Illinois Journal of Mathematics
Volume 47, Number 4, Winter 2003, Pages 1115–1136
S 0019-2082

APPROXIMATION ON THE BOUNDARY AND SETS OF DETERMINATION FOR HARMONIC FUNCTIONS

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ABSTRACT. Let E be a subset of a domain Ω in Euclidean space. This paper deals with the representation, or approximation, of functions on the boundary of Ω by sums of Poisson, Green or Martin kernels associated with the set E, and with the related issue of whether E can be used to determine the suprema of certain harmonic functions on Ω. The results address several questions raised by Hayman.

1. Main results

Let $\mathcal{A}$ be a collection of harmonic functions on the open unit disc D, and let $E \subseteq D$. We call E a *set of determination* for $\mathcal{A}$ if $\sup_E h = \sup_D h$ for all h in $\mathcal{A}$. Bonsall [6] and Hayman and Lyons [14], respectively, have established Theorems A and B below connecting this notion with the representation of appropriate functions on ∂D in terms of the Poisson kernel

$$P(x,y) = \frac{1}{2\pi}\frac{1-\|x\|^2}{\|x-y\|^2} \quad (x \in D;\ y \in \partial D).$$

Let $h^\infty(\Omega)$ denote the collection of bounded harmonic functions on an open set Ω in $\mathbb{R}^n$ $(n \geq 2)$, and $h^1(\Omega)$ the collection of functions of the form $h_1 - h_2$, where h_1 and h_2 are positive and harmonic on Ω. We consider only spaces of real-valued functions in this paper, and use $C^+(K)$ to denote the collection of all (strictly) positive continuous functions on a compact set K. The assertion "$f = \sum f_k$ in L^1" will mean that the series converges to f in the L^1 norm.

THEOREM A. *Let $E \subseteq D$. The following are equivalent:*

(a) *For each $f \in L^1(\partial D)$ and $\varepsilon > 0$ there exist sequences (λ_k) in $\mathbb{R}$ and (x_k) in E such that*

$$f = \sum \lambda_k P(x_k, \cdot) \text{ in } L^1(\partial D) \quad \text{and} \quad \sum |\lambda_k| < \|f\|_{L^1(\partial D)} + \varepsilon. \tag{1}$$

(b) $\sup_E h = \sup_D h$ *whenever* $h \in h^\infty(D)$.

Received October 24, 2002.
2000 *Mathematics Subject Classification.* Primary 31B05.

(c) *Almost every point of ∂D is the nontangential limit of some sequence in E.*

We write $B(x,r)$ for the open ball of centre x and radius r in $\mathbb{R}^n$ $(n \geq 2)$.

THEOREM B. *Let $E \subseteq D$. The following are equivalent:*

(a) *For each $f \in C^+(\partial D)$ there exist sequences (λ_k) in $[0,+\infty)$ and (x_k) in E such that*

$$f(y) = \sum \lambda_k P(x_k, y) \quad (y \in \partial D). \tag{2}$$

(b) $\sup_E h = \sup_D h$ *whenever $h \in h^1(D)$.*

(c) $\int_{E^{\partial D}} \|x-y\|^{-2}\, dx = +\infty$ *for each $y \in \partial D$, where*

$$E^{\partial D} = \bigcup_{x \in E} B\left(x, \frac{1-\|x\|}{2}\right).$$

Hayman [13] subsequently raised the following questions.

(I) Condition (a) of Theorem B implies condition (a) of Theorem A in view of the obvious relationship between the corresponding conditions (b). Now let μ be a measure on a compact set K and let $A \subseteq C^+(K)$. Is it true generally that, if there is a decomposition of the form (2) for every $f \in C^+(K)$ in terms of members of A, then there is a decomposition of the form (1) for every $f \in L^1(\mu)$ in terms of A?

(II) The decomposition (2) implies (using Dini's theorem and writing f as $(\delta + f^+) - (\delta + f^-)$, where $\delta > 0$) that any $f \in C(\partial D)$ can be uniformly approximated by finite sums of the form $\sum \lambda_k P(x_k, y)$, where $\lambda_k \in \mathbb{R}$ and $x_k \in E$ (and, given $\varepsilon > 0$, we can even arrange that $\sum |\lambda_k| < \|f\|_{L^1(\partial D)} + \varepsilon$). Is there a converse result?

The first question is answered positively below. In the context of Theorem A the set A of the next result is $\{P(x,\cdot) : x \in E\}$ and, of course, $\|P(x,\cdot)\|_{L^1(\partial D)} = 1$ for all x. By a measure we always mean a nonnegative Borel measure which assigns a finite value to each compact set.

THEOREM 1. *Let μ be a measure on a compact Hausdorff space K and let $A \subseteq C^+(K)$. Suppose that, for each $f \in C^+(K)$, there exist sequences (λ_k) in $[0,+\infty)$ and (f_k) in A such that $f = \sum \lambda_k f_k$. Then, for each $f \in L^1(\mu)$ and $\varepsilon > 0$, there exist sequences (λ_k) in $\mathbb{R}$ and (f_k) in A such that $f = \sum \lambda_k f_k$ in $L^1(\mu)$ and $\sum |\lambda_k| \, \|f_k\|_{L^1(\mu)} < \|f\|_{L^1(\mu)} + \varepsilon$.*

We will address Question II shortly in the light of the next result. In what follows Ω will denote a domain in $\mathbb{R}^n$ which possesses a Green function $G_\Omega(\cdot,\cdot)$; that is, Ω is connected and, in the case where $n = 2$, the complement of Ω must be nonpolar. Let ν_0 be a measure with compact support, $\operatorname{supp}\nu_0$, contained in Ω. In order to state our results in a general form we will use the notion of the Martin boundary Δ of Ω, an account of which may be found in Chapter 8 of [4]. We denote by Δ_1 the set of minimal boundary points and by $M(\cdot,\cdot)$ the Martin kernel

$$M(x,y)=\lim_{z\to y}\frac{G_\Omega(x,z)}{G_\Omega\nu_0(z)}\quad (x\in\Omega;\ y\in\Delta),$$

where $G_\Omega\nu_0$ is the potential on Ω associated with ν_0. (If $\Omega = D$ and ν_0 is the unit measure on $\{0\}$, then $M(x,y)=2\pi P(x,y)$.) Each positive harmonic function h on Ω has a unique representation of the form $h = M\mu$, where

$$M\mu(x)=\int M(x,y)\ d\mu(y)\quad (x\in\Omega)$$

and μ is a measure on Δ such that $\mu(\Delta\backslash\Delta_1)=0$. In the case where Ω is a Lipschitz domain (or, more generally, a nontangentially accessible domain [16]), Δ can be identified with $\partial\Omega$ and all boundary points are minimal.

If X and Y are normed linear spaces, then we write $X\hookrightarrow Y$ to indicate that there is a continuous injective linear mapping from X into Y.

Theorem 2. *Let $H = M\mu$, where μ is a non-zero measure with compact support $K\subseteq\Delta_1$, and let $(X,\|\cdot\|_X)$ be a Banach space such that $C(K)\hookrightarrow X\hookrightarrow L^1(\mu)$ and $C(K)$ is dense in $(X,\|\cdot\|_X)$. The following conditions on a set $E\subseteq\Omega$ are equivalent:*

(a) *For each $f\in X$ and $\varepsilon>0$, there exist $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ and $x_1,\dots,x_m\in E$ such that*

$$\left\|f-\sum_{k=1}^m\lambda_k M(x_k,\cdot)\right\|_X<\varepsilon\quad\text{and}\quad\sum_{k=1}^m|\lambda_k|\,H(x_k)\le\|f\|_{L^1(\mu)}.$$

(b) $\displaystyle\sup_{x\in E}\frac{|T(M(x,\cdot))|}{H(x)}=\sup_{x\in\Omega}\frac{|T(M(x,\cdot))|}{H(x)}$ *for each T in the dual space X^*.*

Our first application of Theorem 2 involves taking X to be $C(\Delta)$ and making a particular choice of μ.

