek and Kwiatkowski [1990; 1992] studied the centralizer of a class of high-complexity Toeplitz systems. The centralizer of multidimensional symbolic dynamical systems was studied by Hochman [2010]. For example, he showed that the centralizer of a positive-entropy multidimensional shift of finite type contains a copy of every finite group.

The main question this paper seeks to answer is whether there are multidimensional systems with a trivial centralizer and positive entropy. Following the ideas of [Bułatek and Kwiatkowski 1992], which developed this result in one dimension, we establish this result with a constructive proof. We note that there are several constructions of *one-dimensional* Toeplitz systems with trivial centralizers and positive entropy; see, for example, [Donoso et al. 2017].

In Section 3 we present main facts with proofs regarding general G-odometers, where G is a residually finite group. For the reader's convenience, we include the proofs, otherwise scattered across multiple sources. In particular we show that the centralizer group of $\mathbb{Z}^d$-Toeplitz systems embeds into the centralizer group of its maximal equicontinuous factor, which is a $\mathbb{Z}^d$-odometer, and so is Abelian. This result was originally established in [Auslander 1963, Theorem 9] using the techniques of enveloping semigroups. The proof we present in this note follows the approach developed in [Olli 2013].

In Section 4, we construct a class of $\mathbb{Z}^d$-Toeplitz systems that have trivial centralizers. Then in Section 5, we show that this class contains systems of positive entropy, and we provide an explicit construction of a two-dimensional Toeplitz system of positive entropy.

2. Definitions and background

By a *dynamical system* we mean a pair (X, G), where X is a compact topological space and G is a countable discrete group acting on X by homeomorphisms. The action of a group element $g \in G$ on $x \in X$ will be denoted by $g \cdot x = g(x)$. The set $\{g \cdot x \mid g \in G\}$ is called the orbit of the point x. If every orbit of (X, G) is dense, we call the system *minimal*. A system (X, G) is called *equicontinuous* if for all $\varepsilon > 0$

there exists $\delta > 0$ such that for all $x, y \in X$ if $d(x, y) < \varepsilon$, then $d(g \cdot x, g \cdot y) < \delta$ for all $g \in G$. Let (X, G), and (Y, G) be two minimal systems. If there exists a continuous surjection $\pi : X \to Y$ which preserves the action of G, we say that X is an *extension* of Y, and that Y is a *factor* of X. We call π a *factor map*. Given two factor maps π and π', we say that π is *larger* than π' if there exists a third factor map π'' such that $\pi' = \pi'' \circ \pi$. As such, we can discuss the *maximal* factor of a system. It is a known fact that every dynamical system has a maximal equicontinuous factor.

In this paper we are interested in symbolic dynamical systems. We start with a finite set Σ called the alphabet. Say $|\Sigma| = n$. The set of all bi-infinite sequences over Σ is called the full n-shift and is denoted by $\Sigma^{\mathbb{Z}}$. In general, we denote the full d-dimensional n-shift by $\Sigma^{\mathbb{Z}^d}$. This set is endowed with the product topology from the discrete topology in each coordinate. Cylinder sets in which we fix a finite number of coordinates form a basis for the topology. For $x \in \Sigma^{\mathbb{Z}^d}$ we write $x = \{x(v)\}_{v \in \mathbb{Z}^d}$. We call x a $\mathbb{Z}^d$-array. The group $\mathbb{Z}^d$ acts on $\Sigma^{\mathbb{Z}^d}$, denoted by $T^z(x)$ for $z \in \mathbb{Z}^d$ and $x \in \Sigma^{\mathbb{Z}^d}$ as follows: $T^z(x) = \{x(z+v)\}_{v \in \mathbb{Z}^d}$. The orbit of an array is $\{T^v(x) \mid v \in \mathbb{Z}^d\}$. A closed subset $X \subseteq \Sigma^{\mathbb{Z}^d}$ is called a *subshift* if it is closed under the action of $\mathbb{Z}^d$.

For the sake of completeness, we note that symbolic dynamics can be studied over general, discrete groups. In this case, let G be a discrete group. Then Σ^G is acted on by the group G. While in this paper we restrict our study of symbolic dynamics to $\mathbb{Z}^d$-systems, we note that many of the results can be extended to G-systems for more general groups G.

The topological spaces discussed in this note will be topological zero-dimensional compact metric spaces without isolated points, i.e., Cantor sets. Notice that by a theorem of Brouwer [1910] every Cantor set is homeomorphic to the middle-thirds Cantor set, and so all Cantor sets are homeomorphic.

3. Odometers

In this section, we will recall some basic facts about odometers and their almost one-to-one extensions. In particular, we show that the centralizer of an odometer is Abelian, and the centralizer of the almost one-to-one extension of an odometer is also Abelian. These results are mostly known, but are scattered. In particular, the proof of Lemma 3.11 appears in [Veech 1970] and the proof of Proposition 3.12 appears in [Olli 2013]. We present slightly modified proofs for clarity and the reader's convenience.

Definition 3.1. A group G is called *residually finite* if the intersection of all its finite-index normal subgroups is trivial.

