

involve

a journal of mathematics

Computing indicators of Radford algebras

Hao Hu, Xinyi Hu, Linhong Wang and Xingting Wang



Computing indicators of Radford algebras

Hao Hu, Xinyi Hu, Linhong Wang and Xingting Wang

(Communicated by Kenneth S. Berenhaut)

We compute higher Frobenius–Schur indicators of Radford algebras in positive characteristic and find minimal polynomials of these linearly recursive sequences. As a result of the work of Kashina, Montgomery and Ng, we obtain gauge invariants for the monoidal categories of representations of Radford algebras.

1. Introduction

In group theory, the Frobenius–Schur (FS) indicator provides a criterion, depending on its possible values 1, 0, or -1 , for determining whether an irreducible representation of a finite group G is real, complex or quaternionic. This result was generalized to any semisimple Hopf algebra over an algebraically closed field of characteristic zero in [Linchenko and Montgomery 2000]. Kashina, Montgomery and Ng [Kashina et al. 2012] proposed a definition of higher Frobenius–Schur (FS) indicators for an arbitrary finite-dimensional Hopf algebra, which further generalizes the notion given in [Kashina et al. 2006] regarding the regular representation of a semisimple Hopf algebra. Moreover, they proved that these indicators are gauge invariant under gauge equivalence in the sense of [Kassel 1995]. Later, the properties of these indicators were further discussed by Shimizu [2015], who mainly focused on the complex Hopf algebras.

The definition of higher FS indicators of the regular representation of a finite-dimensional Hopf algebra is straightforward by taking the trace of the Sweedler powers followed by the antipode; see [Kashina et al. 2012, Definition 2.1]. But to find their values can be arithmetically challenging over the complex numbers, e.g., in the case of the indicators of Taft algebras; see [Kashina et al. 2012, §3]. Besides Taft algebras, another well-studied Hopf algebra with simple defining relation is the Radford algebra $R(p)$, which was introduced in [Radford 1977, 4.13] and is over a base algebraically closed field of prime characteristic p . It was proved in [Wang and Wang 2014] that $R(p)$ is the only noncommutative and noncocommutative pointed Hopf algebra of dimension p^2 .

MSC2010: 16T05.

Keywords: Hopf algebras, FS indicators, positive characteristic.

In this short note, we find that the higher FS indicators of the Radford algebra $R(p)$ are

$$\{v_n(R(p))\}_{n \geq 1} = \{\underbrace{1, \dots, 1}_{p-1}, 0, \underbrace{1, \dots, 1}_{p-1}, 0, \dots\}.$$

Our approach is via concrete computation involving the left integrals of the Radford algebra and those of its dual Hopf algebra. Our result verified, in the case of the Radford algebra, a theorem by Shimizu [2015, Corollary 4.6] on higher FS indicators over positive characteristic, which states that the sequence of indicators always appears periodically in positive characteristic. As a result of the work of Kashina, Montgomery and Ng, we obtain gauge invariants for the monoidal category of the representation of Radford algebras. Moreover, we also find the minimal polynomial of the sequence of indicators of the Radford algebra.

2. Preliminaries

Throughout, \mathbb{k} is an algebraically closed field, H is a finite-dimensional Hopf algebra over \mathbb{k} . We use the standard notation $(H, m, u, \Delta, \varepsilon, S)$, where $m : H \otimes H \rightarrow H$ is the multiplication map, $u : \mathbb{k} \rightarrow H$ is the unit map, $\Delta : H \rightarrow H \otimes H$ is the comultiplication map, $\varepsilon : H \rightarrow \mathbb{k}$ is the counit map, and $S : H \rightarrow H$ is the antipode. The vector space dual of H is also a Hopf algebra and will be denoted by H^* . The bialgebra maps and antipode of H^* are given by $(m_{H^*}, u_{H^*}, \Delta_{H^*}, \varepsilon_{H^*}, S_{H^*}) = (\Delta^*, \varepsilon^*, m^*, u^*, S^*)$, where $*$ is the transpose. We use the Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. If $f, g \in H^*$, then $fg(h) = \sum f(h_{(1)})g(h_{(2)})$ for any $h \in H$ and $\varepsilon_{H^*}(f) = f(1)$.

2.1. Definition [Montgomery 1993, Definition 2.1.1]. A left integral in H is an element $\Lambda \in H$ such that $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$; a right integral in H is an element $\Lambda' \in H$ such that $\Lambda'h = \varepsilon(h)\Lambda'$ for all $h \in H$. The spaces of left and right integrals are denoted by \int_H^l and \int_H^r , respectively.