Corollary 1. *Let μ_1 be the unique measure on Δ_1 such that $M\mu_1\equiv 1$, and suppose that* $\operatorname{supp}\mu_1=\Delta=\Delta_1$. *The following conditions on a set $E\subseteq\Omega$ are equivalent:*

(a) *For each $f\in C(\Delta)$ and $\varepsilon>0$, there exist $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ and $x_1,\dots,x_m\in E$ such that*

$$\left|f-\sum_{k=1}^m\lambda_k M(x_k,\cdot)\right|<\varepsilon\quad\text{on }\Delta\quad\text{and}\quad\sum_{k=1}^m|\lambda_k|\le\|f\|_{L^1(\mu_1)}.$$

(b) $\sup_E|h|=\sup_\Omega|h|$ *whenever $h\in h^1(\Omega)$.*

In the case where $\Omega = D$ we thus see that any $f \in C(\partial D)$ can be uniformly approximated by finite sums of the form $\sum \lambda_k P(x_k, \cdot)$, with $x_k \in \ E$ and $\sum |\lambda_k| \leq \|f\|_{L^1(\partial D)}$, if and only if

$$\sup_E |h| = \sup_D |h| \quad \text{whenever } h \in h^1(D). \tag{3}$$

The following example answers Question II negatively in this context by showing that (3) is strictly weaker than condition (b) of Theorem B. It would be interesting to obtain an explicit description of the sets E which satisfy (3).

Example 1. Let $z \in \partial D$ and

$$E_\alpha = \left\{ x \in D : 1 - \|x\|^2 < \|z - x\|^\alpha \right\} \quad (\alpha \geq 1).$$

Then condition (c) of Theorem B holds if and only if $\alpha = 1$, whereas (3) holds if and only if $\alpha < 2$. (Details may be found in Section 3.3.)

The punctured disc $\Omega = D\backslash\{0\}$ is an obvious example of where the hypothesis in Corollary 1 fails, since $\operatorname{supp} \mu_1 = \partial D$ but Δ can be identified with $\partial\Omega$. The equivalence of conditions (a) and (b) breaks down in this case since, if $E = \Omega$, then (b) trivially holds but (a) fails (consider the function f valued 1 at 0, and 0 elsewhere on $\partial\Omega$). However, if μ_1 were augmented by the unit mass at 0, then Theorem 2 would show that condition (a) of Corollary 1 is equivalent to

$$\sup_{x \in E} \frac{|h(x)|}{1 + \log \|x\|} = \sup_{x \in \Omega} \frac{|h(x)|}{1 + \log \|x\|} \quad \text{whenever } h \in h^1(\Omega).$$

Our second application of Theorem 2 involves choosing X to be $L^1(\mu)$. It generalizes the equivalence of the first two conditions in Theorem A.

Corollary 2. *Let μ be a non-zero measure with compact support contained in Δ_1 and let $H = M\mu$. The following conditions on a set $E \subseteq \Omega$ are equivalent:*

(a) *For each $f \in L^1(\mu)$ and $\varepsilon > 0$, there exist sequences (λ_k) in $\mathbb{R}$ and (x_k) in E such that*

$$f = \sum_{k=1}^{\infty} \lambda_k M(x_k, \cdot) \ \text{in } L^1(\mu) \ \text{and} \ \sum_{k=1}^{\infty} |\lambda_k| \, H(x_k) < \|f\|_{L^1(\mu)} + \varepsilon.$$

(b) $\sup_E \dfrac{h}{H} = \sup_\Omega \dfrac{h}{H}$ *for each harmonic function h on Ω such that h/H is bounded.*

If K is a compact subset of Δ_1, then we write $h_K^+(\Omega)$ (resp. $h_K(\Omega)$) for the collection of harmonic functions on Ω of the form $M\nu$, where ν is a non-zero measure (resp. a signed measure) such that $\operatorname{supp} \nu \subseteq K$.

THEOREM 3. *Let K be a compact subset of Δ_1, and let $E \subseteq \Omega$. The following are equivalent:*

(a) *For each $f \in C^+(K)$ there exist sequences (λ_k) in $[0, +\infty)$ and (x_k) in E such that $f = \sum_1^\infty \lambda_k M(x_k, \cdot)$ on K.*

(b) *There exists $H \in h_K^+(\Omega)$ such that $\sup_E \dfrac{h}{H} = \sup_\Omega \dfrac{h}{H}$ whenever $h \in h_K(\Omega)$.*

(c) $\sup_E \dfrac{h}{u} = \sup_\Omega \dfrac{h}{u}$ *whenever $h \in h_K(\Omega)$ and $u \in h_K^+(\Omega)$.*

Aikawa (see Theorem 3 of [1]) has given a different version of Theorem 3 for nontangentially accessible domains Ω. Assuming that E is contained in a suitable nontangential approach region for the set K (and that K has more than one point), he asserts that (a) is equivalent to (b) with $H \equiv 1$. This is incorrect, as we explain in Section 4.4: quotients, where the denominator belongs to $h_K^+(\Omega)$, play an essential role here.

If $K \subset \mathbb{R}^n$ is compact and $E \subseteq \mathbb{R}^n \backslash K$, then we define

$$E^K = \bigcup_{x \in E} B\left(x, \frac{1}{2} d_K(x)\right), \quad \text{where} \quad d_K(x) = \operatorname{dist}(x, K).$$

(This is consistent with our previous use of the notation $E^{\partial D}$.) The Poisson kernel for the unit ball B in $\mathbb{R}^n$ is given by

$$P(x,y) = \frac{1}{\sigma_n} \frac{1 - \|x\|^2}{\|x - y\|^n} \quad (x \in B;\ y \in \partial B),$$

where σ_n is the surface area of ∂B. We write $P\mu(x)$ for the Poisson integral, $\int P(x, \cdot) d\mu$, of a measure μ on ∂B.

COROLLARY 3. *Let K be a compact subset of ∂B that contains more than one point and let $E \subseteq B$. Then conditions* (a)–(c) *of Theorem 3* (*with B and $P(\cdot, \cdot)$ in place of Ω and $M(\cdot, \cdot)$*) *are equivalent to:*

(d) $\int_{E^K \cap B} \|x - y\|^{-n} dx = +\infty$ *for all $y \in K$.*

If we take K to be ∂B in Corollary 3, we obtain a generalization of Theorem B (cf. [11]). For other choices of K, the use of E^K, rather than $E^{\partial B}$, is a new and crucial element. In the case of the disc, Theorem 2(a) of Essén [10] asserts (in view of Lemma C in Section 4.2 below) that condition (d), strengthened by the use of $E^{\partial D}$ in place of E^K, is sufficient for condition (b) to hold with $H \equiv 1$. This is again incorrect (see Section 4.4).

Theorem 3 can also be used to obtain the main result of [12], concerning the representation of continuous functions as sums of Green functions, in the same manner that Corollary 5 is deduced from Theorem 4 below.

THEOREM 4. *Let $H = M\mu$, where μ is a non-zero measure with compact support $K \subseteq \Delta_1$, and let $E \subseteq \Omega$. The following are equivalent:*

(a) *For each positive lower semicontinuous function f on K there exist sequences (λ_k) in $[0, +\infty)$ and (x_k) in E such that $f = \sum_1^\infty \lambda_k M(x_k, \cdot)$ almost everywhere (μ).*

(b) $\sup_E \dfrac{h}{H} = \sup_\Omega \dfrac{h}{H}$ *whenever $h \in h_K(\Omega)$ and h/H is bounded above.*

Theorem 4 is clearly intermediate between Corollary 2 and Theorem 3. When $\Omega = B$ and $H \equiv 1$, conditions (b) of Theorem 4 and Corollary 2 are known to be equivalent (see [6], [11]).

