Illinois Journal of Mathematics
Volume 45, Number 2, Summer 2001, Pages 641–656
S 0019-2082

LOWERING THE ASSOUAD DIMENSION BY QUASISYMMETRIC MAPPINGS

JEREMY T. TYSON

Abstract. We study the relationship between the Assouad dimension and quasisymmetric mappings, showing that spaces of dimension strictly less than one can be quasisymmetrically deformed onto spaces of arbitrarily small dimension. We conjecture that this fact holds also for the Hausdorff dimension, and our results yield several corollaries which provide partial support for this conjecture. The proofs make use of connections between Assouad dimension, porosity, and ultrametrics.

1. Introduction

It is well-known that quasiconformal maps can distort Hausdorff dimension. Gehring and Väisälä [10] gave K-dependent estimates for the maximal distortion of the dimension of a subset of $\mathbb{R}^n$ by a K-quasiconformal map. Recently, dilatation-independent results have been obtained for the Hausdorff dimension for quasiconformal maps in $\mathbb{R}^n$ and quasisymmetric maps in metric spaces; see, e.g., [2], [3], [5], [4], [23]. These results show that nontrivial dilatation-independent lower bounds can be given for some sets of dimension greater than or equal to one, while nontrivial dilatation-independent upper bounds do not exist in general. However, none of these papers address the question of lowering dimension for sets with dimension strictly less than one. The author conjectured in [23] that nontrivial dilatation-independent lower bounds do not exist in this case, more precisely, that every set with Hausdorff dimension strictly less than one can always be mapped quasisymmetrically onto sets of arbitrarily small dimension. This conjecture can be formulated in two ways, first, for subsets of the Euclidean space $\mathbb{R}^n$ (with global quasiconformal self-maps of $\mathbb{R}^n$) and second, for general metric spaces (with quasisymmetric maps to other metric spaces).

Received March 1, 2000; received in final form April 10, 2000.
2000 *Mathematics Subject Classification.* Primary 30C65. Secondary 54E40, 54F45, 28A78.
The author is supported by a National Science Foundation Postdoctoral Research Fellowship.

This paper presents some results in the spirit of these conjectures. We establish the Euclidean conjecture for bounded sets in all dimensions $n \geq 1$ and the metric space conjecture in general, but for a different notion of dimension (not the Hausdorff dimension). This notion was first used substantively by Assouad [1], for which reason we use the term Assouad dimension. In Section 3 we prove that subsets of Euclidean spaces with Assouad dimension strictly less than one can be mapped by global quasisymmetric maps onto sets of arbitrarily small dimension and in Section 4 we discuss the case of abstract metric spaces and arbitrary quasisymmetric maps. The case $n = 1$ in Section 3 can be treated by a slightly simpler construction, a variation on a well-known construction used by Kahane [12] which I learned about in a paper of Sjödin [19, pp. 182-183]. While we do not address the case of the Hausdorff dimension in full generality, our results do provide as corollaries some partial answers for the Hausdorff dimension; see Corollary 3.8 and Remark 3.10.

We collect in an appendix a few other elementary observations on the relationship between Assouad dimension and quasisymmetric maps.

For other interesting results regarding the distortion of sets in the real line by quasisymmetric mappings see the papers of Staples and Ward [20] and Buckley, Hanson and MacManus [6]. In contrast with the results of the current paper, these references consider the question of determining when a given set on the real line of positive length (and hence of Hausdorff dimension one) has the property that all of its quasisymmetric images also have positive length. Wu [27] studies the question of which subsets of the real line have the property that every quasisymmetric image has zero length.

I wish to acknowledge Bruce Hanson for some helpful comments regarding the construction in Section 3.

Notation 1.1. We denote the distance function in any metric space by $|x - y|$. We write $B(x, r)$ for the closed ball in the metric space X with center x and radius r. For $Y, Z \subset X$, we write $\operatorname{diam} Y$ for the diameter of Y and $\operatorname{dist}(Y, Z)$ for the distance between Y and Z (with the convention that $\operatorname{dist}(Y, \emptyset) = \infty$). For $0 < \epsilon < 1$, we denote by d^ϵ the *snowflaked metric* $d^\epsilon(x, y) = d(x, y)^\epsilon$. We denote by $\dim_H X$ the Hausdorff dimension of X (see, e.g., [16, Chapter 4]).

An embedding $f : X \to Y$ of metric spaces is called *(L-)bi-Lipschitz* if there exists $L \geq 1$ so that

$$|x - y|/L \leq |fx - fy| \leq L|x - y|$$

for all $x, y \in X$ and f is called *(η-)quasisymmetric* if there exists an increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$\frac{|fx - fy|}{|fx - fz|} \leq \eta\left(\frac{|x - y|}{|x - z|}\right)$$

for all $x, y, z \in X$, $x \neq z$. Every L-bi-Lipschitz map is η-quasisymmetric with $\eta(t) = L^2 t$. The identity map id : $X \to (X, d^\epsilon)$, $0 < \epsilon < 1$, is η-quasisymmetric with $\eta(t) = t^\epsilon$. A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is quasisymmetric if and only if it is quasiconformal.

Following Pansu [17], we define the *conformal dimension* $\mathcal{C}\dim X$ of a metric space X to be the infimum of the values $\dim f(X)$ as f ranges over all quasisymmetric maps of X into another metric space. Here dim can be any notion of dimension; in this paper, we are most interested in the Hausdorff and Assouad dimensions. Similarly, we define the *global conformal dimension* $\mathcal{GC}\dim E$ of a subset $E \subset \mathbb{R}^n$ to be the infimum of the values $\dim f(E)$ as f ranges over all quasiconformal self-maps of $\mathbb{R}^n$ (quasisymmetric if $n = 1$).

