CORRECTION AND COMPLETION OF THE PAPER "GENERALIZATION OF A FORMULA OF HAYMAN"

BY

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1. In a letter of June 2, 1966, Professor L. Schoenfeld made several critical remarks concerning [2]. A few minor printing errors will easily have been corrected by the reader, but I believe that the following two points require correction and/or completion.

(A) The estimate (7) of S_2' on p. 13 depends on all $a_n(x)$ and it is not sufficiently clear how it follows from the assumption of (non-uniform) boundedness made on the $a_n(x)$.

(B) The indications how (14) are obtained are so brief as to mislead the reader concerning the difficulties involved.

In trying to clarify the matter, I observed that a slight strengthening of the assumptions was needed, but once this is done, the result may be stated in a somewhat neater form. In order to settle this matter it seems desirable to restate in full the needed assumptions and the theorem; but, in order to keep these remarks brief, I shall refer to [2] or [4] for all statements proven there and to Harris and Schoenfeld [3] for the proof of the main theorem. The interested reader will have to "translate" quite a few notations, but my version of the corrected proof is rather long and in view of the fact that in the meantime [3] has come out, the publication in toto of another proof does not seem warranted. I wish to thank Professor L. Schoenfeld for having called my attention upon these two points and for a lengthy correspondence with helpful suggestions for the clarification of several others. Some suggestions of referees that permitted streamlining of the presentation are also gratefully acknowledged.

2. In [4] Hayman defines a class H of "admissible" functions, and proves an asymptotic formula for the coefficients of their power series expansion. Here we are concerned with subclasses $F_{V+1} \subset H$ $(V \in Z^+)$ of the Hayman admissible functions, characterized by some additional properties. A function $f \in H$ belongs to F_{V+1} if

(1) for every $k \leq V + 1$ there exist positive constants r_k, ε_k, M_k, such that for $r_k < r < R$, $\varepsilon_k\, a_2(r) \leq |\, a_k(r)\,| \leq M_k\, a_2(r)$ holds; and

(2) for $(0 \leq) r_0 \leq r$, $|z| < R$ and $|z - r| < 2r\delta(r)$ one has $f(z) \neq 0$ and

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$| a_V(z) | < C_V | a_V(r) |$, where $\delta(r)$ (defined in [2] and [3]) satisfies $\delta^3(r) a_2(r) \to 0$ for $r \to R$.

If $f \in F_{V+1}$ for all V we say that $f \in F (= \bigcap_V F_{V+1})$. We recall that $f \in H$ implies (for $r \to R$) that

$$(3) \quad \begin{array}{rl} \text{(i)} & a_1(r), a_2(t) \to \infty \text{ (hence, by (2) and (1), } \delta(r) \to 0 \text{ and } a_k(r) \to \infty \\ & \text{for } 1 \le k \le V + 1); \\ \text{(ii)} & \delta^2(r) a_2(r) \to \infty; \\ \text{(iii)} & \lambda(r) a_2^{1/2}(r) \to 0 \text{ (for definition of } \lambda(r) = \lambda(r; \delta) \text{ see [2]).} \end{array}$$

Let $\Phi_V(r; \delta) = \max (a_2^{1/2}\lambda, a_2^{-(V+9)/6})$; then the following theorem holds.

THEOREM. *If* $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in F_{6m+1}$ *and* $r = r(n)$ *is defined by* (1) *of* [2] *then*

$$(4) \quad \alpha_n = f(r) r^{-n} (2\pi a_2)^{-1/2} \{1 + \pi^{-1/2} \textstyle\sum_{\nu=2}^{3m} (2a_2^{-1})^{\nu} \Gamma(\nu + \tfrac{1}{2}) A_{2\nu}(r) + R\},$$

where $| R | \le K_m \Phi_{6m}(r; \delta)$.

Remarks. The result remains formally correct also for $V \not\equiv 0 \pmod 6$, but then R is of the same order as the last one or two terms of the sum; for $V = 1$, (4) reduces to Hayman's Theorem.