Definition 3.2. Let G be a residually finite group and $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be nested normal subgroups such that $\bigcap G_n = \{0\}$. Let π_n be the natural homomorphism from G/G_n onto G/G_{n-1}; i.e., $\pi_n(hG_n) = hG_{n-1}$ for $h \in G$. The

G-odometer, $\overline{G}$, is the inverse limit

$$\overline{G} = \varprojlim(G/G_i; \pi_i) = \left\{ (g_k)_{k=0}^{\infty} \in \prod_{k=0}^{\infty} G/G_k \;\middle|\; \pi_n(g_n) = g_{n-1} \text{ for all } n \geq 1 \right\}.$$

An element $g \in G$ acts on an element $y = (y_i)_{i=0}^{\infty} \in \overline{G}$ as $g \cdot y = (g \cdot y_i)_{i=0}^{\infty}$. First we prove that G embeds into $\overline{G}$.

Lemma 3.3. *Let $\varphi : G \to \overline{G}$ be defined as $g \mapsto (gG_1, gG_2, \ldots)$. Then φ is an embedding.*

Proof. Notice that φ is a homomorphism. Let $g_1, g_2 \in G$. Suppose

$$\varphi(g_1) = (g_1G_1, g_1G_2, g_1G_3, \ldots) = (g_2G_1, g_2G_2, g_2G_3, \ldots) = \varphi(g_2).$$

So $g_1G_i = g_2G_i$ for all i. Therefore $g_1^{-1}g_2 \in G_i$ for all i, and so $g_1^{-1}g_2 \in \bigcap G_i = \{0\}$. Thus $g_1 = g_2$, which implies that φ is an embedding. □

So we have shown that G embeds into $\overline{G}$ in a natural way. In what follows, we will identify the group G with its image $\varphi(G)$. We now prove that $(\overline{G}, G)$ is minimal.

Lemma 3.4. *The system $(\overline{G}, G)$ is minimal.*

Proof. Consider the identity element, $e \in \overline{G}$. In particular, $e = (G_1, G_2, G_3, \ldots)$. Let $y = (y_i)_{i=0}^{\infty} \in \overline{G}$. So, for each n, we have $y_n = \bar{y}_n G_n$, where $\bar{y}_n \in G$ is a representative of the coset. Note

$$\begin{aligned}
\bar{y}_n \cdot e &= \bar{y}_n(G_1, G_2, G_3, \ldots, G_n, \ldots) \\
&= (\bar{y}_n G_1, \bar{y}_n G_2, \bar{y}_n G_3, \ldots, \bar{y}_n G_n, \ldots) \\
&= (\bar{y}_1 G_1, \bar{y}_2 G_2, \ldots, \bar{y}_n G_n, \ldots) = (y_1, y_2, \ldots, y_n, \ldots).
\end{aligned}$$

So $\bar{y}_n \cdot e$ agrees with y in the first n coordinates. And so we can get arbitrarily close to y as we increase n. It follows that the orbit of e is dense.

Now let $a, b \in \overline{G}$. Note we can find a sequence b_n of elements of $G \subset \overline{G}$ such that $b_n \cdot e \to ab^{-1}$, since e has a dense orbit. Then $(b_n \cdot e) \cdot b \to a$ so $b_n \cdot b \to a$. Therefore b has a dense orbit. □

Definition 3.5 (centralizer). Let (X, G) be a dynamical system. The *centralizer*, $C(G)$, is defined as

$$C(G) = \{\varphi \in \text{Homeo}(X) \mid g\varphi = \varphi g \text{ for all } g \in G\}.$$

That is, the centralizer of a system consists of all homeomorphisms of the system which commute with the group action. It can be checked that this is a group under composition.

Next we show that elements of the centralizer of an odometer act as translations of the odometer.

Lemma 3.6. *Let $\varphi \in C(\overline{G}, G)$. There exists $g_0 \in \overline{G}$ such that $\varphi(x) = x \cdot g_0$ for all $x \in \overline{G}$.*

Proof. Set $g_0 = \varphi(e)$. Let $x \in \overline{G}$. Since the orbit of e is dense, by Lemma 3.4, there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \cdot e \to x$. Since φ is continuous, $\varphi(g_n \cdot e) \to \varphi(x)$. Since $g_n \cdot e \to x$, we have $g_n \to x$. So $\varphi(g_n \cdot e) = \varphi(g_n) \cdot \varphi(e) \to x \cdot \varphi(e)$. Therefore $\varphi(x) = x \cdot \varphi(e) = x \cdot g_0$. □

We are now ready to prove the following proposition. In the following, G is an Abelian group.

Proposition 3.7. *The centralizer $C(\overline{G}, G) = \{\varphi : \overline{G} \to \overline{G} \mid \varphi g = g\varphi \text{ for all } g \in G\}$ of an odometer $\overline{G}$ is isomorphic to $\overline{G}$.*

Proof. Define $\psi : C(\overline{G}, G) \to \overline{G}$ as $\psi(\varphi) = \varphi(e)$ for all $\varphi \in C(\overline{G}, G)$. Let $\varphi_1, \varphi_2 \in C(\overline{G}, G)$. Then

$$\psi(\varphi_1 \circ \varphi_2) = \varphi_1 \circ \varphi_2(e) = \varphi_1(\varphi_2(e)) = \varphi_2(e) \cdot \varphi_1(e) = \varphi_1(e) \cdot \varphi_2(e) = \psi(\varphi_1)\psi(\varphi_2).$$

So ψ is a homomorphism. Let $y \in \overline{G}$. Let $\varphi_y(x) = x \cdot y$ for all $x \in \overline{G}$. Note, for $g \in \overline{G}$, we have $\varphi_y(gx) = g\varphi_y(x)$ so $\varphi_y \in C(\overline{G}, G)$. Also, $\psi(\varphi_y) = y$, so ψ is onto. Suppose $\psi(\varphi_1) = \psi(\varphi_2)$. Then $\varphi_1(e) = \varphi_2(e)$. Using Lemma 3.6, we get that for any $x \in \overline{G}$, $\varphi_1(x) = x \cdot \varphi_1(e) = x \cdot \varphi_2(e) = \varphi_2(x)$. Therefore ψ is an isomorphism. □

We now turn our attention to almost one-to-one extensions of odometers.