Fix integers $n \geq 1$ and $b \geq 2$. For each $m \in \mathbb{Z}$, the *b-adic cubes at level m in $\mathbb{R}^n$* are the closed cubes in $\mathbb{R}^n$ of the form

$$Q(j_1, \dots, j_n; m) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \frac{j_k}{b^m} \leq x_k \leq \frac{j_k + 1}{b^m}, k = 1, \dots, n\}$$

for $j_1, \dots, j_n \in \mathbb{Z}$. We write $\mathcal{C}^n_m$ for the collection of cubes $Q(j_1, \dots, j_n; m)$, $j_1, \dots, j_n \in \mathbb{Z}$, and we write $\mathcal{C}^n = \bigcup_{m \in \mathbb{Z}} \mathcal{C}^n_m$. For each $Q \in \mathcal{C}^n_m$, there exist exactly b^n cubes $Q' \in \mathcal{C}^n_{m+1}$ with $Q' \subset Q$ which we call the *children* of Q. In the case $n = 1$, we have b-adic intervals $I = [c, d]$, each of which has b children which we label $I_1, \dots, I_b$, where $I_k = [c + j(d - c)/b, c + (j + 1)(d - c)/b]$. We call I_1 the *first* and I_b the *last child*.

2. Assouad dimension: definition and basic properties

In this section, we define and review basic properties of the Assouad dimension of a metric space. Our basic reference is Luukkainen's paper [14], which contains a comprehensive discussion.

Definition 2.1. Let X be a metric space and let $s \geq 0$. We call X *s-homogeneous* if there exists a finite constant C so that whenever $0 < \alpha \leq \beta < \infty$, the cardinality of any α-discrete set F with diameter at most β is no more than $C(\beta/\alpha)^s$. The constant C is called a *constant of s-homogeneity* for X. Here a subset $Y \subset X$ is said to be *ϵ-discrete*, $\epsilon > 0$, if $|x - y| \geq \epsilon$ whenever $x, y \in Y$, $x \neq y$.

The *Assouad dimension* of X is defined to be the infimum of the values $s \geq 0$ for which X is s-homogeneous (with the convention that $\dim_A X = \infty$ if no such values exist).

Remarks 2.2. (1) A metric space has finite Assouad dimension if and only if it is a doubling space, that is, there exists a finite constant C so that every ball of radius r can be covered by no more than C balls of radius $r/2$.

(2) David and Semmes [7, Definition 5.17] call a metric space *semi-regular of dimension s* if it is s-homogeneous. The concept of homogeneity of a metric space was previously studied by Vol´berg and Konyagin [26] under the name

uniform metric dimension; they showed that a metric space X has finite uniform metric dimension if and only if there exists a doubling measure on X. A simpler proof of this fact was later given by Wu [28].

We review some basic properties of this notion of dimension. For proofs, see Theorem A.5, Theorem 5.2 and Theorem A.13 of [14]. Proposition 2.3(viii) is Assouad's original result [1] and provides one of the most important reasons for the study of this notion of dimension; it characterizes the metric spaces of finite Assouad dimension as precisely the metric spaces all of whose snowflakes embed bi-Lipschitzly into a Euclidean space.[1]

PROPOSITION 2.3. *Let $X = (X, d)$ be a metric space.*

(i) *s-homogeneity and the Assouad dimension of X are bi-Lipschitz invariants;*
(ii) $\dim_A Y \le \dim_A X$ *whenever $Y \subset X$; equality holds if Y is dense in X;*
(iii) *for $0 < \epsilon \le 1$ the Assouad dimension of (X, d^ϵ) is equal to* $\dim_A X/\epsilon$*;*
(iv) $\dim_H X \le \dim_A X$*;*
(v) $\dim_A \cup_{i=1}^n E_i = \max_{i=1}^n \dim_A E_i$ *for any finite collection of sets $E_1, \ldots, E_n \subset X$;*
(vi) *every subset of $\mathbb{R}^n$, $n \ge 1$, with nonempty interior has Assouad dimension n;*
(vii) *for $n \ge 1$, a set $E \subset \mathbb{R}^n$ has* $\dim_A E < n$ *if and only if E is porous in $\mathbb{R}^n$ (see below);*
(viii) *X has finite Assouad dimension if and only if for each $0 < \epsilon < 1$, there exists a bi-Lipschitz embedding of the snowflaked space (X, d^ϵ) into a Euclidean space $\mathbb{R}^N$, where $N = N(\epsilon) < \infty$.*

The inequality in 2.3(iv) can be strict. For example, the space $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ has $0 = \dim_H X < \dim_A X = 1$. Also, 2.3(v) can fail for countably infinite collections; e.g., the Assouad dimension of the set $\mathbb{Q}$ of rational numbers is equal to one by 2.3(ii) and (vi).

In this paper, we will primarily make use of Proposition 2.3(vii) (and related results which we will prove later). Recall that a subset $E \subset \mathbb{R}^n$, $n \ge 1$, is said to be *c-porous*, $c > 0$, if for every ball $B(x, r)$ in $\mathbb{R}^n$ there exists $z \in B(x, r)$ with $B(z, cr) \cap E = \emptyset$. Porous sets are quasisymmetrically invariant: if $E \subset \mathbb{R}^n$, $n \ge 1$, is c-porous and $f : \mathbb{R}^n \to \mathbb{R}^n$ is η-quasisymmetric, then $f(E)$ is c'-porous with $c' = c'(c, \eta)$ (cf. [25]).

Luukkainen's result in Proposition 2.3(vii) sharpens an earlier result of Sarvas [18], who showed that porous sets in $\mathbb{R}^n$ have Hausdorff dimension

[1]The question of characterizing the metric spaces which themselves embed bi-Lipschitzly in a Euclidean space is extremely difficult and still unsolved. Finiteness of the Assouad dimension is insufficient; the first Heisenberg group provides an example of a space of finite Assouad dimension which fails to admit such an embedding. See Remark A.15 of [14].

strictly less than n. Martio and Vuorinen [15] had already shown that porous sets have Minkowski dimension strictly less than n and their proof applies also to the Assouad dimension. (The Minkowski dimension is a notion of dimension intermediate between Hausdorff and Assouad dimension.) Koskela and Rohde [13] prove that the Minkowski dimension is strictly less than n for a more general class of *weakly porous* sets. Some other results relating dimension and porosity for sets and measures in Euclidean spaces can be found in the work of Eckmann and E. and M. Järvenpää [8], [11].