COROLLARY 4. *Let $H = P\mu$, where μ is a measure with compact support $K \subseteq \partial B$ such that K contains more than one point, and let $E \subseteq B$. Then conditions* (a) *and* (b) *of Theorem 4 are equivalent to:*

(c) $\int_{E^K \cap B} \|x - y\|^{-n}\, dx = +\infty$ *for μ-almost every point $y \in \partial B$.*

Corollary 4 improves Corollary 1 in [11], which asserts that a stronger form of condition (c) (that is, with $E^{\partial B}$ in place of E^K) implies a weaker form of condition (a) (namely, condition (a) of Corollary 2).

Our final result in this section relies on the notion of thinness of a set, an account of which may be found in Chapter 7 of [4].

COROLLARY 5. *Let μ be a non-zero measure with compact support $K \subset \Omega$ such that K is polar and contains more than one point. The following conditions on a set $E \subseteq \Omega \backslash K$ are equivalent:*

(a) *For each positive lower semicontinuous function f on K there exist sequences (λ_k) in $[0, +\infty)$ and (x_k) in E such that $f = \sum_1^\infty \lambda_k G_\Omega(x_k, \cdot)$ almost everywhere (μ).*

(b) *For μ-almost every point y the set E^K is non-thin at y.*

Theorems 1–4, together with their associated corollaries, will be established in Sections 2–5, respectively. Section 6 contains some concluding remarks and open questions.

Acknowledgement. This work was supported by EU Research Training Network Contract HPRN-CT-2000-00116. The second author is also partially supported by DGICYT grant BFM2002-00571.

2. Proof of Theorem 1

Theorem 1 is an immediate consequence of the lemma below.

LEMMA 1. *Let μ be a measure on a compact Hausdorff space K and let $A \subseteq C^+(K)$. Suppose that, for each $f \in C^+(K)$, there exist sequences (λ_k) in*

$[0,+\infty)$ and (f_k) in A such that $f = \sum \lambda_k f_k$ almost everywhere (μ). Then, for each $f \in L^1(\mu)$ and $\varepsilon > 0$, there exist sequences (λ_k) in $\mathbb{R}$ and (f_k) in A such that $f = \sum \lambda_k f_k$ in $L^1(\mu)$ and $\sum |\lambda_k| \, \|f_k\|_{L^1(\mu)} < \|f\|_{L^1(\mu)} + \varepsilon$.

To prove the lemma we first observe that, under the stated hypotheses, any positive lower semicontinuous function on K can be expressed almost everywhere (μ) as $\sum \lambda_k f_k$ where $\lambda_k \geq 0$ and $f_k \in A$ for each k. This follows from the fact that such a function can be expressed as the limit of a sequence in $C^+(K)$ which is pointwise strictly increasing. Next we note that, by writing $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, it is sufficient to establish the result for non-negative members of $L^1(\mu)$.

Now let $\varepsilon > 0$ and $f \in L^1(\mu)$, where f takes values in $[0,+\infty)$. By the Vitali-Carathéodory theorem (see, for example, Theorem 2.24 in [18]) there is a positive lower semicontinuous function v_1 on K such that

$$v_1 \geq f \quad \text{and} \quad \int (v_1 - f)\, d\mu < \frac{\varepsilon}{2^2}.$$

Similarly, there exists a positive lower semicontinuous function u_1 on K such that

$$u_1 \geq v_1 - f \quad \text{and} \quad \int (u_1 - v_1 + f)\, d\mu < \frac{\varepsilon}{2^3}.$$

Again, there are positive lower semicontinuous functions v_2 and u_2 on K such that

$$v_2 \geq u_1 - v_1 + f \quad \text{and} \quad \int (v_1 + v_2 - u_1 - f)\, d\mu < \frac{\varepsilon}{2^4}$$

and

$$u_2 \geq v_1 + v_2 - u_1 - f \quad \text{and} \quad \int \{(u_1 + u_2) - (v_1 + v_2) + f\}\, d\mu < \frac{\varepsilon}{2^5}.$$

Proceeding inductively in this way we obtain sequences (u_k) and (v_k) of positive lower semicontinuous functions on K satisfying

$$\int \left\{ \sum_1^m v_k - \sum_1^{m-1} u_k - f \right\} d\mu < \frac{\varepsilon}{2^{2m}} \quad (m \geq 1)$$

and

$$\int \left\{ \sum_1^m u_k - \sum_1^m v_k + f \right\} d\mu < \frac{\varepsilon}{2^{2m+1}} \quad (m \geq 1),$$

the integrands being non-negative in each case. Adding, we see that

$$\int u_m d\mu < \frac{\varepsilon}{2^{2m}} + \frac{\varepsilon}{2^{2m+1}} \quad (m \geq 1),$$

whence

$$\int \sum_1^m u_k \, d\mu < \sum_1^{2m} \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad \int \left\{ \sum_1^m v_k - f \right\} d\mu < \sum_1^{2m-1} \frac{\varepsilon}{2^{k+1}} \quad (m \geq 1).$$

It follows that the sums $u = \sum_1^\infty u_k$ and $v = \sum_1^\infty v_k$, which are positive lower semicontinuous functions on K, satisfy

$$f = v - u \text{ in } L^1(\mu), \quad \|u\|_{L^1(\mu)} < \frac{\varepsilon}{2} \quad \text{and} \quad \|v - f\|_{L^1(\mu)} < \frac{\varepsilon}{2}. \tag{4}$$

By the observation in the opening paragraph there exist sequences $(\lambda_k), (\kappa_k)$ in $[0, +\infty)$ and $(f_k), (g_k)$ in A such that $v = \sum \lambda_k f_k$ and $u = \sum \kappa_k g_k$ almost everywhere (μ), and clearly

$$\|v\|_{L^1(\mu)} = \sum \lambda_k \|f_k\|_{L^1(\mu)} \quad \text{and} \quad \|u\|_{L^1(\mu)} = \sum \kappa_k \|g_k\|_{L^1(\mu)}.$$

The lemma now follows since

$$\begin{aligned}\sum_1^\infty \left\{\lambda_k \|f_k\|_{L^1(\mu)} + \kappa_k \|g_k\|_{L^1(\mu)}\right\} &\le \|v - f\|_{L^1(\mu)} + \|f\|_{L^1(\mu)} + \|u\|_{L^1(\mu)} \\ &< \|f\|_{L^1(\mu)} + \varepsilon,\end{aligned}$$

by (4).

3. Proof of Theorem 2 and corollaries

3.1. Let H, μ, K and X be as in Theorem 2. Suppose first that condition (a) holds and let $x_0 \in \Omega$ and $\varepsilon > 0$. Since $M(x_0, \cdot) \in C(K)$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $x_1, \ldots, x_m \in E$ such that

$$\left\| M(x_0, \cdot) - \sum_{k=1}^m \lambda_k M(x_k, \cdot) \right\|_X < \varepsilon \tag{5}$$

and

$$\sum_{k=1}^m |\lambda_k| \, H(x_k) \le \|M(x_0, \cdot)\|_{L^1(\mu)} = H(x_0). \tag{6}$$

Let $T \in X^*$. From (5) we see that

$$\left| T\left(M(x_0, \cdot)\right) - T\left(\sum_{k=1}^m \lambda_k M(x_k, \cdot)\right) \right| < \varepsilon \|T\|_{X^*},$$

whence

$$\begin{aligned}|T\left(M(x_0, \cdot)\right)| &< \sum_{k=1}^m |\lambda_k| \, |T\left(M(x_k, \cdot)\right)| + \varepsilon \|T\|_{X^*} \\ &\le \left(\sup_{x \in E} \frac{|T\left(M(x, \cdot)\right)|}{H(x)}\right) \sum_{k=1}^m |\lambda_k| \, H(x_k) + \varepsilon \|T\|_{X^*} \\ &\le \left(\sup_{x \in E} \frac{|T\left(M(x, \cdot)\right)|}{H(x)}\right) H(x_0) + \varepsilon \|T\|_{X^*},\end{aligned}$$

by (6). Since $\varepsilon > 0$ and $x_0 \in \Omega$ were arbitrary, condition (b) follows.