Definition 3.8. We say (X, G) is an *almost one-to-one extension* of (Y, G) if there is a factor map $\pi : X \to Y$ such that there is at least one $y \in Y$ so that $\pi^{-1}y$ is singleton. Almost one-to-one extensions of odometers are also called *Toeplitz systems*.

We make use of the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "G"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "G"'] & Y \end{tikzcd}$$

Sometimes the context will deem the action of G on X or Y ambiguous, so we will use $T^g x$ to denote the action of the group element $g \in G$ on $x \in X$ and $S^g y$ to denote the action of g on $y \in Y$. In particular, $\pi \circ T^g = S^g \circ \pi$. If the context is clear, the action of g on a point x will be denoted by $g \cdot x$.

If (X, G) is a minimal almost one-to-one extension of a minimal equicontinuous system (Y, G), then it is known that (Y, G) is the maximal equicontinuous factor of (X, G) [Auslander 1988]. As such, the odometer of which a Toeplitz system (X, G) is an almost one-to-one extension is its maximal equicontinuous factor.

We will be considering almost one-to-one extensions of $\mathbb{Z}^d$-odometers. In this context, we will need the following proposition.

Proposition 3.9. *The centralizer $C(G)$ of the almost one-to-one extension of a $\mathbb{Z}^d$-odometer is Abelian.*

To prove Proposition 3.9, we show that the centralizer of the almost one-to-one extension of an odometer embeds into the centralizer of its maximal equicontinuous factor, which we have already shown to be isomorphic to the odometer, which is Abelian in the case of $G = \mathbb{Z}^d$.

Definition 3.10 [Veech 1970]. Given a dynamical system (X, G) and a metric d compatible with the topology on X, two points $x_1, x_2 \in X$ are called *proximal* if

$$\inf_{g\in G} d(g\cdot x_1, g\cdot x_2) = 0.$$

Lemma 3.11. *Let (X, G) be an almost one-to-one extension of an odometer $(\bar{G}, G)$ via the factor map π. Then points of X are proximal if and only if they are in the same π-fiber.*

Proof. Let $x_1, x_2 \in X$ be in the same π-fiber; i.e., $\pi(x_1) = \pi(x_2)$. Let $y \in \bar{G}$ be such that $\pi^{-1}y$ is a singleton. Since $(\bar{G}, G)$ is minimal, there exists a sequence $\{g_n\}$ of elements in G such that $\lim_{n\to\infty} S^{g_n}\pi x_1 = y$ and so $\lim_{n\to\infty} S^{g_n}\pi x_2 = y$. Since X is compact, there is a subsequence $\{g_k\}$ of $\{g_n\}$ such that $T^{g_k}x_1$ and $T^{g_k}x_2$ converge. Suppose $\lim_{k\to\infty} T^{g_k}x_1 = z$. Applying π, we have

$$\pi z = \lim_{k\to\infty} \pi T^{g_k}x_1 = \lim_{k\to\infty} S^{g_k}\pi x_1 = y.$$

So we also have

$$\lim_{k\to\infty} \pi T^{g_k}x_2 = \lim_{k\to\infty} S^{g_k}\pi x_2 = y.$$

Since $\pi^{-1}y$ is a singleton, we get that $\lim_{k\to\infty} T^{g_k}x_2 = z$. Now,

$$\begin{aligned}\limsup_{k\to\infty} d(T^{g_k}x_1, T^{g_k}x_2) &\le \limsup_{k\to\infty}(d(T^{g_k}x_1, z) + d(z, T^{g_k}x_2))\\ &\le \limsup_{k\to\infty} d(T^{g_k}x_1, z) + \limsup_{k\to\infty} d(z, T^{g_k}x_2) = 0.\end{aligned}$$

So the points x_1 and x_2 are proximal.

Now suppose $x_1, x_2 \in X$ are proximal. Then there is a sequence $\{g_n\} \subseteq G$ such that $\lim_{n\to\infty} T^{g_n}x_1 = \lim_{n\to\infty} T^{g_n}x_2 = z$. Applying π, we have $\lim_{n\to\infty} \pi T^{g_n}x_1 = \lim_{n\to\infty} \pi T^{g_n}x_2 = \pi z$. So $\lim_{n\to\infty} S^{g_n}\pi x_1 = \lim_{n\to\infty} S^{g_n}\pi x_2$, which implies πx_1 and πx_2 are proximal in $\bar{G}$. But $\bar{G}$ has no proximal points, so $\pi x_1 = \pi x_2$. $\square$

Finally, we prove that the centralizers of Toeplitz systems embed in the centralizers of the underlying odometers.