When $n = 1$, 2.3(vii) characterizes the subsets of the line with Assouad dimension strictly less than one as precisely the porous sets. This makes the proof of our main result somewhat simpler and is the reason we begin with that case.

An elementary consequence of 2.3(vii) is a dilatation-dependent upper bound for Assouad dimension which implies that sets of Assouad dimension n in $\mathbb{R}^n$ are preserved under quasiconformal maps. We discuss this in greater detail in the appendix. For now, we merely note that the corresponding result for the Hausdorff dimension is a well-known theorem of Gehring and Väisälä [9], [10], which relies on the higher integrability of the Jacobian of a quasiconformal map. It does not appear that the Assouad dimension can be treated by this method. On the other hand, in contrast with the situation for the Hausdorff dimension, the preservation of sets of dimension n for Assouad dimension is valid even when $n = 1$.

3. Lowering the Assouad dimension of sets in Euclidean space

In this section, we study the question of lowering the Assouad dimension of a fixed set E in $\mathbb{R}^n$ by quasisymmetric homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$.

Our first remark is that in the case $\dim_A E \geq 1$, results of this type for the Assouad dimension are weaker than the corresponding results for the Hausdorff dimension which appear in [17] and [23]: for each $\alpha \in [1, n]$, there exist compact sets E in $\mathbb{R}^n$ with $\mathcal{GC} \dim_H E = \dim_H E = \alpha$. Since the example given in [23] has $\dim_H E = \dim_A E$, it follows that $\mathcal{GC} \dim_A E = \dim_A E = \alpha$ as well. We thus restrict ourselves to the case $0 \leq \dim_A E < 1$.

The results of this section imply that no similar example exists in the case $0 < \dim_A E < 1$ for bounded sets.[2] As mentioned in the introduction, it is not known whether the corresponding results hold for the Hausdorff dimension in any dimension.

For the question of raising the dimension, Bishop [3] has shown that if $E \subset \mathbb{R}^n$, $n \geq 1$, is a compact set with $\dim_H E > 0$, then the supremum of the

[2]The restriction to bounded sets is necessary because the Assouad dimension is only finitely stable and not countably stable; see 2.3(v) and the remarks at the end of this section. I do not know how to treat the unbounded case. However, Corollary 3.8 holds for all porous sets, bounded or unbounded.

values $\dim_H f(E)$ over all quasisymmetric self-maps f of $\mathbb{R}^n$ is equal to n. I do not know if this result also holds for the Assouad dimension.

We now state the main theorem of this section.

THEOREM 3.1. *Let $n \geq 1$ and let $E \subset \mathbb{R}^n$ be a bounded set. Then $\mathcal{GC} \dim_A E$ is either equal to zero or greater than or equal to one.*

The proof of Theorem 3.1 is slightly different in the case $n = 1$ than in the case $n \geq 2$ and so will be given in two parts. In the case $n = 1$, Theorem 3.1 reads as follows: if $\dim_A E = 1$ then $\dim_A f(E) = 1$ for all quasisymmetric maps $f : \mathbb{R} \to \mathbb{R}$, while if $\dim_A E < 1$ then there exist quasisymmetric maps $f : \mathbb{R} \to \mathbb{R}$ for which $\dim_A f(E)$ is arbitrarily close to zero. The first of these facts was mentioned in the previous section and its proof will be deferred to the appendix. Thus Theorem 3.1 will follow once we have proved the following: if $E \subset \mathbb{R}^n$, $n \geq 1$, is a bounded set with $\dim_A E < 1$ and $\epsilon > 0$, then there exists a quasisymmetric map $f : \mathbb{R}^n \to \mathbb{R}^n$ for which $\dim_A f(E) < \epsilon$.

Our proof of the $n = 1$ part of Theorem 3.1 will actually show something extra, namely, that the quasisymmetric maps which reduce the dimension of E can be taken to be the identity map on the complement of any interval which contains E. This feature will be important in our proof of Corollary 3.8.

Suppose then that $n \geq 1$ and that $E \subset \mathbb{R}^n$ has Assouad dimension strictly less than one. Then by Proposition 2.3(vii), the set E is porous in $\mathbb{R}^n$, i.e., every cube contains a subcube of comparable size which avoids E. To prove Theorem 3.1, it will be helpful to reformulate the porosity condition in a manner that gives greater control over the location of the omitted subcube.

DEFINITION 3.2. Fix integers $n \geq 1$, $b \geq 2$ and $k \in \{0, 1, \ldots, b^n - 1\}$. Let $E \subset \mathbb{R}^n$. We call E *(b,k)-sparse* if no more than k of the children of any b-adic cube Q meet E. We call the remaining $b^n - k$ cubes (which are disjoint from E) the *omitted cubes* of Q. We call E *sparse* if it is (b,k)-sparse for some b and k.

Note that if E is (b,k)-sparse for some b and k, then E is also (b^j, k^j)-sparse for each $j \in \mathbb{N}$.

PROPOSITION 3.3. *Let $E \subset \mathbb{R}^n$. Then there exists an integer $b \geq 2$ so that E is (b,k)-sparse for some $k < b$ if and only if E is s-homogeneous for some $s < 1$. In the case $n = 1$ a third equivalent condition is that E is c-porous for some $c > 0$.*

Note that the second sentence of Proposition 3.3 is a consequence of Proposition 2.3(vii).

Proof. First, suppose that E is (b,k)-sparse with $k < b$; we show that E is s-homogeneous with $s = \log k / \log b$. Choose $0 < \alpha \leq \beta < \infty$ and let F

be a finite subset of E which is α-discrete and has diameter at most β. We estimate the cardinality of F. Choose integers l and m so that $b^{m-1}\sqrt{n} \le \beta < b^m\sqrt{n}$ and $b^l < \alpha \le b^{l+1}$. Then F is contained in at most 2^n b-adic cubes $Q_0, Q_1, \dots, Q_{2^n}$ of size b^m. Furthermore, any two distinct points of F lie in different b-adic cubes of size b^{l-1}. Thus the cardinality of F is no more than $2^n k^{m-l+1}$. By our choice of l and m,

$$m - l \le 2 + \frac{\log(\beta/\alpha) - \log\sqrt{n}}{\log b} \le 2 + \frac{\log(\beta/\alpha)}{\log b}.$$

Hence

$$2^n k^{m-l+1} \le 2^n k^{3+\log(\beta/\alpha)/\log b} = 2^n k^3 (\beta/\alpha)^s$$

which shows that E is s-homogeneous with homogeneity constant $C = 2^n k^3$. Thus

$$\dim_A E \le \frac{\log k}{\log b}. \tag{3.4}$$

Now, assume that $E \subset \mathbb{R}^n$ is s-homogeneous for some $s < 1$ with homogeneity constant $C(s)$. To make things easier in our proof of Theorem 3.1, we will now break the proof into two different tracks according whether $n = 1$ or $n \ge 2$. (It would be possible to give a single proof which covers both of these cases, but our subsequent proof of Theorem 3.1 is more easily explained if we use slightly different arguments in each of these two cases.)