2.2. Lemma [Montgomery 1993, Theorem 2.1.3]. *The spaces \int_H^l and \int_H^r are each one-dimensional.*

2.3. Lemma. *Suppose $\lambda \in H^*$. Then λ is a left integral of H^* if and only if $\sum h_{(1)}\lambda(h_{(2)}) = \lambda(h)$ for any $h \in H$. A similar criterion holds for a right integral of H^* , i.e., λ is a right integral of H^* if and only if $\sum \lambda(h_{(1)})h_{(2)} = \lambda(h)$ for any $h \in H$.*

Proof. By definition, λ is a left integral in H^* if and only if $f\lambda = \varepsilon_{H^*}(f)\lambda$ for any linear function $f \in H^*$. That is, $f\lambda(h) = \varepsilon_{H^*}(f)\lambda(h)$ for any $h \in H$. By duality, this is equivalent to $\sum f(h_{(1)})\lambda(h_{(2)}) = f(1)\lambda(h)$ or $f(\sum h_{(1)}\lambda(h_{(2)})) = f(1\lambda(h))$ since f is linear. Note that f is arbitrary in H^* . We have λ is a left integral in H^*

if and only if $\sum h_{(1)}\lambda(h_{(2)}) = \lambda(h)$ for any $h \in H$. The proof for right integrals is the same. \square

2.4. Definition [Kashina et al. 2012, Definition 2.1]. Let n be a positive integer. Suppose $h_1, \dots, h_n \in H$. Then the n -th power of multiplication is defined as

$$m^{(n)}(h_1 \otimes \dots \otimes h_n) = h_1 \cdots h_n.$$

Let $h \in H$. The n -th power of comultiplication is defined to be

$$\Delta^{(n)}(h) = \begin{cases} h, & n = 1, \\ (\Delta^{(n-1)} \otimes \text{id})(\Delta(h)), & n \geq 2. \end{cases}$$

The n -th Sweedler power of h is defined to be

$$P_n(h) = h^{[n]} = \begin{cases} \varepsilon(h)1_H, & n = 0, \\ m^{(n)} \circ \Delta^{(n)}(h), & n \geq 1. \end{cases}$$

The n -th indicator of H is given by

$$v_n(H) = \text{Tr}(S \circ P_{n-1}).$$

In particular, $v_1(H) = 1$ and $v_2(H) = \text{Tr}(S)$.

Let H and K be two finite-dimensional Hopf algebras over \mathbb{k} such that the two representation categories $\text{Rep}(H)$ and $\text{Rep}(K)$ are monoidally equivalent. By [Ng and Schauenburg 2008, Theorem 2.2], $H \cong K^F$, where K^F is a Drinfeld twist by a gauge transformation F on H which satisfies some 2-cocycle conditions. Then H and K are said to be *gauge equivalent* Hopf algebras.

2.5. Theorem [Kashina et al. 2012, Theorem 2.2, Corollary 2.6]. *The sequence $\{v_n(H)\}$ is an invariant of the gauge equivalence class of Hopf algebras of H ; that is, if H and K are gauge equivalent then $\{v_n(H)\} = \{v_n(K)\}$. Suppose $\lambda \in H^*$ and $\Lambda \in H$ are both left integrals (or both right integrals) such that $\lambda(\Lambda) = 1$. Then*

$$v_n(H) = \lambda(\Lambda^{[n]})$$

for all positive integers n .

2.6. Proposition [Shimizu 2015, Corollary 4.6]. *Suppose $\text{char } \mathbb{k} > 0$. Then, for any finite-dimensional Hopf algebra H over \mathbb{k} , the sequence $\{v_n(H)\}$ is periodic.*

2.7. Definition. A sequence $\{a_n\}_{n \geq 1}$ is linearly recursive if there exists a nonzero polynomial $f(x) = f_0 + f_1x + f_{m-1}x^{m-1} + f_mx^m$ such that

$$f_0a_n + f_1a_{n+1} + \dots + f_ma_{m+n} = 0$$

for any positive integer n . In such a case, we say that $\{a_n\}_{n \geq 1}$ satisfies the polynomial $f(x)$. The monic polynomial of the least degree satisfied by a linearly recursive sequence is called the minimal polynomial of the sequence.