Proposition 3.12. *Let (X, G) be an almost one-to-one extension of a G-odometer (Y, G). Every element $\varphi \in C(X, G)$ determines $\psi_\varphi \in C(Y, G)$ such that the following diagram commutes*:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\psi_\varphi"'] & Y \end{tikzcd}$$

Additionally, this relationship is an embedding; *i.e.*, $\psi_{\varphi_1} = \psi_{\varphi_2} \Rightarrow \varphi_1 = \varphi_2$.

Proof. Let $\varphi \in C(X, G)$. Let $x_1, x_2 \in X$ be proximal. So $\pi x_1 = \pi x_2$. Since x_1 and x_2 are proximal, $\inf_{g\in G} d(g \cdot x_1, g \cdot x_2) = 0$. Thus $\inf_{g\in G} d(\varphi(g \cdot x_1), \varphi(g \cdot x_2)) = 0$, which, by Lemma 3.11, implies that $\varphi(x_1), \varphi(x_2)$ are proximal. So φ preserves the proximal relationship, and so it preserves the π-fibers. Define $\psi_\varphi : Y \to Y$ as $\psi_\varphi = \pi \circ \varphi \circ \pi^{-1}$. This map is well-defined because φ preserves the π-fibers. Suppose $\psi_\varphi(y_1) = \psi_\varphi(y_2)$ for $y_1, y_2 \in Y$. So $\pi \circ \varphi \circ \pi^{-1}(y_1) = \pi \circ \varphi \circ \pi^{-1}(y_2)$, and thus $\varphi \circ \pi^{-1}(y_1)$ and $\varphi \circ \pi^{-1}(y_2)$ are in the same π-fibers. Since φ preserves the π-fibers, $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are in the same π-fibers, and so it is clear that $y_1 = y_2$. Therefore ψ_φ is one-to-one. Also, ψ_φ is continuous, so it is a homeomorphism; i.e., $\psi_\varphi \in C(Y, G)$.

Now suppose $\psi_{\varphi_1} = \psi_{\varphi_2}$. Let $y \in Y$ be such that $\pi^{-1} y = \{x\}$ is a singleton. Then $\varphi_1(x) = \pi^{-1}(\psi_{\varphi_1}(y))$ and $\varphi_2(x) = \pi^{-1}(\psi_{\varphi_2}(y))$. Since φ_i preserves π-fibers for $i \in \{1, 2\}$, these are singletons. In particular, $\varphi_1(x) = \varphi_2(x)$. So it is clear then that $g \cdot \varphi_1(x) = g \cdot \varphi_2(x)$ for all $g \in G$, and so $\varphi_1(g \cdot x) = \varphi_2(g \cdot x)$ for all $g \in G$. But every orbit is dense, so φ_1 and φ_2 agree on a dense subset of X, and hence agree everywhere. □

Finally we prove Proposition 3.9.

Proof. We have shown in Proposition 3.12 that $C(X, G)$ embeds into $C(Y, G)$ and by Proposition 3.7 $C(Y, G)$ is Abelian, so $C(X, G)$ is Abelian. □

4. $\mathbb{Z}^d$-Toeplitz systems

In this section, we study Toeplitz systems over $\mathbb{Z}^d$ and generalize the construction of Bułatek and Kwiatkowski. In particular, we present a class of Toeplitz systems over $\mathbb{Z}^d$ with a trivial centralizer and positive entropy.

Let $x \in \Sigma^{\mathbb{Z}^d}$. Note that the topological closure of the orbit of x, $\overline{O(x)}$, is closed and T-invariant. So $(\overline{O(x)}, T)$ is a subshift. This is called the *orbit closure* of x.

Definition 4.1. The centralizer of a symbolic dynamical system is called *trivial* if every element of the centralizer is T^g for some $g \in \mathbb{Z}^d$.

For $x \in \Sigma^{\mathbb{Z}^d}$, $\sigma \in \Sigma$, and a subgroup $Z \subset \mathbb{Z}^d$, define

$$\mathrm{Per}(x, Z, \sigma) = \{w \in \mathbb{Z}^d \mid x(w+z) = \sigma \text{ for all } z \in Z\},$$
$$\mathrm{Per}(x, Z) = \bigcup_{\sigma \in \Sigma} \mathrm{Per}(x, Z, \sigma).$$

We say that $x \in \Sigma^{\mathbb{Z}^d}$ is a *Toeplitz array* if for every $v \in \mathbb{Z}^d$, there exists a finite-index subgroup $Z \subseteq \mathbb{Z}^d$ (note that Z is necessarily isomorphic to $\mathbb{Z}^d$) such that $v \in \mathrm{Per}(x, Z)$.

It can be shown that the orbit closure of a Toeplitz array is an almost one-to-one extension of a $\mathbb{Z}^d$-odometer. For details, the reader is referred to Theorem 7 and Proposition 21 in [Cortez 2006]. In fact, almost one-to-one extensions of odometers are exactly those systems which are orbit closures of Toeplitz arrays. In particular, defining a Toeplitz system as the orbit closure of a Toeplitz array is equivalent to Definition 3.8.

Definition 4.2. Given a finite alphabet Σ, a *patch* is a pair $(P, \mathcal{L})$, where $P \subseteq \mathbb{Z}^d$ and $\mathcal{L} : P \to \Sigma$ is a labeling of P. For the purposes of this paper, we will only consider rectangular patches (blocks) which can be defined by d vectors parallel to the coordinate axes.

