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THE SPECTRUM OF A SUPERSTABLE OPERATOR AND COANALYTIC FAMILIES OF OPERATORS

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Abstract. We first show that for an infinite dimensional Banach space X, the unitary spectrum of any superstable operator is countable. In connection with descriptive set theory, we show that if X is separable, then the set of stable operators and the set of power bounded operators are Borel subsets of $L(X)$ (equipped with the strong operator topology), while the set $\mathcal{S}'(X)$ of superstable operators is coanalytic. However, $\mathcal{S}'(X)$ is a Borel set if X is a superreflexive and hereditarily indecomposable space. On the other hand, if X is superreflexive and X has a complemented subspace with unconditional basis or, more generally, if X has a polynomially bounded and not superstable operator, then the set $\mathcal{S}'(X)$ is non Borel.

1. Introduction

The main part of this work is devoted to some aspects of the relationship between descriptive set theory and the geometry of Banach spaces. We study the topological complexity of some natural families of operators in an infinite dimensional Banach space.

An operator T on a Banach space X is stable if the orbit $\mathcal{O}_T(x) = \{T^n(x), n \in \mathbb{N}\}$ is relatively compact for every $x \in X$, i.e., if the set $\{T^n, n \in \mathbb{N}\}$ is relatively compact in $L(X)$ for the strong operator topology $\mathcal{S}_{op}$. This is the case for any operator $T \in L(X)$ such that $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in X$. For example, the left-shift operator on $\ell^p(\mathbb{N})$, where $1 \le p \le \infty$, has this property. It follows from the Banach-Steinhaus theorem that every stable operator is power bounded. However, the converse is false as the example of the right-shift operator on $\ell_2(\mathbb{N})$ shows.

Arendt and Batty ([AB], [NR]), and independently Lyubich and Phong ([LP], [NR]) have shown that a power bounded operator T on a reflexive Banach space X is stable if the unitary spectrum of T (denoted by $\sigma_1(T)$) is

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countable. One can show that the converse is not true and that reflexivity is necessary.

More recently, Nagel and Räbiger [NR] have introduced the notion of superstability.

Definition 1.1. A bounded operator T on a Banach space X is superstable if, for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultrapower $T_{\mathcal{U}}$ is stable on the ultrapower $X_{\mathcal{U}}$ where (see [He]):

- $X_{\mathcal{U}}$ is the quotient space $\ell^\infty(X)/\mathcal{C}_{\mathcal{U}}(X)$.
- $\ell^\infty(X)$ is the Banach space of bounded sequences in X.
- $\mathcal{C}_{\mathcal{U}}(X)$ is the closed subspace of sequences converging to zero along $\mathcal{U}$.
- For $\bar{x} = (x_n)_{n\in\mathbb{N}}+\mathcal{C}_{\mathcal{U}}(X)$, $\|\bar{x}\| = \mathcal{U}-\lim_n \|x_n\|$ and $T_{\mathcal{U}}(\bar{x}) = (Tx_n)_{n\in\mathbb{N}} + \mathcal{C}_{\mathcal{U}}$.

Let T be an operator on a Banach space X and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. Since T and its ultrapower $T_{\mathcal{U}}$ have the same spectrum (see [Sc]) and since the ultrapower $X_{\mathcal{U}}$ is reflexive if X is a superreflexive space, the above-mentioned results of Arendt and Batty [AB], and Lyubich and Phong [LP] extend to superstable operators as follows: for a superreflexive Banach space, any power bounded operator T with countable unitary spectrum $\sigma_1(T)$ is superstable. Actually, Nagel and Räbiger showed that the converse is also true.

Theorem 1.2. *Let T be a power bounded operator on a superreflexive Banach space X. Then $\sigma_1(T)$ is countable if and only if T is superstable.*

The superreflexivity of X is necessary in Theorem 1.2. Indeed, take $X = \ell^p(\ell_n^\infty)$ with $1 \le p \le \infty$ and $T((x_n)_n) = (T_n x_n)_n$, where T_n is the multiplication operator by $(\alpha_1, \dots, \alpha_n)$ on ℓ_n^∞ and $(\alpha_n)_n$ an increasing sequence in $]0,1[$. Then $\sigma_1(T)$ is not superstable (see [NR]). However, the superreflexivity of X is not necessary for the reverse implication of Theorem 1.2; the following section is devoted to the proof of this result.

It is clear that, in general, superstability is stronger than stability, even if X is superreflexive. Indeed, take $X = \ell^2(\mathbb{N})$ and the left-shift operator S. A natural problem is to determine for which spaces superstability is strictly stronger. This leads us to study the topological complexity of these operators and to establish a descriptive set hierarchy of certain naturally occurring sets stemming from these operators. The interest of results of this type lies in the fact that a theorem stating that the set of operators belonging to a certain class is coanalytic and non Borel places a strong restriction on alternative characterizations of this class, eliminating, in a single stroke, many conjectured equivalences. Of course, proving a theorem of this sort generally requires much greater insight into the class in question than would be required for the construction of an isolated counterexample.

Recently, the topological nature of certain examples of families belonging to various domains of analysis has been considered, and many connections between descriptive set theory and analysis have been found; for harmonic analysis, we refer to [KL] (see also [BKL]), and for convex analysis we refer to [Bo], [BGK], and [DGS].

Let $L(X)$ be the space of bounded operators on a separable Banach space X. This space is a standard Borel space when equipped with the strong operator topology S_{op} (see Proposition 3.1). We show easily that the subset of stable operators is a Borel set, but the case of the set of superstable operators $\mathcal{S}'(X)$ requires various notions in descriptive set theory, in particular, the coanalytic rank properties (see [KL] or [C]) and the *entropy tree*. We begin with the characterization of superstable operators in terms of well founded trees on $\mathbb{N}$ whose height defines a coanalytic rank on $\mathcal{S}'(X)$ (so that $\mathcal{S}'(X)$ is a coanalytic set). For a superreflexive and separable Banach space X, we use the characterization of superstable operators by the countability of their unitary spectrum and the Cantor derivation to define another coanalytic rank on $\mathcal{S}'(X)$. In particular, we show that the map $\sigma_1 : T \mapsto \sigma_1(T)$ is a S_{op}-Borel map. This gives classes of separable and superreflexive Banach spaces for which $\mathcal{S}'(X)$ is a coanalytic non-Borel set, namely spaces with complemented subspaces having an unconditional basis and, more generally, spaces having a polynomially bounded and non-superstable operator. Finally, we consider the hereditarily indecomposable and superreflexive Banach spaces, for which $\mathcal{S}'(X)$ is a Borel set.

2. The spectrum of a superstable operator

In this section we prove that Theorem 1.2 extends to any Banach space.

THEOREM 2.1. *The unitary spectrum of any superstable operator on an infinite dimensional Banach space is countable.*

The first part of the proof is similar to the argument given in [NR]. However, the remainder of the proof is different from that of [NR], since we observe that the identification of the dual $(X_{\mathcal{U}})^*$ of an ultrapower with the ultrapower $(X^*)_{\mathcal{U}}$, valid when X is superreflexive, can more generally be replaced by the canonical embedding of the latter space in the former.

The proof depends on the following crucial proposition which shows, in particular, that for a diffuse probability measure μ on the unit circle, the set $\{f_n : \lambda \mapsto \lambda^n, n \in \mathbb{N}\} \subset L^1(\mathbb{T}, \mu)$ cannot be covered by a finite number of balls with diameter less than 1.