$n = 1$: In this case, we let $b \ge 3$ be a large odd integer, whose exact value will be determined later in the proof. Let I be a b-adic interval with children $I_1, \dots, I_b$ and consider the children $I_2, I_4, \dots, I_{b-3}, I_{b-1}$. Suppose that E meets all of these sets. Then E contains a finite set F of cardinality $(b-1)/2$ which is α-discrete, $\alpha = |I|/b$, and has diameter at most $\beta = |I|$. By the s-homogeneity,

$$(b-1)/2 \le C(\beta/\alpha)^s = Cb^s$$

which leads to a contradiction if b is chosen so large that $Cb^s < (b-1)/2$. If this is the case, then one of the children $I_2, I_4, \dots, I_{b-3}, I_{b-1}$ of I is disjoint from E, i.e., E is $(b, b-1)$-sparse.

$n \ge 2$: In this case, we let $b \ge 2$ be a large even integer whose exact value will again be determined momentarily. Let Q be a b-adic cube in $\mathbb{R}^n$ with children $Q_1, \dots, Q_{b^n}$. These children can be divided into 2^n subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{2^n}$, each of which contains $(b/2)^n$ cubes, in such a way that the distance between any two of the cubes in a family $\mathcal{F}_i$ is at least $s(Q)/b$, where $s(Q)$ denotes the side length of Q. Then arguing as above we see that the number of cubes in $\mathcal{F}_i$, $i = 1, 2, \dots, 2^n$, which can meet the set E is at most

$$C(s)\left(\frac{\operatorname{diam}(Q)}{s(Q)/b}\right)^s = C(s)(b\sqrt{n})^s$$

and so the number of children of Q which can meet E is at most $C(s)2^n(b\sqrt{n})^s$. This shows that E is $(b, C'b^s)$-sparse, where $C' = C(s)2^n n^{s/2}$, and establishes the result if b is chosen so large that $C'b^s < b$. $\square$

Remark 3.5. The same argument can be used to relate sparseness to any upper bound for the Assouad dimension. More precisely, one can show that if $E \subset \mathbb{R}^n$, $n \geq 1$, and $s_0 \in (0, n]$, then there exists an integer $b \geq 2$ so that E is (b, k)-sparse for some $k < b^{s_0}$ if and only if E is s-homogeneous for some $s < s_0$. However, in what follows we only use this with $s_0 = 1$.

Proof of Theorem 3.1 when $n = 1$. Let $E \subset \mathbb{R}$ be a bounded set satisfying $\dim_A E < 1$. By a preliminary bi-Lipschitz map, we may assume that E lies in the unit interval $[0, 1]$ and has diameter equal to one. By Proposition 3.3, E is $(b, b-1)$-sparse for some sufficiently large odd integer b. Moreover, the proof of Proposition 3.3 shows something extra, namely, that the omitted subinterval of each b-adic interval I can always be chosen not to be either the first or last child of I. For each b-adic interval I in $[0, 1]$, denote this omitted subinterval by I'.

For each $N = 1, 2, \ldots$, we will construct a quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$ which is equal to the identity map on $\mathbb{R} \setminus [0, 1]$ and takes E onto a weakly $(b^N(b-1), b-1)$-sparse set $f(E)$. By Proposition 3.3, specifically (3.4), it follows that

$$\dim_A f(E) \leq \frac{\log(b-1)}{\log(b-1) + N \log b} \tag{3.6}$$

which can be made less than any prescribed $\epsilon > 0$ by choosing N sufficiently large.

It remains to construct the map f. We modify an example of Sjödin [19, Theorem 6].

It is well-known that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is quasisymmetric if and only if the following condition holds for some (equivalently, all) integers $b \geq 2$: there exists a constant $\lambda < \infty$ so that

$$|f(I)| \leq \lambda |f(J)| \tag{3.7}$$

whenever I and J are adjacent b-adic intervals of equal length. Furthermore, the statement is quantitative: if f is η-quasisymmetric then it satisfies (3.7) for all b with $\lambda = \eta(1)$, while if f satisfies (3.7) for some λ and b, then it is η-quasisymmetric with $\eta(t)$ depending only on λ and b.

We construct piecewise linear, increasing functions f_i, $i = 0, 1, \ldots$, on $\mathbb{R}$ which will converge uniformly to the desired function f. Define values $m < 1 < M$ as follows:

$$m = \frac{1}{b^{N-1}(b-1)}, \qquad M = b - \frac{1}{b^{N-1}}.$$

Note that $\frac{1}{b}M + (1 - \frac{1}{b})m = 1$. We proceed inductively to define a sequence of piecewise constant maps ϕ_i. The map ϕ_i will be constant on each b-adic interval of length b^{-i}. (Note that there is some ambiguity here since two adjacent b-adic intervals overlap at a single point on their common boundary. However, the total number of such points is countable and we will eventually be integrating the functions ϕ_i so the choice of what value to assign to these boundary points is irrelevant.)

