Illinois Journal of Mathematics
Volume 45, Number 1, Spring 2001, Pages 113–121
S 0019-2082

PRODUCTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

PATRICK AHERN AND ŽELJKO ČUČKOVIĆ

ABSTRACT. We consider the problem of determining when the product of two Bergman space Toeplitz operators is a Toeplitz operator. In particular, in the case of the zero product $T_f T_g = 0$, we give some conditions that guarantee only the trivial solution.

0. Introduction

In this paper D will denote the unit disc in the complex plane, L^2 the Lebesgue space with respect to the normalized Lebesgue measure $dA = \frac{1}{\pi} dxdy$ on D, and B the subspace of L^2 consisting of the holomorphic functions on D. For a bounded function u on D we have the Toeplitz operator $T_u : B \to B$ given by $T_u f = P(uf)$, where $P : L^2 \to B$ is the orthogonal projection. The function u is called the symbol of T_u. It is easy to see that if $T_u = 0$ then $u = 0$ almost everywhere (see Property 1 below). However, it is not known whether $T_u T_v = 0$ implies $u = 0$ or $v = 0$. In fact, it is not known for which bounded functions u and v there exists a bounded w with $T_u T_v = T_w$.

In this paper we will restrict ourselves to the case in which the symbols are bounded harmonic functions in D. Actually, we assume slightly more, namely that they are of the form $f_1 + \overline{f}_2$ for some bounded holomorphic functions f_1 and f_2 in D. This is assumed just for convenience; without this assumption the conclusion of Proposition 2 would be slightly weaker.

Assume as above that $f = f_1 + \overline{f}_2$, $g = g_1 + \overline{g}_2$ and $h = h_1 + \overline{h}_2$ with f_i, g_i and h_i bounded and holomorphic. There are some obvious cases in which $T_f T_g = T_h$: If f and g are both holomorphic or both conjugate holomorphic, then $T_f T_g = T_{fg}$. Also, if f or g is constant, then $T_f T_g = T_{fg}$. If any of these four cases holds, we say that $T_f T_g = T_h$ in a trivial way. Otherwise we say that $T_f T_g = T_h$ holds in a non-trivial way. We know of no example of such harmonic symbols f, g and h such that $T_f T_g = T_h$ in a non-trivial way. Our results are the following. In Proposition 1 we give a function-theoretic

Received July 6, 1999; received in final form April 11, 2000.
2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 47B10.

identity involving f, g and h that is equivalent to $T_fT_g = T_h$. Next, using the basic identity of Proposition 1, we give four necessary conditions on f, g and h in order that $T_fT_g = T_h$ in a non-trivial way. The first two conditions say that the symbols must be somewhat smooth and the last two conditions say that they cannot be too smooth. In Proposition 2 we show that if $T_fT_g = T_h$ in a non-trivial way, then f_1 and g_2 lie in the Zygmund class Λ_*. Using this result, we then show that the function $\phi = fg - h$ has a continuous extension to the closed disc and vanishes identically on the boundary. This implies, trivially, that the Toeplitz operator T_ϕ is compact, but much more is true. In Proposition 3 we show that T_ϕ lies in the Schatten class S_r for all $r > \frac{1}{2}$.

Next we turn to results that go somewhat in the other direction. In Proposition 4 we show that if $T_fT_g = T_h$ in a non-trivial way, then f_2 and g_1 cannot both be C^1 up to the boundary. (In fact, we prove a slightly more general result.) Finally, in Proposition 5 we show that if $T_fT_g = 0$ in a non-trivial way then all of the functions f_i, g_i are cyclic vectors for the backward shift in the Hardy space H^2. This can be interpreted as saying that the functions f_i and g_i cannot be too smooth, because it rules out polynomials and even rational functions, which are known to be non-cyclic.

Notice that if $T_fT_g = T_h$ in a trivial way, then $fg = h$; in particular, fg is harmonic. In Lemma 4.2 of [3] there is a characterization of all harmonic functions f and g such that fg is harmonic. This characterization includes the cases when f and g are holomorphic and when f and g are conjugate holomorphic, and the case when at least one of f and g is constant, but there are other cases as well, such as $f = f_1 + \overline{f}_2$ and $g = f_1 - \overline{f}_2$, where f_1 and f_2 are any bounded holomorphic functions. It is natural to try to find symbols f, g and h of this type such that $T_fT_g = T_h$ in a non-trivial way. In the corollary to Proposition 2 we show that this approach will not work. In fact, we show that if $T_fT_g = T_h$ in a non-trivial way, then fg is not harmonic. Our proof of this result uses the main result of [1]. It would be interesting to have a more elementary proof of the corollary of Proposition 2.