PROPOSITION 2.2. *Let μ be a diffuse probability measure on the unit circle $\mathbb{T}$. For every $n \in \mathbb{Z}$, consider on $\mathbb{T}$ the function $f_n(\lambda) = \lambda^n$. Then the set $\{f_n, n \in \mathbb{N}\}$ is not relatively compact in $L^1(\mathbb{T}, \mu)$.*

Proof. Assume that the set $\{f_n, n \in \mathbb{N}\}$ is relatively compact in $L^1(\mu)$. By considering the conjugate, it follows that the set $\{f_n, n \in \mathbb{Z}\}$ is also relatively compact in $L^1(\mu)$. Let $0 < \varepsilon < 1$ fixed. Then there exists $N \in \mathbb{N}^*$ and a subdivision $(I_j)_{1\leq j\leq N}$ of $\mathbb{Z}$ such that

$$\forall j \in \{1, ..., N\}, \quad \operatorname{diam}\{f_n, n \in I_j\} < \varepsilon.$$

Recall that the *upper density* $d^*(I)$ of a subset I of $\mathbb{Z}$ is defined by

$$d^*(I) = \limsup_{m\to+\infty} \frac{1}{2m+1} \operatorname{card}\big(I \cap [-m, m]\big).$$

Since $(I_j)_{1\leq j\leq N}$ is a subdivision of $\mathbb{Z}$, it follows that there exists $j_0 \in \{1, ..., N\}$ such that $d^*(I_{j_0}) = \rho > 0$. For two distinct elements $p, q \in I_{j_0}$ we have

$$\begin{aligned} \varepsilon > \|f_p - f_q\|_{L_1(\mu)} &=< \mu, |f_p - f_q| > \\ &=< \mu, |f_q|\, |f_{p-q} - 1| > \\ &=< \mu, |f_{p-q} - 1| > \\ &\geq |< \mu, f_{p-q} > - < \mu, 1 >| \\ &= |\hat{\mu}(p - q) - 1|\,. \end{aligned}$$

Thus, setting $I = I_{j_0} - q$, we get that $|\hat{\mu}(n)| > 1 - \varepsilon$ for all $n \in I$, and $d^*(I) = \rho$. It follows that

$$\begin{aligned} \frac{1}{2m+1} \sum_{-m}^{m} |\hat{\mu}(n)|^2 &\geq \frac{1}{2m+1} \sum_{n\in[-m,m]\cap I} |\hat{\mu}(n)|^2 \\ &> \frac{(1-\varepsilon)^2}{2m+1} \operatorname{card}\left([-m, m] \cap I\right). \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{2m+1} \sum_{-m}^{m} |\hat{\mu}(n)|^2 \geq \rho(1-\varepsilon)^2 > 0.$$

This yields a contradiction to Wiener's theorem (see [Ka, p. 42] or [Kr, p. 96]), according to which $\lim_{n\to\infty}(1/(2m+1)) \sum_{-m}^{m} |\hat{\mu}(n)|^2 = 0$ if μ is a diffuse probability measure on the unit circle $\mathbb{T}$. Hence we have proved Proposition 2.2. □

Proof of Theorem 2.1. Let X be a Banach space and T a power bounded operator on X such that its unitary spectrum $\sigma_1(T)$ is uncountable. (If T is not bounded then by the Banach-Steinhaus theorem it is clear that T can not be superstable.) We want to show that T is not superstable. First, by using the equivalent norm $|||x||| := \sup_{n\in\mathbb{N}} \|T^n x\|$, we can assume that T is a contraction. Moreover, since the unitary spectrum of any ultrapower $T_{\mathcal{U}}$ of the operator T contains only eigenvalues of $T_{\mathcal{U}}$ (see [Sc] or [NR, Proposition 2.2]), we can also assume (by taking $T_{\mathcal{U}}$ instead of T and noting that the superstability of T is equivalent to the superstability of $T_{\mathcal{U}}$) that $\sigma_1(T)$ is composed entirely of eigenvalues.

Nagel and Räbiger [NR] showed that there exists a dense sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \sigma_1(T)$ such that the problem is equivalent to the non-superstability of an isometric multiplicator operator M, corresponding to the values $\{\lambda_n\}_{n\in\mathbb{N}}$, on a Banach sequence lattice Z. Then take a diffuse probability measure on $\mathbb{T}$ supported on $\sigma_1(M) = \overline{\{\lambda_k : k \in \mathbb{N}\}}$ (see [Se, 19.7.6] and [Se, 8.5.5]) which, by the weak*-density of the atomic measures, we can write as

$$\mu = w^* - \lim_{n\to\infty} \sum_{k=0}^{n} \alpha_{n,k}\delta_{\lambda_k},$$

where $\sum_{k=0}^{n} \alpha_{n,k} = 1$, $\alpha_{n,k} \geq 0$, and δ_{λ_k} is the Dirac measure at λ_k. By using the decomposition of probability vectors (see [TJ] or [NR, Lemma 3.4]), Nagel and Räbiger constructed the points of norm 1

$$z_n = \sum_{k=1}^{n} \beta_{n,k} e_k \in Z \qquad \text{and} \qquad \phi_n = \sum_{k=1}^{n} \gamma_{n,k} e_k^* \in Z^*,$$

where $\beta_{n,k}, \gamma_{n,k} \in \mathbb{R}^+$ are such that $\alpha_{n,k} = \beta_{n,k}\gamma_{n,k}$, $\{e_k\}_{k\in\mathbb{N}}$ is the canonical basis of Z, and $\{e_k^*\}_{k\in\mathbb{N}}$ its dual basis. In order to show that M is not superstable, we consider an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and the points

$$z = (z_n)_{n\in\mathbb{N}} + \mathcal{C}_{\mathcal{U}}(Z) \in Z_{\mathcal{U}} \quad \text{and} \quad \phi = (\phi_n)_{n\in\mathbb{N}} + \mathcal{C}_{\mathcal{U}}(Z^*) \in (Z^*)_{\mathcal{U}}$$

of norm 1. Even if Z is not superreflexive, we still have $\phi \in (Z_{\mathcal{U}})^*$ since $(Z^*)_{\mathcal{U}} \subseteq (Z_{\mathcal{U}})^*$. From the definitions of M, Z and μ, we get that, for any integers p and q,

$$\begin{aligned} < \phi, |M_{\mathcal{U}}^p(z) - M_{\mathcal{U}}^q(z)| > &= \mathcal{U} - \lim_n \sum_{k=0}^{n} \alpha_{n,k} \, |\lambda_k^p - \lambda_k^q| \\ &= \mathcal{U} - \lim_n < \mu_n, |f_p - f_q| > \\ &= < \mu, |f_p - f_q| >, \end{aligned}$$

where f_n is the map defined on $\mathbb{T}$ by $f_n(\lambda) = \lambda^n$. Hence

$$\|M_{\mathcal{U}}^p(z) - M_{\mathcal{U}}^q(z)\| \geq \|f_p - f_q\|_{L^1(\mu)} .$$

Since the set $\{f_n, n \in \mathbb{N}\}$ is not relatively compact in $L^1(\mu)$ (see Proposition 2.2), it follows from this last inequality that the orbit $\mathcal{O}_{M_{\mathcal{U}}}(z)$ is not relatively compact in the ultrapower $Z_{\mathcal{U}}$. Hence M, and therefore T, are not superstable. □

3. Introduction to the topological complexity of some operator families

In the following sections we use classical definitions and results from descriptive set theory (see [C], [Ke], [KL]). We denote by X an infinite dimensional Banach space, and we equip the space $L(X)$ of bounded operators on X with the strong operator topology S_{op}. In this section we study the position of the operator sets introduced in the previous sections in the descriptive set hierarchy. The case of superstable operators is more complex and will be investigated in the remaining sections. We first show that the strong operator topology on $L(X)$ is adequate for this purpose.