Put $\phi_0(t) \equiv 1$. Assume that $\phi_0, \phi_1, \ldots, \phi_{i-1}$ have been defined and consider all of the b-adic intervals of length b^{-i+1} which lie in the unit interval $[0, 1]$. For $t \in [0, 1]$, let $\phi_i(t) = M \cdot \phi_{i-1}(t)$ if t lies in the omitted subinterval I' of one of these b-adic intervals I and let $\phi_i(t) = m \cdot \phi_{i-1}(t)$ otherwise.[3] We next define ϕ_i on the $2i-2$ b-adic intervals of lenth b^{-i} which are immediately adjacent to $[0, 1]$. For $j = 1, 2, \ldots, i-1$, let $\phi_i(t) = m^{-i+j}$ if $t \in (1 + (j-1)b^{-i}, 1 + jb^{-i})$ or $t \in (-jb^{-i}, -(j-1)b^{-i})$. Finally, let $\phi_i(t) = 1$ for all other $t \in \mathbb{R}$, i.e., for $t \in [1 + (i-1)b^{-i}, \infty)$ or $t \in (-\infty, -(i-1)b^{-i}]$.

Next, for $t \in \mathbb{R}$, define

$$f_i(t) = \int_0^t \phi_i(s)\, ds.$$

Then f_i is a continuous and increasing function on $\mathbb{R}$ which is equal to the identity map on $(-\infty, -(i-1)b^{-i}) \cup (1 + (i-1)b^{-i}, \infty)$. One easily checks that $|f_i(I)| \le \lambda |f_i(J)|$ for each i and adjacent b-adic intervals I and J of equal length, where

$$\lambda = \max\{1/m, M/m\} = M/m = (b-1)(b^N - 1).$$

(The fact that the omitted subintervals are never first or last is crucial here.) Now $f_0, f_1, \ldots$ is a sequence of self-maps of $\mathbb{R}$ which are all uniformly quasisymmetric. For fixed choices of η and a compact set $K \subset \mathbb{R}$, the collection $\mathcal{F}$, consisting of all η-quasisymmetric homeomorphisms f from $\mathbb{R}$ to $\mathbb{R}$ which are equal to the identity off of K, is compact in the compact-open topology. Hence there exists a subsequence of the given sequence $f_0, f_1, \ldots$ which converges uniformly to an increasing η-quasisymmetric function f on $\mathbb{R}$. It is straightforward to verify that the image set $f(E)$ is $(b^N(b-1), (b-1))$-sparse which, as we have already noted, implies (3.6). Furthermore, f is equal to the identity on $(-\infty, -1) \cup (1, \infty)$. □

As a corollary, we deduce that porous sets have global conformal Hausdorff dimension zero.

Corollary 3.8. *Let $E \subset \mathbb{R}$ be a porous set. Then $\mathcal{GC}\dim_H E = 0$.*

---

[3] Note that we must modify the map on *every* b-adic interval, even those which have been identified as omitted subintervals at an earlier stage. Of course, if I is such an interval, then all of its children are also omitted and so we may choose I' to be any child which is not the first or last.

To prove this corollary, note that E can be written as the disjoint union of an ordered collection of bounded porous sets E_ν, $\nu \in \mathbb{Z}$, so that $L \le \operatorname{diam} E_\nu \le 3L$ and $L/C \le \operatorname{dist}(E_\nu, E_{\nu+1}) \le L$ for some fixed $C < \infty$ and $L > 0$. To see this, fix any value $L > 0$ and apply the porosity condition to the intervals $[2\nu L, (2\nu+1)L]$, $\nu \in \mathbb{Z}$. Now by Theorem 3.1 quasisymmetric maps f_ν can be found which make the dimensions of the sets E_ν arbitrarily small, and by Lemma 3.9 these maps can be "glued together" to make a map $f : \mathbb{R} \to \mathbb{R}$ which makes the dimension of the entire set E arbitrarily small. (This is the point at which the argument breaks down for the Assouad dimension, since it makes use of the countable stability of the Hausdorff dimension.) Note that this proof shows that Corollary 3.8 holds more generally for sets which satisfy the porosity condition uniformly locally, that is, for all intervals I with length at most δ for some fixed $\delta > 0$.

Lemma 3.9. *Let E_ν, $\nu \in \mathbb{Z}$, be an ordered collection of compact sets in $\mathbb{R}$ satisfying the conditions*

$$L/C \le \operatorname{dist}(E_\nu, E_{\nu+1}) \le L \le \operatorname{diam} E_\nu \le CL$$

for all ν, where $L > 0$ and $1 \le C < \infty$ are fixed. Let $\{f_\nu\}$ be a collection of η-quasisymmetric self-maps of $\mathbb{R}$ for which $f_\nu(x) = x$ whenever $x \notin [a_\nu, b_\nu]$, where $[a_\nu, b_\nu]$ is the smallest closed interval containing E_ν. Then the map $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} f_\nu(x), & x \in [a_\nu, b_\nu], \\ x, & x \in \mathbb{R} \setminus \cup_{\nu=-\infty}^{\infty} [a_\nu, b_\nu], \end{cases}$$

is η'-quasisymmetric for some η' depending only on η and C.

For ease of exposition, we postpone the proof of this lemma and continue with the proof of Theorem 3.1 in the case $n \ge 2$. For $x \in \mathbb{R}^n$ and $s > 0$, we denote by $Q(x, s)$ the cube centered at x with side length s.

Proof of Theorem 3.1 when $n \ge 2$. Let E be a bounded subset of $\mathbb{R}^n$, $n \ge 2$, with $\dim_A E < 1$. As before, we may reduce by a preliminary bi-Lipschitz mapping to the case $E \subset [0,1]^n$; furthermore, we may assume that E is closed by 2.3(ii). We choose $s < 1$ so that E is s-homogeneous; by Proposition 3.3, E is $(b, C'b^s)$-sparse, where $C' = C(s)2^n n^{s/2}$ and b is an integer which is chosen so large that $C'b^s \le b/3$ and $b^s \ge (C')^2$.

We construct a quasiconformal self-map f of $\mathbb{R}^n$ for which $\dim_A f(E) \le \frac{3}{4}s$; this clearly suffices to establish the result. The basic construction will again involve an induction on the level of b-adic subdivision of the unit cube $[0,1]^n$. Let $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map.