Conversely, suppose that condition (b) holds, and let A denote the closure in $(X, \|\cdot\|_X)$ of the set of finite sums of the form $\sum \lambda_k M(x_k, \cdot)$, where $\lambda_k \in \mathbb{R}$, $x_k \in E$ and $\sum |\lambda_k| H(x_k) \leq 1$. Thus A is $\|\cdot\|_X$-closed and convex. Let $g \in X \backslash A$. By a well-known separation theorem there exist $T \in X^*$ and $a \in \mathbb{R}$ such that

$$T(f) \leq a \text{ for all } f \in A \quad \text{and} \quad T(g) > a. \tag{7}$$

Since $\pm M(x, \cdot)/H(x) \in A$ whenever $x \in E$,

$$|T(M(x, \cdot))| \leq aH(x) \tag{8}$$

whenever $x \in E$. By hypothesis this inequality therefore holds for all $x \in \Omega$. Since $T \in X^*$, and the inclusion map from $C(K)$ into X is continuous, $T\big|_{C(K)}$ is a bounded linear functional on $C(K)$. Thus, by the Riesz representation theorem, there is a signed measure ν on K such that $T(f) = \int f \, d\nu$ for all $f \in C(K)$. In particular, by (8), the harmonic function $M\nu$ satisfies $|M\nu| \leq aM\mu$ on Ω. It follows from the uniqueness of the Martin representation, and the fact that $K \subseteq \Delta_1$, that $-a\mu \leq \nu \leq a\mu$, and so

$$|T(f)| \leq a \|f\|_{L^1(\mu)} \text{ for all } f \in C(K).$$

Since $C(K)$ is dense in $(X, \|\cdot\|_X)$ and $X \hookrightarrow L^1(\mu)$, we thus have $T(g) \leq a \|g\|_{L^1(\mu)}$ which, together with (7), shows that $\|g\|_{L^1(\mu)} > 1$. Thus we conclude that

$$\left\{ f \in X : \|f\|_{L^1(\mu)} \leq 1 \right\} \subseteq A. \tag{9}$$

Finally, suppose that $f \in X \backslash \{0\}$ and $\varepsilon > 0$. Then $\|f\|_{L^1(\mu)} \neq 0$. In view of (9), there exist $\lambda'_1, \ldots, \lambda'_m \in \mathbb{R}$ and $x_1, \ldots, x_m \in E$ such that

$$\left\| \frac{f}{\|f\|_{L^1(\mu)}} - \sum_{k=1}^{m} \lambda'_k M(x_k, \cdot) \right\|_X < \frac{\varepsilon}{\|f\|_{L^1(\mu)}}$$

and $\sum_1^m |\lambda'_k| H(x_k) \leq 1$. Hence condition (a) holds with $\lambda_k = \lambda'_k \|f\|_{L^1(\mu)}$.

3.2. Corollary 1 follows on taking $\mu = \mu_1$ and $X = C(\Delta)$ in Theorem 2, whence $H \equiv 1$ and X^* can be identified with the collection of all signed measures on Δ.

3.3. The first assertion of Example 1 is easily verified by direct calculation, so it remains to deal with the second assertion. If $\alpha \geq 2$, then (3) clearly fails for $h = P(\cdot, z)$. Suppose, on the other hand, that $\alpha < 2$. If $h \in h^1(D)$ and $\sup_E |h| < +\infty$, then h must be of the form $h_1 + cP(\cdot, z)$, where $h_1 \in h^\infty(D)$ and $c \in \mathbb{R}$. Since $P(\cdot, z)$ is unbounded on E, we have $c = 0$ and (3) now follows (for example, by Theorem A).

3.4. For Corollary 2 we take $X = L^1(\mu)$, whence X^* can be identified with $L^\infty(\mu)$ and condition (b) of Theorem 2 can be reformulated as condition (b) of the corollary. It remains to compare the corresponding conditions (a).

First suppose that condition (a) of Theorem 2 holds, let $f \in L^1(\mu)$ and $\varepsilon > 0$. Then there exist $\lambda_{1,1}, \ldots, \lambda_{1,m_1} \in \mathbb{R}$ and $x_{1,1}, \ldots, x_{1,m_1} \in E$ such that

$$\|f - g_1\|_{L^1(\mu)} < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{k=1}^{m_1} |\lambda_{1,k}|\, H(x_{1,k}) \le \|f\|_{L^1(\mu)},$$

where $g_1 = \sum_{k=1}^{m_1} \lambda_{1,k} M(x_{1,k}, \cdot)$. Applying our hypothesis to $f - g_1$, we then see that there exist $\lambda_{2,1}, \ldots, \lambda_{2,m_2} \in \mathbb{R}$ and $x_{2,1}, \ldots, x_{2,m_2} \in E$ such that

$$\|f - (g_1 + g_2)\|_{L^1(\mu)} < \frac{\varepsilon}{2^2} \quad \text{and} \quad \sum_{k=1}^{m_2} |\lambda_{2,k}|\, H(x_{2,k}) < \frac{\varepsilon}{2},$$

where $g_2 = \sum_{k=1}^{m_2} \lambda_{2,k} M(x_{2,k}, \cdot)$. Considering next $f - (g_1 + g_2)$ and then proceeding inductively, we obtain a sequence (g_j), where g_j is of the form $\sum_{k=1}^{m_j} \lambda_{j,k} M(x_{j,k}, \cdot)$ and $x_{j,k} \in E$, such that

$$\left\| f - \sum_{j=1}^{l} g_j \right\|_{L^1(\mu)} < \frac{\varepsilon}{2^l} \quad \text{and} \quad \sum_{k=1}^{m_l} |\lambda_{l,k}|\, H(x_{l,k}) < \frac{\varepsilon}{2^{l-1}} \quad (l \ge 2).$$

Thus

$$f = \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} \lambda_{j,k} M(x_{j,k}, \cdot) \quad \text{in} \quad L^1(\mu)$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{m_j} |\lambda_{j,k}|\, H(x_{j,k}) < \|f\|_{L^1(\mu)} + \varepsilon,$$

so condition (a) of Corollary 2 holds.

Conversely, suppose that condition (a) of Corollary 2 holds, and let $f \in L^1(\mu)\backslash\{0\}$ and $\varepsilon \in (0, \|f\|_{L^1(\mu)})$. Then there exist sequences (λ_k) in $\mathbb{R}$ and (x_k) in E such that

$$\left(1 - \frac{\varepsilon}{2\,\|f\|_{L^1(\mu)}}\right) f = \sum_1^\infty \lambda_k M(x_k, \cdot) \quad \text{in} \quad L^1(\mu)$$

and

$$\sum_1^\infty |\lambda_k|\, H(x_k) < \left(1 - \frac{\varepsilon}{2\,\|f\|_{L^1(\mu)}}\right) \|f\|_{L^1(\mu)} + \frac{\varepsilon}{2} = \|f\|_{L^1(\mu)}.$$

Thus we can choose m such that

$$\left\| \left(1 - \frac{\varepsilon}{2\,\|f\|_{L^1(\mu)}}\right) f - \sum_1^m \lambda_k M(x_k, \cdot) \right\|_{L^1(\mu)} < \frac{\varepsilon}{2},$$

whence

$$\left\| f - \sum_{1}^{m} \lambda_k M(x_k, \cdot) \right\|_{L^1(\mu)} < \varepsilon,$$

and condition (a) of Theorem 2 holds.