Given a patch $(P, \mathcal{L})$, we denote the coordinate closest to the origin in Cartesian space by $P[0]$. Any other location in the patch is denoted by $P[i]$, where $i \in \mathbb{Z}^d$ is a vector pointing to that location, as referenced from $P[0]$. A square block within P is denoted by $P[i-l, i+k]$, where $k, l \in \mathbb{Z}$ and is the (hyper)cube in P located between $P[i - l\bar{1}]$ and $P[i + k\bar{1}]$, where $\bar{1} = (1, 1, \dots, 1)$.

For a finite block D in d dimensions, we denote the size of D along the i-th dimension as $|D|_i$. Note that the left-most and bottom-most entry of D is identified with $D(0, 0, \dots, 0)$.

We now show how Toeplitz arrays can be constructed over an alphabet Σ borrowing ideas from [Downarowicz 2005].

Let $\{p_{t,i}\}_{t=0}^{\infty}$, $1 \le i \le d$, be d sequences of positive integers such that $p_{0,i} \ge 2$ and $p_{t,i}$ divides $p_{t+1,i}$ for all $1 \le i \le d$. Define $\lambda_{t+1,i} = p_{t+1,i}/p_{t,i}$ and $\lambda_{0,i} = p_{0,i}$ for all $1 \le i \le d$ and $t \ge 0$.

Specify blocks A_t as follows:

(1) $|A_t|_i = p_{t,i}$.

(2) Some spaces in A_t are filled with elements from Σ and others are left unfilled. The unfilled spaces are called *holes*.

(3) The block A_{t+1} is the concatenation of $\lambda_{t+1,i}$ copies of A_t along the i-th dimension for all $1 \le i \le d$, where some holes are filled by symbols from Σ.

(4) For every $(i_1, i_2, \dots, i_d) \in \mathbb{N}^d$ there exists a $t \geq 0$ such that $A_t(i_1, i_2, \dots, i_d) \in \Sigma$ and $A_t(p_{t,1} - i_1, p_{t,2} - i_2, \dots, p_{t,d} - i_d) \in \Sigma$.

Denote by ω_t the periodic tiling of $\mathbb{Z}^d$ by the block A_t with the bottom-left corner of A_t appearing at the origin. Set $\omega = \lim \omega_t$. The fourth condition assures that $\omega \in \Sigma^{\mathbb{Z}^d}$. We will additionally assume that p_t is the smallest period of ω_t, which ensures that ω is a nonperiodic Toeplitz array.

Essentially, in this construction we build finite blocks, each of which contains multiple copies of the block built in the previous step. As we copy these blocks, we fill in some of the holes, and leave some of them as holes. As we continue this process forever, we will have a Toeplitz array covering $\mathbb{Z}^d$.

Example 4.3 (one-dimensional Toeplitz array [Downarowicz 2005]). We will construct a Toeplitz array over $\mathbb{Z}$ from the alphabet $\Sigma = \{0, 1\}$. Let $\{p_t\} = \{2, 4, 8, 16, \dots\}$ and so $\lambda_t = 2$ for all $t \geq 0$. Let $A_0 = 0\_$, where the $\_$ symbol indicates a hole. To get A_1, we copy A_0 twice and fill in some of the holes. Say $A_1 = 0\underline{1}0\_$. The underline indicates a hole that was filled in at that step. In each step we will have two holes. For this construction, at each step we will alternately fill in the first hole with 0 and 1. Let the limiting sequence of this process be ω. Continuing, we have

$$A_2 = 010\underline{0}010\_,$$

$$A_3 = 0100010\underline{1}0100010\_,$$

$$A_4 = 010001010100010\underline{0}010001010100010\_,$$

$$\vdots$$

$$\omega = 0100010101000100010001010100010101000101010001000100010101000100\dots,$$

and so we have a Toeplitz array ω. The orbit closure of this point is a Toeplitz system.

Example 4.4 (two-dimensional Toeplitz array). Again we will use the alphabet $\Sigma = \{0, 1\}$ and we will construct a Toeplitz array over $\mathbb{Z}^2$. Let $\{p_{t,1}\} = \{p_{t,2}\} = \{2, 4, 8, 16, \dots\}$. Then $\lambda_{t,1} = \lambda_{t,2} = 2$ for all $t \geq 0$.

Let

$$A_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & \blacksquare \\ \hline \end{array}, \quad A_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 0 & \blacksquare & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & \blacksquare \\ \hline \end{array}, \quad A_2 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & \blacksquare & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & \blacksquare & 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & \blacksquare & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 0 & \blacksquare \\ \hline \end{array}.$$

The black squares indicate where the holes are. Continuing this process, we will have a coloring of the whole plane, which will be a Toeplitz array, say ω.

We call subblocks of A_{t+1} which coincide with indices of the location of concatenated A_t blocks *t-blocks*. We note that ω consists of the concatenation of t-blocks in all directions for any t, where all t-blocks agree in all locations except for where the holes were. In Example 4.4, the thick lines in A_1 indicate the 0-blocks, and the thick lines in A_2 indicate the 1-blocks.

Now we introduce a condition on constructing Toeplitz arrays which will give rise to Toeplitz systems with a trivial centralizer.