2.8. Proposition [Kashina et al. 2012, Proposition 2.7]. *The sequence $\{v_n(H)\}$ is linearly recursive and the degree of its minimal polynomial is at most $(\dim H)^2$. The minimal polynomial is also a gauge invariant; that is, if H and K are gauge equivalent, then $\{v_n(H)\}$ and $\{v_n(K)\}$ have the same monic minimal polynomial.*

Next, we consider a free bialgebra \mathfrak{B} and the comultiplication of certain monomials in \mathfrak{B} . This information will be used later in our computation of indicators of $R(p)$.

2.9. Definition. Let $\mathfrak{B} = \mathbb{k}\langle g, x \rangle$ be the free \mathbb{k} -algebra on two generators g and x . Equipped with the comultiplication and the counit given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1 \quad \text{and} \quad \varepsilon(x) = 0,$$

the free algebra becomes the free bialgebra $(\mathfrak{B}, \Delta, \varepsilon)$. Let $C_{k,l}$ denote the sum of all monomials with k g 's and l x 's, and $C_{0,0} = 1$ and $C_{k,l} = 0$ if k or $l < 0$ by convention.

2.10. Lemma. *In the free bialgebra \mathfrak{B} , we have*

- (a) $C_{k,l} = g C_{k-1,l} + x C_{k,l-1} = C_{k-1,l} g + C_{k,l-1} x$.
- (b) $\Delta(x^n) = \sum_{k \geq 0} C_{k,n-k} \otimes x^k$ for $n \geq 0$.
- (c) $\Delta(C_{p,q}) = \sum_{k \geq 0} C_{p+k,q-k} \otimes C_{p,k}$.

Proof. Part (a) is clear, since the leftmost (rightmost) factor of any monomial in the sum $C_{k,l}$ is either g or x . For (b), we use induction. When $n = 0$,

$$\sum_{k \geq 0} C_{k,n-k} \otimes x^k = C_{0,0} \otimes 1 = 1 \otimes 1 = \Delta(1).$$

When $n = 1$,

$$\sum_{k \geq 0} C_{k,n-k} \otimes x^k = C_{0,1} \otimes 1 + C_{1,0} \otimes x = x \otimes 1 + g \otimes x = \Delta(x).$$

Suppose $\Delta(x^n) = \sum_{k \geq 0} C_{k,n-k} \otimes x^k$. Then

$$\begin{aligned} \Delta(x^{n+1}) &= \Delta(x^n) \Delta(x) = \left(\sum_{k \geq 0} C_{k,n-k} \otimes x^k \right) \cdot (x \otimes 1 + g \otimes x) \\ &= \sum_{k \geq 0} C_{k,n-k} x \otimes x^k + \sum_{k \geq 1} C_{k-1,n-k+1} g \otimes x^k \\ &= \sum_{k \geq 0} C_{k,n-k} x \otimes x^k + \sum_{k \geq 1} (C_{k,n-k+1} - C_{k,n-k} x) \otimes x^k \\ &= \sum_{k \geq 0} C_{k,(n+1)-k} \otimes x^k. \end{aligned}$$

To show (c), we use the fact that

$$(\Delta \otimes \text{id})(\Delta(x^n)) = (\text{id} \otimes \Delta)(\Delta(x^n)).$$

By (b), we have

$$(\Delta \otimes \text{id})(\Delta(x^n)) = \sum_{k \geq 0} \Delta(C_{k,n-k}) \otimes x^k = \sum_{p+q=n} \Delta(C_{p,q}) \otimes x^p.$$

On the other hand,

$$\begin{aligned} (\text{id} \otimes \Delta)(\Delta(x^n)) &= \sum_{l \geq 0} C_{l,n-l} \otimes \Delta(x^l) = \sum_{l \geq 0} C_{l,n-l} \otimes \left(\sum_{p \geq 0} C_{p,l-p} \otimes x^p \right) \\ &= \sum_{p \geq 0} \sum_{l \geq p} C_{l,n-l} \otimes C_{p,l-p} \otimes x^p \\ &= \sum_{p+q=n} \left(\sum_{l-p=k \geq 0} C_{p+k,q-k} \otimes C_{p,k} \right) \otimes x^p. \end{aligned}$$

It then follows that $\Delta(C_{p,q}) = \sum_{k \geq 0} C_{p+k,q-k} \otimes C_{p,k}$. □