Another operator that arises in the study of Toeplitz operators is the Berezin transform, defined for any integrable function f on D by the formula

$$B(f)(z) = (1 - |z|^2)^2 \int_D \frac{f(\zeta)}{|1 - z\overline{\zeta}|^4} dA(\zeta).$$

We also have the kernel functions k_w for each $w \in D$ defined by $k_w(z) = 1/(1 - z\overline{w})^2$.

We will also make use of the Hankel operator $H_u : B \to B^\perp$ defined by $H_u(v) = (I - P)(uv)$.

We now list some simple and well known properties of Toeplitz operators.

Properties. *1. If $T_u = 0$ then $u = 0$ almost everywhere.*

2. If f is holomorphic, then $T_uT_f = T_{uf}$, and $T_{\overline{f}}T_u = T_{\overline{f}u}$ for any u.

3. $T_u^* = T_{\overline{u}}$.
4. If f *is holomorphic and not identically zero, then* T_f *is one-to-one.*
5. If $g \in B$ *and* $w \in D$ *then* $P(\overline{g}k_w) = \overline{g}(w)k_w$.

A good reference for Properties 2–5 is Axler's survey [2]. Property 1 does not seem to have been stated explicitly in the literature, but it is very easy to prove: If $T_u = 0$, then u is orthogonal to all polynomials (in z and $\overline{z}$); hence $u = 0$ almost everywhere, since such polynomials are dense in L^2.

We conclude this section with the following simple lemma.

LEMMA 1. *Suppose that* $T_f T_g = T_h$ *in a non-trivial way, where* f, g *and* h *are as above. Then* g *is not holomorphic and* f *is not conjugate holomorphic.*

Proof. If g is holomorphic, then by Property 2, $T_f T_g = T_{fg} = T_h$ and hence by Property 1, $h = fg$. If we take the Laplacian of both sides of this identity and use the fact that h is harmonic then we see immediately that either g is constant or f is holomorphic as well. If $\overline{f}$ is holomorphic then again by Properties 2 and 1 we have $fg = h$, and this implies that either f is constant or $\overline{g}$ is holomorphic. □

We note here that Lemma 1 says that if $T_f T_g = T_h$ in a non-trivial way, then neither of the functions f_1, g_2 can be constant. We will use this fact in what follows.

We end this introduction by mentioning the analog of the above problem for the Hardy space H^2. Brown and Halmos [5] have shown that, in the case of Toeplitz operators on H^2, $T_f T_g = T_h$ implies that either g is holomorphic or f is conjugate holomorphic.

1. The basic identity

In this section we prove the identity on which our other results are based.

PROPOSITION 1. *Suppose that* f, g *and* h *are as above. Then the following are equivalent:*

(a)
$$T_f T_g = T_h;$$

(b)
$$\begin{aligned} &f_1(z)g_1(z) + \overline{f}_2(z)\overline{g}_2(z) + f_1(z)\overline{g}_2(z) + B(\overline{f}_2 g_1)(z) \\ &\quad = h_1(z) + \overline{h}_2(z) \quad \text{for } z \in D; \end{aligned}$$

(c)
$$\begin{aligned} &f_1(z)g_1(z) + \overline{f}_2(w)\overline{g}_2(w) + f_1(z)\overline{g}_2(w) \\ &\qquad + (1 - z\overline{w})^2 \int_D \frac{\overline{f}_2(\zeta)g_1(\zeta)}{(1 - z\overline{\zeta})^2(1 - \overline{w}\zeta)^2} dA(\zeta) \\ &\quad = h_1(z) + \overline{h}_2(w) \quad \text{for all } (z, w) \in D \times D. \end{aligned}$$

Proof. Clearly (c) implies (b). We will show first that (a) is equivalent to (c), and then that (b) implies (c). We have $T_f T_g = T_h$ if and only if $T_f T_g k_w = T_h k_w$ for all $w \in D$. Using Property 5 above we see that

$$T_g k_w = P(g_1 k_w + \overline{g}_2 k_w) = g_1 k_w + \overline{g}_2(w) k_w.$$

It then follows from another application of Property 5 that

$$\begin{aligned} T_f T_g k_w &= P((f_1 + \overline{f}_2)(g_1 k_w + \overline{g}_2(w) k_w)) \\ &= f_1 g_1 k_w + \overline{g}_2(w) f_1 k_w + \overline{g}_2(w) \overline{f}_2(w) k_w + P(\overline{f}_2 g_1 k_w). \end{aligned}$$