Condition $(*)$. We say a Toeplitz array satisfies the condition $(*)$ if:

- Every t-block in A_{t+1} is composed of either A_t where no hole remaining from A_t is filled in or A_t with all holes filled.
- The perimeter of A_{t+1} is composed of t-blocks which are all filled in.
- For every $i \in \mathbb{Z}^d$ such that $A_t[i]$ is a hole, there are two t-blocks B_1 and B_2 with $B_1[i] \neq B_2[i]$.

Let $e_1, e_2, \ldots, e_d$ be the generators of $\mathbb{Z}^d$. For $1 \leq i \leq d$, let T_i denote a shift by the vector e_i. In this context, the shift action on the system can be considered d independent shift actions; i.e., $T^g = T^{(g_1,g_2,\ldots,g_d)} = T_1^{g_1} \times T_2^{g_2} \times \cdots \times T_d^{g_d}$.

Theorem 4.5. *Let ω be a Toeplitz array satisfying the condition $(*)$. Then the centralizer $C(T)$ of $(\overline{O(\omega)}, T)$ is trivial.*

Proof. Let $(\bar{G}, T_1 \times T_2 \times \cdots \times T_d)$ be the maximal equicontinuous factor of $(\overline{O(\omega)}, T)$. Denote by $\pi : (\overline{O(\omega)}, T) \to (\bar{G}, T_1 \times T_2 \times \cdots \times T_d)$ the almost one-to-one factor map. Let $S \in C(T)$. By Proposition 3.12, this determines an element $S' \in C(\bar{G}, T_1 \times T_2 \times \cdots \times T_d)$ which acts as a translation by some element $h \in \bar{G}$, by Lemma 3.6. By a result of [Hedlund 1969], we note S is determined by a block code f of window size $k \in \mathbb{N}$. In particular, if $u \in \overline{O(\omega)}$ and $z = S(u)$, then

$$z[i] = f(u[i-k, i+k]) \quad \text{for all } i \in \mathbb{Z}^d. \tag{1}$$

In particular, the automorphism determines what to put in a specific location by looking at a block around that location in the preimage. Increasing k if necessary, we can assume that S^{-1} is also determined by a block code of the same window size k.

Note $\bar{G}$ is a product odometer, so $h = (h_1, h_2, \ldots, h_d)$, where $h_i = \sum_{t=0}^{\infty} h_{t,i} p_{t-1,i}$ for $1 \leq i \leq d$ with $0 \leq h_{t,i} \leq \lambda_{t,i} - 1$, $p_{-1,i} = 1$. Each h_i is an element of the one-dimensional odometer occurring in the i-th coordinate of h. Let $m_{t,i} = \sum_{j=0}^{t} h_{j,i} p_{j-1,i}$ and $m_t = (m_{t,1}, m_{t,2}, \ldots, m_{t,d}) \in \mathbb{Z}^d$. For each $g \in G$, denote by X_g the preimage $\pi^{-1}(\{g\})$ under the factor map π. Then $S(X_g) = X_{g+h}$.

We claim that for all $1 \leq i \leq d$ either $m_{t,i} \leq k$ or $m_{t,i} \geq p_{t,i} - k - 1$.

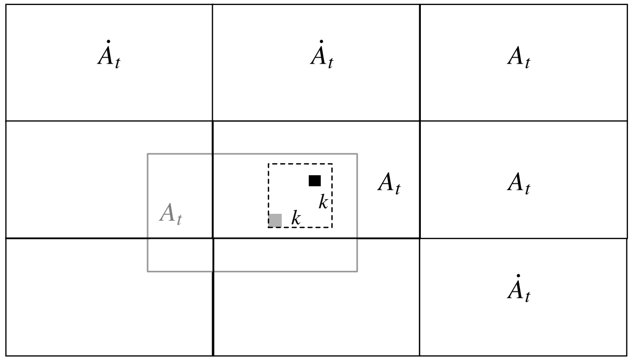

Figure 1. The two-dimensional case of the argument in the proof of Theorem 4.5. $\dot{A}_t$ indicates A_t blocks with all holes filled and the solid black and gray squares indicate holes in x_{t+1} and y_{t+1}, respectively.

Let $x \in \overline{O(\omega)}$ and $y = S(x)$. Suppose that x has a t-block appearing at a location $x[i]$. Then by the construction of Toeplitz subshifts and almost one-to-one extensions, y necessarily has a t-block at the location $y[i-m_t]$. Note that for every $t \geq 1$ the array x can be written as a concatenation of $(t+1)$-blocks, which are made of t-blocks. Recall that all t-blocks are the same, except they may disagree where the holes are located. Denote by x_{t+1} the d-dimensional array over the alphabet $\Sigma \cup \{\text{hole}\}$ consisting of copies of the block A_{t+1} at the locations where they appear in x. Note that $\lim x_t = x$. Similarly, we can define y_{t+1}, a $(t+1)$-block skeleton of the array y. Note that x_{t+1} and y_{t+1} are p_{t+1}-periodic and shifted by the vector m_{t+1} relative to each other. Let A denote any $(t+1)$-block of y. This block looks like A_{t+1}, which in turn is the concatenation of A_t blocks both filled and not. Thus, we can view both y_{t+1} and x_{t+1} as the concatenations of copies of A_t and copies of $\dot{A}_t$, filled versions of A_t. The t-blocks in x_{t+1} and y_{t+1} appear shifted by the vector m_t.