4. Proof of Theorem 3 and Corollary 3

4.1. Bonsall and Walsh (see the proof of Theorem 10 (ii)⇒(i) in [7]) have used the Hahn-Banach theorem to show that (b) implies (a) in Theorem B, and their argument is easily adapted to show that (b) implies (a) in Theorem 3. Since (c) clearly implies (b), it remains to check that (a) implies (c).

To do this, let $h = M\nu \in h_K$ and $u = M\tau \in h_K^+$, let $x_0 \in \Omega$ and suppose that $h/u \leq a$ on E. Assuming condition (a), there exist sequences (λ_k) in $[0, +\infty)$ and (x_k) in E such that

$$M(x_0, y) = \sum_{1}^{\infty} \lambda_k M(x_k, y) \quad (y \in K).$$

Integration of this equation with respect to each of ν and τ yields

$$h(x_0) = \sum_{1}^{\infty} \lambda_k h(x_k) \leq a \sum_{1}^{\infty} \lambda_k u(x_k) = au(x_0),$$

and condition (c) follows from the arbitrary nature of x_0.

4.2. Before turning to the proof of Corollary 3 we present some lemmas. We will use $C(a, b, \dots)$ to denote a positive constant, depending at most on $a, b, \dots$, not necessarily the same on any two occurrences.

Lemma A. *Let $z \in \partial B$, $r \in (0, 2)$ and $\rho \in (0, 1)$. Then there is a positive constant $C(n, \rho)$ with the following property: if g, h are positive harmonic functions on $B(z, r) \cap B$ that tend to 0 at all points of $B(z, r) \cap \partial B$, then*

$$\frac{g(x)}{h(x)} \leq C(n, \rho) \frac{g(y)}{h(y)} \quad (x, y \in B(z, \rho r) \cap B).$$

Further, $C(n, \rho) \to 1$ as $\rho \to 0+$.

Lemma A is a special case of the uniform boundary Harnack principle (see Aikawa [2]). An elementary proof of its halfspace analogue, from which it may be deduced using the Kelvin transform, can be found in Lemma 8.5.1 of [4]. Actually, we will use it in the following modified form.

Lemma A′. *Let $x_0 \in B$ and $\rho \in (0, 1)$. Then there is a positive constant $C(n, \rho)$ with the following property: if $g, h \in h_K^+(B)$, then*

$$\frac{g(x)}{h(x)} \leq C(n, \rho) \frac{g(y)}{h(y)} \quad (x, y \in B(x_0, \rho d_K(x_0)) \cap B). \tag{10}$$

Further, $C(n,\rho) \to 1$ *as* $\rho \to 0+$.

It is enough to verify Lemma A′ when $0 < \rho < 1/3$, since iterative application then yields the general case. For such ρ, (10) follows from the classical Harnack inequality if $d_K(x_0) < 3(1 - \|x_0\|)$, while if $d_K(x_0) \geq 3(1 - \|x_0\|)$, then

$$B(x_0, d_K(x_0)/3) \subset B\left(x_0/\|x_0\|, 2d_K(x_0)/3\right) \subset \mathbb{R}^n \backslash K$$

and we can instead appeal to Lemma A with $z = x_0/\|x_0\|$.

If $K \subset \mathbb{R}^n$ is compact and $E \subseteq \mathbb{R}^n \backslash K$, then we define

$$E_\rho^K = \bigcup_{x \in E} B\left(x, \rho d_K(x)\right) \quad (0 < \rho \leq 1).$$

Thus $E^K = E_{1/2}^K$.

Lemma B. *Let* $K \subset \mathbb{R}^n$ *be compact and* $E \subseteq \mathbb{R}^n \backslash K$ *be bounded, and let* $0 < \rho < 1$. *Then there is a countable subset* $A = \{x_k : k \geq 1\}$ *of* E *such that the constituent balls of* $A_{\rho/5}^K$ *are disjoint and* $E_{\rho/5}^K \subseteq A_\rho^K$.

Lemma B is a well-known covering lemma (see, for example, pp. 9,10 of [20]).

The next result is due to Dahlberg [8] (see also Beurling [5], Maz'ya [17] and Section 7.4 of the book [3]).

Lemma C. *Let* $E \subseteq B$ *be Borel measurable and* $z_0 \in \partial B$. *If there is a positive superharmonic function* u *on* B *such that*

$$\inf_E \frac{u}{P(\cdot, z_0)} > \inf_B \frac{u}{P(\cdot, z_0)},$$

then

$$\int_E \|x - z_0\|^{-n} \, dx < +\infty.$$

Lemma 2. *Let* $K \subseteq \partial B$ *be compact, let* z_0 *be a limit point of* K *and* $0 < \rho < 1$. *The following conditions on a set* $E \subseteq B$ *are equivalent:*

(a) *There is a measure* ν *on* K *such that* $\nu(\{z_0\}) = 0$ *and* $P\nu \geq P(\cdot, z_0)$ *on* E.

(b) $\int_{E_\rho^K \cap B} \|x - z_0\|^{-n} \, dx < +\infty$.

To prove Lemma 2, we first suppose that condition (a) holds. From Lemma A′ we see that $P\nu \geq C(n,\rho) P(\cdot, z_0)$ on $E_\rho^K \cap B$, and since $\nu(\{z_0\}) = 0$, condition (b) follows, by Lemma C.

Conversely, suppose that condition (b) holds. By Lemma B there is a countable subset $A = \{x_k : k \geq 1\}$ of E such that the constituent balls of

$A^K_{\rho/5}$ are disjoint and $E^K_{\rho/5} \subseteq A^K_\rho$. For each k we can choose $z_k \in K\backslash\{z_0\}$ such that

$$\|x_k - z_k\| < 2d_K(x_k), \tag{11}$$

since z_0 is a limit point of K. We now define

$$\mu = \sum \frac{\{d_K(x_k)\}^n}{\|x_k - z_0\|^n} \delta_{z_k},$$

where δ_z denotes the unit mass at z. Since $A^K_{\rho/5} \subseteq E^K_\rho$, we see from condition (b) that

$$\sum_k \frac{\{\rho d_K(x_k)/5\}^n}{\{\|x_k - z_0\| + \rho d_K(x_k)/5\}^n} < +\infty,$$

and so μ is a finite measure. Further,

$$P\mu(x_k) \geq \frac{1}{\sigma_n} \frac{1 - \|x_k\|^2}{\|x_k - z_k\|^n} \frac{\{d_K(x_k)\}^n}{\|x_k - z_0\|^n} > 2^{-n} P(x_k, z_0) \quad (k \geq 1),$$

by (11). Hence $P\mu \geq C(n,\rho)P(\cdot, z_0)$ on $A^K_\rho \cap B$ by Lemma A′. Since $E \subseteq A^K_\rho$ and $\mu(\{z_0\}) = 0$, condition (a) follows easily. Lemma 2 is now proved.

We note that, because condition (a) of Lemma 2 is independent of the value of ρ, the same must be true of condition (b).

4.3. We now prove Corollary 3. Suppose firstly that condition (d) holds, let $h_1, h_2, u \in h^+_K(B)$ and $h = h_1 - h_2$, and suppose that $h/u \leq a$ on E. Further, let $0 < \rho < 1$. It follows from Lemma A′ that

$$(a + |a|)u + h_2 \geq C(n,\rho)(h_1 + |a|\, u) \tag{12}$$

on $E^K_\rho \cap B$. Condition (d) and Lemma C together imply that $E^K \cap B$, and hence $E^K_\rho \cap B$, is not minimally thin at any point of K (we refer to Chapter 9 of [4] for this concept and the associated topology). Using the prefix "mf" to indicate that a limit concept is taken with respect to this minimal fine topology, we thus have

$$\text{mf } \limsup_{x \to y} \frac{(a + |a|)u + h_2}{h_1 + |a|\, u} \geq C(n,\rho) \quad (y \in K),$$

and it now follows from a general minimum principle (Theorem 9.3.7 of [4]) that (12) holds on all of B. Since $C(n,\rho) \to 1$ as $\rho \to 0+$, we conclude that $h/u \leq a$ on B, and condition (c) is established.