To establish a hierarchy from a topological point of view, it is natural to work in a Polish space, i.e., a space homeomorphic to a separable metrizable complete space. However, it is also possible to work in a standard Borel space, i.e., a space Borel-isomorphic to a Borel set of a Polish space. (Hence all the notions and properties in a Polish space can be transferred to the standard Borel space.) The space $L(X)$ equipped with the strong operator topology is not a Polish space (since it is not a Baire space). However, if X is a separable Banach space, $\big(L(X), S_{op}\big)$ can be shown to be Borel-isomorph to a Borel set of the Polish space $X^{\mathbb{N}}$ equipped with the norm product topology, via the map

$$\begin{array}{rcl} \varphi : \big(L(X), S_{op}\big) & \longrightarrow & \big(X^{\mathbb{N}}, \mathcal{P}\big) \\ T & \longmapsto & (Tz_n)_{n\in\mathbb{N}}, \end{array}$$

where $\{z_n, n \in \mathbb{N}\}$ is a dense $\mathbb{Q}$-vector space in X. Hence we have the following result.

Proposition 3.1. *For every separable Banach space X, $\big(L(X), S_{op}\big)$ is a standard Borel space.*

The following lemma yields, in particular, the continuity of the map : $T \to T^n$ on the bounded subsets of the power bounded operator space endowed with the topology S_{op}, and provides a collection of open and closed sets in $\big(L(X), S_{op}\big)$.

Lemma 3.2. *Let X be a Banach space. The multiplication map on $L(X)$, i.e., the map $(R,T) \in L(X) \times L(X) \longmapsto R \cdot T \in L(X)$, is jointly continuous for the strong operator topology on bounded subsets of $L(X)$.*

Proof. The result is immediate from the inequality

$$\|(RT - R_0T_0)x\| \leq \|R\| \,.\, \|(T - T_0)x\| + \|(R - R_0)(T_0x)\| \,,$$

where R_0, T_0, R, $T \in L(X)$ and $x \in X$. □

Notations. We denote by $L_{pb}(X)$, $\mathcal{S}(X)$, and $\mathcal{S}'(X)$, respectively, the sets of power bounded, stable, and superstable operators on X. By $B(x,\varepsilon)$

and $\overline{B}(x,\varepsilon)$ we denote, respectively, the open and the closed balls of center x and radius ε.

PROPOSITION 3.3. *Let X be a separable infinite-dimensional Banach space. Then the set $L_{pb}(X)$ of all power bounded operators and the set $\mathcal{S}(X)$ of all stable operators on X are Borel subsets of $\big(L(X), S_{op}\big)$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset in the closed unit ball of X. It is obvious that the set $L_{pb}(X)$ is a Borel set since this set can be written as countable intersection and union of closed sets.

Let T be an operator on X. The operator T is stable if and only if $T \in L_{pb}(X)$ and

$$(1) \qquad \forall n \in \mathbb{N}, \text{ the orbit } \mathcal{O}_T(x_n) \text{ is } \|.\|\text{-relatively compact in } X.$$

Observe that condition (1) is equivalent to

$$\forall n \in \mathbb{N}\ \forall m \in \mathbb{N}^*\ \exists N_{(n,m)} \in \mathbb{N}\ :\ \mathcal{O}_T(x_n) \subseteq \bigcup_{j=0}^{N_{(n,m)}} B(T^j x_n, \frac{1}{m}),$$

or, in other words,

$$\begin{aligned} T \in & \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}^*} \bigcup_{N\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \Big\{T \in L(X):\ T^k x_n \in \bigcup_{j=0}^{N} B(T^j x_n, \frac{1}{m})\Big\} \\ & = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}^*} \bigcup_{N\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{0\le j\le N} \Big\{T \in L(X):\ \big\|T^k x_n - T^j x_n\big\| < \frac{1}{m}\Big\}. \end{aligned}$$

Hence condition (1) is Borel, and so is the set $\mathcal{S}(X)$. □

4. Coanalytic ranks on the set of superstable operators

In this section we focus on the set $\mathcal{S}'(X)$ of superstable operators. To study the complexity of this set, we use several different methods which require many tools from descriptive set theory and some Banach space results. We introduce several coanalytic ranks for the family of superstable operators and exhibit spaces X on which this family is Borel and other spaces on which it is not Borel. Let us recall notations, properties and methods of construction of coanalytic ranks (see [KL] and [Z]).

NOTATIONS. We denote by $\mathcal{P}_f(J)$, $\mathcal{P}_\infty(\mathbb{N})$, $\mathbb{N}^{<\mathbb{N}}$ and $\mathcal{T}$, respectively, the set of finite subsets of J, the set of infinite subsets of $\mathbb{N}$, the set of finite sequences in $\mathbb{N}$, and the set of trees on $\mathbb{N}$. We denote by ω_1 the first uncountable ordinal.

PROPERTIES 4.1. Let δ be a coanalytic rank on a coanalytic subset C of a Polish space P. Then we have:

(a) For all $\alpha < \omega_1,\ B_\alpha := \{x \in C : \delta(x) \leq \alpha\}$ is a Borel set.
(b) If $A \subseteq C$ is an analytic set, then there exists $\alpha < \omega_1$ such that $A \subseteq B_\alpha$.

In particular, C is Borel if and only if δ is bounded on C by a countable ordinal.

Proposition 4.2. *Let δ be a coanalytic rank on the coanalytic subset C of a Polish space P. Let P' be another Polish space, and let $\psi : P' \longrightarrow P$ be a Borel map. Then $\delta \circ \psi$ is a coanalytic rank on $\psi^{-1}(C)$.*

Since the height (denoted by h) of the trees is a coanalytic rank, we deduce from the topology of the set $\mathcal{T}$ of trees on $\mathbb{N}$ the following result.

Proposition 4.3. *Let P be a Polish space and let ψ be a map from P into the set $\mathcal{T}$ of trees on $\mathbb{N}$. If, for every $s \in \mathbb{N}^{<\mathbb{N}}$, the set $\bar{s} = \{x \in P : s \in \psi(x)\}$ is Borel, then $C = \{x \in P : \psi(x)$ is well founded$\}$ is a coanalytic subset with $h \circ \psi$ as a coanalytic rank.*

4.1. Rank derived from the entropy trees. Our next lemma is the key to describing the topological nature of the set of superstable operators for any separable Banach space (see Theorem 4.6).

Lemma 4.4. *Let X be a Banach space, and let $T \in L(X)$. The following assertions are equivalent:*
(i) T is not a superstable operator.
(ii) There exists $\varepsilon > 0$ and $J \in \mathcal{P}_\infty(\mathbb{N})$ such that for all $F \in \mathcal{P}_f(J)$ there exists $x_F \in B_X$ so that the set $\{T^j(x_F) : j \in F\}$ is ε-separated.

