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On a connection between local rings and
their associated graded algebras

Justin Hoffmeier and Jiyoung Lee



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We study a class of local rings and a local adaptation of the homogeneous property for graded rings. While the rings of interest satisfy the property in the local case, we show that their associated graded k -algebras do not satisfy the property in the graded case.

1. Introduction and preliminaries

Let $Q = k[[X_1, X_2, \dots, X_n]]$ denote the power series ring in n variables over the field k . Let J be an ideal in Q . For an element $b \in J$, the initial form of b is the homogeneous finite sum of lowest-degree terms of b , denoted by b^* . Let $Q^g = k[X_1, X_2, \dots, X_n]$ denote the polynomial ring in n variables over the field k . The initial ideal of J is the ideal in Q^g generated by all of the initial forms of J and is denoted by $\text{In}(J)$. That is,

$$\text{In}(J) = \left\{ \sum_{i=1}^m a_i b_i^* \mid a_i \in Q^g, b_i \in J, 1 \leq i \leq m \right\}.$$

Computations in $\text{In}(J)$ are not always straightforward. The following example is intended to help illustrate some of the nuances of $\text{In}(J)$.

Example 1.1. Let $Q = k[[X, Y]]$ and $J = (x^2 + y^3, xy)$. Since

$$(x^2 + y^3)(-x^2y + x^4y^5 + x^{13} + \dots) \quad \text{and} \quad xy(x^3 + xy^3 - x^5y^4 + \dots)$$

are in J , we have that the initial form of their sum

$$(-x^4y + x^4y + x^2y^4 - x^2y^4 + x^6y^5 - x^6y^5 + x^4y^8 + x^{15} + x^{13}y^3 + \dots)^* = x^4y^8$$

is in $\text{In}(J)$.

Describing $\text{In}(J)$ is not as simple as finding the initial forms of the generators of J . The next example is adapted from [Eisenbud 1995], although similar examples can be found in several other texts.

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Example 1.2. Let $Q = k[[X, Y]]$ and $J = (x^2 + y^3, xy)$. Then $(x^2 + y^3)^* = x^2$ and $(xy)^* = xy$, but $\text{In}(J) = (x^2, y^4, xy)$. In [Lemma 2.5](#), we provide a method to prove this fact for a more general class of rings.

Let R be a commutative local ring with maximal ideal \mathfrak{m} and residue field k . By the Cohen structure theorem, the completion of any local ring can be written as a quotient of a regular local ring by an ideal. Hence, if R is a complete local ring then $R = Q/J$, where $J \subseteq (X_1, X_2, \dots, X_n)^2$.

Definition 1.3. Let R be a complete local ring with a minimal Cohen presentation $R = Q/J$, where $J = (f_1, f_2, \dots, f_l)$ with $f_i \in Q$ for $1 \leq i \leq l$. If f_i^* has degree t for each i then R is t -homogeneous.

In [\[Hoffmeier and Şega 2017\]](#) the authors give a more general version of the above definition. They go on to show that knowing a ring is t -homogeneous is helpful for identifying various homological properties. Indeed, [Theorem 2.5](#) of that paper establishes that the t -homogeneous property plays an important role connecting these homological traits of local rings.

Let $J = (f_1, f_2, \dots, f_l) \subseteq Q^{\mathfrak{g}}$ be the ideal generated by polynomials f_i in $Q^{\mathfrak{g}}$ for $1 \leq i \leq l$. If each of the f_i is homogeneous of degree t then the quotient $R = Q^{\mathfrak{g}}/J$ is a t -homogeneous graded k -algebra.

The associated graded ring of R with respect to the maximal ideal is the direct sum

$$R^{\mathfrak{g}} = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

This notation is consistent with $Q^{\mathfrak{g}}$. That is, for the local ring $Q = k[[X_1, X_2, \dots, X_n]]$, we have $Q^{\mathfrak{g}} = k[X_1, X_2, \dots, X_n]$. Furthermore, if $R = Q/J$ then $R^{\mathfrak{g}} = Q^{\mathfrak{g}}/\text{In}(J)$.