Conversely, suppose that condition (d) fails. Then there exists $z_0 \in K$ such that

$$\int_{E^K_\rho \cap B} \|x - z_0\|^{-n}\, dx < +\infty. \tag{13}$$

If z_0 is an isolated point of K, then (13) implies that $z_0 \notin \overline{E}$. If we take $u = P(\cdot, z_0)$ and $h = -P(\cdot, z_1)$, where $z_1 \in K\backslash\{z_0\}$, then $u \in h_K^+(\Omega)$, $h \in h_K(\Omega)$ and we see that condition (c) also fails, since

$$\inf_E \frac{P(\cdot, z_1)}{P(\cdot, z_0)} > 0 \quad \text{whereas} \quad \inf_B \frac{P(\cdot, z_1)}{P(\cdot, z_0)} = 0.$$

On the other hand, if z_0 is not an isolated point of K, then (13) allows us to choose ν as in condition (a) of Lemma 2. Taking $u = P(\cdot, z_0)$ and $h = -P\nu$, we again find that condition (c) of the corollary fails since

$$\inf_E \frac{P\nu}{P(\cdot, z_0)} \geq 1 \quad \text{whereas} \quad \inf_B \frac{P\nu}{P(\cdot, z_0)} = \nu(\{z_0\}) = 0.$$

4.4. Here we justify the remarks made in Section 1 concerning work of Aikawa and Essén that is related to Theorem 3 and Corollary 3. (Zhang [21] has also noted these errors.)

Firstly, it is not possible to replace condition (b) of Theorem 3 by the condition

$$\sup_E h = \sup_\Omega h \quad \text{whenever } h \in h_K(\Omega). \tag{14}$$

For example, let U denote the closed upper halfplane, let $\Omega = D$, $K = U \cap \partial D$ and $E = U \cap D$, and let h be the harmonic measure of K in D. Then condition (a) of Theorem 3 holds, in view of Corollary 3. However, $-h \in h_K(D)$ and $\sup_D(-h) = 0$ whereas $\sup_E(-h) = -1/2$.

Secondly, this example also shows that condition (d) of Corollary 3, even when E^K is replaced by $E^{\partial B}$, does not imply (14).

5. Proof of Theorem 4 and corollaries

5.1. First suppose that condition (a) of Theorem 4 holds and let $h \in h_K(\Omega)$, where h/H is bounded above. Further, let $a = \sup_E h/H$ and let b be a non-negative upper bound for h/H on Ω. Then $h = bH - M\tau$, where τ is a measure with $\operatorname{supp}\tau \subseteq K$. Let $x_0 \in \Omega$. By condition (a), there exist sequences (λ_k) in $[0, +\infty)$ and (x_k) in E such that $M(x_0, \cdot) = g$ almost everywhere (μ), where $g = \sum_1^\infty \lambda_k M(x_k, \cdot)$. Since g is lower semicontinuous, $M(x_0, \cdot) \geq g$ on K and so

$$H(x_0) = \sum_1^\infty \lambda_k H(x_k) \quad \text{and} \quad M\tau(x_0) \geq \sum_1^\infty \lambda_k M\tau(x_k).$$

Hence

$$h(x_0) = bH(x_0) - M\tau(x_0) \le \sum_1^\infty \lambda_k \{bH(x_k) - M\tau(x_k)\}$$
$$= \sum_1^\infty \lambda_k h(x_k) \le a \sum_1^\infty \lambda_k H(x_k) = aH(x_0),$$

and condition (b) follows from the arbitrary nature of x_0.

For the converse we adapt an argument of Hoffman and Rossi [15], and recall the following corollary of the Krein-Smulian theorem from that paper.

Lemma D. *Let A be a convex subset of ℓ^∞. The following are equivalent:*

(i) *A is weak* closed.*

(ii) *If $\{(a_k^{(m)})_{k\ge 1} : m \in \mathbb{N}\}$ is a bounded subset of A and $a_k^{(m)} \to a_k$ as $m \to \infty$ for each k, then $(a_k) \in A$.*

Now suppose that condition (b) of Theorem 4 holds, let (x_k) be a dense sequence of points in E and let $x_0 \in \Omega$. We define A to be the set of bounded real sequences (a_k) with the following property: there exists $h \in h_K(\Omega)$ such that h/H is bounded above, $h(x_k) \le a_k H(x_k)$ for each k, and $h(x_0) = 0$. Clearly A is convex. We now use Lemma D to check that A is weak* closed. Let $\{(a_k^{(m)})_{k\ge 1} : m \in \mathbb{N}\}$ be a subset of A such that $\|(a_k^{(m)})_{k\ge1}\|_\infty \le c$ for each m, where $c \in \mathbb{R}$, and $a_k^{(m)} \to a_k$ as $m \to \infty$ for each k. For each m let h_m be a member of $h_K(\Omega)$ corresponding to $(a_k^{(m)})_{k\ge1}$ in the definition of A. By condition (b) and the density of (x_k) in E, we see that $h_m - cH \le 0$ on Ω. Since $h_m(x_0) = 0$ for each m, the compactness property of harmonic functions shows that there is a subsequence (h_{m_j}) which converges pointwise to a harmonic function which must belong to $h_K(\Omega)$. It is now clear that $(a_k) \in A$, so A is weak* closed by Lemma D.

We note also from condition (b) that $(-1)_{k\ge1} \notin A$. Hence, by a standard separation theorem, there exists $(b_k) \in \ell^1\backslash\{0\}$ such that $\sum a_k b_k \ge 0$ whenever $(a_k) \in A$. Since $0 \in h_K(\Omega)$, it follows that A contains every sequence (a_k) of non-negative numbers, and so $b_k \ge 0$ for each k. Now let $L \subseteq K$ be a Borel set and define

$$h = M(\mu|_L) - M(\mu|_L)(x_0)\frac{H}{H(x_0)}.$$

Since $\pm((h/H)(x_k)) \in A$, we have

$$\sum_{k=1}^\infty b_k \left\{ \frac{M(\mu|_L)(x_k)}{H(x_k)} - \frac{M(\mu|_L)(x_0)}{H(x_0)} \right\} = 0,$$

whence

$$\int_L \left\{ \sum \frac{b_k}{H(x_k)} M(x_k, \cdot) - \frac{M(x_0, \cdot)}{H(x_0)} \sum b_k \right\} d\mu = 0 \quad (L \subseteq K).$$

Thus

$$M(x_0,\cdot)=\frac{H(x_0)}{\sum b_k}\sum\frac{b_k}{H(x_k)}M(x_k,\cdot)\quad\text{almost everywhere }(\mu).$$

In view of the arbitrary nature of x_0, condition (a) now follows using the simple case of Theorem 3 where $E=\Omega$.