Proof. Let $T \in L(X)$. By definition, T is not superstable if and only if there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and $\bar{x} \in X_\mathcal{U}$ with $\|\bar{x}\| \leq 1$ such that the orbit $\mathcal{O}_{T_\mathcal{U}}(\bar{x}) := \{T^n_\mathcal{U} : n \in \mathbb{N}\}$ is not relatively compact. The non-compactness of $\overline{\mathcal{O}_{T_\mathcal{U}}(\bar{x})}$ is equivalent to

$$\exists \varepsilon > 0 :\ \forall I \in \mathcal{P}_f(\mathbb{N})\ \exists n \notin I : T^n_\mathcal{U}(\bar{x}) \notin \bigcup_{i \in I} B\big(T^i_\mathcal{U}(\bar{x}), \varepsilon\big)$$

or

$$\exists \varepsilon > 0 :\ \forall k \in \mathbb{N}\ \exists m_k > k : T^{m_k}_\mathcal{U}(\bar{x}) \notin \bigcup_{i=0}^{k} B\big(T^i_\mathcal{U}(\bar{x}), \varepsilon\big). \tag{2}$$

Using this last assertion, we construct a strictly increasing sequence $\{n_k, k \in \mathbb{N}\}$ as follows: For $k = 0$, we let n_0 be the integer m_0 given by (2), and for $k \geq 1$, we define n_k inductively by setting $n_k = m_{n_{k-1}}$. Thus (with $J = \{n_k, k \in \mathbb{N}\}$), (2) implies the existence of $\varepsilon > 0$ and $J \in \mathcal{P}_\infty(\mathbb{N})$ such that

$$\{T^j_\mathcal{U}(\bar{x}) : j \in J\} \text{ is } \varepsilon\text{-separated.} \tag{3}$$

For any $(x_n)_{n\in\mathbb{N}}$ in the class $\bar{x}$, chosen in the unit ball B_X of X, (3) becomes

$$\forall i,j \in J \text{ with } i \neq j: \quad \mathcal{U} - \lim_n \left\|T^j x_n - T^i x_n\right\| > \varepsilon,$$

which is equivalent to

$$\forall i,j \in J \text{ with } i \neq j\ \exists A \in \mathcal{U}: \quad \forall n \in A,\ \left\|T^j x_n - T^i x_n\right\| > \varepsilon,$$

or

$$\forall F \in J^{<\mathbb{N}} \exists A_F \in \mathcal{U}: \quad \forall n \in A_F, \{T^j(x_n) : j \in F\} \text{ is } \varepsilon\text{-separated.}$$

In particular,

$$\forall F \in J^{<\mathbb{N}},\ \exists x_F \in B_X: \quad \{T^j(x_F) : j \in F\} \text{ is } \varepsilon\text{-separated.}$$

For the proof of the converse, let $\varepsilon > 0$ and $J \in \mathcal{P}_\infty(\mathbb{N})$ such that

$$\forall F \in \mathcal{P}_f(J)\ \exists x_F \in B_X: \quad \{T^j(x_F) : j \in F\} \text{ is } 2\varepsilon\text{-separated.}$$

We put $J = \{j_i : i \in \mathbb{N}\}$ (where the j_i are in strictly increasing order) and

$$F_n = \{j_0, j_1, ..., j_n\}, \quad E_n = J \setminus F_n.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing the subsets E_n. We have

$$\forall n \in \mathbb{N}\ \exists x_n \in B_X: \quad \{T^j(x_n) : j \in F_n\} \text{ is } 2\varepsilon\text{-separated.}$$

Let $\bar{x}$ be the class of $(x_n)_{n\in\mathbb{N}}$ in $X_{\mathcal{U}}$. We claim that the set $\{T^j_{\mathcal{U}}(\bar{x}) : j \in J\}$ is ε-separated (so that T is not superstable).

Let $j_p, j_q \in J$ with $p < q$. Then $j_p, j_q \in F_n$ for all $n \geq q$. It follows from the definition of x_n that

$$\left\|T^{j_p} x_n - T^{j_q} x_n\right\| > 2\varepsilon, \quad \forall n \geq q.$$

Hence, $\mathcal{U} - \lim_n \left\|T^{j_p} x_n - T^{j_q} x_n\right\| > \varepsilon$, i.e., $\left\|T^{j_p}_{\mathcal{U}} \bar{x} - T^{j_q}_{\mathcal{U}} \bar{x}\right\| > \varepsilon$. □

For an operator T and any real number $\varepsilon > 0$, we consider the "entropy tree"

$$\mathcal{A}(T,\varepsilon) = \Big\{F \in \mathcal{P}_f(\mathbb{N}) : |F| \leq 1 \text{ or } \exists\, x \in B_X : \{T^j x : j \in F\} \text{ is } \varepsilon\text{-separated}\Big\}.$$

It follows from Lemma 4.4 that an operator T is not superstable if and only if there exists an $\varepsilon > 0$ such that the tree $\mathcal{A}(T,\varepsilon)$ has an infinite branch J or, equivalently, that the tree $\mathcal{A}(T,\varepsilon)$ is not well founded.

Proposition 4.5. *Let X be a Banach space and let T be a bounded operator on X. The following assertions are equivalent:*

(a) *T is superstable.*
(b) *For all $\varepsilon > 0$, the entropy tree $\mathcal{A}(T,\varepsilon)$ is well founded.*
(c) *We have $\eta(T) := \sup_{\varepsilon>0} h\big(\mathcal{A}(T,\varepsilon)\big) < \omega_1$.*

Proof. It only remains to check the equivalence of (b) and (c). However, this is clear since (b) is equivalent to the assertion that for all $n \in \mathbb{N}^*$, $\mathcal{A}(T, 1/n)$ is well founded. □

We now extend the index η, defined above on $L(X)$ by $\eta(T) = \omega_1$, to the case when T is not superstable.

THEOREM 4.6. *Let X be a separable Banach space. We consider $L(X)$ equipped with the strong operator topology S_{op}. Let η be the index on $L(X)$ defined above. Then we have:*

(a) *The set $\mathcal{S}'(X)$ of superstable operators on X is a coanalytic subset.*
(b) *η is a coanalytic rank on $\mathcal{S}'(X)$.*
(c) *$T \in \mathcal{S}'(X)$ if and only if $\eta(T) < \omega_1$.*
(d) *$\mathcal{S}'(X)$ is a Borel subset of $\big(L(X), S_{op}\big)$ if and only if $\eta(X) := \sup_{T\in\mathcal{S}'(X)} \eta(T) < \omega_1$.*

Proof. Assertion (c) is part of Proposition 4.5. Let T be a bounded operator on X. We have

$$\eta(T) = \sup_{n\in\mathbb{N}^*} h\big(\mathcal{A}(T, \frac{1}{n})\big).$$

We define the natural tree $\mathcal{A}(T)$ on $\mathbb{N}$ containing all the trees $\mathcal{A}(T, \frac{1}{n})$, $n \in \mathbb{N}^*$, as the set of all $\sigma \in \mathcal{P}_f(\mathbb{N})$ such that

$$|\sigma| = 0 \text{ or } \ \sigma = (p, F) \text{ with } p \in \mathbb{N}^* \text{ and } F \in \mathcal{A}(T, 1/p).$$

It is not difficult to show that the map $\mathcal{A} : T \in L_{pb}(X) \longmapsto \mathcal{A}(T)$ satisfies the assumptions of Proposition 4.3. Then the set

$$\begin{aligned} C :=& \big\{T \in L_{pb}(X) : \mathcal{A}(T) \text{ is well founded}\big\} \\ =& \Big\{T \in L_{pb}(X) : \mathcal{A}(T, \frac{1}{n}) \text{ is well founded for all } n \in \mathbb{N}^*\Big\} \end{aligned}$$

is S_{op}–coanalytic in $L_{pb}(X)$, and has $h\circ\mathcal{A}$ as a coanalytic rank, where $h\circ\mathcal{A}(T)$ is the height of the tree $\mathcal{A}(T)$. It follows from the Proposition 4.5 that

$$C = \mathcal{S}'(X) \cap L_{pb}(X) = \mathcal{S}'(X).$$

Hence the set $\mathcal{S}'(X)$ of superstable operators on X is S_{op}–coanalytic in $L_{pb}(X)$, and therefore also in $L(X)$, since $L_{pb}(X)$ is a S_{op}-Borel subset of $L(X)$. The index η is a coanalytic rank on $\mathcal{S}'(X)$ since $\eta = h \circ \mathcal{A}$. Assertion (d) follows immediately from Properties 4.1 of the coanalytic ranks. □

One cannot decide whether the set $\mathcal{S}'$ of superstable operators is a true coanalytic set or only a Borel set. In the sequel we give natural classes of Banach spaces where $\mathcal{S}'(X)$ is a true coanalytic set, and a class where this set is Borel.