2.11. Lemma. *In the free bialgebra \mathfrak{B} , we have*

$$(g^i x^j)^{[n+1]} = \sum_{0 \leq k_1 + \dots + k_n \leq j} g^i C_{k_1 + \dots + k_n, j - (k_1 + \dots + k_n)} g^i C_{k_1 + \dots + k_{n-1}, k_n} \dots g^i C_{k_1, k_2} g^i C_{0, k_1}.$$

Proof. By induction on n , using [Lemma 2.10](#), it is easy to see that

$$\Delta^{(n+1)}(C_{p,q}) = \sum_{0 \leq k_1 + \dots + k_n \leq q} C_{p+k_1 + \dots + k_n, q - (k_1 + \dots + k_n)} \otimes C_{p+k_1 + \dots + k_{n-1}, k_n} \otimes \dots \otimes C_{p+k_1, k_2} \otimes C_{p, k_1}.$$

Therefore, we have

$$\begin{aligned} (g^i x^j)^{[n+1]} &= m^{(n+1)} (\Delta^{(n+1)}(g^i) \Delta^{(n+1)}(x^j)) \\ &= m^{(n+1)} (g^i \otimes \dots \otimes g^i) \left(\sum_{k \geq 0} \Delta^{(n)}(C_{k, j-k}) \otimes x^k \right) \\ &= \sum_{0 \leq k_1 + \dots + k_n \leq j} g^i C_{k_1 + \dots + k_n, j - (k_1 + \dots + k_n)} \dots g^i C_{k_1, k_2} g^i C_{0, k_1}. \quad \square \end{aligned}$$

3. Radford algebras

In this section, the base field \mathbb{k} is algebraically closed of prime characteristic p .

3.1. The Radford algebra $R(p)$ [1977, 4.13] was first discussed over a base field \mathbb{k} of prime characteristic p , and was proved in [Wang and Wang 2014] to be the only noncommutative and noncocommutative pointed Hopf algebra of dimension p^2

over \mathbb{k} . In fact, one can write $R(p)$ as the quotient Hopf algebra \mathfrak{B}/\mathcal{R} , where the ideal \mathcal{R} of \mathfrak{B} is generated by

$$g^p - 1, \quad x^p - x, \quad [g, x] - (g^2 - g) \quad (\mathcal{R})$$

if $p > 2$, or

$$g^2 - 1, \quad x^2 - x, \quad [g, x] - (1 - g)$$

if $p = 2$. It is straightforward to check that the Radford algebra $R(p)$ has dimension p^2 and the linear basis can be chosen as $\{g^i x^j \mid 0 \leq i, j \leq p - 1\}$. We denote by $c_{k,l}$ the image of $C_{k,l}$ (the sum of all monomials with k g 's and l x 's in \mathfrak{B}) in $R(p)$ under the projection $\mathfrak{B} \rightarrow \mathfrak{B}/\mathcal{R} = R(p)$. It follows from (\mathcal{R}) that, for $0 \leq k, l \leq p - 1$,

$$c_{k,l} = \binom{k+l}{k} g^k x^l + \sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq j \leq l-1}} a_{ij} g^i x^j \quad \text{for some } a_{ij} \in \mathbb{k}. \quad (1)$$

Moreover, the Radford algebra $R(p)$ is self-dual. The dual basis of $(R(p))^*$ to the chosen basis $\{g^i x^j \mid 0 \leq i, j \leq p - 1\}$ of $R(p)$ is $\{\delta_{g^i x^j} \mid 0 \leq i, j \leq p - 1\}$, where $\delta_{g^i x^j}$ are characteristic functions, that is,

$$\delta_{g^i x^j}(g^m x^n) = \begin{cases} 1 & \text{if } m = i, n = j, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Lemma. *For the Radford algebra $R(p)$, the integral spaces are given by*

$$\begin{aligned} \int_{R(p)}^l &= \mathbb{k} \left(\sum_{0 \leq i \leq p-1} g^i \right) \left(\sum_{1 \leq i \leq p-1} (-1)^i x^i \right), \\ \int_{R(p)}^r &= \mathbb{k} \left(\sum_{1 \leq i \leq p-1} x^i \right) \left(\sum_{0 \leq i \leq p-1} g^i \right). \end{aligned}$$

For the dual Hopf algebra $(R(p))^$, the integral spaces are given by*

$$\int_{(R(p))^*}^l = \mathbb{k} \delta_{g x^{p-1}} \quad \text{and} \quad \int_{(R(p))^*}^r = \mathbb{k} \delta_{x^{p-1}},$$