5.2. In this section we establish two lemmas in preparation for the proof of Corollary 4.

LEMMA 3. *Let μ be a measure with compact support K in $\mathbb{R}^n$ and suppose that $\mu(I)>0$, where $I\subseteq K$ and I contains no isolated points of K. Then there is a subset J of I such that $\mu(J)>0$ and $\mu(U\backslash J)>0$ for every open set U which intersects K.*

To see this, let $\mathcal{I}_0$ be the collection of open cubes of the form $(a_1,a_1+2)\times\cdots\times(a_n,a_n+2)$, where $(a_1,\ldots,a_n)\in\mathbb{Z}^n$, and, for each $m\in\mathbb{N}$, let

$$\mathcal{I}_m=\left\{2^{-m}Q:Q\in\mathcal{I}_0\right\},\quad\text{where}\quad 2^{-m}Q=\left\{2^{-m}x:x\in Q\right\}.$$

Next, let $Q_{m,1},\ldots,Q_{m,k_m}$ be the members of $\mathcal{I}_m$ which intersect K, and let $k\in\{1,\ldots,k_m\}$. If $\mu(Q_{m,k}\backslash I)=0$ then, because I contains no isolated point of K, we can choose a Borel subset $L_{m,k}$ of $Q_{m,k}\cap I$ such that

$$0<\mu(L_{m,k})<2^{-m-k}\mu(I). \tag{15}$$

On the other hand, if $\mu(Q_{m,k}\backslash I)>0$, then we define $L_{m,k}=\emptyset$. In either case,

$$\mu\left((Q_{m,k}\backslash I)\cup L_{m,k}\right)>0. \tag{16}$$

If we now define

$$J=I\backslash\left(\bigcup_{m=1}^{\infty}\bigcup_{k=1}^{k_m}L_{m,k}\right),$$

we see from (15) that

$$\mu(J)\geq\mu(I)-\sum_{m=1}^{\infty}\sum_{k=1}^{k_m}\mu(L_{m,k})>0.$$

Further, if U is an open set which intersects K, then we can choose m and k such that $Q_{m,k}\subset U$ and $Q_{m,k}\cap K\neq 0$, and it follows from (16) that

$$\mu(U\backslash J)\geq\mu\left((Q_{m,k}\backslash I)\cup L_{m,k}\right)>0.$$

Lemma 3 is now proved.

LEMMA 4. *Let H,μ,K and E be as in Corollary 4 and let*

$$I=\left\{z\in K:\int_{E^K\cap B}\frac{dx}{\|x-z\|^n}<+\infty\right\}.$$

If $\mu(I) > 0$, then there is a non-negative function $f \in L^1(\mu)$ such that $f = 0$ on a set of positive μ-measure and $P(f\mu) \geq H$ on E.

To prove Lemma 4, we suppose that $\mu(I) > 0$. If I contains an isolated point z_0 of K, then $z_0 \notin \overline{E}$ since $\int_{E^K \cap B} \|x - z\|^{-n}\, dx$ is finite. We note by hypothesis that $K \backslash \{z_0\} \neq \emptyset$, and the functions

$$\left\{ \frac{P(\cdot, z_0)}{P(\cdot, z)} : z \in K \backslash \{z_0\} \right\}$$

are uniformly bounded above on E, since

$$\frac{P(x, z_0)}{P(x, z)} = \left\{ \frac{\|x - z\|}{\|x - z_0\|} \right\}^n \leq \left\{ \frac{\|x - z\|}{\operatorname{dist}(z_0, E)} \right\}^n \qquad (x \in E;\, z \in \partial B).$$

Hence $P\left(\mu_{|K \backslash \{z_0\}}\right)/H$ has a positive lower bound, c say, on E. It thus suffices in this case to define $f = c^{-1}$ on $K \backslash \{z_0\}$ and $f(z_0) = 0$.

It remains to consider the case where I contains no isolated points of K. By Lemma 3 there is a subset J of I such that $\mu(J) > 0$ and $\mu(U \backslash J) > 0$ for every open set U which intersects K. Further, we can choose J so that the set of numbers

$$\left\{ \int_{E^K \cap B} \frac{dx}{\|x - z\|^n} :\ z \in J \right\}$$

is bounded above by a positive constant C. By Lemma B there is a countable subset $A = \{x_k : k \geq 1\}$ of E such that the constituent balls of $A^K_{1/10}$ are disjoint and $E^K_{1/10} \subseteq A^K$. We define

$$f = \chi_{K \backslash J} + \sum_k \frac{a_k}{\mu(I_k)} \chi_{I_k},$$

where χ_L is the characteristic function valued 1 on the set L and 0 elsewhere,

$$I_k = B\left(x_k, 2d_K(x_k)\right) \cap (K \backslash J)$$

(our choice of J ensures that $\mu(I_k) > 0$) and

$$a_k = \{d_K(x_k)\}^n \int_J \frac{d\mu(z)}{\|x_k - z\|^n}.$$

Since $A^K_{1/10} \subseteq E^K$, we have

$$\begin{aligned} \sum_k \int_{B(x_k, d_K(x_k)/10) \cap B} \int_J \frac{d\mu(z)}{\|x - z\|^n}\ dx &= \int_J \int_{A^K_{1/10} \cap B} \frac{dx}{\|x - z\|^n}\ d\mu(z) \\ &\leq C\mu(J), \end{aligned}$$

and hence

$$\int_K f d\mu = \mu(K \backslash J) + \sum_k \{d_K(x_k)\}^n \int_J \frac{d\mu(z)}{\|x_k - z\|^n} < +\infty.$$

Finally,

$$\begin{aligned} P(f\mu)(x_k) &\geq \frac{1-\|x_k\|^2}{\sigma_n}\Bigg\{\int_{K\backslash J}\frac{d\mu(z)}{\|x_k-z\|^n} \\ &\qquad\qquad + \frac{\{d_K(x_k)\}^n}{\mu(I_k)}\int_J \frac{d\mu(z)}{\|x_k-z\|^n}\int_{I_k}\frac{d\mu(z)}{\|x_k-z\|^n}\Bigg\} \\ &\geq \frac{1-\|x_k\|^2}{2^n\sigma_n}\int_K \frac{d\mu(z)}{\|x_k-z\|^n} \\ &= 2^{-n}H(x_k) \quad (k\geq 1), \end{aligned}$$

by the definition of I_k. Thus $P(f\mu) \geq C(n)H$ on $A^K \cap B$ by Lemma A$'$. Since $E \subseteq A^K \cap B$ and $f = 0$ on J, the lemma follows on replacing f by $f/C(n)$.

5.3. We now prove Corollary 4. Suppose firstly that condition (c) holds and let $h \in h_K(B)$, where h/H is bounded above (by $b > 0$, say) on B. Further, suppose that $h/H \leq a$ on E, and let $0 < \rho < 1$. It follows from Lemma A$'$ that

$$(b+|a|)H \leq C(n,\rho)\left\{(a+|a|)H + (bH-h)\right\} \tag{17}$$

on $E^K_\rho \cap B$. Condition (c) and Lemma C together imply that at μ-almost every point of K the set $E^K_\rho \cap B$ is not minimally thin. As in the proof of Corollary 3 it follows that (17) holds on all of B and, letting $\rho \to 0+$, we conclude that $h/H \leq a$ on B. Hence condition (b) holds.

Conversely, if condition (c) fails, then we can choose f as in Lemma 4. Thus condition (b) is seen to fail on putting $h = -P(f\mu)$ and noting that $\inf_B P(f\mu)/P\mu = 0$.

5.4. The following analogue of Lemma 4 will be required for the proof of Corollary 5. We note from Lemma 2 of [12] that, if E^K_ρ is thin at $z \in K$ for some $\rho \in (0,1)$, then it is thin at z for all $\rho \in (0,1)$.

Lemma 5. *Let K be a compact polar subset of Ω that contains more than one point, let $E \subseteq \Omega\backslash K$ and $I = \{z \in K : E^K$ is thin at $z\}$. If $\mu(I) > 0$, then there is a non-negative function $f \in L^1(\mu)$ such that $f = 0$ on a set of positive μ-measure and $G_\Omega(f\mu) \geq G_\Omega\mu$ on E.*

We will prove Lemma 5 in the case where $n \geq 3$; minor modifications are required when $n = 2$. Given an open set V such that $K \subset V \subset \Omega$, it is enough to establish the above inequality at points of $E \cap V$: Harnack's inequalities then show that, provided $f \neq 0$ in $L^1(\mu)$, the quotient $G_\Omega(f\mu)/G_\Omega\mu$ has a positive lower bound on the remainder of E. For this reason we may assume that $B(x, 2d_K(x)) \subset \Omega$ for all $x \in E$ and that there is a positive constant c

such that

$$G_\Omega(x,y) \geq c\,\|x-y\|^{2-n} \quad (x,y \in E_1^K \cup K). \tag{18}$$

If I contains an isolated point z_0 of K, then

$$r^{1-n}\sigma\left(E^K \cap \partial B(z_0,r)\right) \to 0 \quad \text{as} \quad r \to 0+,$$

where σ denotes surface area measure on $\partial B(z_0,r)$. Hence $z_0 \notin \overline{E}$ and the quotient $G_\Omega(\mu\left|_{K\backslash\{z_0\}}\right.)/G_\Omega\mu$ has a positive lower bound b on E, so it suffices to define $f = b^{-1}$ on $K\backslash\{z_0\}$ and $f(z_0) = 0$. (We assumed that K contains more than one point.)