We now state [\[Hoffmeier and Şega 2017, Lemma 1.3\]](#), which also provides further motivation for the terminology given in [Definition 1.3](#).

Lemma 1.4. *Let R be a complete local ring. If $R^{\mathfrak{g}}$ is a t -homogeneous k -algebra, then R is a t -homogeneous local ring.*

Hoffmeier and Şega [\[2017, Remark 1.4\]](#) also provide a counterexample to show that the converse of the lemma does not hold. We now reproduce this example.

Example 1.5. Let $Q = k[[X, Y]]$, $J = (x^2 + y^3, xy)$, and $R = Q/J$. Then R is 2-homogeneous. However, $R^{\mathfrak{g}} = Q^{\mathfrak{g}}/\text{In}(J) = k[X, Y]/(x^2, y^4, xy)$, which is not 2-homogeneous.

It is significant that the converse of [Lemma 1.4](#) does not hold. Otherwise, the t -homogeneous property of a local ring R would depend only on its associated graded k -algebra $R^{\mathfrak{g}}$, making the connections between the homological properties of R alluded to above (stated in [\[Hoffmeier and Şega 2017, Theorem 2.5\]](#)) also related to $R^{\mathfrak{g}}$. The main goal of this note is to identify a larger class of rings for

which the converse of the lemma fails, which consequently further distinguishes the homological nature of local rings from properties of their associated graded k -algebras. We achieve this in the next section by generalizing [Example 1.5](#).

Further motivation for our result is the fact that [Example 1.5](#) is stated without proof in [[Hoffmeier and Şega 2017](#)] and is therefore further explained by the proof of our more general result.

Remark 1.6. Connections between a local ring and its associated graded algebra have been well documented throughout the literature of commutative algebra. For example, if R^g is Cohen–Macaulay then R is Cohen–Macaulay and if R^g is Gorenstein then R is Gorenstein; see, e.g., [[Achilles and Avramov 1982](#)]. The text [[Bruns and Herzog 1993](#)] also states several of these results and is a good reference for other topics that appear in this note. In his survey on the subject, Fröberg [[1987](#)] states that “A local ring is at least as nice as its associated graded ring.” Our results provide another example that makes the inequality Fröberg alludes to strict.

2. Unassociated t -homogeneous local rings

In this section we prove our main result. We begin with a definition.

Definition 2.1. Let R be a t -homogeneous local ring. If R^g is not a t -homogeneous graded k -algebra then we say that R is unassociated t -homogeneous.

Theorem 2.2. Let $J = (x^2 + y^t, xy) \subseteq Q = k[[X, Y]]$ with $t \geq 3$ and set $R = Q/J$. Then R is unassociated 2-homogeneous.

Remark 2.3. Note that by setting $t = 3$ in [Theorem 2.2](#), we recover the result in [Example 1.5](#).

We now provide two lemmas which will be used in the proof of the theorem.

Lemma 2.4. Let $J = (x^2 + y^t, xy) \subseteq Q = k[[X, Y]]$ with $t \geq 3$. Then y^t is not in $\text{In}(J)$.

Proof. Suppose $y^t \in \text{In}(J)$. Then

$$y^t = \sum_{i=1}^m a_i b_i^*,$$

where $a_i \in Q^g$, $b_i \in J$, and $1 \leq i \leq m$. For each i , let $b_i = c_i(x^2 + y^t) + d_i(xy)$ with $c_i, d_i \in Q$. Hence,

$$y^t = \sum_{i=1}^m a_i(c_i(x^2 + y^t) + d_i(xy))^*.$$

Since the sum equals y^t , the terms of the sum that are factors of xy either cancel or are dropped by taking the lowest-degree terms. Therefore,

$$y^t = \sum_{i=1}^m a_i(c_i(x^2 + y^t))^*.$$

Since $t \geq 3$, we have $(c_i(x^2 + y^t))^* = c_i^*x^2$ for each i , where c_i^* is the finite sum of lowest-degree terms of c_i . Hence

$$y^t = \sum_{i=1}^m a_i c_i^* x^2,$$

which is a contradiction. \square

Lemma 2.5. *Let $J = (x^2 + y^t, xy) \subseteq Q = k\llbracket X, Y \rrbracket$ as in Lemma 2.4. Then $\text{In}(J) = (x^2, y^{t+1}, xy)$.*