Proof. Note that $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, and ε is linear. To show that the element $\Lambda = (\sum_{0 \leq i \leq p-1} g^i) (\sum_{1 \leq i \leq p-1} (-1)^i x^i)$ is a left integral in $R(p)$, it is sufficient to show that $g\Lambda = \Lambda$ and $x\Lambda = 0$. The first equation is obvious. To show the second, one can check that $[x, g^i] = i g^i (1 - g)$. Hence we have

$$\left[x, \sum_{i=1}^{p-1} g^i \right] = \sum_{i=1}^{p-1} i g^i (1 - g) = \sum_{i=1}^{p-1} i g^i - \sum_{j=2}^p (j-1) g^j = g + \sum_{i=2}^{p-1} g^i + g^p = \sum_{i=0}^{p-1} g^i,$$

and so

$$\begin{aligned} x\Lambda &= x\left(\sum_{0\leq i\leq p-1} g^i\right)\left(\sum_{1\leq i\leq p-1} (-1)^i x^i\right) \\ &= \left(\sum_{i=0}^{p-1} g^i\right)(x+1)\left(\sum_{1\leq i\leq p-1} (-1)^i x^i\right) = \left(\sum_{i=0}^{p-1} g^i\right)(x^p - x) = 0. \end{aligned}$$

Therefore, Λ is a left integral in $R(p)$.

To show that the characteristic function $\delta_{gxp^{p-1}}$ is a left integral in $R(p)^*$, it is sufficient, by Lemma 2.3, to verify that

$$\sum h_{(1)}\delta_{gxp^{p-1}}(h_{(2)}) = \delta_{gxp^{p-1}}(h) \quad \text{for } h = g^i x^j \in R(p) \text{ with } 0 \leq i, j \leq p-1.$$

By Lemma 2.10, we have $\Delta(g^i x^j) = (g^i \otimes g^i)\Delta(x^j) = \sum_{k=0}^j g^i c_{k, j-k} \otimes g^i x^k$. Hence

$$\begin{aligned} \sum h_{(1)}\delta_{gxp^{p-1}}(h_{(2)}) &= \sum_{k=0}^j (g^i c_{k, j-k} \cdot \delta_{gxp^{p-1}}(g^i x^k)) \\ &= \begin{cases} gc_{p-1,0} = 1 & \text{if } i = 1, j = k = p-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\delta_{gxp^{p-1}}(h) = \delta_{gxp^{p-1}}(g^i x^j) = \begin{cases} 1 & \text{if } i = 1, j = p-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\delta_{gxp^{p-1}}$ is a left integral in $R(p)^*$. The statements on right integrals can be shown similarly. □

3.3. Theorem. *The higher FS indicators of the Radford algebra $R(p)$ are given by*

$$v_n(R(p)) = \begin{cases} 1 & \text{if } n \not\equiv 0 \pmod{p}, \\ 0 & \text{if } n \equiv 0 \pmod{p}. \end{cases}$$

Proof. By Lemma 3.2, we choose the left integral $\lambda = \delta_{gxp^{p-1}}$ of the dual Hopf algebra $(R(p))^*$, and the left integral $\Lambda = (\sum_{0\leq i\leq p-1} g^i)(\sum_{1\leq i\leq p-1} (-1)^i x^i)$ of $R(p)$. It is clear that $\lambda(\Lambda) = 1$. By Theorem 2.5, we have

$$v_{n+1}(R(p)) = \lambda(\Lambda^{[n+1]}) = \delta_{gxp^{p-1}}\left(\sum_{0\leq i, j\leq p-1} (-1)^j (g^i x^j)^{[n+1]}\right).$$

By Lemma 2.11 and (1), one sees that, for any $0 \leq i, j \leq p-1$,

$$(g^i x^j)^{[n+1]} \in \text{Span}(g^k x^l \mid 0 \leq k \leq p-1, 0 \leq l \leq j).$$

Hence,

$$v_{n+1}(R(p)) = \delta_{g x^{p-1}} \left(\sum_{0 \leq i \leq p-1} (g^i x^{p-1})^{[n+1]} \right).$$