Fix $t > 0$ such that $p_{t-1,i} > 2k+1$ for every $i = 1, \dots, d$. Fix $j \in \mathbb{Z}^d$ such that $y_{t+1}[j]$ is a hole. Note that this hole would correspond to a hole in A_t. Then the hypercube $x_{t+1}[j-k, j+k]$ must also contain a hole. For otherwise, $x_{t+1}[j-k, j+k]$ would be the same for every j with $j \equiv i \bmod p_t$ and, thus, $y[j] = S(x)[j] = f(x[j-k, j+k])$ would be the same for every such j. This would contradict the last property of the condition $(*)$. Applying the same argument to S^{-1}, we see that if $x_{t+1}[j]$ is a hole for some $j \in \mathbb{Z}^d$, then $y[j-k, j+k]$ must also contain a hole.

This argument is demonstrated for the two-dimensional case and for the forward-looking centralizers in Figure 1.

Now, the t-blocks in the arrays x_{t+1} and y_{t+1} are shifted by the vector m_t relative to each other. At the same time, by the argument above, the filled t-blocks $\dot{A}_t$ in x_{t+1} and y_{t+1} must appear under each other and can be shifted by a vector of length at most k. Since p_t is the smallest period of ω_t, we conclude that for all $1 \le i \le d$ either $m_{t,i} \le k$ or $m_{t,i} \ge p_{t,i} - k - 1$. It follows that h_i is an integer for every $i = 1, \ldots, d$. So, $h = S'(0) = T^h(0)$; i.e., S' and T^h agree on one point. Furthermore, S' agrees with the action of T^h on the entire orbit of 0, which is dense. Therefore, $S' = T^h$.

Let α be in the orbit of ω in $(\overline{O(\omega)}, T)$; i.e., $\alpha = T^g \omega$ for some $g \in \mathbb{Z}^d$. Note

$$\pi S(\alpha) = \pi S(T^g \omega) = S' \pi(T^g \omega) = S' T^g(0) = T^h T^g(0) = \pi T^h T^g \omega = \pi T^h(\alpha).$$

So $S(\alpha)$ and $T^h(\alpha)$ are in the same π-fiber. Since α is in the orbit of ω, it has a unique preimage under π. Therefore $S(\alpha) = T^h(\alpha)$. And so S and T^h agree on the entire orbit of ω, which is dense. So $S = T^h$. □

5. Positive-entropy Toeplitz subshift

We now construct an explicit example of a two-dimensional Toeplitz subshift which has positive entropy. This example is constructed so that it obeys the condition $(*)$, thus ensuring that it has a trivial centralizer.

Let $h > 0$ and choose an integer l_0 such that $\log(l_0 - 1) \le h \le \log(l_0)$. For $i \ge 0$, let $\varepsilon_i > 0$ and $\{\varepsilon_i\}$ be such that $\sum_{i=0}^{\infty} \varepsilon_i < h/2$.

We note that for any l and any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ sufficiently large such that

$$\frac{\log(l^{n^2})}{(n+2)^2} \ge \log(l) - \varepsilon \tag{2}$$

since $(n/(n+2))^2 \to 1$.

Let q_0 be chosen so that

$$\frac{\log({l_0}^{q_0^2})}{(q_0+2)^2} \ge \log(l_0) - \frac{\varepsilon_0}{2}.$$

Also require $q_0^2 \ge l_0$. Define $l_1 = l_0^{q_0^2}$. We notice that there are $l_0^{q_0^2}$ square blocks of side length q_0 over the alphabet $\{0, 1, \ldots, l_0 - 1\}$. We enumerate these blocks as $B_i^{(0)}$ for $0 \le i \le l_1 - 1$. Furthermore, we require that $B_0^{(0)}$ and $B_1^{(0)}$ contain every letter from the alphabet. Let $C_i^{(0)}$ be the square block of side length $q_0 + 2$ with the block $B_i^{(0)}$ surrounded by a 0 in the top left corner, a 1 in the bottom right corner, and 0's below the main diagonal and 1's above it, as in the diagram below. We will denote this as $C_i^{(0)} = 0B_i^{(0)}1$ for $0 \le i \le l_1 - 1$:

$$C_i^{(0)} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & \multicolumn{2}{c|}{} & \vdots \\ \cline{1-1}\cline{4-4} \vdots & \multicolumn{2}{c|}{\smash{\raisebox{1.5ex}{$B_i^{(0)}$}}} & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array}\,.$$

For $k \geq 1$, define $l_k = l_{k-1}^{q_{k-1}^2}$ and let q_k be such that

$$\frac{\log({l_k}^{q_k^2})}{(q_k+2)^2} \geq \log(l_k) - \frac{\varepsilon_k}{2}. \tag{3}$$

Additionally, require that $q_k^2 \geq l_k$. Let $B_i^{(k)}$ be all the square blocks of side length q_k over the alphabet $\{0, 1, \ldots, l_k - 1\}$ for $0 \leq i \leq l_{k+1} - 1$. Require that $B_0^{(k)}$ and $B_1^{(k)}$ contain every letter from the alphabet. Let $C_i^{(k)} = 0B_i^{(k)}1$ for $0 \leq i \leq l_{k+1} - 1$.