Since $T_h k_w = h_1 k_w + \overline{h}_2(w) k_w$, as above, we see that $T_f T_g = T_h$ if and only if

$$\begin{aligned} f_1(z) g_1(z) + \overline{f}_2(w) \overline{g}_2(w) + f_1(z) \overline{g}_2(w) + \frac{1}{k_w(z)} P(\overline{f}_2 g_1 k_w)(z) \\ = h_1(z) + \overline{h}_2(w), \end{aligned}$$

for all $(z, w) \in D \times D$. To complete the proof that (a) is equivalent to (c) it suffices to observe that

$$\frac{1}{k_w(z)} P(\overline{f}_2 g_1 k_w)(z) = (1 - z\overline{w})^2 \int_D \frac{\overline{f}_2(\zeta) g_1(\zeta)}{(1 - z\overline{\zeta})^2 (1 - \overline{w}\zeta)^2} dA(\zeta).$$

To show that (b) implies (c), we consider the holomorphic function defined in the bi-disc by the formula

$$\begin{aligned} F(z, w) = f_1(z) g_1(z) + \overline{f}_2(\overline{w}) \overline{g}_2(\overline{w}) + f_1(z) \overline{g}_2(\overline{w}) \\ + (1 - zw)^2 \int_D \frac{\overline{f}_2(\zeta) g_1(\zeta)}{(1 - z\overline{\zeta})^2 (1 - w\zeta)^2} dA(\zeta) - h_1(z) - \overline{h}_2(\overline{w}). \end{aligned}$$

Assuming (b), F is identically zero on the set $\{(z, \overline{z}) : z \in D\}$, and hence is identically zero in $D \times D$. So $F(z, \overline{w}) \equiv 0$, which is the statement (c). □

Next we record a corollary to Proposition 1 that will be used in the proof of Proposition 4.

Corollary. *If $T_f T_g = T_h$ in a non-trivial way, then f is not holomorphic and g is not conjugate holomorphic.*

Proof. If f were holomorphic then we would have $\overline{f}_2 = C$, where C is a constant. Since the Berezin transform reproduces holomorphic functions, part (b) of the proposition says that $fg = h$. As before, if we take the Laplacian of this identity we see that either f is constant or g is holomorphic. The case when g is conjugate holomorphic is similar. □

2. Necessary conditions

Proposition 2. *Suppose that $T_fT_g = T_h$ in a non-trivial way. Then we have:*

(i) *f_1 and g_2 lie in the Zygmund class Λ_*.*
(ii) *$\phi = fg - h$ extends to a continuous function on $\overline{D}$ and is identically zero on $\partial\overline{D}$.*

Proof. Since g_2 is not constant, there is a $k \geq 1$ such that $g_2^{(k)}(0) \neq 0$. Now, differentiating the identity of part (c) of Proposition 1 k times with respect to $\overline{w}$, and letting $w = 0$, we obtain

$$f_1(z)\overline{g}_2^{(k)}(0) + C + \int_D \frac{S(z,\zeta)}{(1-z\overline{\zeta})^2}\overline{f}_2(\zeta)g_1(\zeta)dA(\zeta) \equiv 0,$$

where

$$S(z,\zeta) = \frac{\partial^k}{\partial\overline{w}^k}(\frac{1-z\overline{w}}{1-\zeta\overline{w}})^2|_{w=0}.$$

To evaluate $S(z,\zeta)$, we note that

$$(\frac{1-z\overline{w}}{1-\zeta\overline{w}})^2 = (1-z\overline{w})^2\sum(k+1)(\zeta\overline{w})^k.$$

Multiplying out and collecting terms we get

$$(\frac{1-z\overline{w}}{1-\zeta\overline{w}})^2 = 1 + (\zeta - z)\sum_{k=1}^{\infty}[k(\zeta - z) + \zeta + z]\zeta^{k-2}\overline{w}^k.$$