Suppose k_1, \dots, k_n are nonnegative integers such that $\sum_{i=1}^n k_i = m$. Recall that the multinomial coefficients are given by

$$\binom{m}{k_1, \dots, k_n} := \frac{(m!)}{(k_1!) \cdots (k_n!)}.$$

Assume that $n \geq 1$. Set $k_{n+1} = p - 1 - k_1 - \cdots - k_n$. By Lemma 2.11, we have

$$\begin{aligned} & \sum_{0 \leq i \leq p-1} (g^i x^{p-1})^{[n+1]} \\ &= \sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq k_1, \dots, k_n \leq p-1}} (g^i c_{k_1 + \dots + k_n, k_{n+1}} g^i c_{k_1 + \dots + k_{n-1}, k_n} \cdots g^i c_{k_1, k_2} g^i c_{0, k_1}) \\ &= \sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq k_1, \dots, k_n \leq p-1}} \left(\binom{p-1}{k_1 + \dots + k_n} \binom{k_1 + \dots + k_n}{k_n} \cdots \binom{k_1 + k_2}{k_1} \right) \\ & \quad g^i (g^{k_1 + \dots + k_n} x^{k_{n+1}}) g^i (g^{k_1 + \dots + k_{n-1}} x^{k_n}) \cdots g^i (g^{k_1} x^{k_2}) g^i (x^{k_1}) \\ &= \sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq k_1, \dots, k_n \leq p-1}} \binom{p-1}{k_1, \dots, k_{n+1}} g^\kappa x^{p-1}, \end{aligned}$$

where $\kappa = (n+1)i + nk_1 + (n-1)k_2 + \cdots + k_n$. Therefore,

$$v_{n+1}(R(p)) = \sum_{0 \leq k_1, \dots, k_{n+1} \leq p-1} \binom{p-1}{k_1, \dots, k_{n+1}} \delta_{g x^{p-1}} \left(\sum_{i=0}^{p-1} g^\kappa x^{p-1} \right).$$

Suppose the indices k_1, k_2, \dots, k_n are fixed. Then the inner summation of the above equation becomes

$$\sum_{0 \leq i \leq p-1} g^\kappa x^{p-1} = \begin{cases} p(g^{(nk_1 + (n-1)k_2 + \dots + k_n)} x^{p-1}) = 0 & \text{if } p \mid n+1, \\ (1 + g + \dots + g^{p-1})x^{p-1} & \text{if } p \nmid n+1. \end{cases}$$

In a conclusion, by Fermat's little theorem and for $n \geq 1$, we have

$$v_{n+1}(R(p)) = \begin{cases} 0 & \text{if } p \mid n+1, \\ \sum_{k_1, \dots, k_{n+1}} \binom{p-1}{k_1, \dots, k_{n+1}} = (n+1)^{p-1} = 1 & \text{if } p \nmid n+1. \end{cases}$$

Note that $v_1(R(p)) = 1$. Therefore, we showed that

$$\{v_n(R(p))\}_{n \geq 1} = \{\underbrace{1, \dots, 1}_{p-1}, 0, \underbrace{1, \dots, 1}_{p-1}, 0, \dots\}.$$

□

3.4. Proposition. *The minimal polynomial of the sequence $\{v_n(R(p))\}$ is*

$$f(x) = x^p - 1.$$

Proof. The first $p + 1$ terms of $\{v_n(R(p))\}$ are $1, \dots, 1, 0, 1$. The degree of the minimal polynomial cannot be less than p . Otherwise, $\{v_n(R(p))\}$ satisfies a polynomial $f(x) = f_0 + f_1x_1 + \dots + f_{p-1}x^{p-1}$. Then

$$A[f_0 \ f_1 \ \dots \ f_{p-1}]^T = 0,$$

where A is the matrix with 0's on the antidiagonal and 1's elsewhere. Note that the determinant of A is $p - 1$ or $-(p - 1)$. This implies that $f_0 = f_1 = \dots = f_{p-1} = 0$, a contradiction. Hence the degree of the minimal polynomial is at least p . One can verify that $\{v_n(R(p))\}$ satisfies the polynomial $f(x) = x^p - 1$. \square

Acknowledgements

We began this work in an undergraduate research project at the University of Pittsburgh, and we would like to express our gratitude to the math department for hosting a visit of X. Wang in Spring 2016. We are grateful to the referee for careful reading.