Proof. First, we show that $(x^2, y^{t+1}, xy) \subseteq \text{In}(J)$. It is sufficient to show that $x^2, y^{t+1}, xy \in \text{In}(J)$, which is clear since

$$x^2 = (x^2)^*, \quad xy = (xy)^*, \quad y^{t+1} = (y(x^2 + y^t) - x(xy))^*.$$

Next, we show that $\text{In}(J) \subseteq (x^2, y^{t+1}, xy)$. Let $g \in \text{In}(J)$. Then

$$g = a_1 F_1^* + a_2 F_2^* + \cdots + a_n F_n^*,$$

where $a_i \in k[X, Y]$ and $F_i \in J$ for $1 \leq i \leq n$. Therefore, it suffices to show if $F \in J$ then $F^* \in (x^2, y^{t+1}, xy)$. Let $\alpha, \beta \in k\llbracket X, Y \rrbracket$ such that $F = \alpha(x^2 + y^t) + \beta xy$. Then

$$F^* = (\alpha x^2 + \alpha y^t + \beta xy)^* = \alpha x^2 + b y^t + c xy$$

for some $a, b, c \in k[X, Y]$. If $b = 0$ then $F^* = \alpha x^2 + c xy \in (x^2, y^{t+1}, xy)$.

Assume $b \neq 0$. Since F^* is homogeneous, b is homogeneous and may be written as $b = px^n + qy$, where $p \in k$, $q \in k[X, Y]$, and n is a nonnegative integer. Therefore,

$$F^* = \alpha x^2 + c xy + px^n y^t + qy^{t+1}.$$

If $p = 0$ then we again have the needed form.

Assume $p \neq 0$ and consider two cases for n .

Case (i): Assume $n \geq 1$. Then $px^n y^t = px^{n-1} y^{t-1} (xy)$. Hence,

$$F^* = \alpha x^2 + qy^{t+1} + (c + px^{n-1} y^{t-1}) xy$$

has the needed form.

Case (ii): Assume $n = 0$. Since we have already shown $(x^2, y^{t+1}, xy) \subseteq \text{In}(J)$, we have

$$F^* - \alpha x^2 - qy^{t+1} - c xy = px^n y^t \in \text{In}(J).$$

Since $n = 0$ and $p^{-1} \in k$ we have $y^t \in \text{In}(J)$, which contradicts Lemma 2.4. Therefore, case (ii) does not occur. \square

Remark 2.6. A common approach to working with $\text{In}(J)$ is to invoke the use of Gröbner bases. However, we opt for the more elementary method presented above.

We are now ready to prove the theorem.

Proof of Theorem 2.2. Since R is artinian, it is complete. Since $(x^2 + y^t)^* = x^2$ and $(xy)^* = xy$, we know R is 2-homogeneous. As noted above,

$$R^g = Q^g / \text{In}(J).$$

By Lemma 2.5, $\text{In}(J) = (x^2, y^{t+1}, xy)$. Hence, R^g is not a graded k -algebra and the theorem follows. \square

References

- [Achilles and Avramov 1982] R. Achilles and L. L. Avramov, “Relations between properties of a ring and of its associated graded ring”, pp. 5–29 in *Seminar D. Eisenbud/B. Singh/W. Vogel, II*, Teubner-Texte Math. **48**, Teubner, Leipzig, 1982. [MR](#) [Zbl](#)
- [Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge Univ. Press, 1993. [MR](#) [Zbl](#)
- [Eisenbud 1995] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995. [MR](#) [Zbl](#)
- [Fröberg 1987] R. Fröberg, “Connections between a local ring and its associated graded ring”, *J. Algebra* **111**:2 (1987), 300–305. [MR](#) [Zbl](#)
- [Hoffmeier and Şega 2017] J. Hoffmeier and L. M. Şega, “Conditions for the Yoneda algebra of a local ring to be generated in low degrees”, *J. Pure Appl. Algebra* **221**:2 (2017), 304–315. [MR](#) [Zbl](#)

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