From this we see that $S(z,\zeta)$ is $\zeta - z$ times a polynomial in ζ and z, and we conclude that

$$f_1(z) = A + \int_D \frac{P(z,\zeta)(\zeta - z)\overline{f}_2(\zeta)g_1(\zeta)}{(1-z\overline{\zeta})^2}dA(\zeta),$$

where P is a polynomial. If we differentiate this expression twice we obtain the estimate

$$|f_1''(z)| \leq C(\int_D \frac{|\zeta - z|}{|1-z\overline{\zeta}|^4}dA(\zeta) + \int_D \frac{1}{|1-z\overline{\zeta}|^3}dA(\zeta)).$$

Here the constant C incorporates a bound for the sup norms of f_2 and g_1. If we use the fact that $|\zeta - z| \leq |1 - z\overline{\zeta}|$ then standard estimates show that $|f_1''(z)| \leq C/(1-|z|)$, and this in turn is equivalent to the statement that $f_1 \in \Lambda_*$ (see, for example, Theorem 5.3 of [6]). The proof that $g_2 \in \Lambda_*$ is very similar. This completes the proof of (i).

For the proof of (ii) we start with part (b) of Proposition 1:

$$f_1g_1 + \overline{f}_2\overline{g}_2 + f_1\overline{g}_2 + B(\overline{f}_2g_1) = h_1 + \overline{h}_2.$$

Since the Berezin transform reproduces harmonic functions, we can rewrite this as

$$B(\overline{f}_2 g_1 + f_1 g_1 + \overline{f}_2 \overline{g}_2 - h_1 - \overline{h}_2) = -f_1 \overline{g}_2,$$

or $B(u) = -f_1\overline{g}_2$, where u denotes the function in the argument of the operator B. By part (i) of the proposition, the right hand side is continuous on $\overline{D}$. Since u is clearly in the algebra generated by the bounded harmonic functions it follows that u itself has a continuous extension to $\overline{D}$ (see Corollary 3.11 of [3]). But $fg - h = u + f_1\overline{g}_2$, so $fg - h$ has a continuous extension to $\overline{D}$. To see that $fg - h = 0$ on ∂D we note that, since u is continuous on $\overline{D}$, it follows that $B(u) = u$ on ∂D. But $B(u) + f_1\overline{g}_2 = 0$ in D, and so by continuity $fg - h = u + f_1\overline{g}_2 = 0$ on ∂D. This completes the proof. □

Corollary. *If $T_f T_g = T_h$ in a non-trivial way, then fg is not harmonic.*

Proof. Suppose that fg is harmonic. By Proposition 2 we have $fg = h$ almost everywhere on ∂D. Hence $fg = h$ in D, since fg and h are harmonic. A slight rewriting of part (b) of Proposition 1 gives

$$fg + B(\overline{f}_2 g_1) - \overline{f}_2 g_1 = h.$$

Since $fg = h$ we obtain $B(\overline{f}_2 g_1) = \overline{f}_2 g_1$. The main result of [1] then implies that $\overline{f}_2 g_1$ is harmonic, from which it easily follows that either f_2 or g_1 is constant. That is, either f is holomorphic or g is conjugate holomorphic. Applying the corollary to Proposition 1 completes the proof. □

Proposition 3. *If $T_f T_g = T_h$ in a non-trivial way and $\phi = fg - h$, then we have:*

(i) *T_ϕ is in the Schatten class S_r for all $r > \frac{1}{2}$.*
(ii) *$\sum(\frac{1}{|R_k|}\int_{R_k} |\phi|^2 dA)^r < \infty$ for all $r > \frac{1}{2}$, where $\{R_k\}$ is a partition of D into hyperbolically equal-sized Carleson half-squares and $|R_k|$ denotes the area measure of R_k.*

Proof. We will use the well known formula $T_{fg} - T_f T_g = H^*_{\overline{f}} H_g$, valid for any bounded symbols f and g. In our case, $H_g = H_{\overline{g}_2}$ and $H_{\overline{f}} = H_{\overline{f}_1}$ so that, under the assumptions of Proposition 3, $T_{fg-h} = H^*_{\overline{f}_1} H_{\overline{g}_2}$. In [4] it was shown that if F is holomorphic and $1 < p < \infty$ then $H_{\overline{F}} \in S_p$ if and only if

$$\int_D |F'(z)|^p (1 - |z|)^{p-2} dA(z) < \infty.$$

We have seen in the proof of Proposition 2 that if $F = f_1$ or $F = g_2$ then $|F''(z)| \le C/(1 - |z|)$. This implies in a standard way that

$$\int_D |F'(z)|^p (1 - |z|)^{p-2} dA(z) < \infty$$

for all $1 < p < \infty$. This means that $H_{\overline{f}_1}$ and $H_{\overline{g}_2}$ are in S_p for all $1 < p < \infty$. Hence the product $H^*_{\overline{f}_1} H_{\overline{g}_2}$ is in S_r for all $r > \frac{1}{2}$.