4.2. Rank derived from the Cantor derivation. We now apply the characterization of a superstable operator in spectral terms (see Section 1). Consider the natural map

$$\begin{array}{rcl} \sigma_1 : L(X) & \longrightarrow & \mathcal{K}(\mathbb{T}) \\ T & \longmapsto & \sigma_1(T), \end{array}$$

which associates to an operator its unitary spectrum, where $\mathcal{K}(\mathbb{T})$ denotes the set of compact subsets of the torus $\mathbb{T}$. It is well known that $\mathcal{K}(\mathbb{T})$, endowed with the Hausdorff topology, is a compact metric space, where the Borel structure is generated by the family

$$\Big\{\{K \in \mathcal{K}(\mathbb{T}) : K \cap V \neq \emptyset\} : \ V \text{ open in } \mathbb{T}\Big\}.$$

We first show that this map is rather regular.

Proposition 4.7. *For any separable Banach space X, the map $\sigma_1 : L(X) \longrightarrow \mathcal{K}(\mathbb{T})$, which to an operator associates its unit spectrum, is Borel when $L(X)$ is equipped with the strong operator topology.*

Proof. Since $\mathcal{K}(\mathbb{T})$ is endowed with the Hausdorff topology, it is enough to show that, for all open V in the torus $\mathbb{T}$, the subset $E_V = \{T \in L(X) : \sigma_1(T) \cap V \neq \emptyset\}$ is Borel in $\big(L(X), S_{op}\big)$. Since $V \subset \mathbb{T}$, we have

$$E_V = \big\{T \in L(X) : \ \sigma(T) \cap V \neq \emptyset\big\} = P_{L(X)}(\Omega),$$

where $\Omega = \big\{(T, \lambda) \in L(X) \times V : \ \lambda \in \sigma(T)\big\}$ and $P_{L(X)}$ denotes the canonical projection from $L(X) \times \mathbb{T}$ to $L(X)$. By [SR], E_V is Borel if Ω is a Borel set with K_σ sections. For $T \in L(X)$, the vertical section of the set $\Omega \subseteq L(X) \times \mathbb{T}$ over T is

$$\Omega(T) = \big\{\lambda \in \mathbb{T} : \ (T, \lambda) \in \Omega\big\} = \big\{\lambda \in \mathbb{T} : \ \lambda \in V \cap \sigma(T)\big\} = \sigma(T) \cap V.$$

Thus, $\Omega(T)$ is a K_σ subset of $\mathbb{T}$. To prove that Ω is a Borel set, we consider

$$\Delta = \big\{(T, \lambda) \in L(X) \times \mathbb{T} : \ \lambda \in \sigma(T)\big\}.$$

Since $\Omega = \Delta \ \cap \ L(X) \times V$, to complete the proof it is enough to show that Δ is a Borel subset of $L(X) \times \mathbb{T}$. We have $\Delta = A \cup B$ with

$$A = \big\{(T, \lambda) \in L(X) \times \mathbb{T} : \ T - \lambda I \ \text{ is not an isomorphism onto its range}\big\},$$
$$B = \big\{(T, \lambda) \in L(X) \times \mathbb{T} : \ (T - \lambda I)(X) \text{ is not dense in } X\big\}.$$

Indeed, if $T - \lambda I$ is an isomorphism onto its range, then $(T - \lambda I)(X)$ is a closed subspace and necessarily strict, since $\lambda \in \sigma(T)$. We now show that A and B are Borel sets.

Since X is separable, there exists a countable and dense subset $\mathcal{D}$ of the sphere S_X of X, and there exists a dense sequence $\{x_n\}_{n \in \mathbb{N}}$ in X. An element

$(T,\lambda) \in L(X) \times \mathbb{T}$ is in A if and only if

$$\exists (z_n)_{n\in\mathbb{N}} \subseteq S_X : \quad \lim_{n\to\infty} \|(T-\lambda I)z_n\| = 0,$$

that is to say

$$\exists (z_n)_{n\in\mathbb{N}} \subseteq \mathcal{D},\ \forall k \geq 1\ \exists N_k \in \mathbb{N}\ \forall n \geq N_k : \quad \|(T-\lambda I)z_n\| < \frac{1}{k}.$$

By taking the sequence $(z_{N_k})_{k\in\mathbb{N}}$, this, in turn, is equivalent to

$$\exists (z_n)_{n\in\mathbb{N}} \subseteq \mathcal{D},\ \forall k \geq 1\ \exists N_k \in \mathbb{N}\ : \quad \|(T-\lambda I)z_{N_k}\| < \frac{1}{k},$$

or simply

$$\forall k \geq 1,\ \exists x \in \mathcal{D} : \quad \|Tx - \lambda x\| < \frac{1}{k}.$$

Hence

$$A = \bigcap_{k\geq 1} \bigcup_{x\in\mathcal{D}} \left\{(T,\lambda) \in L(X) \times \mathbb{T} : \ \|Tx - \lambda x\| < \frac{1}{k}\right\}.$$

The set A is thus a G_δ set for the strong operator topology. We have

$$\begin{aligned}
B &= \Big\{(T,\lambda) \in L(X) \times \mathbb{T} /\ (T-\lambda I)(X) \text{ is not dense in } X\Big\} \\
&= \bigcup_{y\in S_X} \bigcup_{k\in\mathbb{N}^*} \Big\{(T,\lambda) \in L(X) \times \mathbb{T} : \ \|y - (T-\lambda I)x\| \geq \frac{1}{k} \ \text{ for all } \ x \in X\Big\} \\
&= \bigcup_{y\in \mathcal{D}} \bigcup_{k\in\mathbb{N}^*} \Big\{(T,\lambda) \in L(X) \times \mathbb{T} : \ \|y - (T-\lambda I)x_n\| \geq \frac{1}{k} \ \text{ for all } \ n \in \mathbb{N}\Big\} \\
&= \bigcup_{y\in \mathcal{D}} \bigcup_{k\in\mathbb{N}^*} \bigcap_{n\in\mathbb{N}} \Big\{(T,\lambda) \in L(X) \times \mathbb{T} : \ \|y - (T-\lambda I)x_n\| \geq \frac{1}{k}\Big\}.
\end{aligned}$$

Hence B is an F_σ set for the strong operator topology. □

Consider now the Cantor derivation on $\mathcal{K}(\mathbb{T})$ which, to each $K \in \mathcal{K}(\mathbb{T})$ associates the set $K' := K \setminus \{\text{isolated points of } K\}$. By transfinite induction the derivative $K^{(\alpha)}$ is well-defined for every ordinal α. We then consider the ordinal index δ_c on $\mathcal{K}(\mathbb{T})$ defined from the Cantor derivation by

$$\delta_c(K) = \begin{cases} \inf\{\alpha \text{ ordinal } : K^{(\alpha)} = \emptyset\} & \text{if the infimum exists,} \\ \omega_1 & \text{otherwise.} \end{cases}$$

It is well known that δ_c is a coanalytic rank on the true coanalytic set $\mathcal{D}(\mathbb{T})$ of countable compact subsets of $\mathbb{T}$ (see [Ke]).