3. The verification that the function $f(z)$ of the second part satisfies the conditions of the theorem, (actually, for every V so that $f \in F$) proceeds as follows. The conditions for $f \in H$ are verified in [4]. Next one recalls that the zeros t of $\Xi(t)$ are of the form $t = t_0 + iY$, $| Y | < \frac{1}{2}$, and that the zeros z of $f(z)$ satisfy $z = -t^2$. Consequently, $z = -(t_0^2 - Y^2) - 2iYt_0$ and $\operatorname{Re} z < -(t_0^2 - \frac{1}{4}) < 0$, because (see [5]) $t_0 > 14$. It follows that $f(z) \ne 0$ in $\operatorname{Re} z > 0$. Next, for each fixed $c > 0$, the following estimates hold uniformly for $| \arg z | < \pi - c$, $\nu = 1, 2, \cdots, V + 1$ ((14) of [2])

$$(14) \quad a_\nu(z) = 2^{-\nu-1} z^{1/2} \log \frac{e^{\nu-1} z^{1/2}}{2} + \frac{(-1)^\nu}{6} z^{-1/2} (2^{-\nu} - \tfrac{1}{2} z^{-1/2}) + \gamma_\nu + O(z^{-3/2})$$

where $\gamma_1 = \frac{7}{8}$, $\gamma_\nu = 0$ for $\nu \ge 2$.

Using (14) of [2], one observes that (1) holds, e.g., with $\varepsilon_k = 2^{1-k}$ and $M_k = 1$. From (14) also follows for $\varepsilon > 0$ arbitrarily small, $\delta(r) = r^{-1/6} \log^{-1/2} r$ and $| z - r | < 2r\delta$, that

$$\left| \frac{a_k(z)}{a_k(r)} \right| < \frac{1 + \varepsilon}{1 - \varepsilon} \frac{1 + (2/\log r)\,(k - 1 - \log(2\pi) + \frac{1}{2}\pi + \delta)}{1 + (2/\log r)\,(k - 1 - \log 2\pi)} < 2,$$

say, so that (2) holds with $C_V = 2$, and also $\delta^3 a_2 \to 0$, so that all conditions for $f \in F$ hold; finally $\lambda(r) = \exp(-r^{1/6}/32)$ (see [2], p. 19) and the proof is completed as in [2] starting on p. 19, line 23.

4. In order to obtain (14) of [2], one follows the indications on p. 18, line 3. From (13), $\log f(z) = f_1(z) + f_2(z) + f_3(z)$, with

$$f_1(z) = \tfrac{1}{2}z^{1/2}\log(z^{1/2}/2\pi e) + \tfrac{7}{8}\log z + \tfrac{1}{4}\log(\pi/2),$$
$$f_2(z) = J(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4}), \qquad f_3(z) = \log\zeta(z^{1/2} + \tfrac{1}{2}).$$

One computes the corresponding $a_\nu^{(j)}(z)$ $(j = 1, 2, 3)$ as follows: For $|\arg z| < \pi$,

$$a_1^{(1)} = \tfrac{1}{4}z^{1/2}\log(z^{1/2}/2\pi) + \tfrac{7}{8}, \qquad a_2^{(1)} = \tfrac{1}{8}z^{1/2}\log(ez^{1/2}/2\pi)$$

and, by induction on ν,

$$a_\nu^{(1)} = 2^{-\nu-1}z^{1/2}\log(e^{\nu-1}z^{1/2}/2\pi).$$

For each fixed $c > 0$, uniformly for $|\arg z| < \pi - c$,

$$a_1^{(3)} = z\frac{\zeta'}{\zeta}(z^{1/2} + \tfrac{1}{2})\cdot\tfrac{1}{2}z^{-1/2} = \tfrac{1}{2}z^{1/2}\textstyle\sum_{n=1}^{\infty}\Lambda(n)n^{-z^{1/2}-1/2},$$
$$a_2^{(3)} = -\tfrac{1}{4}z\textstyle\sum_{n=1}^{\infty}\Lambda(n)n^{-z^{1/2}-1/2}(\log n - z^{-1/2})$$

and, by induction,

$$a_\nu^{(3)} = (-1)^{\nu-1}z^{\nu/2}2^{-\nu}\textstyle\sum_{n=1}^{\infty}\Lambda(n)n^{-z^{1/2}-1/2}P_{\nu-1}(\log n, z^{-1/2}),$$

where $P_k(x, y)$ is a homogeneous polynomial, monic in x, of degree k; consequently,

$$a_\nu^{(3)} \sim (-1)^{\nu-1}z^{\nu/2}2^{-\nu}(\log 2)^\nu 2^{-z^{1/2}-1/2} = o(z^{-K}) \quad (z\to\infty, K \text{ arbitrarily large}).$$