In [8] there is a characterization of the non-negative symbols whose Toeplitz operators lie in the Schatten classes. The main theorem of [8] says that if $\psi \geq 0$, then $T_\psi \in S_p$ if and only if

$$\sum\left(\frac{1}{|R_k|}\int_{R_k} \psi dA\right)^p < \infty,$$

where $\{R_k\}$ is partition of D into hyperbolically equal-sized Carleson half-squares. Our symbol $\phi = fg - h$ is not necessarily non-negative, but we can apply the theorem in the following way. Suppose $u = \sum_1^n \overline{\alpha}_i \beta_i$, where the α_i and the β_i are bounded holomorphic functions. Then we have

$$T_{\phi u} = \sum T_{\overline{\alpha}_i} T_\phi T_{\beta_i}.$$

So $T_{\phi u} \in S_r$ for all $r > \frac{1}{2}$ for any such u. But we can take $u = \overline{\phi}$ and conclude that

$$\sum\left(\frac{1}{|R_k|}\int_{R_k} |\phi|^2 dA\right)^r < \infty,$$

for all $r > \frac{1}{2}$. □

Corollary. *We have* $\int_D \frac{|\phi(z)|^{2r}}{(1-|z|^2)^2} dA(z) < \infty$, *for all* $r > \frac{1}{2}$.

Proof. Fix $1/2 < r < 1$. By Hölder's inequality we have $\int_{R_k} |\phi|^{2r} dA \leq (\int_{R_k} |\phi|^2 dA)^r |R_k|^{1-r}$. This yields $\frac{1}{|R_k|}\int_{R_k} |\phi|^{2r} dA \leq (\frac{1}{|R_k|}\int_{R_k} |\phi|^2 dA)^r$. Now for $z \in R_k$, $(1-|z|^2)^2$ is of the order of $|R_k|$. Using this fact and summing on k, we arrive at the conclusion of the corollary. □

Proposition 4. *Suppose that* $T_f T_g = T_h$ *in a non-trivial way. Then there does not exist a subset* $E \subset \partial D$ *of positive measure such that* f_2' *and* g_1' *have continuous extensions to each point of* E.

Proof. Suppose there is a set $E \subset \partial D$ of positive measure such that f_2' and g_1' have continuous extensions to each point of E. By condition (b) of Proposition 1 we have

$$f_1(z)g_1(z) + \overline{f}_2(z)\overline{g}_2(z) + f_1(z)\overline{g}_2(z) + B(\overline{f}_2 g_1)(z) = h(z).$$

Applying the invariant Laplacian to this equation and using the fact that the invariant Laplacian commutes with the Berezin transform (see [7]), we obtain

$$(1-|z|^2)^2 f_1'(z)\overline{g}_2'(z) + (1-|z|^2)^2 \int_D \frac{(1-|\zeta|^2)^2 \overline{f}_2'(\zeta) g_1'(\zeta)}{|1-z\overline{\zeta}|^4} dA(\zeta) \equiv 0.$$

Dividing by $(1-|z|^2)^2$ we get

$$f_1'(z)\overline{g}_2'(z)+\int_D \frac{(1-|\zeta|^2)^2\overline{f}_2'(\zeta)g_1'(\zeta)}{|1-z\overline{\zeta}|^4}dA(\zeta)\equiv 0.$$

Since we are assuming that $T_fT_g=T_h$ in a non-trivial way, it follows from the corollary to Proposition 1 that neither f_2' nor g_1' can be identically zero. From this it follows that there is a subset of E of positive measure on which neither $\overline{f}_2'$ nor g_1' vanishes. (What we are using here is a very special (and easy) case of the theorem of Privalov that says that if a meromorphic function in the unit disc has non tangential limit zero on a set of positive measure on ∂D then it is identically zero; see [9] or Theorem 1.9 of [10].) It follows that there is a set of positive measure on which $\mathrm{Re}(\overline{f}_2'g_1')$ (or $\mathrm{Im}(\overline{f}_2'g_1')$) is never zero, and finally a set of positive measure on which this function is positive (or negative) and, in fact, bounded away from zero. Let us suppose, say, that there is a set of positive measure (which we will continue to call E) on which $u=\mathrm{Re}\overline{f}_2'g_1'>\epsilon>0$. Take a point $e^{it}\in E$. Then there is a number $\eta>0$ such that if $|\zeta-e^{it}|<\eta$ then $u(\zeta)>\epsilon$. Then we have