Proposition 4.8. *Let X be a separable and superreflexive Banach space. Consider on $L(X)$ the index δ derived from the Cantor derivation defined by*

$$\delta(T) := \delta_c[\sigma_1(T)] = \begin{cases} \inf\{\alpha \textit{ ordinal } : [\sigma_1(T)]^{(\alpha)} = \emptyset\} & \textit{if the infimum exists,} \\ \omega_1 & \textit{otherwise.} \end{cases}$$

If $L(X)$ is equipped with the strong operator topology, then we have:

(a) *The set $\mathcal{S}'(X)$ of superstable operators is coanalytic.*
(b) *δ is a coanalytic rank on $\mathcal{S}'(X)$.*
(c) *T is superstable if and only if $\delta(T) < \omega_1$ and $T \in L_{pb}(X)$.*
(d) *$\mathcal{S}'(X)$ is Borel set if and only if $\delta(X) := \sup_{T \in \mathcal{S}'(X)} \delta(T) < \omega_1$.*

Proof. Consider the Borel map σ_1 defined in Proposition 4.7. Denote by $\widetilde{\sigma_1}$ the restriction of σ_1 to the set $L_{pb}(X)$ of power bounded operators on X. The map $\widetilde{\sigma_1}$ is a Borel map and $\big(L_{pb}(X), S_{op}\big)$ is a standard Borel space, since $L_{pb}(X)$ is a S_{op}-Borel subset of $L(X)$. Since δ_c is a coanalytic rank on $\mathcal{D}(X)$, it follows from Proposition 4.2 that $\delta_c \circ \widetilde{\sigma_1}$ is a coanalytic rank on the coanalytic set $\widetilde{\sigma_1}^{-1}\big(\mathcal{D}(\mathbb{T})\big)$. Now observe that on $L_{pb}(X)$, $\delta_c \circ \widetilde{\sigma_1} = \delta$ and that $\widetilde{\Psi}^{-1}\big(\mathcal{D}(\mathbb{T})\big) = \mathcal{S}'(X)$, since every power bounded operator on X is superstable if and only if its unit spectrum is countable (see Theorem 1.2). This proves assertions (a) and (b). Assertions (c) and (d) follow from classic properties of the coanalytic rank (see Properties 4.1). □

The next corollary follows from Proposition 4.8, Properties 4.1, and Proposition 3.3.

Corollary 4.9. *Let X be a separable and superreflexive Banach space. Then either there exists a stable but not superstable operator on X, or there is an ordinal $\alpha < \omega_1$ such that $[\sigma_1(T)]^{(\alpha)} = \emptyset$ for every stable operator T.*

Theorem 4.6 and Proposition 4.8 imply, in particular, that for a separable and superreflexive Banach space there is an equivalence between $\delta(T) < \omega_1$ and $\eta(T) < \omega_1$, for every superstable operator T on X. In fact, more is known about the relation between these two ranks (see [KL] or [Z]).

Corollary 4.10. *For a separable and superreflexive Banach space X, there exist two maps Γ_1 and Γ_2 on the set ω_1 of countable ordinals such that, for every superstable operator T on X,*

$$\eta(T) \leq \alpha \Longrightarrow \delta(T) \leq \Gamma_1(\alpha),$$
$$\delta(T) \leq \alpha \Longrightarrow \eta(T) \leq \Gamma_2(\alpha).$$

4.3. Using the classical entropy index. Consider now the well-known entropy index ρ which characterizes relative compactness. Given a Banach space X, a subset E of X and $\varepsilon > 0$, we have

$$\rho(E,\varepsilon) := \inf\Big\{n \in \mathbb{N} : \ \exists (x_i)_{1\leq i\leq n} \subset E \text{ such that } E \subseteq \bigcup_{i=1}^{n} B(x_i,\varepsilon)\Big\}.$$

Hence an operator T on X is stable if and only if

$$\forall \varepsilon > 0, \forall x \in B_X \ : \ \rho(\mathcal{O}_T(x),\varepsilon) < +\infty.$$

Consider then the following uniform version of this condition:

$$\forall \varepsilon > 0, \quad \rho_\varepsilon(T) := \sup_{x \in B_X} \rho\big(\mathcal{O}_T(x), \varepsilon\big) < +\infty.$$

PROPOSITION 4.11. *Let X be a separable Banach space. If T is a bounded operator on X such that*

$$\forall \varepsilon > 0, \quad \rho_\varepsilon(T) < +\infty, \tag{4}$$

then T is superstable. Moreover, the set of operators with the above condition is S_{op}-Borel in $L(X)$.

Proof. Let T be a non-superstable operator on X. Then, by Lemma 4.4, there exists $\varepsilon > 0$ and $J \in \mathcal{P}_\infty(\mathbb{N})$ such that, for all finite subsets F of J, the set $\{T^j(x_F) : j \in F\}$ is ε-separated for a certain $x_F \in B_X$. In particular,

$$\exists \varepsilon > 0 \ \exists J \in \mathcal{P}_\infty(\mathbb{N}) \text{ such that } \ \forall F \in \mathcal{P}_f(J) : \ \rho_\varepsilon(T) \geq |F|.$$

Since F is arbitrarily large, it follows that with ε as above, $\rho_\varepsilon(T) = +\infty$.

To prove the second assertion, take a dense sequence $\{x_n\}_{n \in \mathbb{N}}$ in B_X. Observe that the set of operators satisfying (4) coincides with the set of power bounded operators T such that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall n \in \mathbb{N}, \ \rho\big(\mathcal{O}_T(x_n), \varepsilon\big) < N.$$

This is a Borel condition. Indeed, $\rho\big(\mathcal{O}_T(x_n), \varepsilon\big) < N$ is equivalent to the existence of a subset J in $\mathbb{N}$ with $|J| < N$ such that

$$\mathcal{O}_T(x_n) \subseteq \bigcup_{j \in J} B(T^j x_n, \varepsilon). \tag{5}$$

Condition (5) is S_{op}-Borel since it says that for all $k \in \mathbb{N}$ there exists $j \in J$ such that $\left\|T^k x_n - T^j x_n\right\| < \varepsilon$, and since $T \in L_{pb}(X)$. □

REMARKS. The set of operators satisfying (4) is, in general, different from the set of superstable operators, since the latter set can be non-Borel (see the next section). From Theorem 4.6 and Propositions 4.8 and 4.11 we find:

- For a separable Banach space X and $T \in L_{pb}(X)$,

$$\rho(T, \varepsilon) < +\infty, \ \forall \varepsilon > 0 \implies \eta(T) < \omega_1.$$

- If, moreover, X is superreflexive, then

$$\rho(T, \varepsilon) < +\infty, \ \forall \varepsilon > 0 \implies \delta(T) < \omega_1.$$

5. Spaces where $\mathcal{S}'$ is a non-Borel set

We now know that the set $\mathcal{S}'$ of superstable operators is S_{op}–coanalytic and the Borel character of this set depends on the coanalytic ranks introduced previously. In this section, we will exhibit classes of Banach spaces where $\mathcal{S}'$ is not Borel, and hence different from the set $\mathcal{S}$, and such that the introduced indices δ and η are arbitrarily large.