The most laborious part is the computation of $a_\nu^{(2)}$. One has (see, e.g. [1, p. 166–167])

$$J(Z) = \sum_{\mu=1}^{m}\frac{B_{2\mu}}{2\mu(2\mu-1)}Z^{1-2\mu} + (-1)^m Z^{-2m-1}\frac{1}{\pi}\int_0^\infty\frac{u^{2m}}{1+(u/Z)^2}\log(1-e^{-2\pi u})^{-1}\,du,$$

where the convergence is clearly uniform for $\operatorname{Re} Z > c' > 0$, and thus for $|\arg z| < \pi$ if $Z = \tfrac{1}{2}z^{1/2} + \tfrac{1}{4}$. In particular, setting

$$g(u, z) = u^2\left\{1 + \left(\frac{2u}{z^{1/2}+\tfrac{1}{2}}\right)^2\right\}^{-1},$$

one has

$$J(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4}) = \tfrac{1}{2}B_2(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4})^{-1} - (\tfrac{1}{2}z^{1/2} + \tfrac{1}{4})^{-3}\cdot\frac{1}{\pi}\int_0^\infty g(u, z)\log(1-e^{-2\pi u})^{-1}\,du.$$

On account of the uniform convergence of the integral (which is majorized by $\int_0^\infty u^2 \log (1 - e^{-2\pi u})^{-1} du$) and that of its formal derivatives, one may differentiate under the integral sign and obtain

$$a_1^{(2)}(z) = z\frac{d}{dz} J(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4}) = -\frac{1}{12(z^{1/2} + \frac{1}{2})} + \frac{1}{24(z^{1/2} + \frac{1}{2})^2}$$
$$+ \left(\frac{12}{(z^{1/2} + \frac{1}{2})^3} - \frac{6}{(z^{1/2} + \frac{1}{2})^4}\right) I_1 - \left(\frac{32}{(z^{1/2} + \frac{1}{2})^5} - \frac{16}{(z^{1/2} + \frac{1}{2})^6}\right) I_2,$$

where

$$I_t = I_t(z) = \frac{1}{\pi}\int_0^\infty g^t(u, z) \log (1 - e^{-2\pi u})^{-1} du.$$

In general, one obtains, by finite induction on ν,

$$a_\nu^{(2)} = \sum_{k=1}^{\nu+1} \alpha_k^{(\nu)} (z^{1/2} + \tfrac{1}{2})^{-k} + \sum_{t=1}^{\nu} \{\sum_{k=1}^{\nu+1} \beta_{t,k}^{(\nu)} (z^{1/2} + \tfrac{1}{2})^{-2t-k}\} I_t(z).$$

One observes that all integrals $I_t(z)$ are majorized by the integrals

$$\frac{1}{\pi}\int_0^\infty \sum_{i=1}^{t} u^{2i} \log (1 - e^{-2\pi u})^{-1} du.$$

Consequently,

$$a_\nu^{(2)} = \alpha_1^{(\nu)}(z^{1/2} + \tfrac{1}{2})^{-1} + \alpha_2^{(\nu)}(z^{1/2} + \tfrac{1}{2})^{-2} + O(z^{-3/2})$$
$$= \alpha_1^{(\nu)} z^{-1/2} + \rho^{(\nu)} z^{-1} + O(z^{-3/2}),$$

the constant implied by the O-symbol depending, of course, on ν. One observes, that

$$\rho^{(\nu)} = \alpha_2^{(\nu)} - \tfrac{1}{2}\alpha_1^{(\nu)}, \quad \alpha_1^{(\nu)} = -\tfrac{1}{2}\alpha_1^{(\nu-1)} \quad \text{and} \quad \alpha_2^{(\nu+1)} = -\alpha_2^{(\nu)} + \tfrac{1}{4}\alpha_1^{(\nu)};$$

hence, using $\alpha_1^{(1)} = -\frac{1}{12}$, one obtains

$$\alpha_1^{(\nu)} = (-1)^\nu (6.2^\nu)^{-1},$$

and

$$\rho^{(\nu)} = \alpha_2^{(\nu)} - \tfrac{1}{2}\alpha_1^{(\nu)} = -(\alpha_2^{(\nu-1)} - \tfrac{1}{2}\alpha_1^{(\nu-1)})$$
$$= -\rho^{(\nu-1)} = (-1)^{\nu-1}\rho^{(1)} = (-1)^{\nu-1}/12.$$

Consequently, as $|z| \to \infty$, for fixed ν,

$$\alpha_\nu^{(2)} = \frac{(-1)^\nu}{6.2^\nu z^{1/2}} + \frac{(-1)^{\nu-1}}{12z} + O(z^{-3/2}).$$

Adding the results for $a_\nu^{(j)}(z)$ ($j = 1, 2, 3$), one obtains (14).

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