Consider first the collection of level one b-adic subcubes of $[0,1]^n$. Suppose that E meets some of these cubes, say $Q_1, \dots, Q_{\nu_1}$, where $\nu_1 \le C'b^s$. Fix one of these cubes Q_i and arrange the remaining cubes in $\mathcal{C}_1^n$ in "shells" $\mathcal{S}_j$,

$j = 1, 2, \dots$ (e.g., for each j, $\mathcal{S}_j$ consists of all cubes in $\mathcal{C}_1^n \setminus \{Q_i\}$ which are adjacent to cubes in $\mathcal{S}_{j-1}$ but not contained in any of the collections $\mathcal{S}_k$, $k \le j-1$). Since $C'b^s$ is negligible in comparison with b, there must exist a shell $\mathcal{S}_{j(i)}$, $1 \le j(i) \le b/2$, consisting entirely of cubes which are disjoint from E.

Consider the set $A(1) = [0,1]^n \setminus \cup_{i=1}^{\nu_1} \mathcal{S}_{j(i)}$. By construction, $A(1)$ can be decomposed into subsets $B(1,1), \dots, B(1,p_1)$, $p_1 \le \nu_1$, where each set $B(1,k)$ consists of a union of level one b-adic cubes and any two subsets $B(1,k)$ and $B(1,k')$ are separated by a layer (of thickness at least one) of level one cubes. The various sets $B(1,k)$, $k = 1, \dots, p_1$ have disjoint $b/3$-neighborhoods $U(1,k)$ and so there exists a quasiconformal mapping $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

(i) f_1 is equal to f_0 on $\mathbb{R}^n \setminus (U(1,1) \cup \cdots \cup U(1,p_1))$;
(ii) on each $B(1,k)$, $k = 1, \dots, p_1$, f_1 is a conformal scaling which shrinks the set $B(1,k)$ into one of its constituent subcubes (the exact choice is irrelevant);
(iii) f_1 is quasiconformal in $\cup_{k=1}^{p_1} U(1,k) \setminus B(1,k)$.

We repeat this construction inside all of the level one subcubes contained in $\cup_{k=1}^{p_1} B(1,k)$, generating a set $A(2)$ decomposed into sets $B(2,1), \dots, B(2,p_2)$, $p_2 \le \nu_2 \le (C'b^s)^2$, disjoint $b^2/3$-neighborhoods $U(2,1) \dots, U(2,p_2)$ of the sets $B(2,1), \dots, B(2,p_2)$, and a quasiconformal map f_2. In a similar fashion, we construct for each $m \in \mathbb{N}$ the data $A(m)$, $B(m,k)$, $U(m,k)$, $p_m \le \nu_m \le (C'b^s)^m$ and f_m. The maps f_m are coherent in the sense that $f_m = f_{m-1}$ on the set $\mathbb{R}^n \setminus (U(m,1) \cup \cdots \cup U(m,p_m))$. Furthermore, the dilatations of these maps are uniformly bounded.

Since $E = \cap_{m=1}^\infty A(m)$, we may pass to a limit map $f : \mathbb{R}^n \to \mathbb{R}^n$ which is a homeomorphism and is K-quasiconformal (for some fixed K) on the set $\mathbb{R}^n \setminus E$. Finally, E is removable for quasiconformal maps since the (Hausdorff) dimension of E is strictly less than $n-1$ [24, Theorem 35.1].

The only remaining thing to verify is that the Assouad dimension of E is in fact decreased by the map f, but this is straightforward. Indeed, in each b^2-adic cube, $f(E)$ is contained in at most $C'b^s$ children, that is, $f(E)$ is $(b^2, C'b^s)$-sparse. By (3.4),

$$\dim_A f(E) \le \frac{s \log b + \log C'}{2 \log b} \le \frac{3}{4} s$$

since $b^s \ge (C')^2$. The proof is complete. □

Remark 3.10. The preceding argument can clearly be applied to a wider class of subsets of $\mathbb{R}^n$ than just the sets with Assouad dimension strictly less than one, but it is difficult to formulate a precise statement. For example, recall that for each $\alpha \ge 1$ and $n \ge 2$ there are sets $E \subset \mathbb{R}^n$ with

$\dim_A E = \mathcal{GC}\dim_A E = \alpha$. Thus we cannot give a more general statement whose hypotheses involve only the value of the Assouad dimension.

As with the $n = 1$ case, the preceding proof also shows that certain sets $E \subset \mathbb{R}^n$ have global conformal Hausdorff dimension zero. For example, any set E which is (b, k)-sparse with $k \le b/3$ has this property.

Proof of Lemma 3.9. All of the maps f_ν satisfy condition (3.7) with $\lambda = \eta(1)$ for all adjacent intervals $I = [x - t, x]$ and $J = [x, x + t]$ of equal length, that is,

$$\frac{1}{\lambda} \le \frac{f_\nu(x+t) - f_\nu(x)}{f_\nu(x) - f_\nu(x-t)} \le \lambda \tag{3.11}$$

for all $x \in \mathbb{R}$ and $t > 0$. It suffices to verify that (3.11) holds for all x and $t > 0$ when f_ν is replaced by f and λ is replaced by a constant $\lambda' = \lambda'(\lambda, C)$. This is immediate if $\{x - t, x, x + t\}$ intersects at most one of the intervals $[a_\nu, b_\nu]$, since in that case $f(y) = f_\nu(y)$ for $y = x - t, x, x + t$. Suppose that $x - t \in [a_\mu, b_\mu]$, $x \in [a_\nu, b_\nu]$, and $x + t \in [a_\xi, b_\xi]$ with $\mu < \nu < \xi$. Then

$$\begin{aligned} \frac{L}{C}(\nu-\mu) &\le \frac{L}{C}(\nu-\mu) + L(\nu-\mu-1) \le t \\ &\le L(\nu-\mu) + CL(\nu-\mu+1) \le 3CL(\nu-\mu) \end{aligned} \tag{3.12}$$

and a similar inequality holds with $\nu - \mu$ replaced by $\xi - \nu$. Then

$$\begin{aligned} \frac{f(x+t) - f(x)}{f(x) - f(x-t)} &\le \frac{b_\xi - a_\nu}{a_\nu - b_\mu} \le \frac{CL(\xi-\nu+1) + L(\xi-\nu)}{L(\nu-\mu-1) + (L/C)(\nu-\mu)} \\ &\le 3C^2\frac{\xi-\nu}{\nu-\mu} \le 9C^4 \end{aligned}$$

by (3.12). A similar argument shows that

$$\frac{f(x+t) - f(x)}{f(x) - f(x-t)} \ge \frac{1}{9C^4}.$$

The case when $\{x - t, x, x + t\}$ intersects exactly two of the intervals $[a_\nu, b_\nu]$ is similar and will be left to the reader. $\square$

4. Lowering the Assouad dimension of abstract metric spaces

When can the Assouad dimension of a given metric space be lowered by a quasisymmetric map?