Consider the following operation on finite blocks. Let $\{A_1, A_2, \ldots, A_n\}$ be square blocks of the same side length $|A|$ over some alphabet. Let B be a square block over an alphabet containing $\{1, 2, \ldots, n\}$. We define the block

$$C = \{A_1, A_2, \ldots, A_n\} * B$$

as $C[i, j] = A_{B[i.j]}$. In particular, C will be a square block of side length $|B| \cdot |A|$.

Our goal is to construct a sequence of blocks $\{A_t\}$ that defines a system of k-blocks and that satisfies the condition $(*)$. We proceed as follows: Let $A_i^{(0)} = C_i^{(0)}$ and

$$A_i^{(k)} = \{A_0^{(k-1)}, A_1^{(k-1)}, \ldots, A_{l_k-1}^{(k-1)}\} * C_i^{(k)}.$$

We note that since $C_0^{(1)}$ and $C_1^{(1)}$ have every letter of the alphabet $\{0, 1, \ldots, l_1-1\}$, the blocks $A_0^{(1)}$ and $A_1^{(1)}$ will have every 0-block as a subblock. Similarly, $C_0^{(2)}$ and $C_1^{(2)}$ contain every letter in $\{0, 1, \ldots, l_2 - 1\}$ and so the blocks $A_0^{(2)}$ and $A_1^{(2)}$ will contain every 1-block as a subblock. In general, we note that each block $A_i^{(k)}$ for $i = 0, 1$ has every $(k-1)$-block as a subblock.

We let

$$A_0 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & \multicolumn{2}{c|}{} & \vdots \\ \cline{1-1}\cline{4-4} \vdots & \multicolumn{2}{c|}{-} & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array},$$

where the side length of the square box A_0 is $q_0 + 2$, and the dash in the center square indicates a square of side length q_0 consisting of all holes. We note that $A_i^{(0)}$, $i = 0, \ldots, l_1 - 1$, are 0-blocks corresponding to A_0.

Inductively, define

$$A_{k+1} = \begin{array}{|c|c|c|c|c|} \hline A_0^{(k)} & A_1^{(k)} & \cdots & A_1^{(k)} & A_1^{(k)} \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline A_0^{(k)} & A_0^{(k)} & \cdots & A_0^{(k)} & A_1^{(k)} \\ \hline \end{array},$$

where there is a square block consisting of q_k^2 copies of A_k surrounded by $4q_k+4$ copies of $A_i^{(k)}$ for $i=0$ or 1 on each side positioned similarly to 0's and 1's in A_0. Notice that $A_0^{(k)}$ and $A_1^{(k)}$ have no holes, so all the holes are contained in the middle block of A_k blocks. Note that $A_i^{(k)}$, $i=0,\ldots,l_k-1$, are the k-blocks corresponding to the pattern A_k.

Let ω be the limiting array from the above process. Note that $\omega\in\{0,\ldots,l_0-1\}^{\mathbb{Z}^2}$ and ω satisfies the condition $(*)$.

Proposition 5.1. *The Toeplitz system $(\overline{O(\omega)},T)$ has positive entropy.*

Proof. Define $\lambda_k=q_k+2$ and $p_k=\lambda_1\lambda_2\cdots\lambda_k$. Let h_ω be the entropy of $(\overline{O(\omega)},T)$ and let $\Theta(n)$ be the number of square blocks of side length n appearing in ω. We note that
$$h_\omega=\lim_{n\to\infty}\frac{\log(\Theta(n))}{n^2}=\lim_{k\to\infty}\frac{\log(\Theta(p_k))}{p_k^2},$$
by switching to a subsequence.

There are l_{k+1} many k-blocks. We note that every block A_k contains every $(k-1)$-block as a subblock. This is because the blocks $C_i^{(k)}$ for $i=0$ or $i=1$ contain every letter of the alphabet in them. This means that as we do the shuffling process described above, the blocks $A_i^{(k)}$ for $i=0$ or $i=1$ contain every single block $A_i^{(k-1)}$ for $0\le i\le l_k-1$. Furthermore, since k-blocks are squares of side length p_k, there are at least as many blocks of side length p_k occurring in ω as there are k-blocks. Specifically, square blocks of length p_k can occur at any position within ω, while k-blocks only occur at specific positions. Hence we have
$$\Theta(p_k)\ge l_{k+1}. \tag{4}$$

So we have
$$h_\omega\ge\limsup_{k\to\infty}\frac{\log(l_{k+1})}{p_k^2}. \tag{5}$$

By (3) we have
$$\frac{\log(l_{k+1})}{\lambda_k^2}\ge\log(l_k)-\frac{\varepsilon_k}{2}.$$

It then follows by (2) that
$$\frac{\log(l_{k+1})}{p_k^2}\ge\frac{\lambda_k^2(\log(l_k)-\varepsilon_k/2)}{p_k^2}=\frac{\log(l_k)-\varepsilon_k/2}{p_{k-1}^2}\ge\frac{\log(l_k)}{p_{k-1}^2}-\varepsilon_k.$$

Continuing, we have
$$\frac{\log(l_{k+1})}{p_k^2}\ge h-\sum_{i=0}^{k}\varepsilon_i.$$

Taking the limit as $k\to\infty$, from (5), we have $h_\omega\ge h/2>0$.

It is a basic fact that every Toeplitz system is minimal, so this system is minimal. It is either finite or uncountable, and since it has positive entropy, it cannot be finite. So this is an infinite minimal Toeplitz system. □

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