5.1. Superreflexive spaces with unconditional basis.

THEOREM 5.1. *Let X be a superreflexive Banach space with an unconditional basis. Then the set of superstable operators is a coanalytic non-Borel subset of $\big(L(X), S_{op}\big)$.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an unconditional basis of X. Since $|||\sum a_n e_n||| := \sup_{|\varepsilon_n|=1} \|\sum a_n \varepsilon_n e_n\|$ yields an equivalent norm on X (see [LT]), we may assume that $\{e_n\}_{n\in\mathbb{N}}$ is 1-unconditional. Let us take again the S_{op}-Borel map (see Proposition 4.7)

$$\begin{array}{rcl} \widetilde{\sigma_1} : L_{pb}(X) & \longrightarrow & \mathcal{K}(\mathbb{T}) \\ T & \longmapsto & \sigma_1(T). \end{array}$$

Since X is superreflexive (and separable), it follows from Theorem 1.2 that $\widetilde{\sigma_1}\big(\mathcal{S}'(X)\big) \subseteq \mathcal{D}(\mathbb{T})$. We now show that $\widetilde{\sigma_1}\big(\mathcal{S}'(X)\big) = \mathcal{D}(\mathbb{T})$.

Let $K = \{\lambda_n : n \in \mathbb{N}\}$ be a countable and compact subset of $\mathbb{T}$. Consider the operator T on X defined by $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$. Then T is a power bounded operator since $\|T\| \leq 1$ (see [LT]), and it is clear that $K \subseteq \sigma(T) \cap \mathbb{T}$. Conversely, consider $\lambda \in \mathbb{T} \setminus K$ and let $\varepsilon = \mathrm{dist}(\lambda, K) > 0$. For $n \in \mathbb{N}$ define

$$Re_n := \frac{1}{(\lambda - \lambda_n)} e_n.$$

Since $\{e_n\}_{n\in\mathbb{N}}$ is an unconditional basis and the set $\{1/(\lambda - \lambda_n) : n \in \mathbb{N}\}$ is bounded by $1/\varepsilon$, R is a bounded operator on X (see [LT]). Moreover,

$$R(\lambda I - T) = (\lambda I - T)R = I.$$

Thus, $\lambda \notin \sigma_1(T)$, and therefore $K = \sigma_1(T) = \widetilde{\sigma_1}(T)$. Hence $\widetilde{\sigma_1}\big(\mathcal{S}'(X)\big) = \mathcal{D}(\mathbb{T})$.

On the other hand, it is well known that the set $\mathcal{D}(\mathbb{T})$ is coanalytic non-Borel, and hence, by Lustin's separation theorem, not analytic. It follows then that $\mathcal{S}'(X)$ is a non-Borel set, since the image by a Borel map of a Borel set is analytic. □

COROLLARY 5.2. *Let X be a separable and superreflexive Banach space having a complemented subspace with unconditional basis. Then the set of superstable operators is coanalytic non-Borel in $\big(L(X), S_{op}\big)$.*

Proof. In view of the above proof, we only need to show that, for any $K \in \mathcal{D}(\mathbb{T})$, there exists $\widetilde{T} \in L(X)$ such that $K = \widetilde{\sigma_1}(\widetilde{T})$. We put $X = Y \oplus Z$, where Y is a complemented subspace with unconditional basis. By the proof of Theorem 5.1, there exists $T \in L(Y)$ such that $\sigma_1(T) = K$. It is clear that the operator $\widetilde{T} := T \oplus \frac{1}{2} Id_Z$ on X has the desired property. □

5.2. Spaces with polynomially bounded operators. Polynomially bounded operators might exhibit more properties than simple power bounded operators. In particular, we will use such operators to give an interesting category of Banach spaces in which the set of superstable operators is non-Borel. We first recall some definitions and establish some preliminary results.

DEFINITION 5.3. An operator T on a Banach space is polynomially bounded whenever there exists a constant C such that, for every polynomial P, we have

$$\|P(T)\| \leq C \, \|P\|_\infty \, ,$$

where $\|P\|_\infty = \sup\{|P(z)| : z \in \mathbb{T}\}$.

In particular, any contraction on a Hilbert space has this property. Let $A[D]$ be the disc algebra, i.e., the space of functions that are holomorphic on the unit open disc $D \setminus \mathbb{T}$ and continuous on D, equipped with the norm

$$\|f\|_\infty := \sup_{|z|\leq 1} |f(z)| = \sup_{|z|=1} |f(z)| \, .$$

Since the polynomials are $\|.\|_\infty$–dense in $A[D]$, the operator $f(T)$ is well defined for every C-polynomially bounded operator T on a Banach space and for every $f \in A[D]$, and we have $\|f(T)\|_{L(X)} \leq C \, \|f\|_\infty$.

The following lemma extends the classical spectral theorem (for holomorphic functions) to all functions in the disc algebra.

LEMMA 5.4. *Let T be a polynomially bounded operator on a Banach space. Then, for every function f in the disc algebra, we have*

$$\sigma\big(f(T)\big) = f\big(\sigma(T)\big).$$

Proof. Note first that the spectral radius of T is at most 1 since T is, in particular, power bounded. Hence the spectrum $\sigma(T)$ is contained in D. Fix $f \in A[D]$ and, for $r \in [0,1[$, put $f_r(z) = f(rz)$. Then f_r is a holomorphic function on the neighborhood of D, and so on the neighborhood of $\sigma(T)$. It follows from the spectral theorem that

$$\sigma\big(f_r(T)\big) = f_r\big(\sigma(T)\big). \tag{6}$$

Note also that

$$\lim_{r\to 1} \|f_r - f\|_\infty = 0 \text{ and } \lim_{r\to 1} \|f_r(T) - f(T)\| = 0, \tag{7}$$

since f is uniformly continuous on the compact set D, and since T is a C-polynomially bounded operator.

Consider the space $\mathcal{K}(\mathbb{C})$ of compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance d; i.e., for any compact sets K_1 and K_2 in $\mathbb{C}$ we have

$$d\big(K_1, K_2\big) = \max\Big\{ \sup_{z\in K_2} \operatorname{dist}(z, K_1) \; ; \; \sup_{z\in K_1} \operatorname{dist}(z, K_2)\Big\}.$$

In particular,

$$d\Big(f\big(\sigma(T)\big), f_r\big(\sigma(T)\big)\Big) \leq \max_{\lambda\in\sigma(T)} |f(\lambda) - f_r(\lambda)| \leq \|f - f_r\|_\infty .$$

Hence in the space $\big(\mathcal{K}(\mathbb{C}), d\big)$,

$$\lim_{r\to 1} f_r\big(\sigma(T)\big) = f\big(\sigma(T)\big). \tag{8}$$

On the other hand, we have

$$\lim_{r\to 1} \sigma\big(f_r(T)\big) = \sigma\big(f(T)\big). \tag{9}$$

Indeed, this follows from (7) and from Aupetit's theorem (see [Au, p. 48]) on the continuity in an abelian subalgebra of the map that associates to an operator its spectrum, by considering the abelian subalgebra $A_T := \{g(T) : g \in A[D]\}$ of $L(X)$.

The lemma follows immediately from (6), (8) and (9). □

Before stating the main theorem of this section, we recall the following theorem of Fatou (see [Ho, p. 80]).