It thus remains to deal with the case where I contains no isolated points of K. By Lemma 3 we can choose $J \subseteq I$ such that $\mu(J) > 0$ and $\mu(U\backslash J) > 0$ for any open set U which intersects K. Next, by Lemma B there is a countable subset $A = \{x_k : k \geq 1\}$ of E such that $A_{1/10}^K$ is a disjoint union of balls and $E_{1/10}^K \subseteq A^K$. The set $A_{1/20}^K$ is thin at each point of I, so the regularized reduced function $\widehat{R}_{G_\Omega\mu}^{A_{1/20}^K}$, of $G_\Omega\mu$ relative to $A_{1/20}^K$ in Ω, is the potential of a measure ν such that

$$\operatorname{supp}\nu \subseteq \left(\bigcup_k \partial B\left(x, \frac{d_K(x_k)}{20}\right)\right) \cup K.$$

Further, $\nu \leq \mu$ on K by the Riesz decomposition theorem, since K is polar and so $G_\Omega\mu - G_\Omega(\nu\left|_K\right.)$ has a non-negative superharmonic extension from $\Omega\backslash K$ to all of Ω. Also, $\nu(I) = 0$ by Theorem 1.XI.14 of [9]. We define

$$f = \chi_{K\backslash J} + \sum_k \frac{a_k}{\mu(I_k)}\chi_{I_k},$$

where

$$I_k = B\left(x_k, 2d_K(x_k)\right) \cap (K\backslash J) \quad \text{and} \quad a_k = \nu\left(\partial B\left(x_k, \frac{d_K(x_k)}{20}\right)\right).$$

Then

$$\int_K f\, d\mu = \mu(K\backslash J) + \sum_k a_k \leq \mu(K) + \nu(\Omega) < +\infty$$

since the spheres $\partial B\left(x_k, d_K(x_k)/20\right)$ are disjoint. Also, by (18),

$$\begin{aligned} G_\Omega(f\mu)(x_j) &\geq G_\Omega\left(\mu\left|_{K\backslash J}\right.\right)(x_j) + c\sum_k \frac{a_k}{\left\{\|x_j - x_k\| + 2d_K(x_k)\right\}^{n-2}} \\ &\geq C(n) G_\Omega\nu(x_j) \\ &= C(n) G_\Omega\mu(x_j) \quad (j \geq 1), \end{aligned}$$

since $G_\Omega\nu = G_\Omega\mu$ on $A_{1/20}^K$. Thus $G_\Omega(f\mu) \geq C(n)G_\Omega\mu$ on A^K by Harnack's inequalities and, since $E \subseteq A^K$, the lemma follows on multiplication of f by a suitable constant.

5.5. We note from Theorem 9.5.1 of [4] that, if $K \subset \Omega$ is polar, then the Martin boundary of $\Omega\backslash K$ is $\Delta \cup K$, the set of minimal Martin boundary points of $\Omega\backslash K$ is $\Delta_1 \cup K$ and the Martin kernel for $\Omega\backslash K$ with pole $y \in K$ is of the form $G_\Omega(\cdot, y)/G_\Omega\nu_0(y)$ (provided that the support of the reference measure ν_0 is disjoint from K). We thus have the following immediate consequence of Theorem 4.

Corollary 6. *Let μ be a non-zero measure with polar compact support $K \subset \Omega$, and let $E \subseteq \Omega\backslash K$. Then condition* (a) *of Corollary 5 holds if and only if*

$$\inf_E \frac{G_\Omega \nu}{G_\Omega \mu} = \inf_\Omega \frac{G_\Omega \nu}{G_\Omega \mu} \quad \textit{for every measure } \nu \textit{ on } K.$$

We now deduce Corollary 5. Suppose firstly that condition (b) holds, let ν be a measure on K and suppose that $G_\Omega\nu/G_\Omega\mu \geq a$ on E. We choose an open set V such that $K \subset V \subset \Omega$ and $F_1^K \subset \Omega$, where $F = E \cap V$. By Harnack's inequalities,

$$G_\Omega \nu \geq C(n,\rho) a G_\Omega \mu \quad \text{ on } F_\rho^K.$$

At μ-almost every point of K the quotient $G_\Omega\nu/G_\Omega\mu$ has a fine limit given by the Radon-Nikodým derivative $d\nu/d\mu$ (see Theorems 9.4.6 and 9.5.1 in [4]). Thus $\nu \geq C(n,\rho)a\mu$ by condition (b), and since $C(n,\rho) \to 1$ as $\rho \to 0+$, we have $G_\Omega\nu/G_\Omega\mu \geq a$ on Ω. Condition (a) now follows from Corollary 6.

Conversely, if condition (b) fails, then we can choose f as in Lemma 5. Clearly

$$\inf_E \frac{G_\Omega(f\mu)}{G_\Omega\mu} \geq 1 \quad \text{and} \quad \inf_\Omega \frac{G_\Omega(f\mu)}{G_\Omega\mu} = 0,$$

so condition (a) fails, by Corollary 6.

6. Concluding remarks

6.1. Hayman (see Section 2.4 of [13]) has also asked about sets $E \subseteq D$ such that every $f \in C(\partial D)$ can be uniformly approximated by finite linear combinations of $\{P(x, \cdot) : x \in E\}$. We make the following general observation, which is analogous to Proposition 3 of Sakai [19].

Proposition 1. *Let K be a compact subset of Δ_1 and let $E \subseteq \Omega$. The following are equivalent:*

(a) *For each $f \in C(K)$ and $\varepsilon > 0$ there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $x_1, \ldots, x_m \in E$ such that*

$$\left| f - \sum_{k=1}^m \lambda_k M(x_k, \cdot) \right| < \varepsilon \quad \textit{on } K.$$

(b) *If $h \in h_K(\Omega)$ and $h = 0$ on E, then $h \equiv 0$.*

To see this, let A denote the closure, in $C(K)$, of the set of finite sums of the form $\sum \lambda_k M(x_k, \cdot)$, where $\lambda_k \in \mathbb{R}$ and $x_k \in E$. If condition (a) holds, then $A = C(K)$. Thus, if $M\nu = 0$ on E for some signed measure ν on K, then we have $\nu = 0$, and condition (b) follows. On the other hand, if condition (a) fails, then the Hahn-Banach theorem shows that there is a signed measure ν on K such that $\int f \, d\nu = 0$ for all $f \in A$ yet $\nu \neq 0$, so condition (b) also fails.

We remark that, if $\left(\overline{E}\right)^\circ \neq \emptyset$, then condition (b) holds by the analyticity of harmonic functions. Thus condition (a) is essentially different from the corresponding conditions in Theorems A and B, which depend only the nature of E near ∂D.

6.2. The following questions remain open.

Problem 1. Find an explicit characterization of the sets $E \subseteq D$ which satisfy (3).

Problem 2. Condition (b) of Theorem 4 clearly implies condition (b) of Corollary 2. Under what circumstances are they equivalent? As noted in Section 1, equivalence is known to hold when $\Omega = B$ and $H \equiv 1$.

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