As in Section 3, the theory in the case $\dim_A X \ge 1$ follows from the theory for the Hausdorff dimension. In fact, by [17] and [23] exactly the same result holds: for each $\alpha \ge 1$, there exist compact metric spaces X with $\mathcal{C}\dim_A X = \mathcal{C}\dim_H X = \dim_A X = \dim_H X = \alpha$.

Note also that the analog to Bishop's result on raising dimension is trivial in the metric space setting in light of the example of the snowflaked spaces (X, d^ϵ), $0 < \epsilon < 1$.

We are thus left to consider the case of lowering the dimension for spaces X with $\dim_A X < 1$, and this case is covered by the following result:

THEOREM 4.1. *Let X be a metric space with* $\dim_A X < 1$. *Then* $\mathcal{C}\dim_A X = 0$.

Recall that every space with Hausdorff dimension strictly less than one is totally disconnected. Having Assouad dimension strictly less than one implies the following quantitative version of total disconnectivity.

DEFINITION 4.2 ([7], Section 15). Let X be a metric space. We call X *uniformly disconnected* if there exists a constant $C < \infty$ so that every ball $B(x,r)$ in X contains a closed set A satisfying $B(x,r/C) \subset A \subset B(x,r)$ and $\operatorname{dist}(A, X \setminus A) \geq r/C$.

In Lemma 15.2 of [7], David and Semmes prove that any metric space with Assouad dimension strictly less than one is uniformly disconnected.[4] This result can be used to give an elementary proof of Theorem 4.1:

Proof of Theorem 4.1. Let X be a uniformly disconnected metric space. It is well-known (see, for example, Proposition 15.13 of [7]) that the metric on X is bi-Lipschitz equivalent to an ultrametric, e.g., a distance function d_1 which satisfies the stronger inequality $d_1(x,y) \leq \max\{d_1(x,z), d_1(z,y)\}$ for all $x,y,z \in X$. Indeed, let $d_1(x,y)$ be the infimum of the values $\gamma > 0$ for which there exists a finite sequence of points $x = x_0, x_1, \ldots, x_m = y$ with $d(x_{i-1}, x_i) < \gamma$ for each i. Then d_1 is an ultrametric on X satisfying

$$d_1(x,y) \leq |x-y|$$

and the uniform disconnectivity of X implies that

$$d_1(x,y) \geq d(x,y)/L$$

for some $L < \infty$.

By the bi-Lipschitz equivalence of Assouad dimension, $\dim_A(X, d_1) = \dim_A X$. Moreover, since d_1 is an ultrametric, (X, d_1^ϵ) is a metric space for *all* $0 < \epsilon < \infty$. Since $\dim_A(X, d_1^\epsilon) = \dim_A X/\epsilon$, we see (by taking ϵ very large) that $\mathcal{C}\dim_A X = 0$. The proof is complete. $\square$

5. Appendix

This appendix contains several miscellaneous facts relating Assouad dimension and quasisymmetric maps.

Our first result gives dilatation-dependent bounds on the change in Assouad dimension under a power quasisymmetric map. Recall that a quasisymmetric

[4] In fact, [7, Lemma 15.2] is stated with slightly stronger hypotheses, but an analysis of the proof reveals that the only condition used is s-homogeneity for some $s < 1$.

map f between metric spaces is called *p-power quasisymmetric* if the distortion function η can be chosen to be of the form

$$\eta(t) = C_0 \max(t^p, t^{1/p})$$

for some constant $C_0 < \infty$.

Proposition 5.1. *Let $f : X \to Y$ be a p-power quasisymmetric homeomorphism between metric spaces, $p \geq 1$. Then*

$$\dim_A X/p \leq \dim_A Y \leq p \dim_A X. \tag{5.2}$$

The corresponding result for the Hausdorff dimension is a well-known consequence of the Hölder continuity of quasisymmetric maps on (uniformly perfect) metric spaces [22, Theorem 3.18]. Note that this approach cannot be used to prove this result for the Assouad dimension which (in general) need not be well-behaved under Hölder or even Lipschitz maps. (See Section A.6.2 in [14] for an example of a Lipschitz homeomorphism of a space of Assouad dimension one onto a space of infinite Assouad dimension.)

Corollary 5.3. *Let X be a metric space with $\dim_A X = 0$ and let $f : X \to Y$ be a power quasisymmetric map onto a metric space Y. Then $\dim_A Y = 0$.*

Quasisymmetric homeomorphisms need not be power quasisymmetric; e.g., consider the unique increasing homeomorphism of $X = \{0\} \cup \{e^{-n!} : n \in \mathbb{N}\} \subset \mathbb{R}$ onto $Y = \{0\} \cup \{1/n! : n \in \mathbb{N}\} \subset \mathbb{R}$ [22, Remark 3.16.1]. Väisälä and Trotsenko [21] have characterized the metric spaces on which all quasisymmetric maps are power quasisymmetric. It is an open problem to characterize the spaces where Corollary 5.3 holds for all quasisymmetric maps; indeed, it may hold for every metric space.

Proof of Proposition 5.1. Let $f : X \to Y$ be as in the theorem. It suffices to prove that $\dim_A X \leq p \dim_A Y$. If $\dim_A Y = \infty$ there is nothing to prove so assume that $\dim_A Y$ is finite. Fix $s < \infty$ so that Y is s-homogeneous; we will show that X is ps-homogeneous.