Theorem 5.5. *Let K be a compact subset of the torus $\mathbb{T}$ with Lebesgue measure zero. Then there exists a function φ in the disc algebra $A[D]$ such that*

$$K = \{z \in D : \varphi(z) = 0\},$$
$$\Re\varphi(z) < 0, \quad \forall z \in D \setminus K.$$

Theorem 5.6. *Let X be a separable and superreflexive Banach space. If there exists a polynomially bounded operator on X which is not superstable, then the set $\mathcal{S}'(X)$ of superstable operators on X is a coanalytic non-Borel subset of $\big(L(X), S_{op}\big)$.*

Proof. Let T be a C-polynomially bounded and non-superstable operator on X. In particular, the spectrum $\sigma_1(T)$ is compact metrizable and uncountable (see Theorem 1.2). Since $\sigma_1(T)$ contains a copy of the Cantor set, it follows that $\sigma_1(T)$ contains scattered compact subsets with arbitrarily large index (derived from the Cantor derivation) (see [Se, §.8]), i.e.,

$$\forall\alpha < \omega_1, \exists F_\alpha \subseteq \sigma_1(T) \cap \mathcal{D}(\mathbb{T}) : F_\alpha^{(\alpha)} \neq \emptyset.$$

Then Fatou's theorem (Theorem 5.5) implies that

$$\forall\alpha < \omega_1,\ \exists\varphi_\alpha \in A[D] : \begin{cases} F_\alpha = \{z \in D : \varphi_\alpha(z) = 0\}, \\ \Re\varphi_\alpha(z) < 0, \quad \forall z \in D \setminus K. \end{cases}$$

If we set $f_\alpha(z) = z \exp\big(\varphi_\alpha(z)\big)$ for every $z \in D$, then f_α belongs to the disc algebra and

$$f_\alpha(z) = z, \quad \forall z \in F_\alpha,$$
$$|f_\alpha(z)| < |z|, \quad \forall z \in D \setminus F_\alpha.$$

Consider $T_\alpha = f_\alpha(T) \in L(X)$. The operator T_α is also C-polynomially bounded. Indeed, since $f_\alpha(D) \subseteq D$ the function f_α^n is in the disc algebra for every $n \in \mathbb{N}$, and hence T_α^n is well defined with

$$\|T_\alpha^n\| = \|f_\alpha^n(T)\| \le C \cdot \|f_\alpha^n\|_\infty \le C.$$

Moreover, $\sigma_1(T_\alpha) = F_\alpha$ for every countable ordinal α. Indeed, by the definitions of F_α and f_α we have

$$\begin{aligned} F_\alpha &= f_\alpha\big(F_\alpha\big) \subseteq f_\alpha\big(\sigma(T)\big), \\ F_\alpha &\subseteq \mathbb{T}, \\ f_\alpha(D) \cap \mathbb{T} &= F_\alpha, \\ \sigma(T) &\subseteq D. \end{aligned}$$

It follows that $f_\alpha(\sigma(T)) \cap \mathbb{T} = F_\alpha$. Since $\sigma\big(T_\alpha\big) = \sigma\big(f_\alpha(T)\big) = f_\alpha\big(\sigma(T)\big)$ (see Lemma 5.4), we obtain

$$\sigma_1(T_\alpha) = \sigma(T_\alpha) \cap \mathbb{T} = f_\alpha(\sigma(T)) \cap \mathbb{T} = F_\alpha \cap \mathbb{T} = F_\alpha.$$

The properties of the sets F_α imply

$$\forall \alpha < \omega_1, \quad \begin{cases} \sigma_1(T_\alpha) \text{ is countable,} \\ [\sigma_1(T_\alpha)]^{(\alpha)} \neq \emptyset. \end{cases}$$

Since X is superreflexive and T_α is a power bounded operator, it follows from Theorem 1.2 and the definition of the coanalytic rank δ (see Proposition 4.8) that

$$\forall \alpha < \omega_1, \quad \begin{cases} T_\alpha \text{ is superstable,} \\ \delta(T_\alpha) \ge \alpha. \end{cases}$$

Hence,

$$\delta(X) := \sup_{R \in \mathcal{S}'(X)} \delta(R) \ge \sup_{\alpha < \omega_1} \delta(T_\alpha) = \omega_1.$$

In view of Proposition 4.8, this proves that the set $\mathcal{S}'(X)$ of superstable operators on X is non-Borel in $\big(L(X), S_{op}\big)$. □

Remarks. Corollary 5.2 is a particular case of Theorem 5.6, since every superreflexive Banach space X having a complemented subspace with unconditional basis admits a polynomially bounded and non-superstable operator.

Indeed, let $X = Y \oplus Z$, where Y is a subspace having an unconditional basis $\{e_k\}_{k\in\mathbb{N}}$. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence in the torus $\mathbb{T}$ such that the closure

$\overline{\{\lambda_k : k \in \mathbb{N}\}}$ is not countable. Consider the multiplication operator M on Y defined by

$$M\big(\sum_{k\in\mathbb{N}} a_k e_k\big) = \sum_{k\in\mathbb{N}} \lambda_k a_k e_k$$

for all $\sum_{k\in\mathbb{N}} a_k e_k \in Y$. For any polynomial P, it is clear that

$$P(M)\big(\sum_{k\in\mathbb{N}} a_k e_k\big) = \sum_{k\in\mathbb{N}} P(\lambda_k) a_k e_k.$$

for all $\sum_{k\in\mathbb{N}} a_k e_k \ \in Y$. Since $\{e_k\}_{k\in\mathbb{N}}$ is an unconditional basis, it follows that

$$\begin{aligned} \left\| P(M)\big(\sum_{k\in\mathbb{N}} a_k e_k\big)\right\| &= \left\|\sum_{k\in\mathbb{N}} P(\lambda_k) a_k e_k\right\| \\ &= \left\|\sum_{k\in\mathbb{N}} |P(\lambda_k) a_k|\, e_k\right\| \\ &\le \|P\|_\infty \left\|\sum_{k\in\mathbb{N}} |a_k|\, e_k\right\| \\ &= \|P\|_\infty \left\|\sum_{k\in\mathbb{N}} a_k e_k\right\|. \end{aligned}$$

Hence $\|P(M)\| \le \|P\|_\infty$, i.e., M is a polynomially bounded operator. Moreover, it is not difficult to check that $\sigma_1(M) = \sigma(M) = \overline{\{\lambda_k : k \in \mathbb{N}\}}$, and that this set is not countable. Hence M is not superstable. It is easy to check that $T := M \oplus \frac{1}{2} Id_Z$ is a polynomially bounded operator on X and that it is not superstable.

6. The case of hereditarily indecomposable spaces

We now exhibit a class of Banach spaces for which the set of superstable operators is Borel. We take a family of Banach spaces that is at the opposite end of the family of spaces having complemented subspaces with an unconditional basis, namely the family of hereditarily indecomposable Banach spaces, introduced by Gowers and Maurey (see [GM]).

Definition 6.1. A Banach space is hereditarily indecomposable ($H.I.$), if it does not have a decomposable subspace. A Banach space is decomposable if it can be written as a topological direct sum of two infinite-dimensional subspaces.

We have the following important theorem (see [GM]).

Theorem 6.2. *If X is a complex H.I. Banach space then every bounded operator T on X can be written as $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is a* *strictly singular operator (as defined in [GM]). Moreover, the spectrum of T is finite or consists of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to λ.*

The following result is a consequence of Theorem 6.2 and Theorem 1.2.

Proposition 6.3. *Let X be a complex $H.I.$ Banach space, and let T be a bounded operator on X. If X is superreflexive, then the following assertions are equivalent:*

(a) *T is power bounded.*
(b) *T is stable.*
(c) *T is superstable.*

The existence of such $H.I.$ and superreflexive Banach spaces is shown in [Fe]. The following corollary is obvious.

Corollary 6.4. *Let X be a complex $H.I.$ and superreflexive Banach space. Then the set $\mathcal{S}'(X)$ of superstable operators on X is S_{op}-Borel in $L(X)$.*

Spaces of this type are examples of Banach spaces in which the heights of the entropy trees of superstable operators are uniformly bounded by a countable ordinal. It is an open question whether this holds for an arbitrary H.I. Banach space.

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