$$\begin{aligned}\mathrm{Re}&\int_D \frac{(1-|\zeta|^2)^2\overline{f}_2'(\zeta)g_1'(\zeta)}{|1-z\overline{\zeta}|^4}dA(\zeta)\\ &=\int_{|\zeta-e^{it}|<\eta}\frac{(1-|\zeta|^2)^2u(\zeta)}{|1-z\overline{\zeta}|^4}dA(\zeta)+\int_{|\zeta-e^{it}|>\eta}\frac{(1-|\zeta|^2)^2u(\zeta)}{|1-z\overline{\zeta}|^4}dA(\zeta)\end{aligned}$$

Now let $z\to e^{it}$. Then the second integral stays bounded, but the first integral is greater than

$$\epsilon\int_{|\zeta-e^{it}|<\eta}\frac{(1-|\zeta|^2)^2}{|1-z\overline{\zeta}|^4}dA(\zeta).$$

This integral is in turn equal to

$$\int_D \frac{(1-|\zeta|^2)^2}{|1-z\overline{\zeta}|^4}dA(\zeta)-\int_{|\zeta-e^{it}|>\eta}\frac{(1-|\zeta|^2)^2}{|1-z\overline{\zeta}|^4}dA(\zeta).$$

Again, the second integral stays bounded as $z\to e^{it}$ and the first one goes to ∞ like $\log(1/(1-|z|))$ as $|z|\to 1$. It now follows that $|f_1'(z)g_2'(z)|=|f_1'(z)\overline{g}_2'(z)|\geq|\mathrm{Re}(f_1'(z)\overline{g}_2'(z))|\to\infty$ as $z\to e^{it}$. This means that the holomorphic function $f_1'g_2'$ continuously takes on the value ∞ on a set of positive measure on ∂D. This is impossible, because then the meromorphic function $1/(f_1'g_2')$ would take on the value zero on this set of positive measure and hence be identically zero. □

Our final result deals with the "zero product" situation, i.e., the case when $T_fT_g=0$. For this to occur in the trivial way means that $fg=0$ and hence that either f or g is identically zero. Recall that a function f in the Hardy space H^2 is said to be non-cyclic for the backward shift if there is an inner function ϕ such that f is orthogonal to the subspace ϕH^2 of H^2. Otherwise f is said to be cyclic for the backward shift. As is well known, f is orthogonal to ϕH^2 if and only if there is a function $F \in H^2$ such that $F(0) = 0$ and such that $\overline{\phi} f = \overline{F}$ almost everywhere on ∂D.

Proposition 5. *If $T_f T_g = 0$ in a non-trivial way, then f_1, f_2, g_1, g_2 are all cyclic vectors for the backward shift.*

Proof. If $T_f T_g = 0$ then, as we have seen above, fg extends continuously to $\overline{D}$ and vanishes on ∂D. Since neither f nor g is identically zero, we conclude that there is a set $E \subset \partial D$ such that E has positive measure, $\partial D - E$ has positive measure and such that $f = 0$ almost everywhere on E and $g = 0$ almost everywhere on $\partial D - E$. If f_1 were non cyclic for the backward shift, there would exist an inner function ϕ and a function $F \in H^2$ such that $\overline{\phi} f_1 = \overline{F}$ almost everywhere on ∂D. Now $\overline{f}_2 = -f_1$ almost everywhere on E so we see that $\overline{\phi f}_2 = -\overline{\phi} f_1 = -\overline{F}$ almost everywhere on E. But this means that $\phi f_2 = -F$ on all of ∂D. This in turn implies that $\overline{\phi} f_1 = -\overline{\phi f}_2$ on ∂D and hence that $f_1 = -\overline{f}_2$ on ∂D. This means that $f = 0$ on ∂D and hence $f = 0$ in D, contrary to the assumptions. The same argument shows that f_2, g_1 and g_2 are also cyclic. □

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Patrick Ahern, Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388

E-mail address: ahern@math.wisc.edu

Željko Čučković, Department of Mathematics, University of Toledo, Toledo, OH 43606

E-mail address: zcuckovi@math.utoledo.edu