Fix $0 < \alpha \leq \beta < \infty$ and let F be an α-discrete set with $\operatorname{diam} F \leq \beta$. Set $F' = f(F) \subset Y$. Then F' is α'-discrete and has diameter at most β' for some constants $0 < \alpha' \leq \beta' < \infty$. (For the second claim, see Theorem 2.5 of [22]; the first is similar and is left to the reader. Note that we are not concerned with determining exact values for α' and β'.) Since f is a homeomorphism, the cardinalities of F and F' are equal.

Choose $x_0, y_0, z_0 \in F$ with $|fx_0 - fz_0| \leq 2\alpha'$ and $|fx_0 - fy_0| \geq \frac{1}{2}\beta'$. Then

$$\frac{\beta'}{\alpha'} \leq 4\frac{|fx_0 - fy_0|}{|fx_0 - fz_0|} \leq 4\eta\left(\frac{|x_0 - y_0|}{|x_0 - z_0|}\right) \leq 4\eta\left(\frac{\beta}{\alpha}\right) = 4C_0\left(\frac{\beta}{\alpha}\right)^p$$

and s-homogeneity of Y implies that the cardinality of F is at most

$$C(s)(\beta'/\alpha')^s \leq C(s)(4C_0)^s(\beta/\alpha)^{ps}$$

which shows that X is ps-homogeneous. □

As mentioned in Section 2, the Euclidean analog of Proposition 5.1 follows from 2.3(vii):

Proposition 5.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, be an η-quasisymmetric homeomorphism and let E in $\mathbb{R}^n$ satisfy $\dim_A E \in (0, n)$. Then*

$$0 < \alpha \leq \dim_A f(E) \leq \beta < n, \tag{5.5}$$

where the constants α and β depend only on n, η, $\dim_A E$, and an s-homogeneity constant $C(s)$ for the set E for some $s \in (\dim_A E, n)$.

The lower bound in (5.5) follows directly from Theorem 5.1 as quasisymmetric maps in Euclidean spaces are always power quasisymmetric. The upper bound in (5.5) follows by combining the quasisymmetric invariance of porosity with Luukkainen's result 2.3(vii). Note that we must include the homogeneity constant $C(s)$ in the data for the upper bound β. This follows from the exact statement of 2.3(vii) given in [14, Theorem 5.2]: a subset of $\mathbb{R}^n$ is porous if and only if it is s-homogeneous for some $s < n$, however, the porosity constant $c > 0$ given depends on s, n, and a homogeneity constant $C = C(s)$.[5] I do not know if the upper bound in (5.5) can be chosen to be independent of the homogeneity constant.

References

[1] P. Assouad, *Plongements Lipschitziens dans* $\mathbf{R}^n$, Bull. Soc. Math. France **111** (1983), 429–448.
[2] Z. Balogh, *Hausdorff dimension distortion of QC mappings on the Heisenberg group*, J. Anal. Math. **83** (2001), 289–312.
[3] C. J. Bishop, *Quasiconformal mappings which increase dimension*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 397–407.
[4] C. J. Bishop and J. T. Tyson, *Locally minimal sets for conformal dimension*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 361–373.
[5] ———, *Conformal dimension of the antenna set*, Proc. Amer. Math. Soc. **129** (2001), 3631–3636.
[6] S. M. Buckley, B. Hanson, and P. MacManus, *Doubling for general sets*, Math. Scand. **88** (2001), 229–245.
[7] G. David and S. Semmes, *Fractured fractals and broken dreams. self-similar geometry through metric and measure*, The Clarendon Press, Oxford University Press, New York, 1997.
[8] J.-P. Eckmann, E. Järvenpää, and M. Järvenpää, *Porosities and dimensions of measures*, Nonlinearity **13** (2000), 1–18.

[5]The homogeneity constant cannot be chosen in general to depend solely on s; e.g., every finite set E is 0-homogeneous but the smallest admissible homogeneity constant C is the cardinality of E.

[9] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[10] F. W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2) **6** (1973), 504–512.

[11] E. Järvenpää and M. Järvenpää, *Porous measures on the real line have packing dimension close to zero*, Preprint, University of Jyväskylä, 2000.

[12] J.-P. Kahane, *Trois notes sur les ensembles parfaits linéaires*, Enseignement Math. (2) **15** (1969), 185–192.

[13] P. Koskela and S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609.

[14] J. Luukkainen, *Assouad dimension: antifractal metrization, porous sets, and homogeneous measures*, J. Korean Math. Soc. **35** (1998), 23–76.

[15] O. Martio and M. Vuorinen, *Whitney cubes, p-capacity, and Minkowski content*, Exposition. Math. **5** (1987), 17–40.

[16] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, Cambridge, 1995.

[17] P. Pansu, *Dimension conforme et sphère à l'infini des variétés à courbure négative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 177–212.

[18] J. Sarvas, *The Hausdorff dimension of the branch set of a quasiregular mapping*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 297–307.

[19] T. Sjödin, *On s-sets and mutual absolute continuity of measures on homogeneous spaces*, Manuscripta Math. **94** (1997), 169–186.

[20] S. G. Staples and L. A. Ward, *Quasisymmetrically thick sets*, Ann. Acad. Sci. Fenn. Math. **23** (1998), 151–168.

[21] D. A. Trotsenko and J. Väisälä, *Upper sets and quasisymmetric maps*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 465–488.

[22] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 97–114.

[23] J. T. Tyson, *Sets of minimal Hausdorff dimension for quasiconformal maps*, Proc. Amer. Math. Soc. **128** (2000), 3361–3367.

[24] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, vol. 229, Springer-Verlag, Berlin, 1971.

[25] ______, *Porous sets and quasisymmetric maps*, Trans. Amer. Math. Soc. **299** (1987), 525–533.

[26] A. L. Vol´berg and S. V. Konyagin, *On measures with the doubling condition*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 666–675.

[27] J.-M. Wu, *Null sets for doubling and dyadic doubling measures*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 77–91.

[28] ______, *Hausdorff dimension and doubling measures on metric spaces*, Proc. Amer. Math. Soc. **126** (1998), 1453–1459.

Department of Mathematics, State University of New York, Stony Brook, NY 11794-3651, USA

E-mail address: `tyson@math.sunysb.edu`