References

- [Kashina et al. 2006] Y. Kashina, Y. Sommerhäuser, and Y. Zhu, *On higher Frobenius–Schur indicators*, Mem. Amer. Math. Soc. **855**, American Mathematical Society, Providence, RI, 2006. [MR](#) [Zbl](#)
- [Kashina et al. 2012] Y. Kashina, S. Montgomery, and S.-H. Ng, “On the trace of the antipode and higher indicators”, *Israel J. Math.* **188**:1 (2012), 57–89. [MR](#) [Zbl](#)
- [Kassel 1995] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer, New York, 1995. [MR](#) [Zbl](#)
- [Linchenko and Montgomery 2000] V. Linchenko and S. Montgomery, “A Frobenius–Schur theorem for Hopf algebras”, *Algebr. Represent. Theory* **3**:4 (2000), 347–355. [MR](#) [Zbl](#)
- [Montgomery 1993] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics **82**, American Mathematical Society, Providence, RI, 1993. [MR](#) [Zbl](#)
- [Ng and Schauenburg 2008] S.-H. Ng and P. Schauenburg, “Central invariants and higher indicators for semisimple quasi-Hopf algebras”, *Trans. Amer. Math. Soc.* **360**:4 (2008), 1839–1860. [MR](#) [Zbl](#)
- [Radford 1977] D. E. Radford, “Operators on Hopf algebras”, *Amer. J. Math.* **99**:1 (1977), 139–158. [MR](#) [Zbl](#)
- [Shimizu 2015] K. Shimizu, “On indicators of Hopf algebras”, *Israel J. Math.* **207**:1 (2015), 155–201. [MR](#) [Zbl](#)
- [Wang and Wang 2014] L. Wang and X. Wang, “Classification of pointed Hopf algebras of dimension p^2 over any algebraically closed field”, *Algebr. Represent. Theory* **17**:4 (2014), 1267–1276. [MR](#) [Zbl](#)

Received: 2016-12-19

Revised: 2017-03-16

Accepted: 2017-04-09

hah73@pitt.edu*Department of Mathematics, University of Pittsburgh,
Pittsburgh, PA, United States*xih40@pitt.edu*Department of Mathematics, University of Pittsburgh,
Pittsburgh, PA, United States*lhwang@pitt.edu*Department of Mathematics, University of Pittsburgh,
Pittsburgh, PA, United States*xingting@temple.edu*Department of Mathematics, Temple University,
Philadelphia, PA, United States*

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Erin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2018 is US \$190/year for the electronic version, and \$250/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

involve

2018

vol. 11

no. 2

Finding cycles in the k -th power digraphs over the integers modulo a prime	181
GREG DRESDEN AND WENDA TU	
Enumerating spherical n -links	195
MADELEINE BURKHART AND JOEL FOISY	
Double bubbles in hyperbolic surfaces	207
WYATT BOYER, BRYAN BROWN, ALYSSA LOVING AND SARAH TAMMEN	
What is odd about binary Parseval frames?	219
ZACHERY J. BAKER, BERNHARD G. BODMANN, MICAH G. BULLOCK, SAMANTHA N. BRANUM AND JACOB E. MCLANEY	
Numbers and the heights of their happiness	235
MAY MEI AND ANDREW READ-MCFARLAND	
The truncated and supplemented Pascal matrix and applications	243
MICHAEL HUA, STEVEN B. DAMELIN, JEFFREY SUN AND MINGCHAO YU	
Hexatonic systems and dual groups in mathematical music theory	253
CAMERON BERRY AND THOMAS M. FIORE	
On computable classes of equidistant sets: finite focal sets	271
CSABA VINCZE, ADRIENN VARGA, MÁRK OLÁH, LÁSZLÓ FÓRIÁN AND SÁNDOR LŐRINC	
Zero divisor graphs of commutative graded rings	283
KATHERINE COOPER AND BRIAN JOHNSON	
The behavior of a population interaction-diffusion equation in its subcritical regime	297
MITCHELL G. DAVIS, DAVID J. WOLLKIND, RICHARD A. CANGELOSI AND BONNI J. KEALY-DICHONE	
Forbidden subgraphs of coloring graphs	311
FRANCISCO ALVARADO, ASHLEY BUTTS, LAUREN FARQUHAR AND HEATHER M. RUSSELL	
Computing indicators of Radford algebras	325
HAO HU, XINYI HU, LINHONG WANG AND XINGTING WANG	
Unlinking numbers of links with crossing number 10	335
LAVINIA BULAI	
On a connection between local rings and their associated graded algebras	355
JUSTIN HOFFMEIER AND JIYOON LEE	